HW1 - CS 555S
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Q1 solution:

Q1:

a. $X \sim N(\mu, \sigma^2)$

i. $F(-3.5)$ when $\mu = 1.0$, $\sigma = 2$

R command: pnorm(-3.5, mean = 1.0, sd = 2)

Ans: 0.00621 $\to \sigma$

ii. $F(0.5)$ when $\mu = -5.0$, sd = 4

R command: pnorm(0.5, mean = -5.0, sd = 4)

Ans: 0.93319

b. $X \sim \chi^2_K$

i. $F(2.5)$ when $K = 15$

R command: pchisq(2.5, df = 15)

Ans: 0.00406

ii. $F(15)$ when $K = 35$

R command: pchisq(15, df = 35)

Ans: 0.00818

c. The CDF, $F(x)$, is the left tail probability: $P(X \leq x)$

The right tail probability is $P(X > x) = 1 - F(x)$

Therefore, CDF + tail probability = 1

Q2 solution:

Problem 6.51

Q2:

a:

$$P(k) = \binom{n}{k} p^{k} (1-p)^{n-k}$$

n → number of trials

p → probability of success.

k → number of success

(a) $n=5$, $p=\frac{1}{3}$, $k=2$

$$P(2) = \binom{5}{2} \left(\frac{1}{3}\right)^2 \left(1-\frac{1}{3}\right)^3 =$$

$$\Rightarrow 10 \cdot \frac{1}{9} \cdot \left(\frac{2}{3}\right)^3$$

$$\Rightarrow \frac{10}{9} \cdot \frac{8}{27} = \frac{80}{243} = \boxed{0.3292}$$

(b) $n=7$, $p=\frac{1}{2}$, $k=3$

$$P(3) = \binom{7}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^4$$

$$\Rightarrow \binom{7}{3} \left(\frac{1}{2}\right)^7 = \frac{35}{128} = \boxed{0.2734}$$

(c) $n=4$, $p=\frac{1}{4}$, $k=2$

$$P(2) = \cancel{\binom{4}{2}} \binom{4}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^2$$

$$= 6 \cdot \frac{1}{16} \cdot \frac{9}{16} = \frac{54}{256} = 0.2109$$

R command: pnorm()

b:

Problem 6.68

a. $P(Z \leq 0.73) = 0.7673$

b. $P(Z \leq 1.8) = 0.9641$

c. $P(Z \geq 0.2) = 1 - P(Z \leq 0.2) = 0.4207$

d. $P(Z \geq -1.5) = 1 - P(Z \leq -1.5) = 0.9332$

e. $P(Z = 1.8) = 0$

f. $P(|Z| \leq 0.25) = \begin{cases} P(Z \leq 0.25) = 0.5987 \\ P(Z \leq -0.25) = 0.4013 \end{cases}$

$\Rightarrow 0.5987 - 0.4013 = \boxed{0.1974}$

Problem 6.67:

