



上海交通大学
SHANGHAI JIAO TONG UNIVERSITY



上海交通大学
约翰·霍普克罗夫特
计算机科学中心

John Hopcroft Center for Computer Science

CS 3330: Combinatorics

Shuai Li & Biaoshuai Tao

John Hopcroft Center, Shanghai Jiao Tong University

<https://shuaili8.github.io>

<https://jhc.sjtu.edu.cn/~bstao/>

<https://shuaili8.github.io/Teaching/CS3330/index.html>

Shuai Li (Week 1-8)

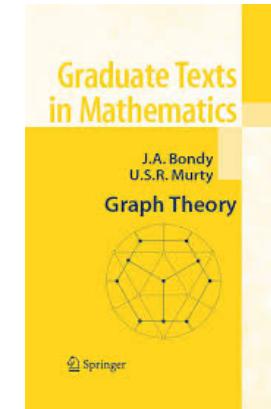
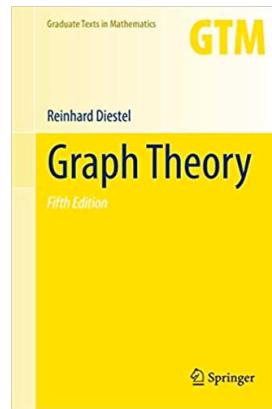
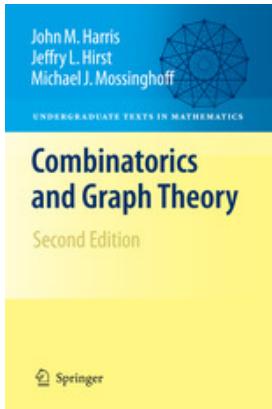
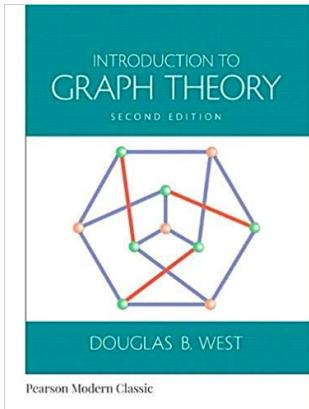
- Position
 - Assistant Professor at John Hopcroft Center since Sep 2019
- Education
 - PhD in Computer Science from the Chinese University of Hong Kong
 - Master in Math from the Chinese Academy of Sciences
 - Bachelor in Math from Zhejiang University
- Research interests
 - Reinforcement learning algorithms and analysis
 - Optimization, algorithms and analysis in machine learning

Biaoshuai Tao (Week 9-16)

- Position
 - Assistant Professor at John Hopcroft Center since Nov 2020
- Education
 - PhD in Computer Science from the University of Michigan
 - Bachelor in Math from Nanyang Technological University
- Research interests
 - Theoretical Computer Science
 - Computational Economics

References

- References:
 - Introduction to Graph Theory, by Douglas West
 - Combinatorics and Graph Theory, by Harris, Hirst and Mossinghoff
 - Graph Theory, Reinhard Diestel
 - Graph Theory, by Bondy and Murty
 - A Course in Combinatorics, J. H. Van Lint



Previous courses

- Discrete Mathematics
 - Basic concepts for graph theory
- Mathematical Foundations of Computer Science (CS 499?)
 - Basic notions and hand shaking lemma
 - Graph isomorphism and graph score
 - Applications of handshake lemma: Parity argument
 - The number of spanning trees
 - Isomorphism of trees
 - Random graphs

Goal

- Knowledge of the basic problems for graph theory
 - Bipartite graphs/Matching/Circuits/Coloring/...
- Knowledge of the important counting related results on graphs
- Familiar with the common proof techniques
- Awareness of the popular applications of graphs in many fields

Grading policy

- Attendance and participation: 5%
- Assignments: 35%
- Midterm exam: 15%
- Entry editing: 5%
- Reading report: 10%
- Final exam: 30%

Honor code

- Discussions are encouraged
- Independently write-up homework and project
- Same reports and homework will be reported

Teaching Assistant

- Ruofeng Yang (杨若峰) (Week 1-8)
 - Email: wanshuiyin@sjtu.edu.cn
 - 1st year PhD Student
 - Research interests on bandit algorithms and optimization
 - Office hour: Tue 7-9 PM
- Yichen Tao (陶浥尘) (Week 9-16)
 - Email: taoyc0904@sjtu.edu.cn
 - Senior undergraduate student
 - Research interests on economics & computation
 - Office hour: Mon 7-9 PM, Activity Room of No.1 Dormitory Building, East District
- Zilong Wang (王子龙) (Week 1-8)
 - Email: wangzilong@sjtu.edu.cn
 - Senior undergraduate student
 - Research interests on bandit and RL algorithms
 - Office hour: Wed 7-9 PM

Course Outline

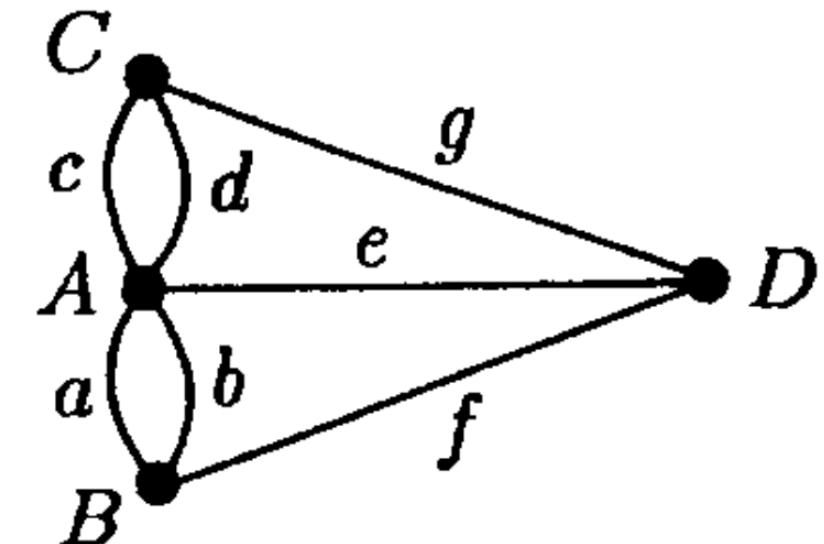
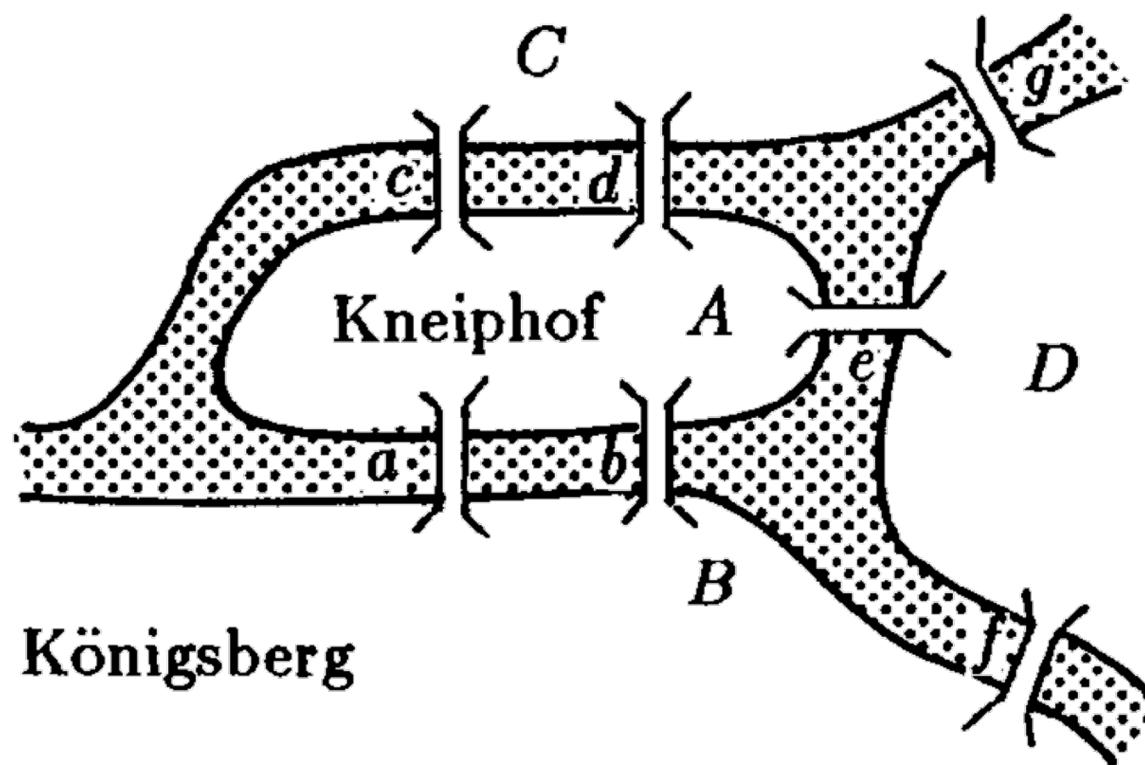
- Basics
 - Graphs, paths and cycles, connectivity, bipartite graphs
- Trees
- Matching
- Connectivity
- Planar Graphs
- Coloring
- Circuits

