

# Lecture 8: Planarity

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# Motivation

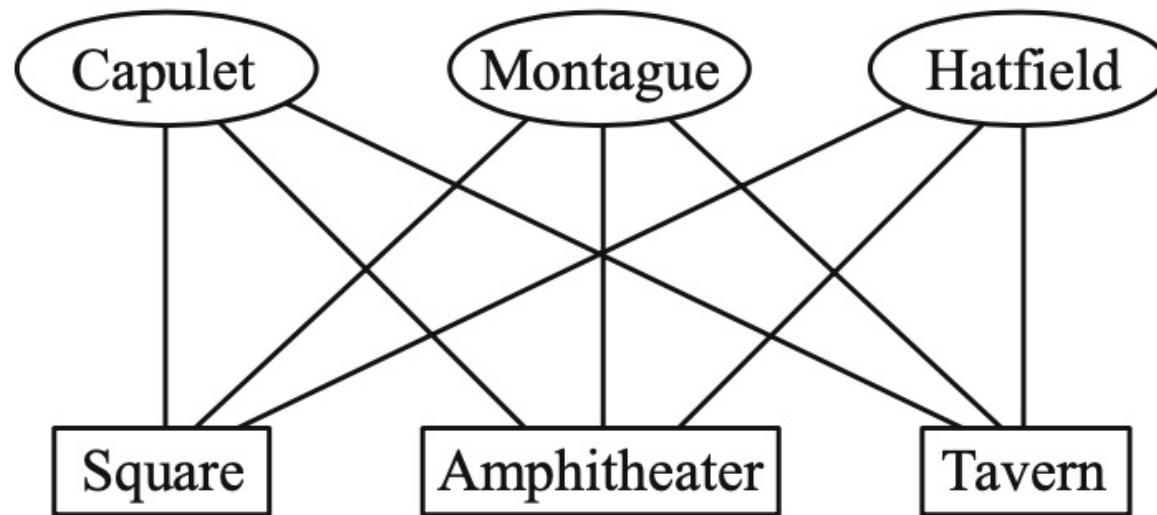


FIGURE 1.72. Original routes.

# Definition and examples

- A graph  $G$  is said to be **planar** if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices
- If  $G$  has no such representation,  $G$  is called **nonplanar**
- A drawing of a planar graph  $G$  in the plane in which edges intersect only at vertices is called a **planar representation** (or a planar embedding) of  $G$

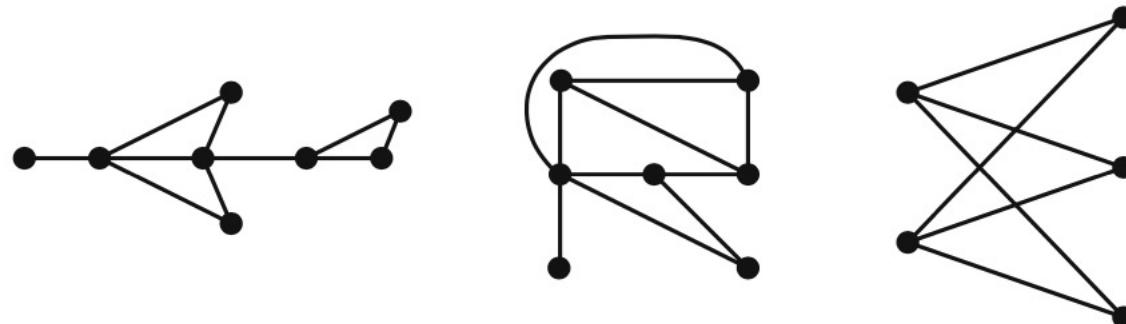
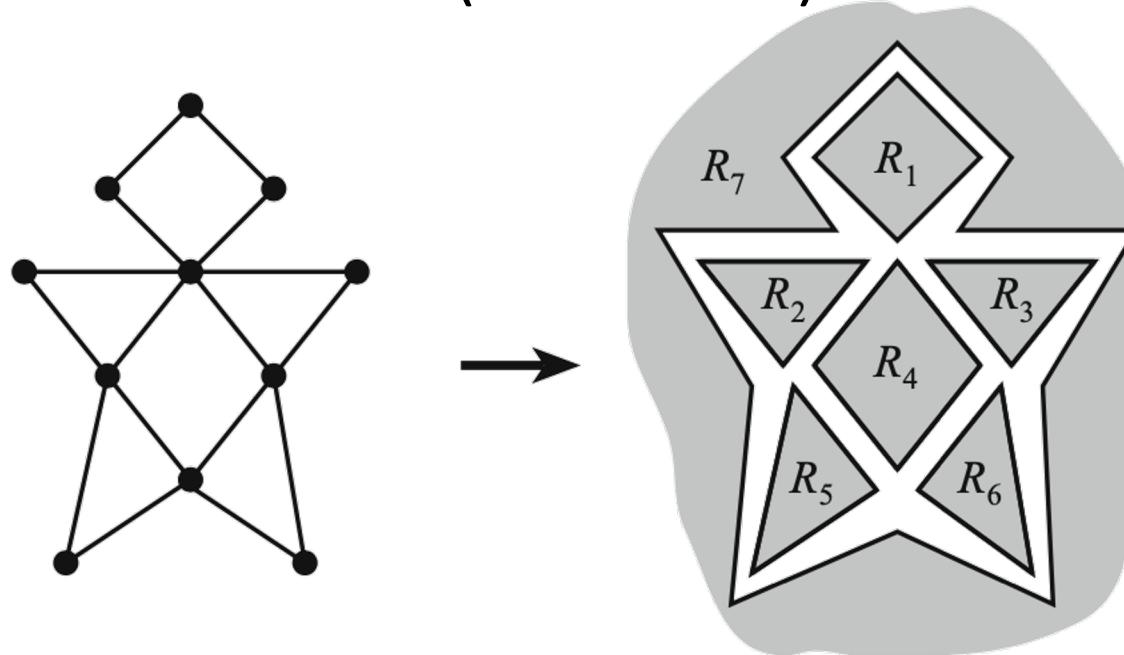