a. $P(-0.81 \leq Z \leq 1.13) = P(Z \leq 1.13) - P(Z \leq -0.81)$

$\Rightarrow 0.8708 - 0.2090 = 0.6618$

b. $P(-0.23 \leq Z \leq 1.6) = P(Z \leq 1.6) - P(Z \leq -0.23)$

$\Rightarrow 0.9452 - 0.4090 = 0.5362$

c. $P(0.53 \leq Z \leq 2.03) = P(Z \leq 2.03) - P(Z \leq 0.53)$

$\Rightarrow 0.9788 - 0.7019 = 0.2769$

d. $P(0.15 \leq Z \leq 1.50) = P(Z \leq 1.50) - P(Z \leq 0.15)$

$\Rightarrow 0.9332 - 0.5596 = 0.3736$

Problem 6.71

a. $P(X \leq 100)$

$$Z = \frac{100-155}{20} = \frac{-55}{20} = -2.75$$

$$P(Z \leq -2.75) = 0.0030$$

$$\Rightarrow 2000 \cdot 0.0030 = \boxed{6}$$

b. $P(120 \leq X \leq 130)$

$$Z_1 = \frac{120-155}{20} = -1.75 \qquad P(Z \leq -1.25) = 0.1056$$

$$Z_2 = \frac{130-155}{20} = -1.25 \qquad P(Z \leq -1.75) = 0.0401$$

$$\Rightarrow 0.1056 - 0.0401 = 0.0655 \cdot 2000 = \boxed{131}$$

c. $P(150 \leq X \leq 175)$

$$Z_1 = \frac{150-155}{20} = -0.25 \qquad P(Z \leq 1.0) = 0.8413$$

$$Z_2 = \frac{175-155}{20} = 1.5 \qquad P(Z \leq -0.25) = 0.4013$$

$$\Rightarrow \cancel{0.4013}\ 0.8413 - 0.4013 = 0.44$$

$$\Rightarrow 0.4400 \cdot 2000 = \boxed{880}$$

d. $P(X \geq 200)$

$$Z = \frac{200-155}{20} = 2.25$$

$$P(Z \geq 2.25) = 1 - P(Z \leq 2.25) =$$

$$\Rightarrow 0.0122 \times 2000 = \boxed{24}$$

problem 6.70

a. $P(5 \leq X \leq 10)$

$Z_1 = \frac{5-8}{4} = -0.75$ $P(Z \leq -0.75) = 0.2266$

$Z_2 = \frac{10-8}{4} = \cancel{0.75}\ 0.5$ $P(Z \leq 0.5) = 06915$

$P = 0.6915 - 0.2266$

b. $P(10 \leq X \leq 15)$

$Z_1 = \frac{10-8}{4} = 0.5$ $P(Z \leq 0.5) = 0.6915$

$Z_2 = \frac{15-8}{4} = 1.75$ $P(Z \leq 1.75) = 0.9599$

$P = 0.9599 - 0.6915 = 0.2684$

c. $P(3 \leq X \leq 9)$

$Z_1 = \frac{3-8}{4} = -1.25$ $P(Z \leq -1.25) = 0.1056$

$Z_2 = \frac{9-8}{4} = 0.25$ $P(Z \leq 0.25) = 0.5987$

$P = 0.5987 - 0.1056 = 0.4931$

d. $P(3 \leq X \leq 7)$

$Z_1 = \frac{3-8}{4} = -1.25$ $P(Z \leq -1.25) = 0.1056$

$Z_2 = \frac{7-8}{4} = -0.25$ $P(Z \leq -0.25) = 0.4013$

$P = 0.4013 - 0.1056 = 0.2957$

e. $p(x \geq 15)$

$$z = \frac{15-8}{4} = 1.75$$

$$p(x \geq 15) = 1 - p(z \leq 1.75) = 1 - 0.9599$$

$$\Rightarrow 0.0401$$

f. $p(X \leq 5)$

$$z = \frac{5-8}{4} = -0.75$$

$$p(z \leq -0.75) = 0.2266$$

Q3 solution:

Q3:

we are given $X_1, X_2, X_3, X_4 \sim N(\mu = 10, \sigma^2 = 4)$ and we need to find out:

$$P\left(\sum_{i=1}^{4} (X_i - \mu)^2 \geq 35\right)$$

each $(X_i - \mu)^2 \sim \sigma^2 \cdot \chi^2_1$, and $\sigma^2 = 4$,

$$\sum_{i=1}^{4} (X_i - \mu)^2 \sim 4 \cdot \chi^2_4 = \frac{\text{Sum}}{4} \sim \chi^2_4 = \frac{35}{4} = \boxed{8.75}$$