Introduction

Seven bridges of Königsberg 七桥问题

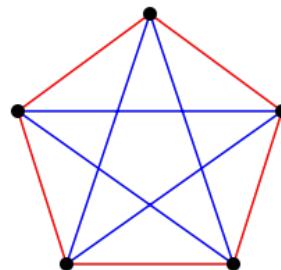
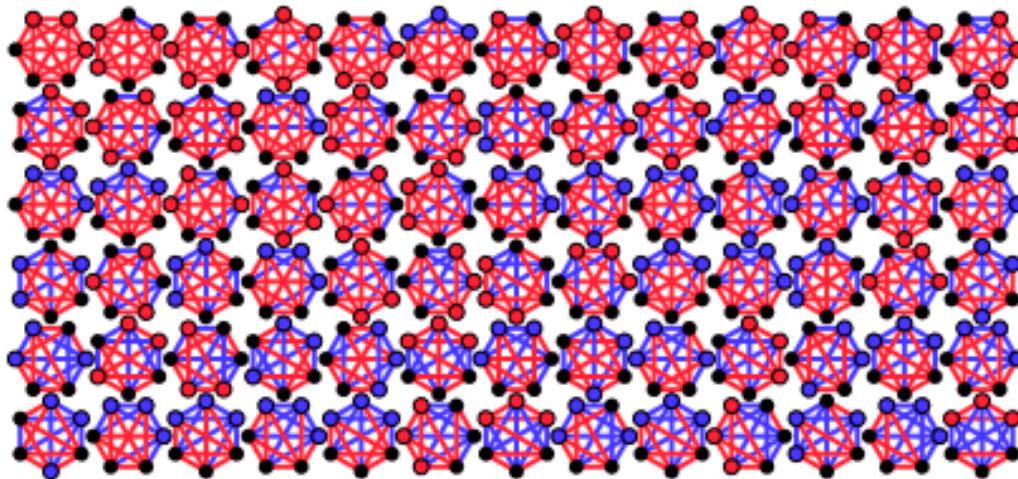
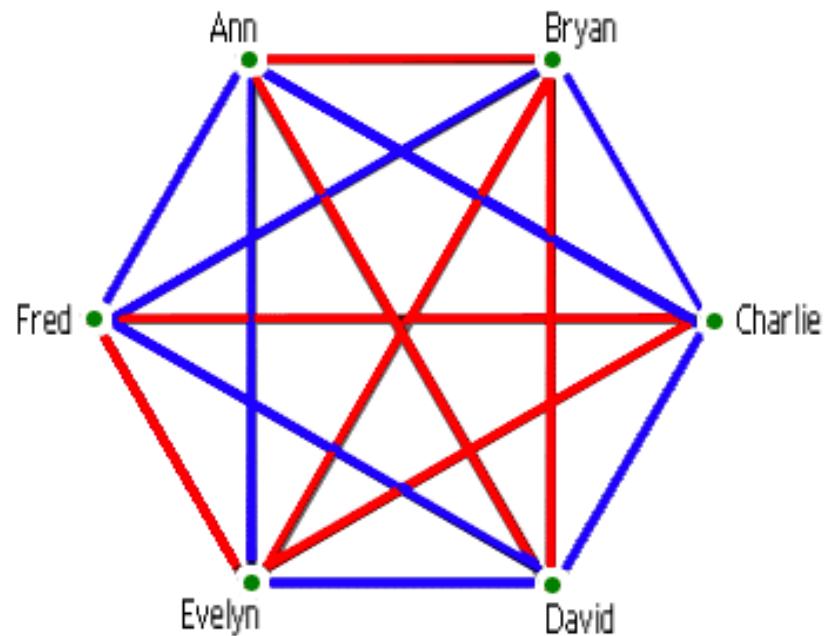


- Leonhard Euler 1736: Is it possible to make a walk through the city, returning to the starting point and crossing each bridge exactly once?



The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?

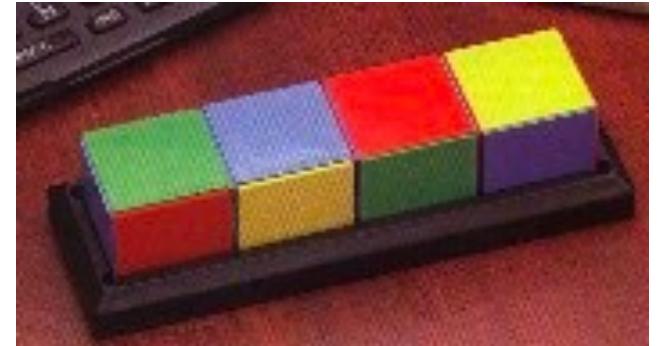


$$\begin{aligned} R(3,3) &= 6 \\ R(3,4) = R(4,3) &= 9 \\ R(3,5) = R(5,3) &= 14 \\ R(3,6) = R(6,3) &= 18 \end{aligned}$$

Examples of general combinatorics problems using graph theory

- Instant Insanity 四色方柱问题

- make a stack of these cubes so that all four colors appear on each of the four sides of the stack

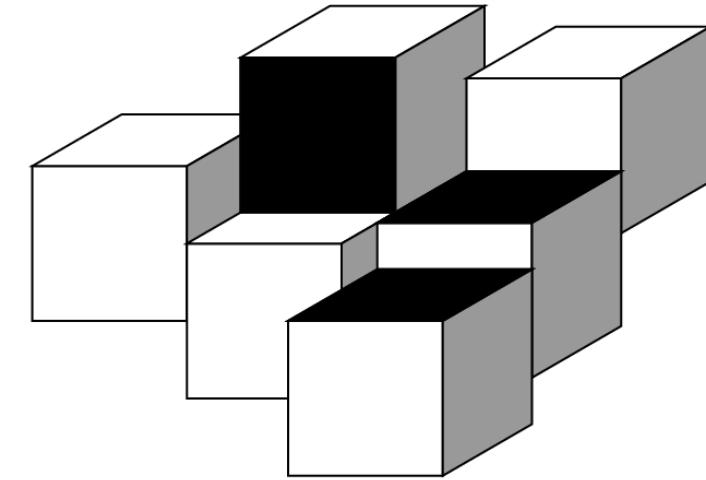
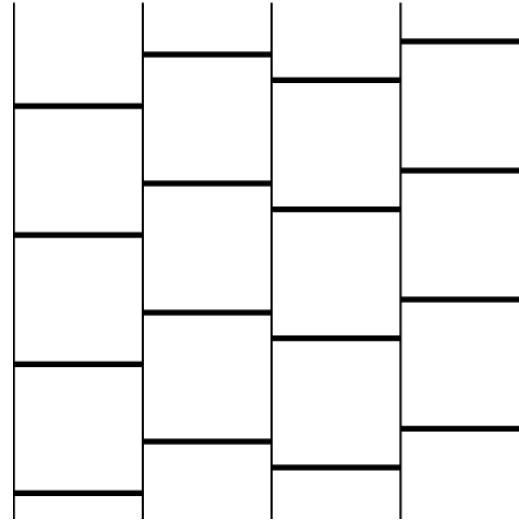


- A set problem

- Let A_1, \dots, A_n be n distinct subsets of the n -set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}$, $1 \leq i \leq n$, are all distinct

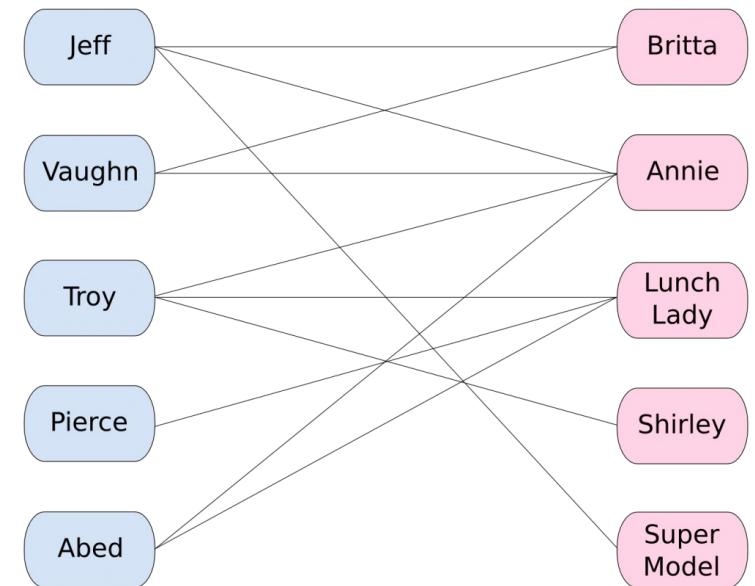
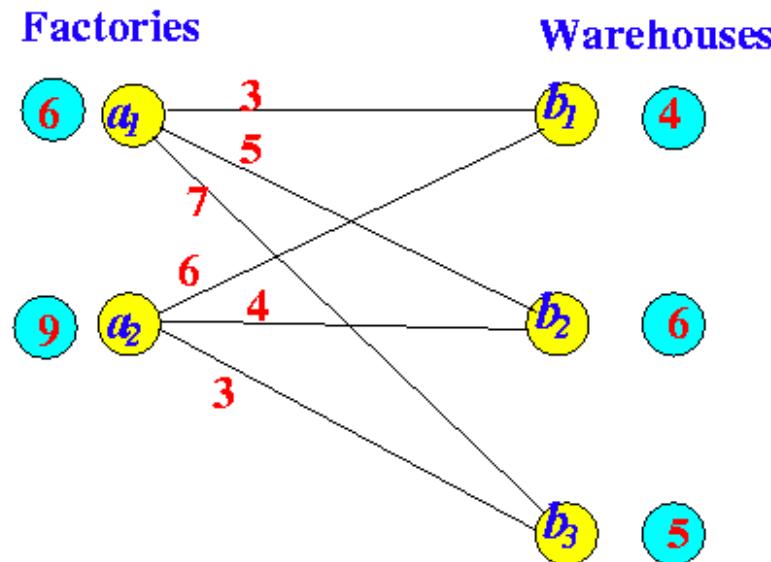
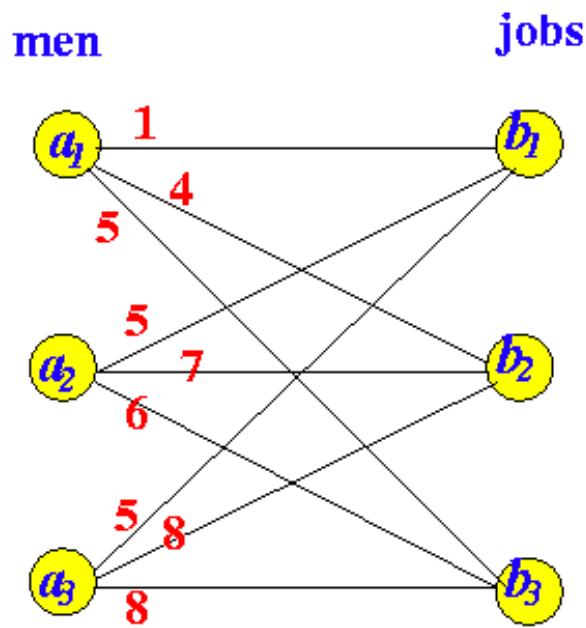
Keller's conjecture

- In 1930, Keller conjectured that any tiling of n -dimensional space by translates of the unit cube must contain a pair of cubes that share a complete $(n - 1)$ -dimensional face