FIGURE 1.73. Examples of planar graphs.

# Face

- Given a planar representation of a graph  $G$ , a **face** is a maximal region (polygonal open set) of the plane in which any two points can be joined by a curve that does not intersect any part of  $G$
- The face  $R_7$  is called the **outer** (or exterior) face



# Face - properties

- An edge can come into **contact** with either one or two faces
- Example:
  - Edge  $e_1$  is only in contact with one face  $S_1$
  - Edge  $e_2, e_3$  are only in contact with  $S_2$
  - Each of other edges is in contact with two faces
- An edge  $e$  **bounds** a face  $F$  if  $e$  comes into contact with  $F$  and with a face **different** from  $F$
- The **bounded degree**  $b(F)$  is the number of edges that bound the face
  - Example:  $b(S_1) = b(S_3) = 3, b(S_2) = 6$

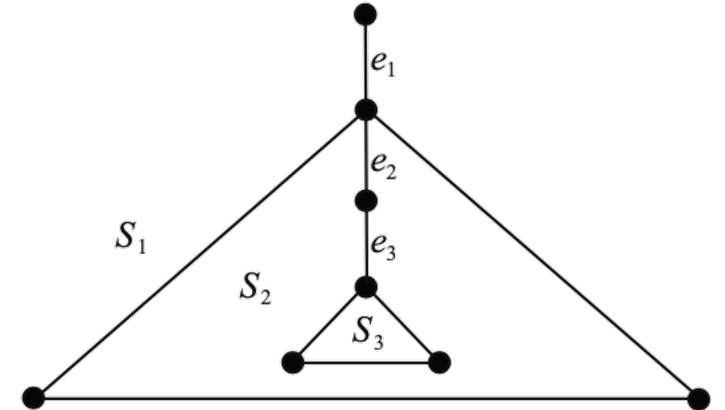


FIGURE 1.76. Edges  $e_1, e_2$ , and  $e_3$  touch one face only.

