# Things you should be able to prove:

These are just some of the things you are responsible to prove.
You are also responsible for all the computational problems and the assigned homework.

## 1. Properties of Matrix Arithmetics

You should know all the properties and be able to prove them. (**page 38**)

## 2. System of equations

- If $\mathbf{A}$ is an $m \times n$ matrix and $x$ is an $n \times 1$ column vector, then the product $\mathbf{A}x$ can be expressed as a linear combination of column vectors of $\mathbf{A}$.
- Two matrices have the same reduced row echelon form if and only if they are row equivalent.
- **(Theorm 1.6.1)** A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.
- **(Problem 1.6.22)** Let $\mathbf{A}x = 0$ be a homogeneous system of $n$ linear equations in $n$ unknowns, and let $Q$ be an invertible $n \times n$ matrix. Show that $Ax = 0$ has just the trivial solution if and only if $(Q\mathbf{A})x = 0$ has just the trivial solution.
- **(Problem 1.6.23)** Let $\mathbf{A}x = b$ be any consistent system of linear equations, and let $x_1$ be a fixed solution. Show that every solution to the system can be wirtten in the form $x = x_1 + x_0$, where $x_0$ is a solution to $\mathbf{A}x = 0$ (the homogeneous system). Show also that every matrix of this form is a solution.
- For a constant $k$ and a matrix $\mathbf{A}$, if $k\mathbf{A} = 0$ then either $k = 0$ or $\mathbf{A} = 0$. (0 is a scalar or a zero matrix of appropriate dimensions.)

## 3. Transpose operation

- If $\mathbf{A}$ and $\mathbf{B}$ are matrices of appropriate dimensions,
  - $(\mathbf{A}^T)^T = \mathbf{A}$.
  - $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.
  - $(\mathbf{A}-\mathbf{B})^T = \mathbf{A}^T - \mathbf{B}^T$.
  - $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$.
  - $(k\mathbf{A})^T = k\mathbf{A}^T$.

## 4. Inverses (Assume all matrices are square and of appropriate dimensions)

- If $\mathbf{A}$ is an invertible matrix,
  - $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
  - $\mathbf{A}^m$ is also invertible and $(\mathbf{A}^m)^{-1} = (\mathbf{A}^{-1})^m = \mathbf{A}^{-m}$.
  - $\mathbf{AB}$ is also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
  - $k\mathbf{A}$ is also invertible and $(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1}$ (note that for us $k^{-1} = \frac{1}{k}$).
- If $\mathbf{B}$ and $\mathbf{C}$ are both inverses of a matrix $\mathbf{A}$ then $\mathbf{B} = \mathbf{C}$. (Uniqueness of the inverse)
- If $\mathbf{A}$ is an idempotent matrix then so is $I - \mathbf{A}$.
- If $\mathbf{A}$ is an idempotent matrix then $(2\mathbf{A} - I)$ is invertible and is its own inverse.
- If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of appropriate dimensions and $\mathbf{BA} = I$ then $\mathbf{B} = \mathbf{A}^{-1}$.

## 5. Symmetry

- if $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices then:
  - $\mathbf{A}^T$ is symmetric.
  - $\mathbf{A} + \mathbf{B}$ is symmetric.
  - $\mathbf{A} - B$ is symmetric.
  - $k\mathbf{A}$ is symmetric.
  - $\mathbf{A}^{-1}$ is symmetric.
  - $\mathbf{AB}$ is symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ commute.
- If $\mathbf{A}^T\mathbf{A} = \mathbf{A}$ then $\mathbf{A}$ is symmetric and $\mathbf{A} = \mathbf{A}^2$.
- If $\mathbf{A}$ is invertible and skew-symmetric then $\mathbf{A}^{-1}$ is skew-symmetric.
- If $\mathbf{A}$ and $\mathbf{B}$ are skew-symmetric then so are $\mathbf{A}^T$, $\mathbf{A} + \mathbf{B}$, $\mathbf{A} - \mathbf{B}$, and $k\mathbf{A}$.
- Every square matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

## 6. Determinant

- If $\mathbf{A}$ is a square matrix and $\mathbf{A}'$ is obtained from $\mathbf{A}$ by switching two row of $\mathbf{A}$ then we have $\det(\mathbf{A}') = -\det(\mathbf{A})$.
  - Note: I know there are probably easier proofs for this theorem but I take this opportunity to use induction proof.