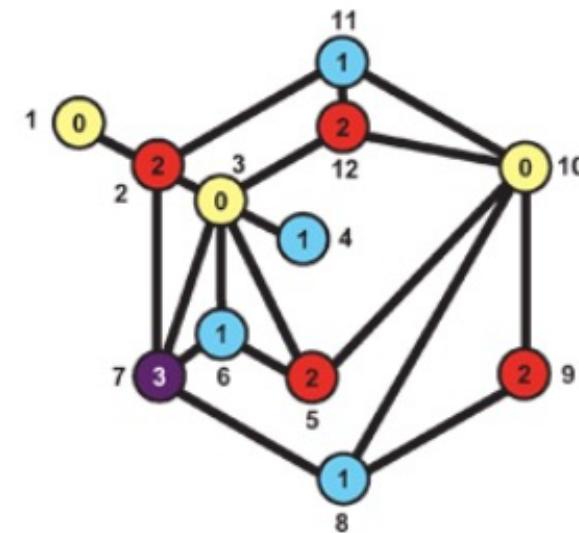
- Corrádi and Szabó transfer it into a graph theory problem
 - Constructing Keller graph
- The conjecture is solved by computer search recently (June 2020)

Assignment problems



Scheduling and coloring

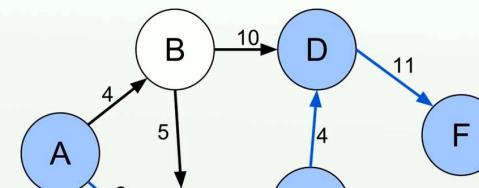
- University examination timetabling
 - Two courses linked by an edge if they have the same students
- Meeting scheduling
 - Two meetings are linked if they have same member



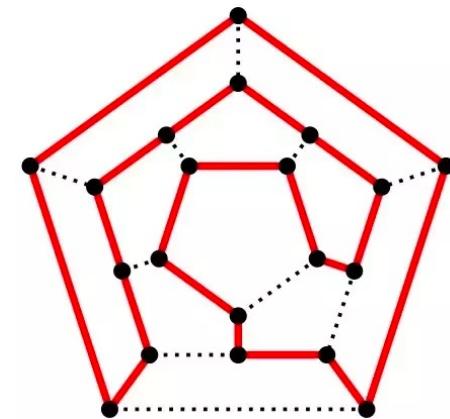
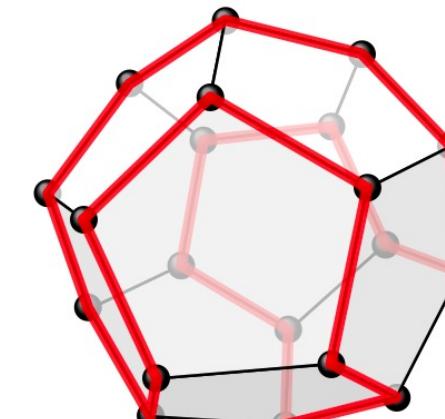
Routes in road networks

- How can we find the shortest route from *A* to *F*?
- If the vertices of the graph represent our house and other places to visit, then we may want to follow a route that visits every vertex exactly once, so as to visit everyone without overstaying our welcome
 - Hamilton circuit

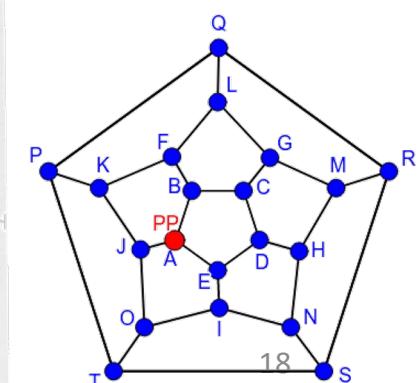
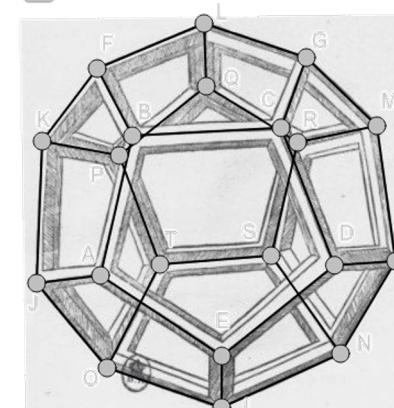
Shortest path problem



https://en.wikipedia.org/wiki/File:Shortest_path_with_direct_weights.svg

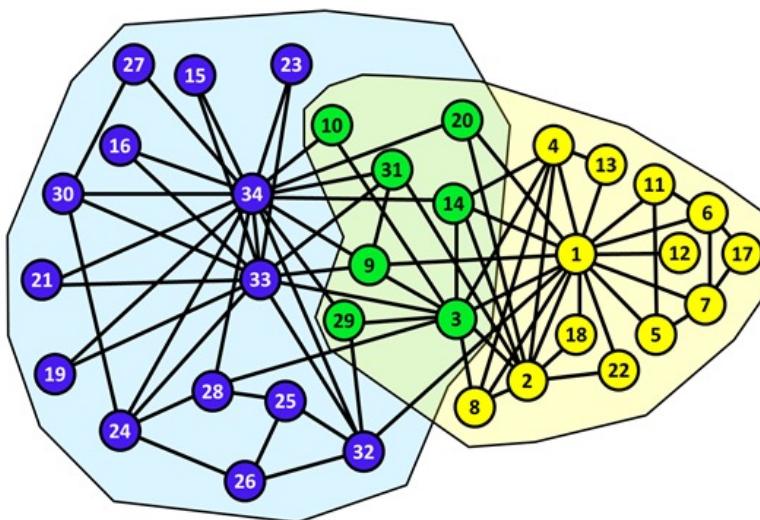
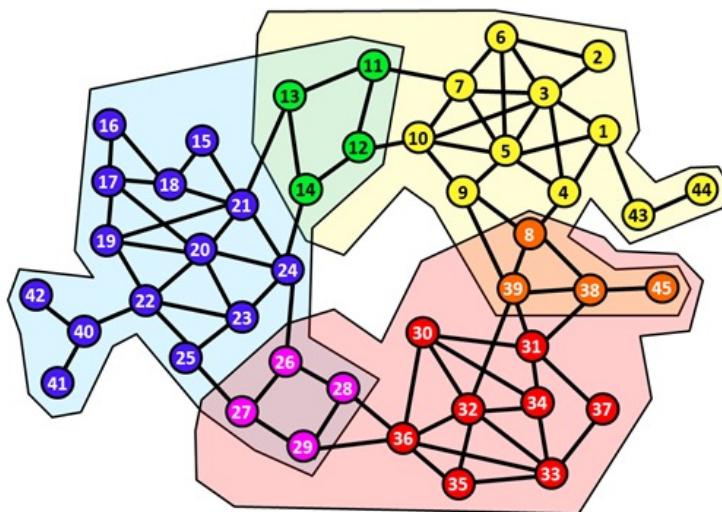


Leonardo da Vinci: DVODECEDRON



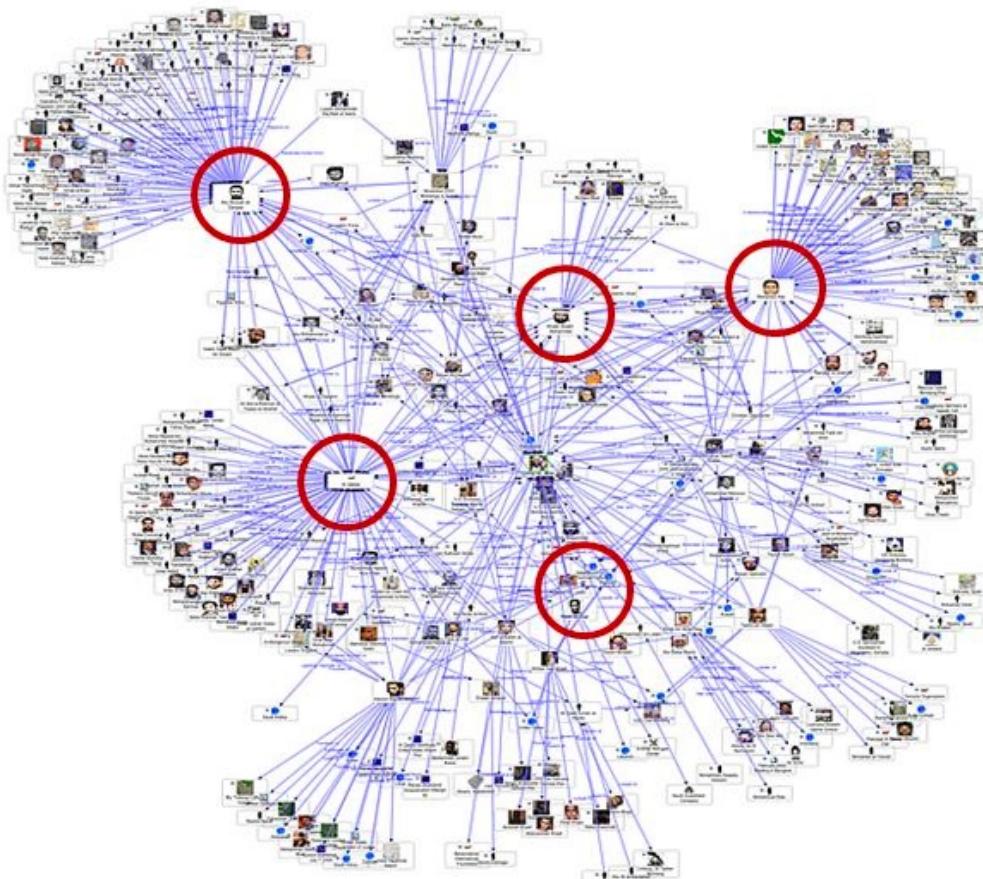
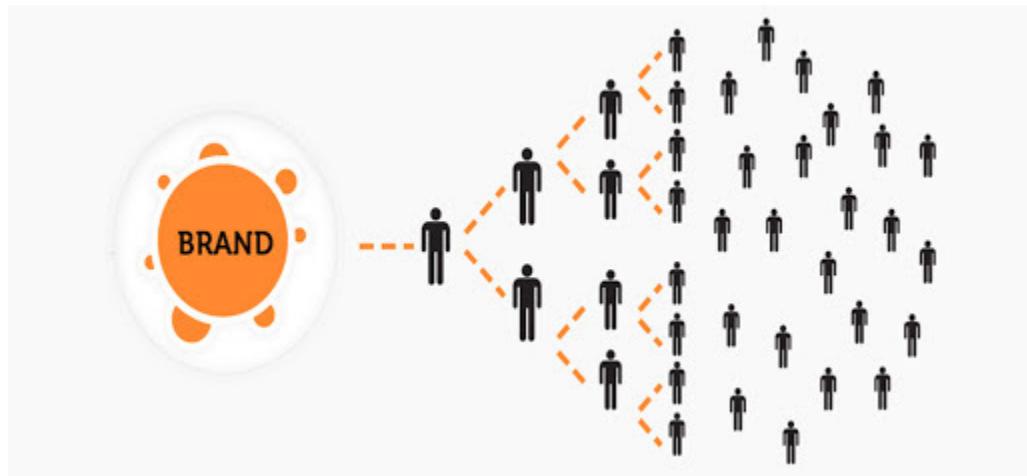
Social network