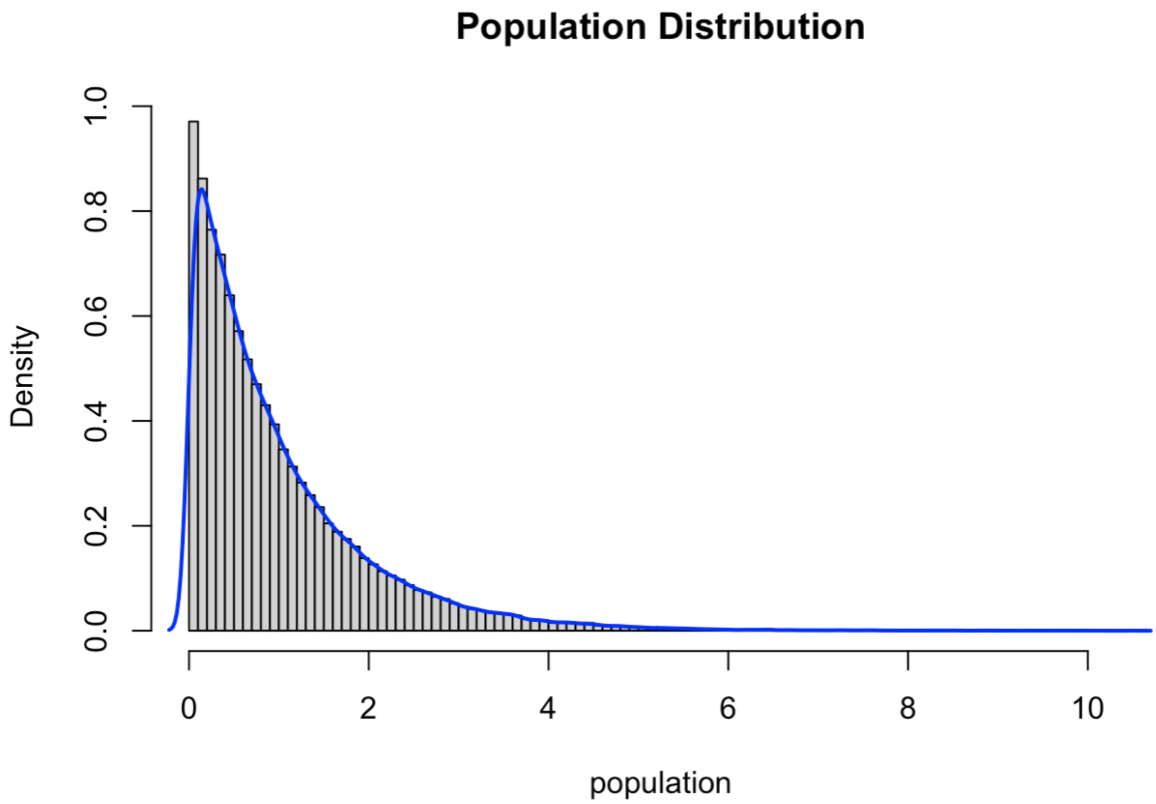
R command: pchisq(8.75, df=4, lower.tail=False)

Ans: 0.0676

There is a chance of 6.76% that the deviation from the mean is as large as 35.0.

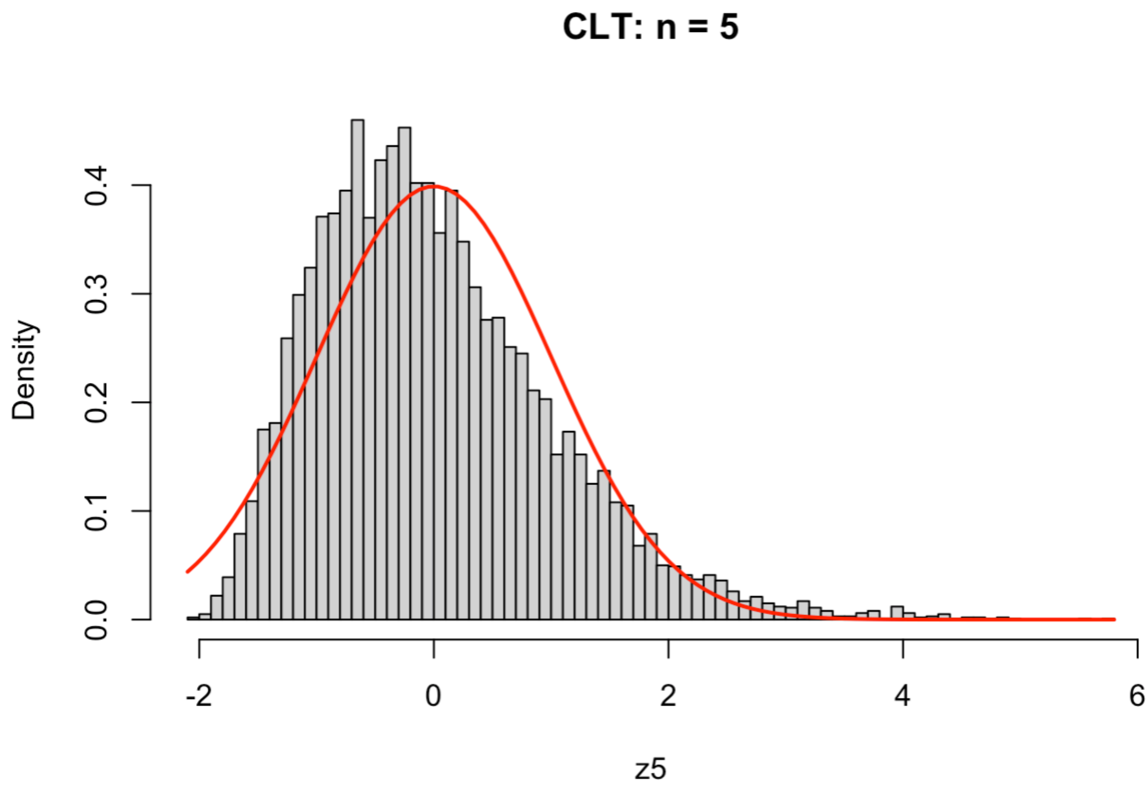
Q5 solution:

```
# a
set.seed(3)
population <- rexp(100000, rate = 1)

# b
hist(population, breaks = 100, probability = TRUE, main = "Population Distribution")
lines(density(population), col = "blue", lwd = 2)
```