    Proof: We prove by induction on the dimension of $\mathbf{A}$. It is clear that the property holds for every $2 \times 2$ matrix (check). Suppose for all $k \times k$ matrix $\mathbf{B}$, $k \geq 2$, we have $\det(\mathbf{B}') = -\det(\mathbf{B})$ where $\mathbf{B}'$ is obtained from $\mathbf{B}$ by switching two rows. Let $\mathbf{A}'$ be a matrix obtained by switching rows $i$ and $j$ of $\mathbf{A}$. Then the determinat of $\mathbf{A}'$ can be obtained by cofactor expansion of $\mathbf{A}'$ along the $p$-th row where $p \neq i, j$. We have:

    $$\det(\mathbf{A}') = (-1)^{p+1} a_{p\,1} M'_{p\,1} + (-1)^{p+2} a_{p\,2} M'_{p\,2} + \cdots + (-1)^{p+k+1} a_{p\,k+1} M'_{p\,k+1}$$

    Expanding $\mathbf{A}$ along the same row we get:

    $$\det(\mathbf{A}) = (-1)^{p+1} a_{p\,1} M_{p\,1} + (-1)^{p+2} a_{p\,2} M_{p\,2} + \cdots + (-1)^{p+k+1} a_{p\,k+1} M_{p\,k+1}.$$

    Minors $M'_{p\,q}$ and $M_{p\,q}$ correspond to the same $k \times k$ submatrices where one is obtained by switiching two of the rows of the other. Hence by induction hypothesis, $M'_{p\,q} = -M_{p\,q}$ and

    $$\begin{aligned}\det(\mathbf{A}') &= (-1)^{p+1} a_{p\,1} M'_{p\,1} + (-1)^{p+2} a_{p\,2} M'_{p\,2} + \cdots + (-1)^{p+k+1} a_{p\,k+1} M'_{p\,k+1} \\ &= (-1)^{p+1} a_{p\,1}(-M_{p\,1}) + (-1)^{p+2} a_{p\,2}(-M_{p\,2}) + \cdots + (-1)^{p+k+1} a_{p\,k+1}(-M_{p\,k+1}) \\ &= -\det(\mathbf{A}).\end{aligned}$$

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- $\det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ ka_i \\ \vdots \\ a_n \end{bmatrix}\right) = k \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}\right)$ for all $i$.
- If $\mathbf{A} \in \mathcal{M}_{n\times n}$ then $\det(k\mathbf{A}) = k^n \det(\mathbf{A})$.
- If one row of a square matrix $\mathbf{A}$ is a nonzero multiple of another row of $\mathbf{A}$ then $\det(\mathbf{A}) = 0$.
- $\det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i + b_i \\ \vdots \\ a_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix}\right) + \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ b_i \\ \vdots \\ a_n \end{bmatrix}\right)$ where $a_i, b_i$ are $n$-dimensional row.
- $\det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_j + ka_i \\ \vdots \\ a_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \\ \vdots \\ a_n \end{bmatrix}\right)$ (Basically the third type elementary row operation does not change the determinant).

- For every square matrix $\mathbf{A}$ and elementary matrix $E$ $\det(E\mathbf{A}) = \det(E)\det(\mathbf{A})$.
- For every square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimensions, $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.
- If $\mathbf{A}$ is invertible, $\det(\mathbf{A}^{-1}) = \dfrac{1}{\det(\mathbf{A})}$.
- Product of a row and the column vector of the cofactors of a different row is zero.
- for an invertible matrix $\mathbf{A}$, $\mathbf{A}^{-1} = \dfrac{\operatorname{adj}(\mathbf{A})}{\det(\mathbf{A})}$.