- Recommendation
- Clustering



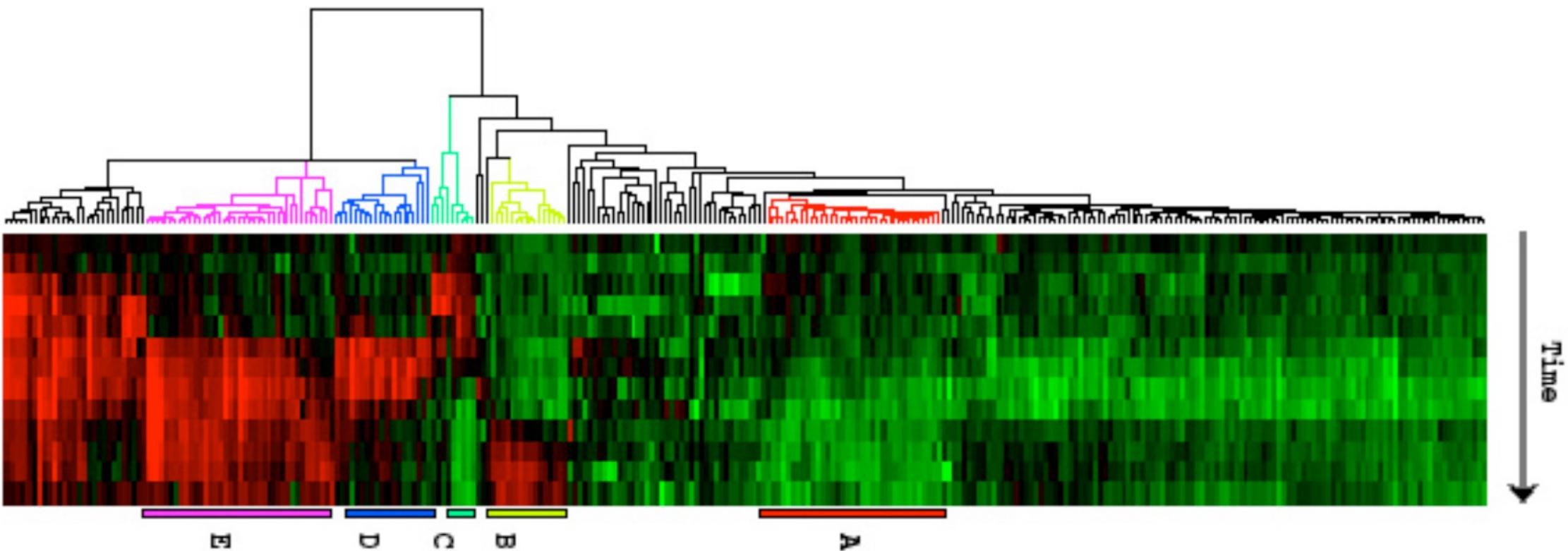
Influence maximization

- Select the best seed set

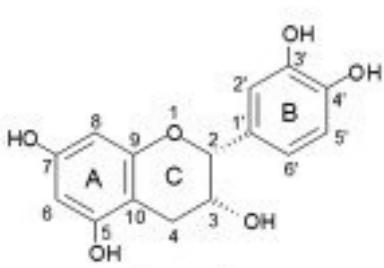


Gene structure

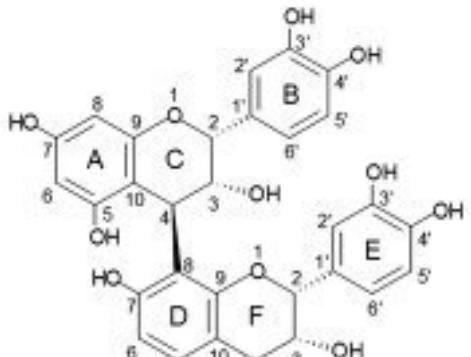
- Tree graph
 - Agglomerative clustering method