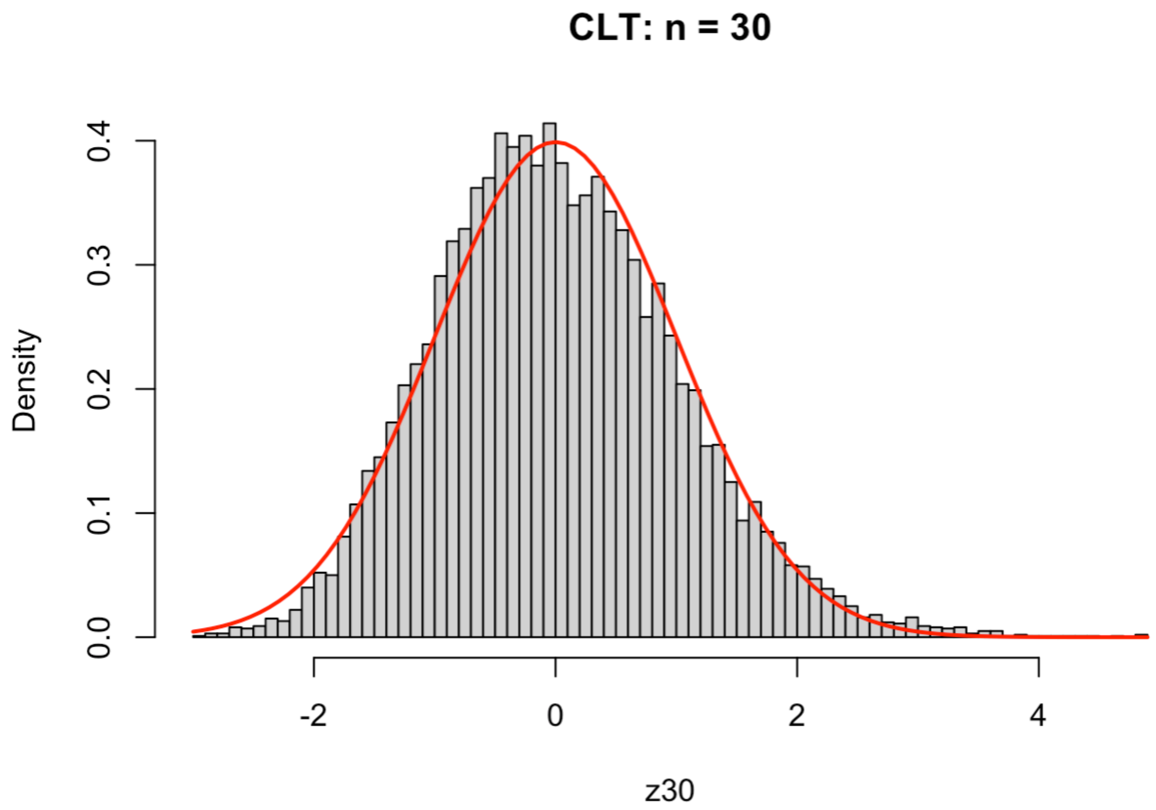



```
# c
mean(population)  # ≈ 1
var(population)   # ≈ 1

# d
get_sample_means <- function(n) {
  replicate(10000, mean(sample(population, n)))
}

means_n5 <- get_sample_means(5)
means_n30 <- get_sample_means(30)
means_n100 <- get_sample_means(100)

# e
standardize <- function(sample_means, n) {
  (sample_means - 1) / (1 / sqrt(n))
}

z5 <- standardize(means_n5, 5)
z30 <- standardize(means_n30, 30)
z100 <- standardize(means_n100, 100)

# Plot for n = 5
hist(z5, breaks = 60, probability = TRUE, main = "CLT: n = 5", col = "lightgray")
```