## 7. Euclidean Vector Spaces

- (Paralellogram Equation) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then

$$||\mathbf{u}+\mathbf{v}||^{\mathbf{2}} + ||\mathbf{u}-\mathbf{v}||^{\mathbf{2}} = \mathbf{2}\left(||\mathbf{u}||^{\mathbf{2}} + ||\mathbf{v}||^{\mathbf{2}}\right)$$

- If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices then

$$\mathbf{Au} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{A}^T\mathbf{v}$$
$$\mathbf{u} \cdot \mathbf{Av} = \mathbf{A}^T\mathbf{u} \cdot \mathbf{v}$$

- Orthogonal decomposition of a vector (find the orthogonal and projection components of a vector with respect to a given vector)
- If $\mathbf{A}$ is an $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.
- The general solution of a consistent linear system $\mathbf{Ax} = \mathbf{b}$ can be obtained by adding any specific solution of $\mathbf{Ax} = \mathbf{b}$ to the general solution of $\mathbf{Ax} = \mathbf{0}$.

## 8. Vector Spaces

- Real-valued functions that are defined an $\mathbb{R}$ form a vector space with addition and scalar multiplication of real numbers.
- The finite intersection of subspaces of a vector space is a subspace of that vector space. The union of subspaces is not necessarily a subspace.
- Span of vectors in a vector space is the smallest subspace of that space.
- The solution set of a homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ in $n$ unknowns is a subspace of $\mathbb{R}^n$.
- If $S = \{\mathbf{v}_1.\mathbf{v}_2, \ldots \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space $V$, then

$$\operatorname{span}\{\mathbf{v}_1.\mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

  if and only if each vector in $S$ is a linear combination of those in $S'$, and each vector in $S'$ is a linear combination of those in $S$.
- A set $S$ with two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is expressible as a linear combination of the other vectors in $S$.
- If Wronskian of $n$ functions is not identically zero, then those functions form a linearly independent set of vectors in $\mathbf{C}^{n-1}(-\infty, \infty)$. (no conclusion if the Wronskian IS identically zero).

- Any set of vectors that has a linearly dependent subset is linearly dependent.
- Any subset of a linearly independent set of vectors is linearly independent.
- (Uniqueness of Basis Representation) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space $V$, then every vector $\mathbf{v}$ in $V$ can be represented in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.
- (no proof) If a vector space has a basis with $n$ vectors, then any set with more than $n$ vectors is linearly dependent and any set with less than $n$ vectors does not span that vector space.
- (no proof) All bases for a finite-dimensional vector space have the same number of vectors.
- (Plus/Minus Theorem, no proof) If $S$ is a linearly dependent set of vectors in a vector space $V$, then we can remove a vector in $S$ that can be expressed as a linear combination of other vectors in $S$ and the span of the resulting set is the same as the span of $S$. On the other hand, if $S$ is a linearly independent set of vectors in $V$ then the union of $S$ and a vector from $V$ that is not in the span of $S$ is also linearly independent.
- If $V$ is an $n$-dimensional vector space and $S$ is a subset of $V$ with exactly $n$ vectors, then $S$ is a basis for $V$ if and only if either $S$ is linearly independent or $S$ spans $V$.
- (no proof) A linearly dependent finite set of vectors in a vector space $V$ that spans $V$ can be made into a basis for $V$ by removing appropriate vectors. A linearly independent finite set of vectors in a vector space $V$ can be made to a basis for $V$ by inserting appropriate vectors from $V$.
- (no proof) If $W$ is a subspace of a finite dimensional vector space $V$, $\dim(W) \leq \dim(V)$ with equality if and only if $W = V$.
- (no proof) If $\mathbf{P}$ is the transition matrix from a basis $\mathbf{B'}$ to a basis $\mathbf{B}$ for a finite dimensional vector space $V$, then $\mathbf{P}$ is invertible and $\mathbf{P^{-1}}$ is the transition matrix from $\mathbf{B}$ to $\mathbf{B'}$.
- (no proof) A system of linear equations $\mathbf{Ax} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of $\mathbf{A}$.
- (no proof) Elementary row operations do not change the null space and row space of a matrix but they do change the column space.
- (no proof) If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then a given set of column vectors of $\mathbf{A}$ is linearly independent if and only if the corresponding column vectors of $\mathbf{B}$ are linearly independent. Also, a given set of column vectors of $\mathbf{A}$ form a basis for the column space of $\mathbf{A}$ if and only if the corresponding column vectors of $\mathbf{B}$ form a basis for the column space of $\mathbf{B}$.
- (no proof) If a matrix $\mathbf{R}$ is in ref, the row vectors and column vectors with the leading 1's form a basis for the row and column spaces of $\mathbf{R}$, respectively.
- (Dimension Theorem for Matrices) If $\mathbf{A}$ is a matrix with $n$ columns, then $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n$.
- Let $\mathbf{A}$ be an $m \times n$ matrix
  - (a) (Overdetermined Case) If $m > n$ then the linear system $\mathbf{Ax} = \mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in $\mathbb{R}^m$. (Correction: your book is wrong; it says $\mathbb{R}^n$ which is not correct)
  - (b) (Underdetermined Case) If $m < n$, then for each vector $\mathbf{b}$ in $\mathbb{R}^m$ the linear system $\mathbf{Ax} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.
- If $\mathbf{A}$ is any matrix, then $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$.