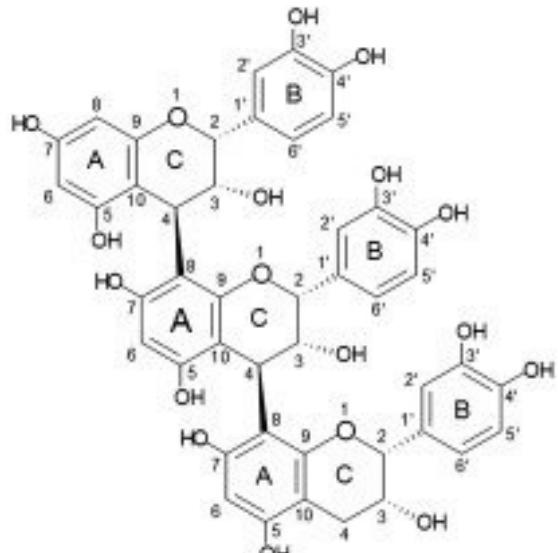
Molecular structure



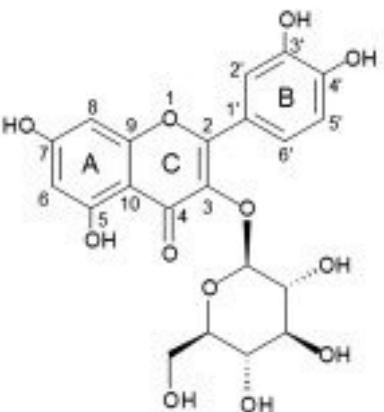
Flavan-3-ol monomer
(-)-epicatechin



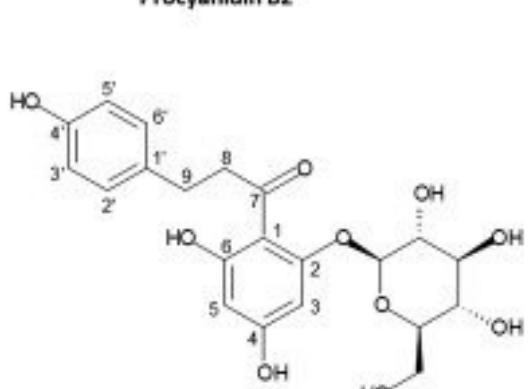
Flavan-3-ol dimer
Procyanidin B2



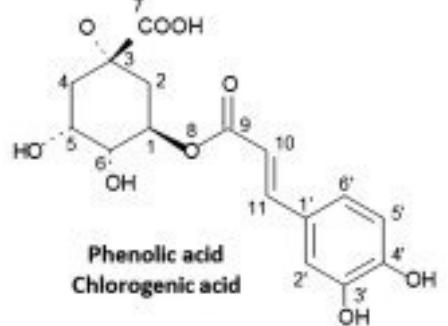
Flavan-3-ol trimer
Procyanidin C1



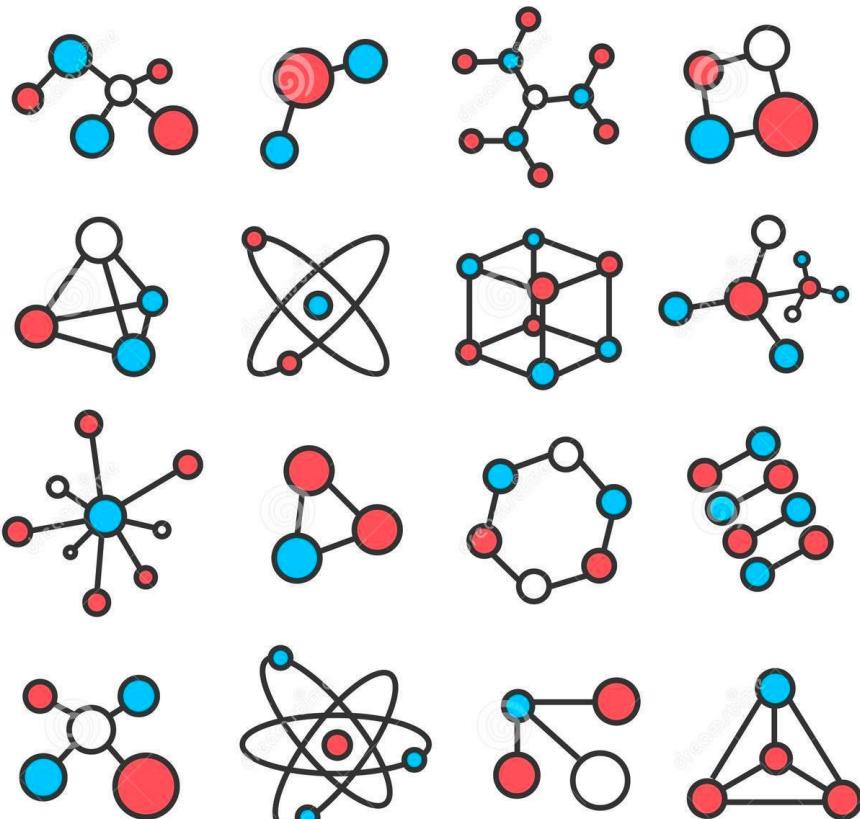
Flavonol
Quercetin 3-glucoside



Dihydrochalcone
Phloridzin



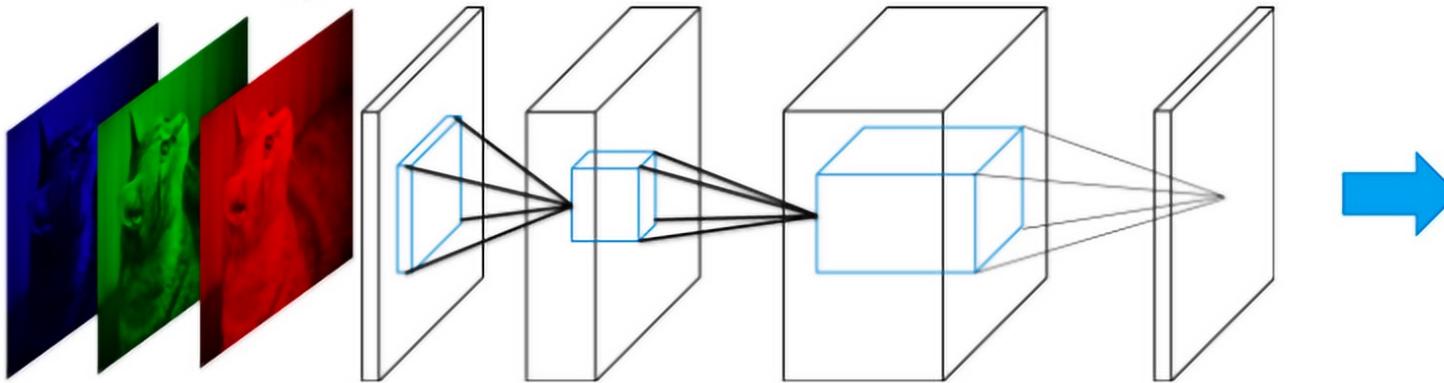
Phenolic acid
Chlorogenic acid



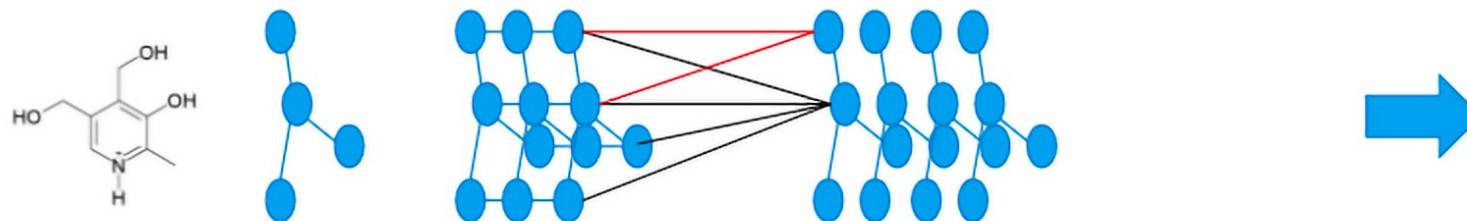
Graph neural network (GNN)

How Graph Convolutions work

CNN on image



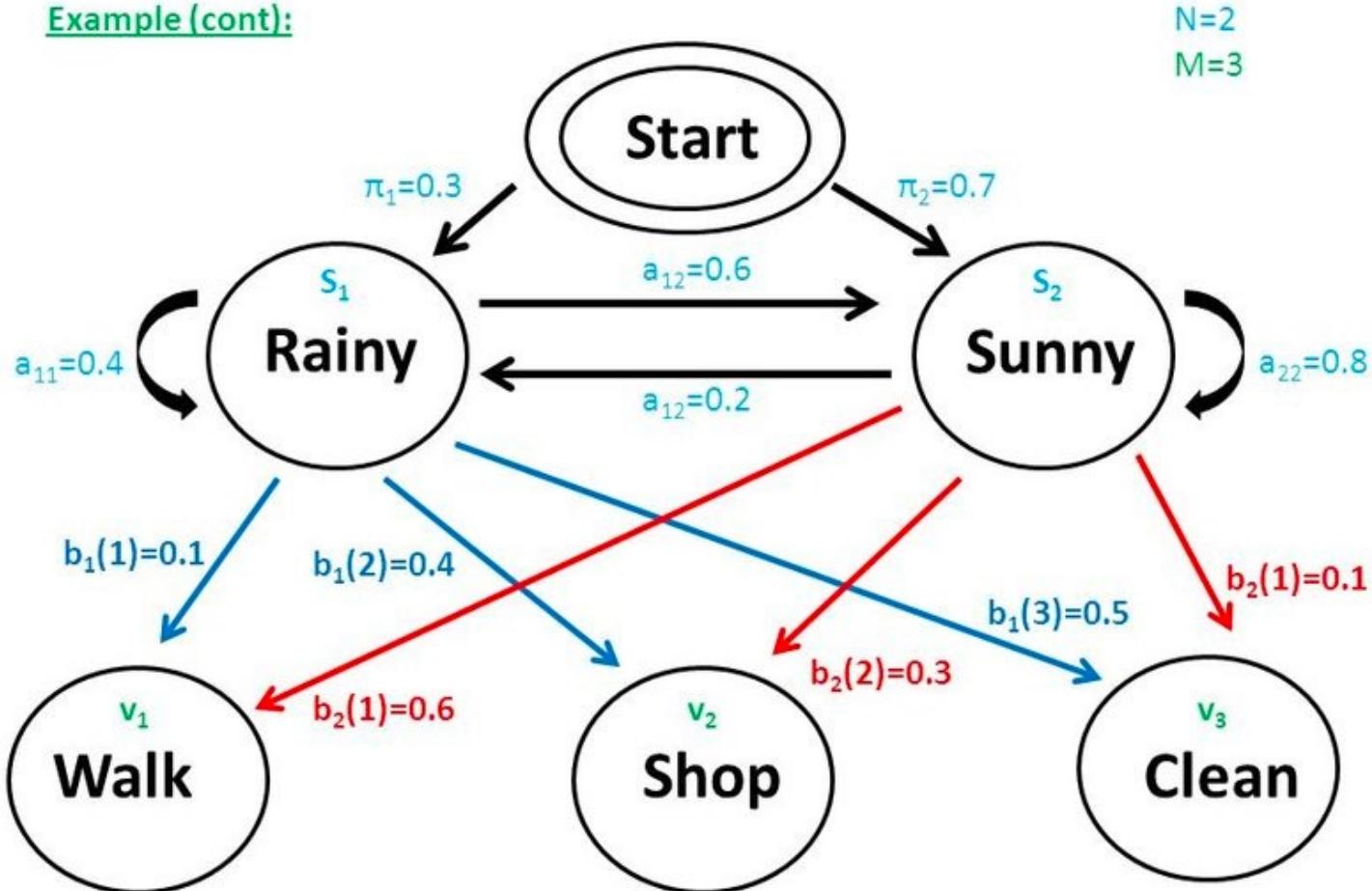
Graph convolution



Convolution “kernel” depends on Graph structure

Hidden Markov Model

Example (cont):



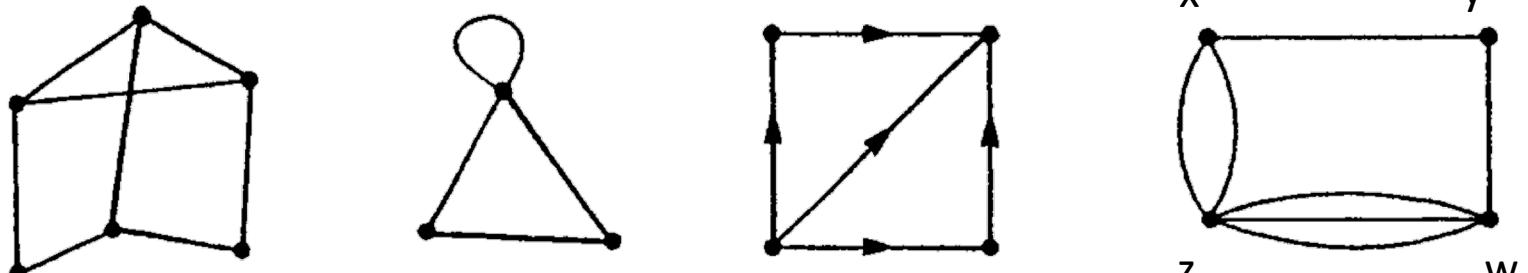
Basics

Graphs

- **Definition** A graph G is a pair (V, E)
 - V : set of vertices
 - E : set of edges
 - $e \in E$ corresponds to a pair of endpoints $x, y \in V$

We mainly focus on
Simple graph:
No loops, no multi-edges

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w



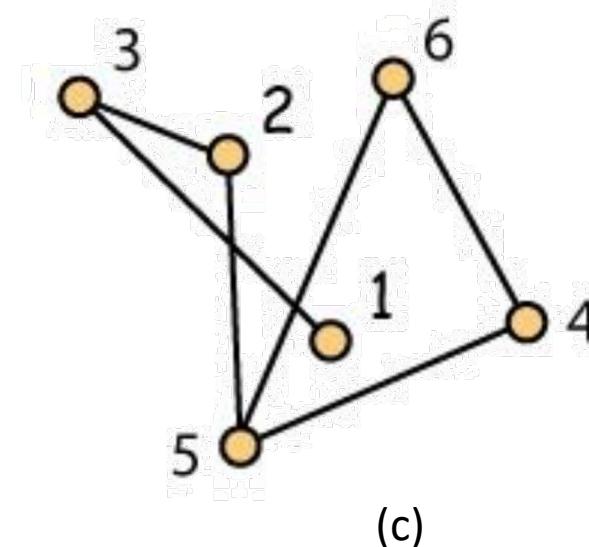
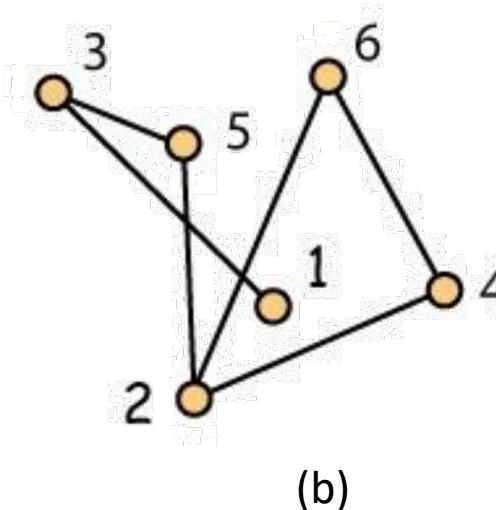
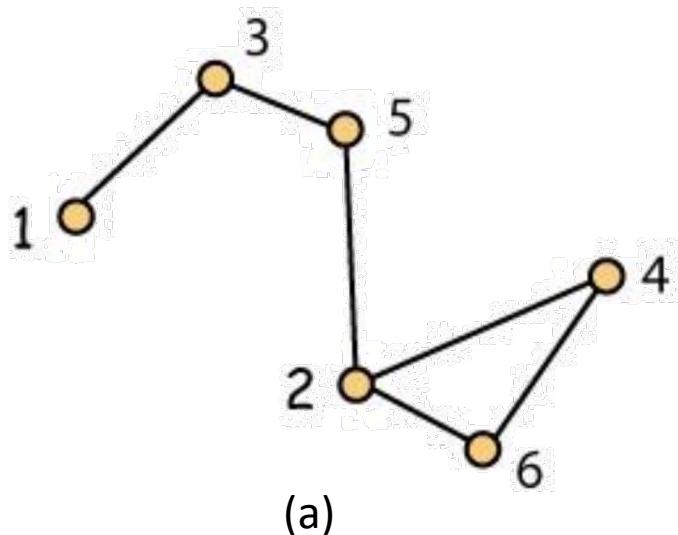
(i) graph (ii) graph with loop (iii) digraph (iv) multiple edges