# Face - properties 2

- The **length** of a face in a plane graph  $G$  is the total length of the closed walk(s) in  $G$  bounding the face
- **Proposition** (6.1.13, W) If  $l(F)$  denotes the length of face  $F$  in a plane graph  $G$ , then  $2|E(G)| = \sum l(F_i)$
- **Theorem** (Restricted Jordan Curve Theorem) A simple closed polygonal curve  $C$  consisting of finitely many segments partitions the plane into exactly two faces, each having  $C$  as boundary

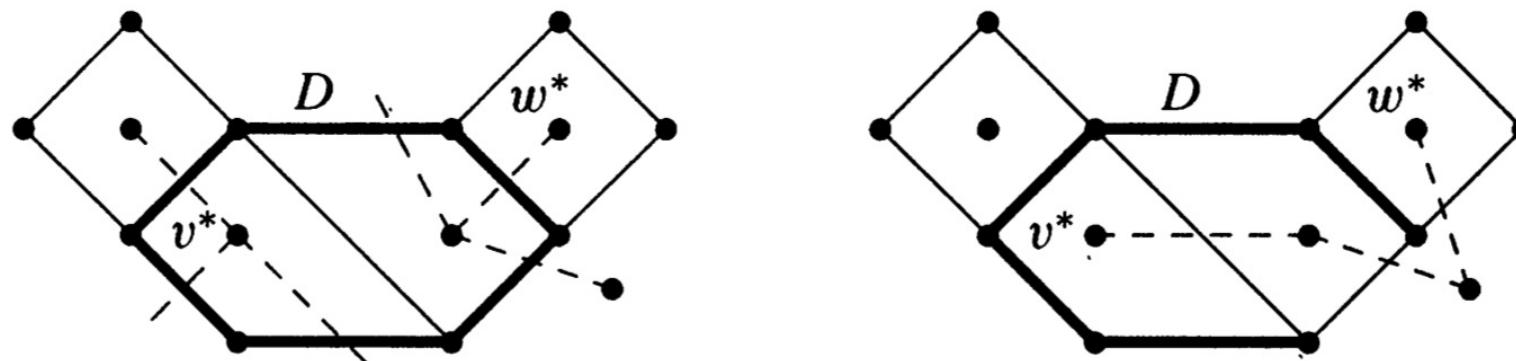
# Bond

- An edge cut may contain another edge cut
- Example:  $K_{1,2}$  or star graphs
- A **bond** is a minimal nonempty edge cut
- **Proposition** (4.1.15, W) If  $G$  is a connected graph, then an edge cut  $F$  is a bond  $\Leftrightarrow G - F$  has exactly two components



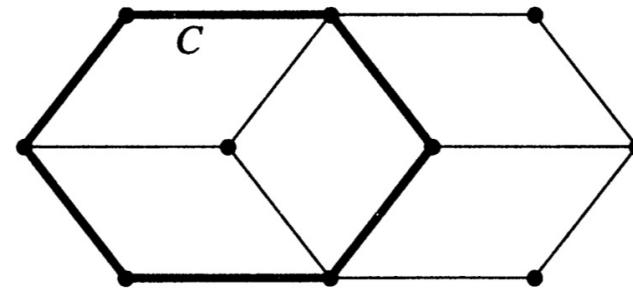
# Dual graph

- The **dual graph**  $G^*$  of a plane graph  $G$  is a plane graph whose vertices are faces of  $G$  and edges are those contacting two faces
- **Theorem** (6.1.14, W) Edges in a plane graph  $G$  form a cycle in  $G \Leftrightarrow$  the corresponding dual edges form a bond in  $G^*$



# Dual graph of bipartite graph

- **Theorem** (6.1.16, W) TFAE
  - (a)  $G$  is bipartite
  - (b) Every face of  $G$  has even length
  - (c) The dual graph  $G^*$  is Eulerian

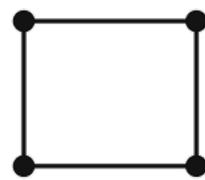


**Theorem** (1.2.18, W, König 1936)

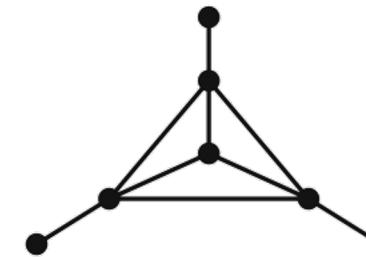
A graph is bipartite  $\Leftrightarrow$  it contains no odd cycle

# The relationship between numbers of vertices, edges and regions

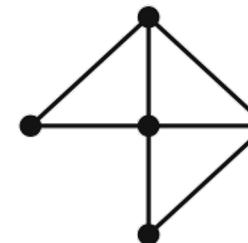
- The number of vertices  $n$
- The number of edges  $m$
- The number of faces  $f$