- If $\mathbf{A}$ is an $m \times n$ matrix, then
  (a) The null space of $\mathbf{A}$ and the row space of $\mathbf{A}$ are orthogonal complements in $\mathbb{R}^n$.
  (b) The null space of $\mathbf{A}^T$ and the column space of $\mathbf{A}$ are orthogonal complements in $\mathbb{R}^m$.

## 9. Transformations

- If $T_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ and $T_{\mathbf{B}} : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and if $T_{\mathbf{A}}(\mathbf{x}) = \mathbf{T_B}(\mathbf{x})$ for every vector $\mathbf{x}$ in $\mathbb{R}^n$, then $\mathbf{A} = \mathbf{B}$.
- Be able to describe and use the reflection, projection, rotation, contraction, dilation, and shear from $\mathbb{R}^2$ to $\mathbb{R}^2$ only.
- Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation, and conversely, every matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation.

## 10. Eigenvalues and Eigenvectors

- (No proof) If $\mathbf{A}$ is an $n \times n$ matrix, TFAE:
  (a) $\lambda$ is an eigenvalue of $\mathbf{A}$.
  (b) The system of equations $\det(\lambda I - \mathbf{A}) = 0$ has nontrivial solutions.
  (c) There is a nonzero vector $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.
  (d) $\lambda$ is a solution of the characteristic equation $\det(\lambda I - \mathbf{A}) = 0$.
- (No proof) If $\mathbf{A}$ is an $n \times n$ triangular matrix (upper, lower, or diagonal) then the eigenvalues of $\mathbf{A}$ are the entries on the main diagonal of $\mathbf{A}$.
- If $k$ is a positive integer, $\lambda$ is an eigenvalue of a matrix $\mathbf{A}$, and $\mathbf{x}$ is a corresponding eigenvector, then $\lambda^k$ is an eigenvalue of $\mathbf{A}^k$ and $\mathbf{x}$ is a correspoinding eigenvector.
- If $\mathbf{A}$ is an $n \times n$ matrix, the following statements are equivalent.
  (a) $\mathbf{A}$ is diagonalizable.
  (b) $\mathbf{A}$ has $n$ linrearly independent eigenvectors.
- (No proof) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a matrix $\mathbf{A}$ corresponding to distinct eigenvalues, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent set.
- (Use above theorem to prove) If an $n \times n$ matrix $\mathbf{A}$ has $n$ distict eigenvalues, then $\mathbf{A}$ is diagonalizable.
- (No proof) If $\mathbf{A}$ is a square matrix, then:
  (a) For every eigenvalue of $\mathbf{A}$, the geometric multiplicity is less than or equal to the algebraic multiplicity.
  (b) $\mathbf{A}$ is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity.