Figure 1.2

Figure 1.1

Graphs: All about adjacency

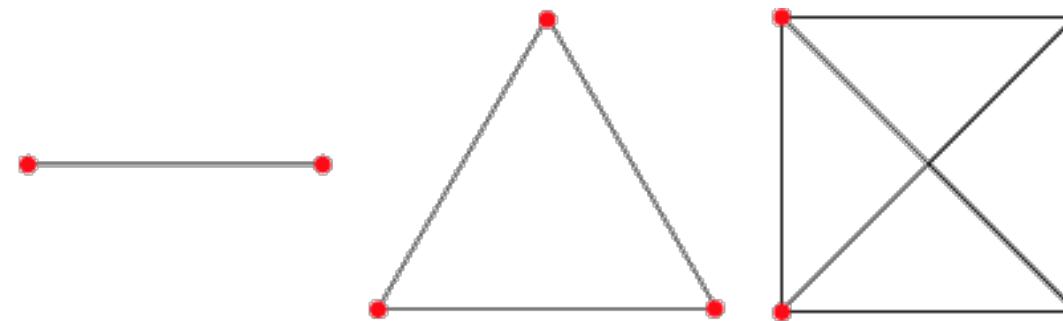
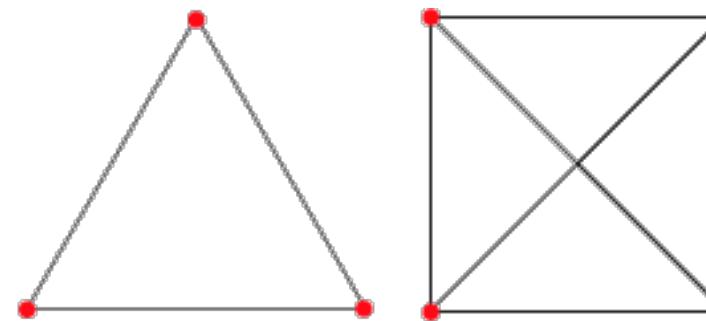
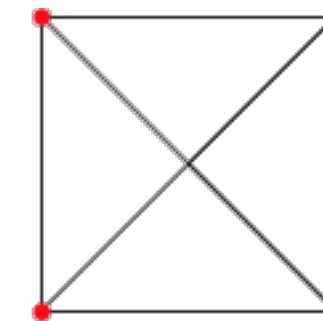
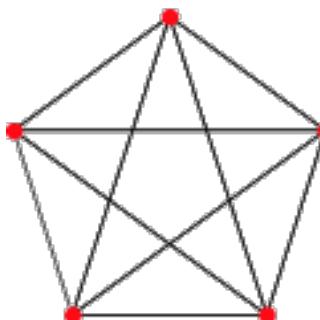
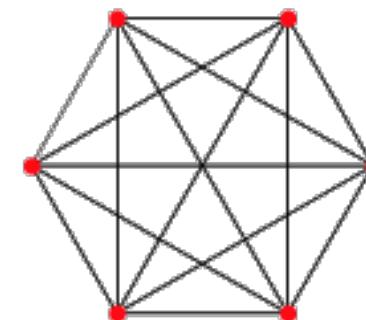
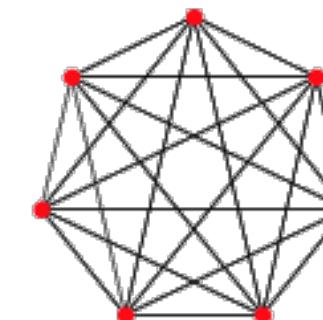
- Same graph or not



- Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection $f: V_1 \rightarrow V_2$ s.t.
$$e = \{a, b\} \in E_1 \Leftrightarrow f(e) := \{f(a), f(b)\} \in E_2$$

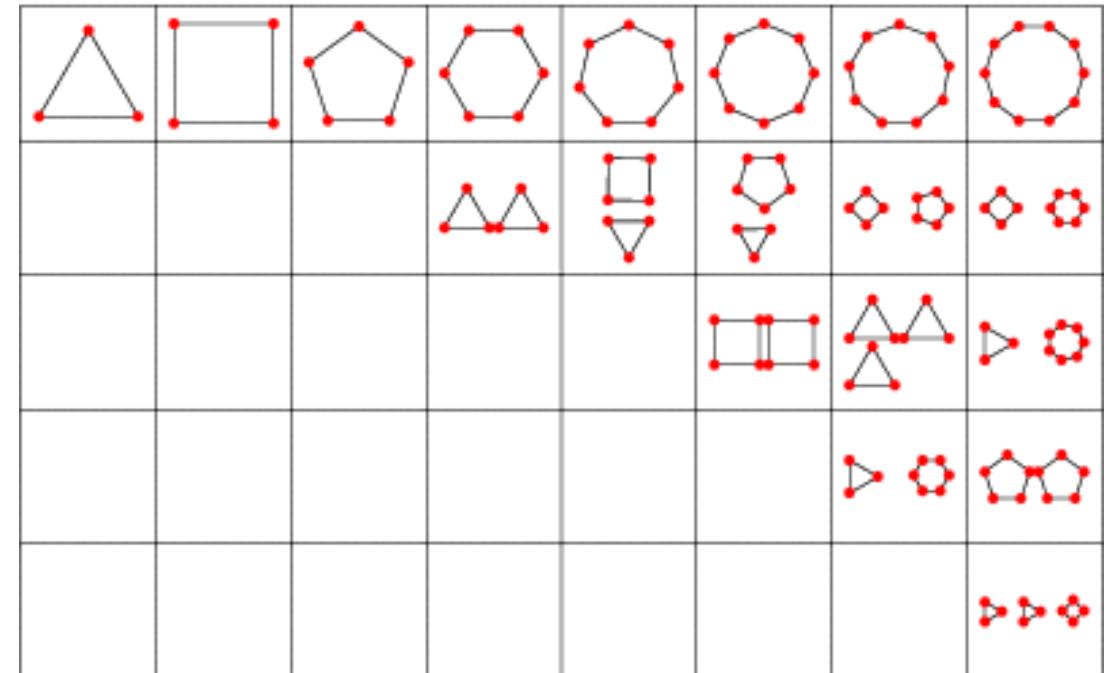
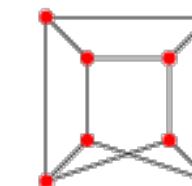
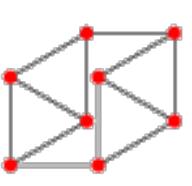
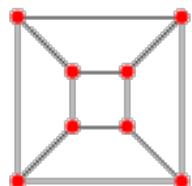
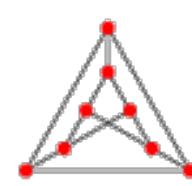
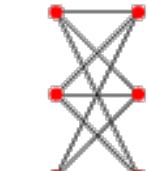
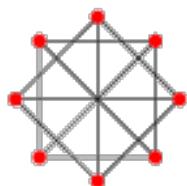
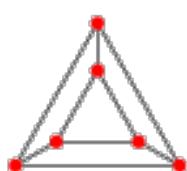
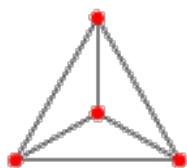
Example: Complete graphs

- There is an edge between every pair of vertices

 K_2  K_3  K_4  K_5  K_6  K_7

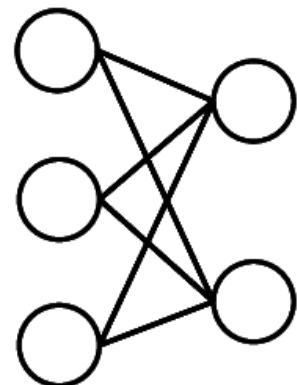
Example: Regular graphs

- Every vertex has the same degree

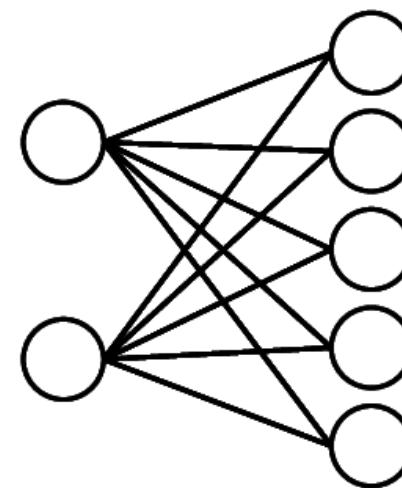


Example: Bipartite graphs

- The vertex set can be partitioned into two sets X and Y such that every edge in G has one end vertex in X and the other in Y
- Complete bipartite graphs