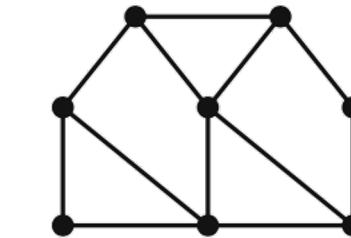
$$\begin{aligned}n &= 4 \\m &= 4 \\f &= 2\end{aligned}$$



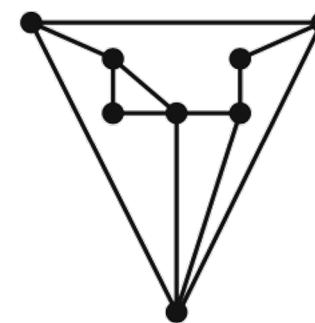
$$\begin{aligned}n &= 7 \\m &= 9 \\f &= 4\end{aligned}$$



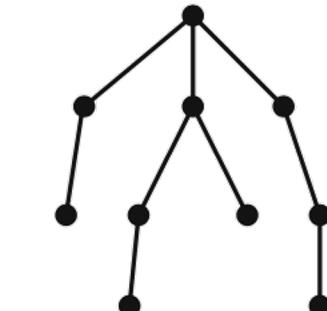
$$\begin{aligned}n &= 5 \\m &= 10 \\f &= 2\end{aligned}$$



$$\begin{aligned}n &= 8 \\m &= 12 \\f &= 6\end{aligned}$$



$$\begin{aligned}n &= 8 \\m &= 12 \\f &= 6\end{aligned}$$



$$\begin{aligned}n &= 10 \\m &= 9 \\f &= 1\end{aligned}$$

# Euler's formula

- **Theorem** (1.31, H; 6.1.21, W; Euler 1758) If  $G$  is a connected planar graph with  $n$  vertices,  $m$  edges, and  $f$  faces, then

$$n - m + f = 2$$

- Need Lemma: (Ex4, S1.5.1, H) Every tree is planar
- (Ex6, S1.5.2, H) Let  $G$  be a planar graph with  $k$  components. Then

$$n - m + f = k + 1$$

$K_{3,3}$  is nonplanar

- **Theorem (1.32, H)**  $K_{3,3}$  is nonplanar

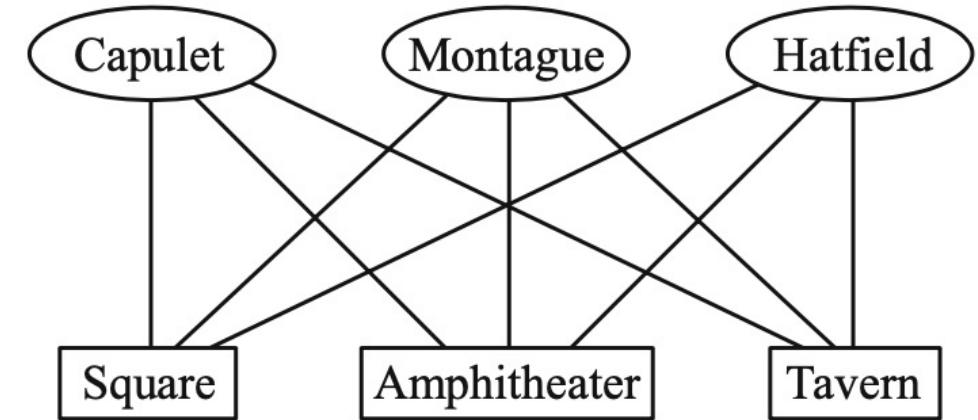


FIGURE 1.72. Original routes.

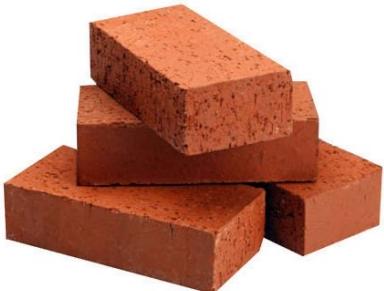
# Upper bound for $m$

- **Theorem** (1.33, H; 6.1.23, W) If  $G$  is a planar graph with  $n \geq 3$  vertices and  $m$  edges, then  $m \leq 3n - 6$ . Furthermore, if equality holds, then every face is bounded by 3 edges. In this case,  $G$  is maximal
- (Ex4, S1.5.2, H) Let  $G$  be a connected, planar,  $K_3$ -free graph of order  $n \geq 3$ . Then  $G$  has no more than  $2n - 4$  edges
- **Corollary** (1.34, H)  $K_5$  is nonplanar
- **Theorem** (1.35, H) If  $G$  is a planar graph , then  $\delta(G) \leq 5$
- (Ex5, S1.5.2, H) If  $G$  is bipartite planar graph, then  $\delta(G) < 4$