## 11. Inner product

- If $\mathbf{u}$ and $\mathbf{v}$ are vectors in a real inner product space $V$, then

$$| < \mathbf{u}, \mathbf{v} > | \leq ||\mathbf{u}||||\mathbf{v}||.$$

- If $W$ is a subspace of an inner product space$V$, then:
  (a) $W^{\perp}$ is a subspace of $V$.
  (b) $W \cap W^{\perp} = \{\mathbf{0}\}$.
- (No proof) If $W$ is a subspace of a finite-dimensional inner product space $V$, then the orthogonal complement of $W^{\perp}$ is $W$; that is
$$(W^{\perp})^{\perp} = W.$$
- For every inner product space $V$, if $\mathbf{w}$ is orthogonal to each of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_r$ then it is orthogonal to every vector in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_r\}$.
- If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a basis for an inner product space $V$, then zero vector is the only vector in $V$ that is orthogonal to all of the basis vectors.
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then $S$ is linearly independent.
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space $V$, and if $\mathbf{u}$ is any vector in $V$, then
$$\mathbf{u} = \frac{< \mathbf{u}, \mathbf{v}_1 >}{||\mathbf{v}_1||^2}\mathbf{v}_1 + \frac{< \mathbf{u}, \mathbf{v}_2 >}{||\mathbf{v}_2||^2}\mathbf{v}_2 + \cdots + \frac{< \mathbf{u}, \mathbf{v}_n >}{||\mathbf{v}_n||^2}\mathbf{v}_n.$$
- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space $V$, and if $\mathbf{u}$ is any vector in $V$, then
$$\mathbf{u} =< \mathbf{u}, \mathbf{v}_1 > \mathbf{v}_1 + < \mathbf{u}, \mathbf{v}_2 > \mathbf{v}_2 + \cdots + < \mathbf{u}, \mathbf{v}_n > \mathbf{v}_n.$$
- Every nonzero finite-dimensional inner product space has an orthonormal basis.
- If $W$ is a finite-dimensional inner product space, then
  (a) Every orthogonal set of nonzero vectors in $W$ can be enlarged to an orthogonal basis for $W$.
  (b) Every orthonormal set in $W$ can be enlarged to an orthonormal basis for $W$.
- (QR-Decomposition) If $\mathbf{A}$ is an $m \times n$ matrix with linearly independent column vectors, then $\mathbf{A}$ can be factored as
$$\mathbf{A} = \mathbf{QR}$$
  where $\mathbf{Q}$ is an $m \times n$ matrix with orthonormal column vectors, and $\mathbf{R}$ is an $n \times n$ invertible upper triangular matrix.
- (Best Approximation Theorem) If $W$ is a finite-dimensional subspace of an inner product space $V$, and if $\mathbf{b}$ is a vector in $V$, then $\text{proj}_W \mathbf{b}$ is the best approximation to $\mathbf{b}$ from $W$ in the sense that
$$||\mathbf{b} - \text{proj}_W \mathbf{b}|| < ||\mathbf{b} - \mathbf{w}||$$
  for every vector $\mathbf{w}$ in $W$. that is different from $\text{proj}_W \mathbf{b}$.

- (No proof) For every linear system $\mathbf{Ax} = \mathbf{b}$, the associated normal system

  $$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$$

  is consistent, and all solutions of it are least squares solutions of $\mathbf{Ax} = \mathbf{b}$. Moreover, if $W$ is the column space of $\mathbf{A}$, and $\mathbf{x}$ is any least squares solution of $\mathbf{Ax} = \mathbf{b}$, then the orthogonal prijection of $\mathbf{b}$ on $W$ is $\text{proj}_W\mathbf{b} = \mathbf{Ax}$.

- If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}$ has linearly independent column vectors if and only if $\mathbf{A}^T\mathbf{A}$ is invertible.

- If $A$ is an $m \times n$ matrix with linearly independent column vectors, and if $\mathbf{A} = \mathbf{QR}$ is a **QR**-decomposition of $\mathbf{A}$, then for each $\mathbf{b}$ in $\mathbb{R}^m$ the system $\mathbf{Ax} = \mathbf{b}$ has a unique least squares solution given by $\mathbf{x} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{b}$.

## 12. Equivalence Statements

If $\mathbf{A}$ is an $n \times n$ matrix, then TFAE:

(a) $\mathbf{A}$ is invertible.