$K_{3,2}$



$K_{2,5}$

Example (1A, L): Peterson graph

- Show that the following two graphs are same/isomorphic

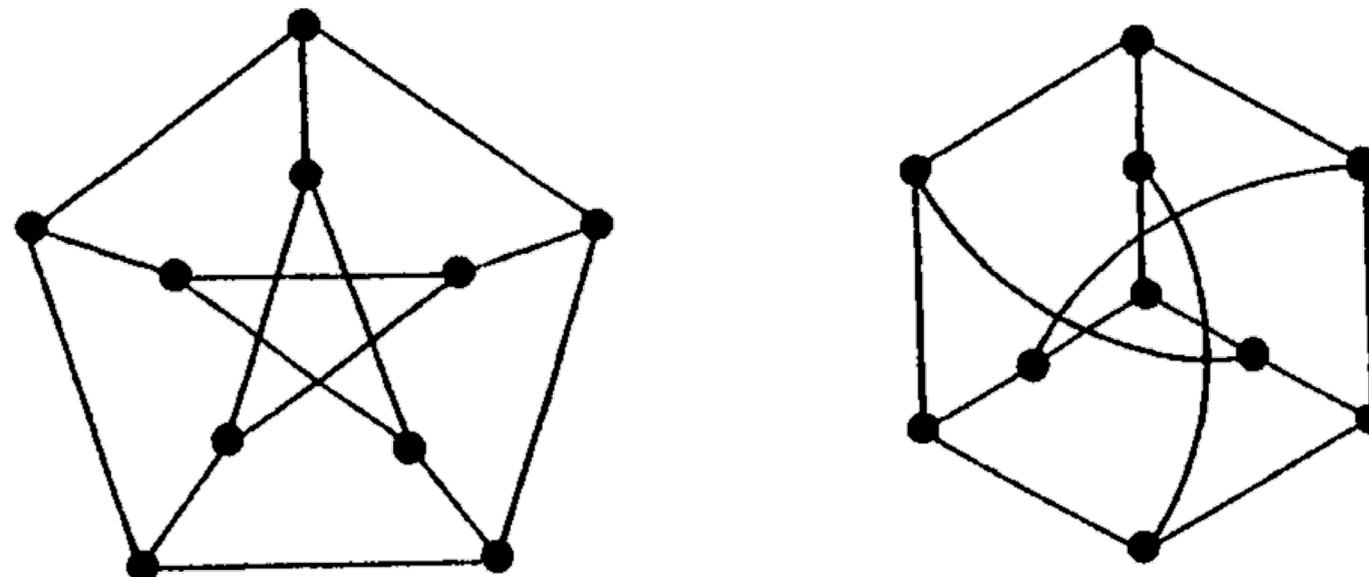
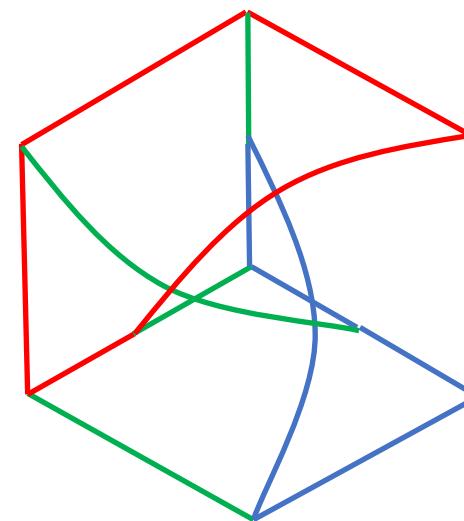
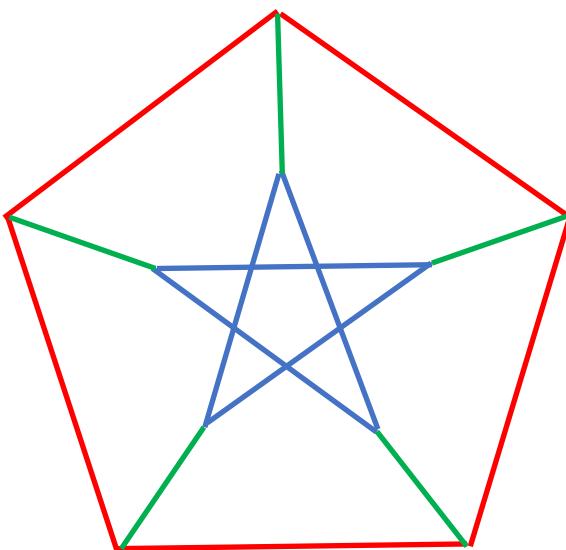


Figure 1.4

Example: Peterson graph (cont.)

- Show that the following two graphs are same/isomorphic



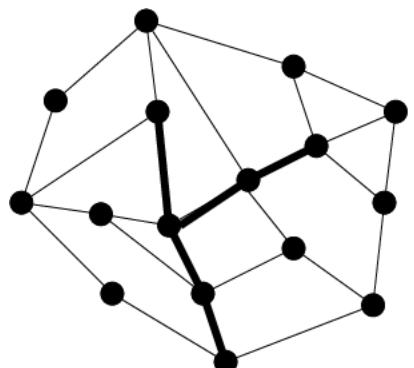
Subgraphs

- A **subgraph** of a graph G is a graph H such that

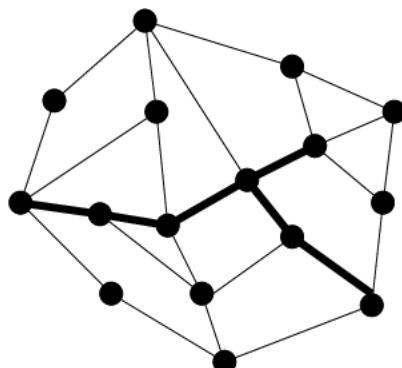
$$V(H) \subseteq V(G), E(H) \subseteq E(G)$$

and the ends of an edge $e \in E(H)$ are the same as its ends in G

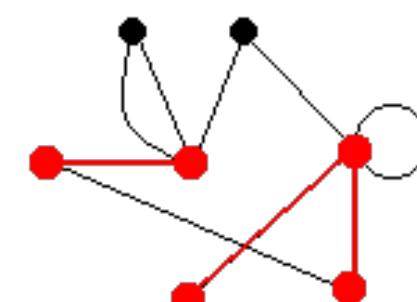
- H is a **spanning subgraph** when $V(H) = V(G)$
- The subgraph of G **induced** by a subset $S \subseteq V(G)$ is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S



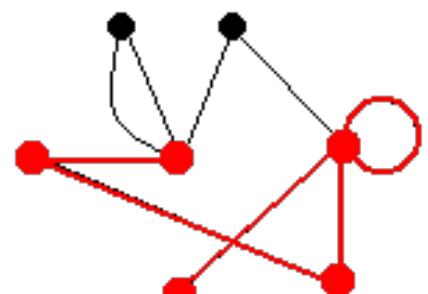
(a)



(b)



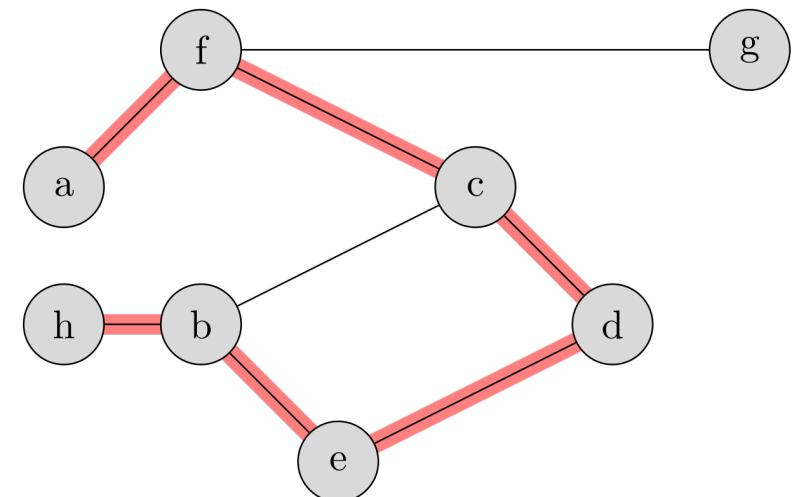
Subgraph (in red)



Induced Subgraph

Paths (路径)

- A **path** is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \dots e_k v_k$ where vertices are all **distinct**
 - Or it can be written as $v_0 v_1 \dots v_k$ in simple graphs
- P^k : path of length k (the number of edges)

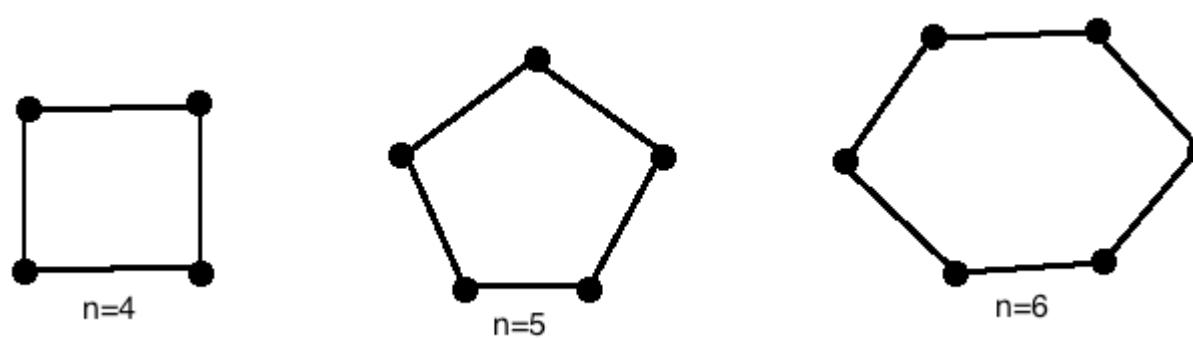


Walk (游走)

- A **walk** is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \dots e_k v_k$
 - The vertices not necessarily distinct
 - The length = the number of edges
- **Proposition** (1.2.5, W) Every u - v walk contains a u - v path

Cycles (环)

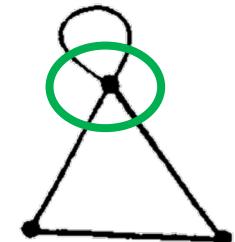
- If $P = x_0x_1 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1}x_0$ is called a **cycle**
- C^k : cycle of length k (the number of edges/vertices)



- **Proposition** (1.2.15, W) Every closed odd walk contains an odd cycle

Neighbors and degree

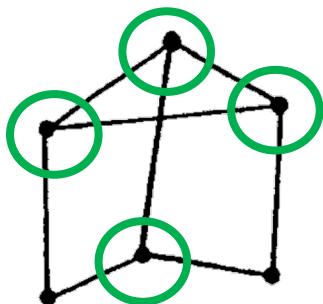
- Two vertices $a \neq b$ are called **adjacent** if they are joined by an edge
 - $N(x)$: set of all vertices adjacent to x
 - **neighbors** of x
 - A vertex is **isolated** vertex if it has no neighbors
- The number of edges incident with a vertex x is called the **degree** of x
 - A **loop** contributes **2** to the degree
- A graph is **finite** when both $E(G)$ and $V(G)$ are finite sets



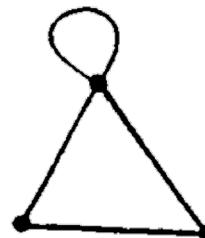
graph with loop

Handshaking Theorem (Euler 1736)