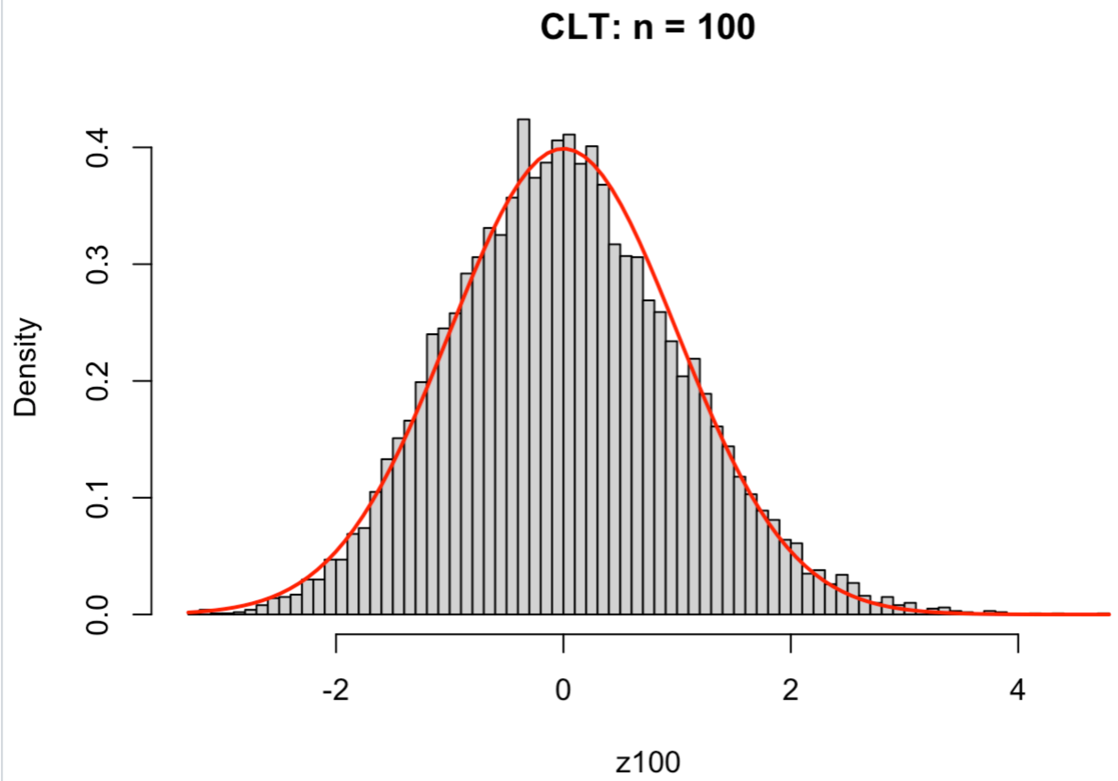
```
curve(dnorm(x), col = "red", lwd = 2, add = TRUE)
```




```
# Plot for n = 30
hist(z30, breaks = 60, probability = TRUE, main = "CLT: n = 30", col = "lightgray")
curve(dnorm(x), col = "red", lwd = 2, add = TRUE)
```




```
# Plot for n = 100
hist(z100, breaks = 60, probability = TRUE, main = "CLT: n = 100", col = "lightgray")
curve(dnorm(x), col = "red", lwd = 2, add = TRUE)
```



Explanation:

As we see the graphs, we get this:

- For n=5: sample means are still skewed
- For n=30: distribution starts to look bell-shaped
- For n=100: the sample means are almost perfectly normal.