(b) $\mathbf{A}x = 0$ has only the trivial solution.

(c) The rref of $\mathbf{A}$ is $I_n$.

(d) $\mathbf{A}$ is expressible as a product of elementary matrices.

(e) $\mathbf{A}x = b$ is consistent for every $n \times 1$ matrix $b$.

(f) $\mathbf{A}x = b$ has exactly one solution for every $n \times 1$ matrix $b$.

(g) $\det(\mathbf{A}) \neq 0$.

(h) The column vectors of $\mathbf{A}$ are linearly independent.

(i) The row vectors of $\mathbf{A}$ are linearly independent.

(j) The column vectors of $\mathbf{A}$ span $\mathbb{R}^n$

(k) The row vectors of $\mathbf{A}$ span $\mathbb{R}^n$

(l) The column vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^n$

(m) The row vectors of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

(n) $\mathbf{A}$ has rank $n$.

(o) $\mathbf{A}$ has nullity 0.

(p) The orthogonal complement of the null space of $\mathbf{A}$ is $\mathbb{R}^n$.

(q) The orthogonal complement of the row space of $\mathbf{A}$ is $\{\mathbf{0}\}$.

(r) The range of $T_{\mathbf{A}}$ is $\mathbb{R}^n$.

(s) $T_{\mathbf{A}}$ is one-to-one.

(t) $\lambda = 0$ is not an eigenvalue of $\mathbf{A}$.

(t) $\mathbf{A}^T\mathbf{A}$ is invertible.

# Things you should know or be able to calculate:

- Gaussian and Gauss-Jordan elimination.
- Row echelon and reduced row echelon form.
- Solve a system using elimination.
- Find the general solution.
- Understand and be able to use the following theorems:
  - If a homogeneous linear system has $n$ unknowns and if rref of its augmented matrix has $r$ nonzero rows then the system has $n-r$ free variables.
  - A homogeneous linear system of more unknowns than equations has infinitely many solutions.
- Different representations of matrix multiplication, $\mathbf{A}_{m\times p}\mathbf{B}_{p\times n}$:
  - $(\mathbf{AB})_{ij} = \sum_{r=1}^{p}(\mathbf{A})_{ir}(\mathbf{B})_{rj}$.
  - If $\mathbf{B} = [b_1 b_2 \dots b_n]$, $\mathbf{AB} = [\mathbf{A}b_1 \mathbf{A}b_2 \dots \mathbf{A}b_n]$.
  - If $\mathbf{A} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$, $\mathbf{AB} = \begin{bmatrix} a_1\mathbf{B} \\ a_2\mathbf{B} \\ \vdots \\ a_m\mathbf{B} \end{bmatrix}$.
- Know the concept of linear combination .
- Find the transpose and the trace.
- Invertible and singular matrices .
- Find the inverse by Gauss-Jordan elimination.
- Find the inverse using the formula.
- Solve a system using inverse of the coefficient matrix.
- Calculate polynomials of matrices.
- Elementary matrices and their inverses.
- Write a matrix as product of elementary matrices.
- Row equivalent matrices.
- Find column vectors $\mathbf{b}$ for which the system $\mathbf{Ax} = \mathbf{b}$ is consistent.
- Definition and properties of a diagonal matrix.
- Multiplication of a matrix by a diagonal matrix from left (multiplies the rows) and right (multiplies the columns).
- Power and inverse of a diagonal matrix.