# Polyhedra

# (Convex) Polyhedra 多面体

- A **Polyhedron** is a solid that is bounded by flat surfaces



# Polyhedra are planar

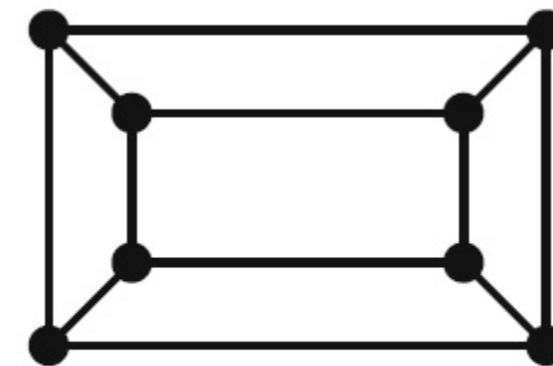
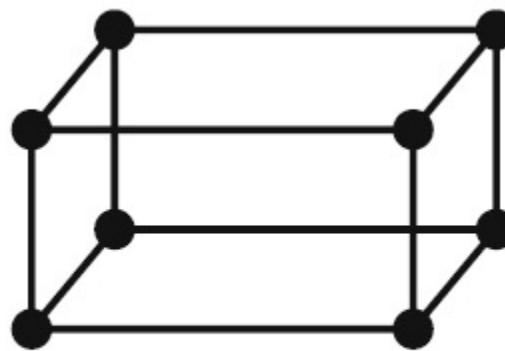
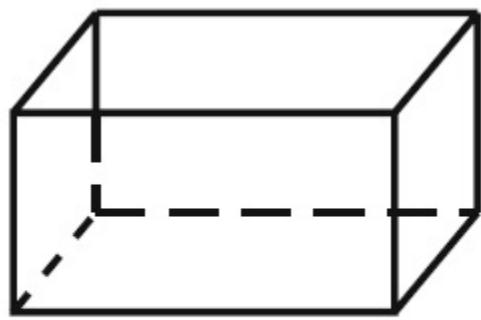


FIGURE 1.81. A polyhedron and its graph.

# Properties

- **Theorem (1.36, H)** If a polyhedron has  $n$  vertices,  $m$  edges, and  $f$  faces, then