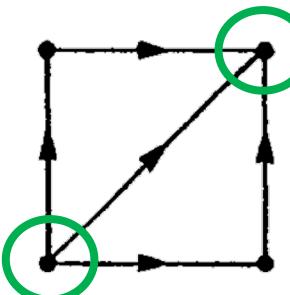
- **Theorem** A finite graph G has an even number of vertices with odd degree



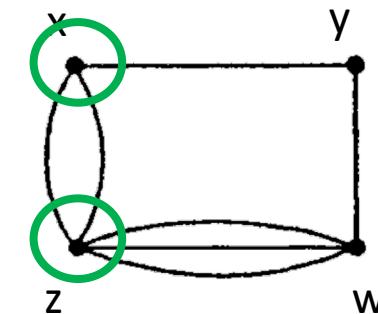
(i) graph



(ii) graph with loop



(iii) digraph



(iv) multiple edges

Figure 1.2

Proof

- **Theorem** A finite graph G has an even number of vertices with odd degree.
- **Proof** The degree of x is the number of times it appears in the right column. Thus

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

Degree

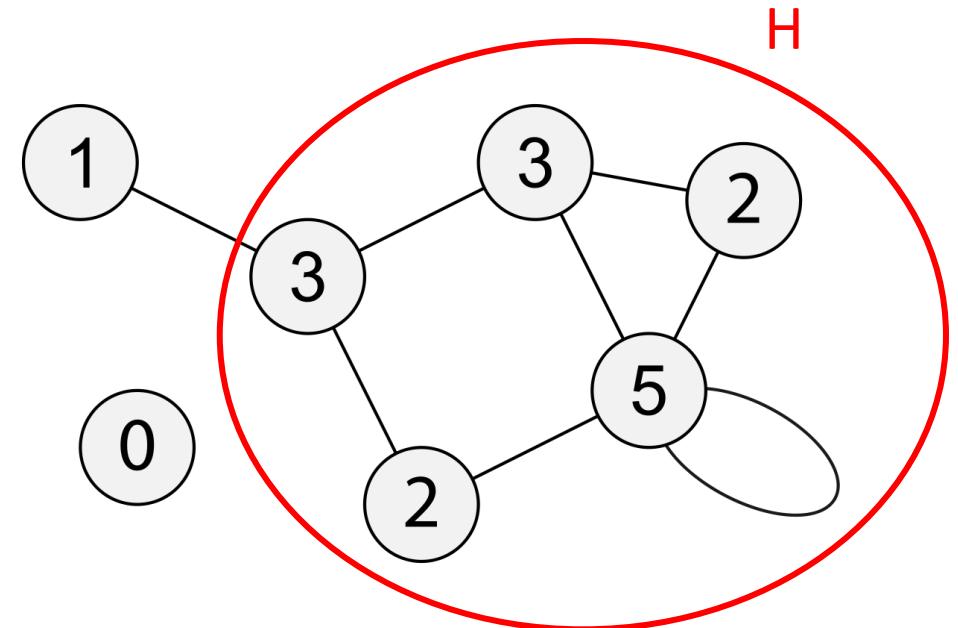
- **Minimal** degree of G : $\delta(G) = \min\{d(v): v \in V\}$
- **Maximal** degree of G : $\Delta(G) = \max\{d(v): v \in V\}$
- **Average** degree of G : $d(G) = \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{2|E|}{|V|}$
- All measure the ‘density’ of a graph
- $d(G) \geq \delta(G)$

Degree (global to local)

- **Proposition** (1.2.2, D) Every graph G with at least one edge has a subgraph H with

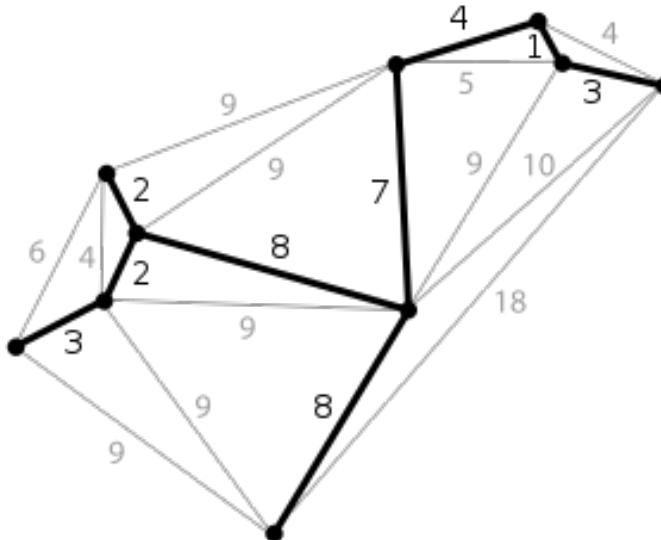
$$\delta(H) > \frac{1}{2}d(H) \geq \frac{1}{2}d(G)$$

- Example: $|G| = 7, d(G) = \frac{16}{7}$
- $\delta(H) = 2, d(H) = \frac{14}{5}$



Minimal degree guarantees long paths and cycles

- **Proposition** (1.3.1, D) Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$, provided $\delta(G) \geq 2$.



Distance and diameter

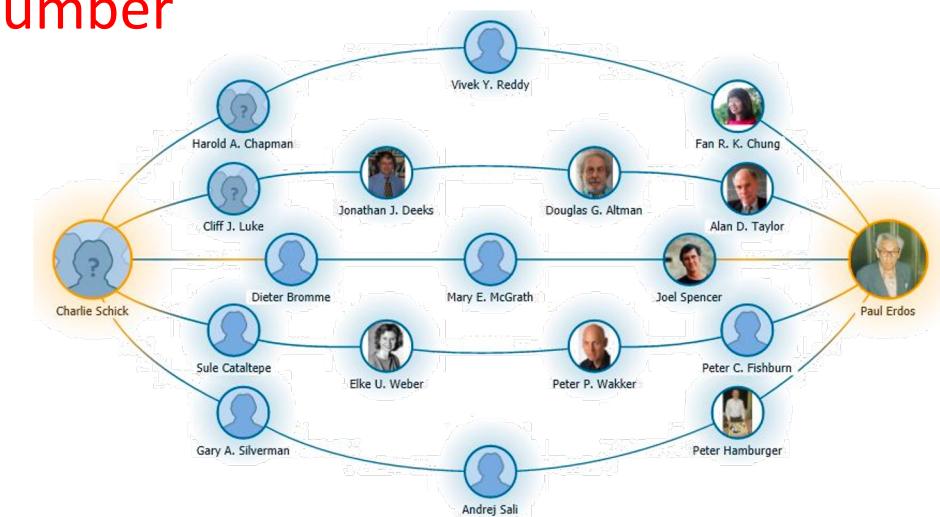
- The **distance** $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x \sim y$ path
 - if no such path exists, we set $d(x, y) := \infty$
- The greatest distance between any two vertices in G is the **diameter** of G

$$\text{diam}(G) = \max_{x, y \in V} d(x, y)$$

Example -- Erdős number



- A well-known graph
 - vertices: mathematicians of the world
 - Two vertices are adjacent if and only if they have published a joint paper
 - The distance in this graph from some mathematician to the vertex Paul Erdős is known as his or her **Erdős number**

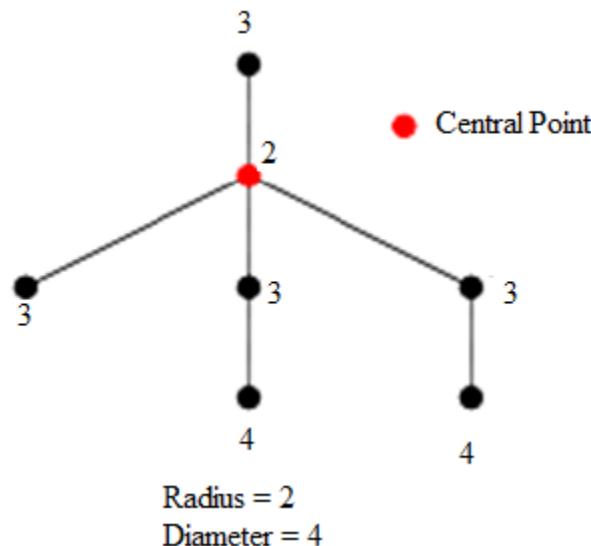


Radius and diameter

- A vertex is **central** in G if its greatest distance from other vertex is smallest, such greatest distance is the **radius** of G

$$\text{rad}(G) := \min_{x \in V} \max_{y \in V} d(x, y)$$

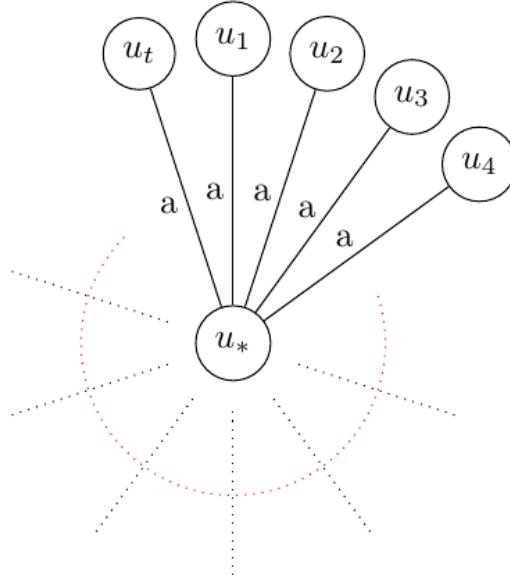
- **Proposition** (1.4, H; Ex1.6, D) $\text{rad}(G) \leq \text{diam}(G) \leq 2 \text{ rad}(G)$



Radius and maximum degree control graph size

- **Proposition** (1.3.3, D) A graph G with radius at most r and maximum degree at most $\Delta \geq 3$ has fewer than $\frac{\Delta}{\Delta-2}(\Delta - 1)^r$.

Figure 1: Star Graph



Summary

- Motivation and applications
- Basic concepts:
 - graph, isomorphism, subgraphs, paths, walks, cycles,
 - Neighbors, degree, distance, diameter, radius
- Examples:
 - Complete/regular/bipartite graphs, Peterson graph
- Theorems:
 - Handshaking
 - Large average degree guarantees dense subgraphs
 - Large minimal degree guarantees long paths and cycles
 - Radius and maximum degree control graph size

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Questions?