- Definition and properties of triangular matrices.
- Definition and properties of symmetric matrices.
- Formal definition of determinant of a matrix.
- Determinant of a $2 \times 2$ and $3 \times 3$ matrix using the arrow method.
- Determinant using the cofactor expansion.
- Properties of determinants.
- Adjoint of a square matrix $\mathbf{A}$ (the matrix of cofactors).
- Cramer's rule.
- Adding and subtracting vectors (parallelogram and triangle rule), and scalar multiplication.
- Components and equivalence of vectors
- Algebraic operations and linear combination of vectors
- Definition of a norm, Euclidean norm, distance, unit vector, and normalizing a vector
- Dot product (Euclidean inner product)
- Angle between two vectors
- Orthogonal vectors and the dot product
- Properties of the dot product (thm 3.2.2, 3.2.3)
- Cauchy-Schwarz Inequaliy
- Paralellogram Equation
- Equation of a line and a plane using the normal vector
- The distance of a point from a line or a plane, and the distance between two parallel lines or planes.
- Vector and parametric equastions of a line in $\mathbb{R}^2$ and $\mathbb{R}^3$ and a plane in $\mathbb{R}^3$.
- Interpretation of the solution to a linear system of equatios with infinitely many solutions as a particular solution point plus the solution space to the homogeneous system.
- Cross product and its properties (thm 3.5.1, 3.5.2)
- Find the area of a parallelogram and the volume of a parallelopiped using the cross product and the scalar triple product, respectively.
- The definition of a vector space , its 10 properties and verifying those properties for a particular set of objects and given operations.
- Examples of sets with operations that do not form a vector space.
- Definition of a subspace.
- Verify a subset of a given vector space is a subspace (check closures).
- The two subspaces every space has and subspaces of $\mathbb{R}^n$.

- A subset of $M_{2\times 2}$ that is NOT a subspace.
- An example to show that finite union of subspaces of a vector space is NOT a subspace.
- Check if a set of vectors spans a particular vector space.
- Definition of linear independence (that's the one on page 191 of your book).
- Verify the linear independence of a set of vectors in a vector space.
- The definition and calculation of Wronskian of $n$ functions.
- Definition of a basis for a vector space.
- Verify that a set of vectors form a basis for a vector space.
- Having a basis for a vector space, find the coordinates of any vector in that vector space relative to that basis.
- The definition of the dimention of a vector space
- Finding a basis and dimention of a solution space
- Given two different bases of a vector space, find the transition matrix from one basis to the other
- Find the components of a vector relative to one basis given its components relative to a differnt basis
- Row space, column space, null space
- Use row operations to find a basis for the row space, column space, null space
- Pick row vectors of a matrix to form a basis for the row space of that matrix.(Note that this is different form the last one)
- Find a vector form of the general solution of $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Ax} = \mathbf{b}$.
- Find a basis for the null space of a matrix.
- Find a basis for a subspace spanned by a given set of vectors.
- Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$, find a subset of these vectors that form a basis for span($S$), and express those vectors that are not in that basis as a linear combination of the basis vectors.
- What is the rank and the nullity.
- Use the dimension theorem to find the number of parameters in a general solution $\mathbf{Ax} = \mathbf{0}$.
- Fundamental spaces.
- Orthogonal complement of a subspace.
- Linear transformations and matrix transformations and their properties.
- Domain, codomain, and range of a transformation.
- Find the standard matrix for a matrix transformation.

- Composition of two or more transformations.
- Commutativity of the composition of transformations.
- Invertibility and one-to-oneness of transformations.
- Definition of eigenvalue, eigenvector, characteristic equation, characteristic polynomial.
- Find eigenvalues and a basis for the eigenspace associated with each eigenvalue.
- Similar matrices, similarity transformation, and similarity invariant properties of matrices (Table 1, pg 306).
- Geometric and algebraic multiplicity of an eigenvalue.
- Properties of an inner product and a norm defined on a vector space.
- Verify a function is an inner product.
- Different examples of inner products (Euclidean, weighted, matrix, and defined on $\mathbf{C}([a, b])$ using definite integrals).
- Define a norm using an inner product.
- Find the norm of a vector given an inner product.
- The angle between two vectors in an inner product space.
- Orthogonality of two vectors.
- Find a basis for the orthogonal complement of a subspace.
- Given a basis for a vector space, use Gram-Schmidt process to find an orthonormal basis for that vector space.
- Find the projection of a vector on a subspace given an orthogonal basis for that subspace.
- What is meant by best approximation
- Find the least squares solution
- Use the concept of finding the least squares solution to find the projection of a vector onto a space.
- Find the matrix of the projection map using $[P] = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ (Example 3 in 6.4)
- Given a set of points, be able to find the best fitting linear, quadratic, and cubic function.