$$n - m + f = 2$$

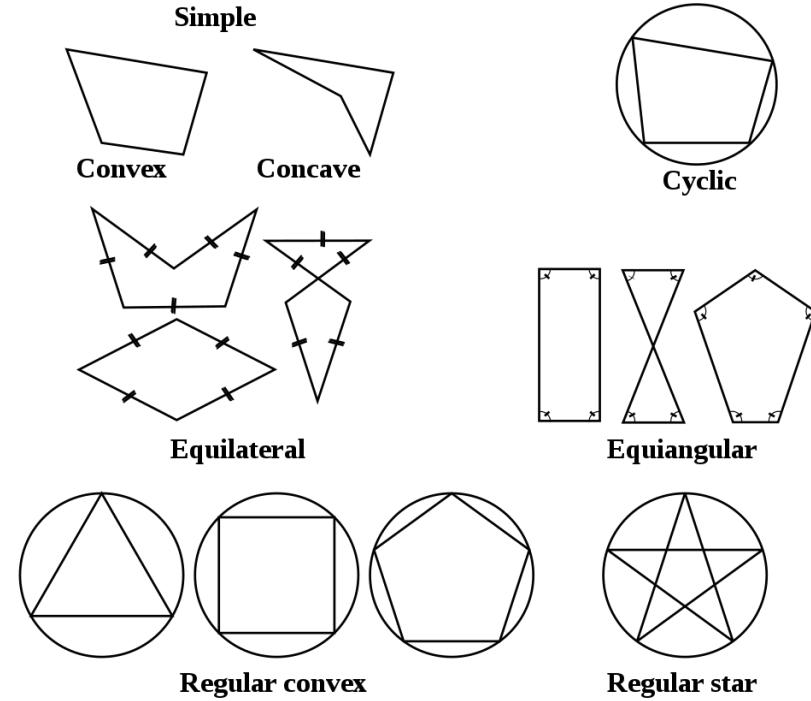
- Given a polyhedron  $P$ , define

$$\rho(P) = \min\{l(F) : F \text{ is a face of } P\}$$

- **Theorem (1.37, H)** For all polyhedron  $P$ ,  $3 \leq \rho(P) \leq 5$

# Regular polyhedron 正多面体

- A **regular polygon** is one that is equilateral and equiangular  
正多边形(cycle), 等边、等角
- A polyhedron is **regular** if its faces are mutually congruent, regular polygons and if the number of faces meeting at a vertex is the same for every vertex  
正多面体  
面是相互全等的、正多边形、点的度数相等



# Regular polyhedron 正多面体

- **Theorem** (1.38, H; 6.1.28, W) There are exactly five regular polyhedral
- 正四面体
- 立方体（正六面体）
- 正八面体
- 正十二面体
- 正二十面体

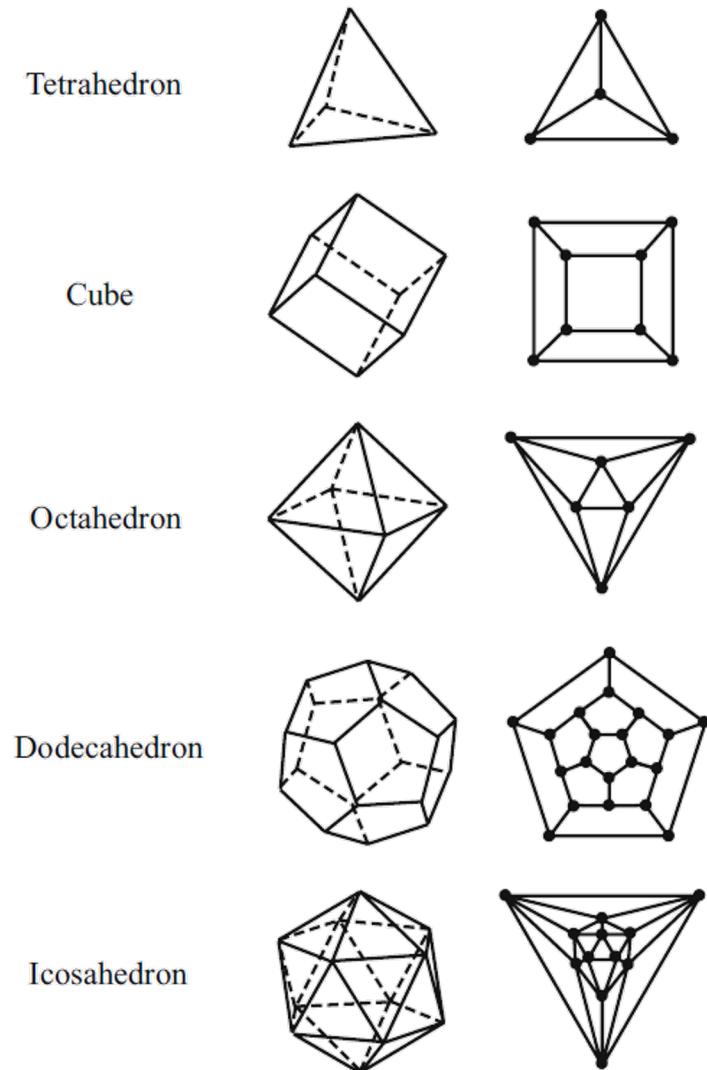


FIGURE 1.82. The five regular polyhedra and their graphical representations.

# Kuratowski's Theorem

# Kuratowski's Theorem

- **Theorem** (1.39, H; Ex1, S1.5.4, H) A graph  $G$  is planar  $\Leftrightarrow$  every subdivision of  $G$  is planar
- **Theorem** (1.40, H; Kuratowski 1930) A graph is planar  $\Leftrightarrow$  it contains no subdivision of  $K_{3,3}$  or  $K_5$

# Summary

- Planarity

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**Questions?**