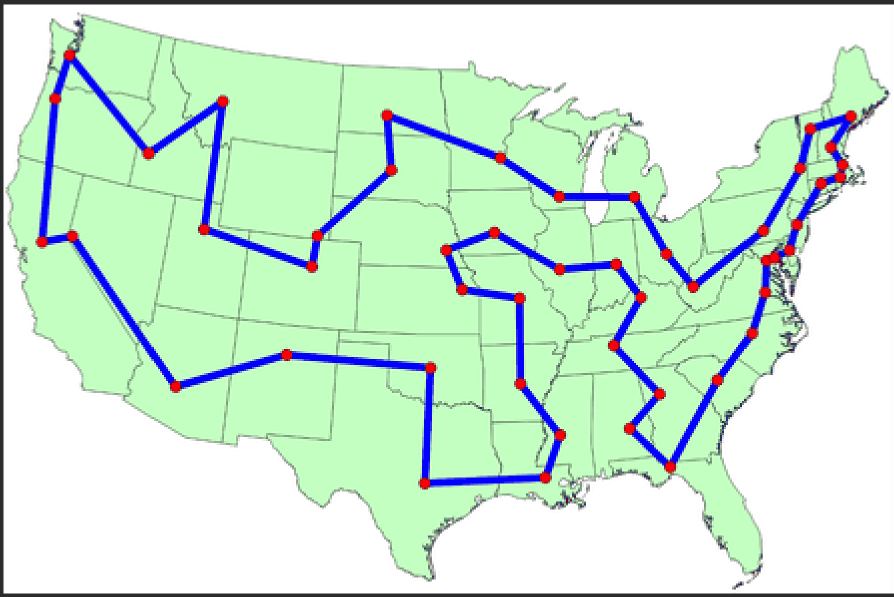
Traveling Salesman Problem



Soltion to 48 States TSP.

Given a list of cities n and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?

ref: http://www.opl.ufc.br/post/tsp/

Parameters

1. The (x,y) coordinates of each city.
2. $Cost_{ij}$: the distance between city i to city j.

What's output?

A sequnce of city to visit, like 1->10->3->28->5....->1

Variables

$$x_{ij} = \begin{cases} 1 & \text{if the route includes a direct link between cities } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

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Objective

minimize the total travel distance:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

Constraint1

each city is arrived at from exactly one other city ----**One In**

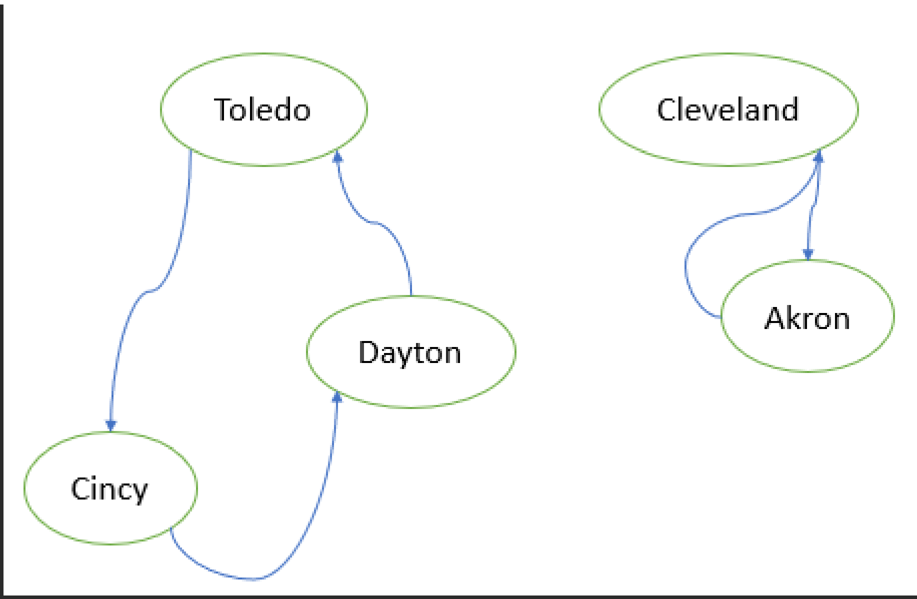
$$\sum_{i=1, i \neq j}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n,$$

Constraint2

from each city there is a departure to exactly one next city. ----**One Out**

$$\sum_{j=1, j \neq i}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n,$$

Subtour



The first subtour goes through nodes Toledo, Cincy, and Dayton, while the second subtour goes through nodes Clevland and Akron.

Note that in this solution each node has exactly two edges in the tour incident to it. But there is no path for the salesman to travel between subtour.

So we must add extra constraints to our model to eliminate these solutions.

Let's define a variable $\mu_i = t$ to represent city i is visited in step t(t=1,2,...,n) of the tour. Define the tour as originating and ending at city 1.

For example, $\mu_{Toledo} = 1, \mu_{Cincy} = 2, \mu_{Dayton} = 3...$

Subtour Cleveland <-> Arkon:

$x_{Clevend,Akron} = 1, \mu_{Cleveland} = 4, \mu_{Akron} = 5$

$=> \mu_{Akron} - \mu_{Cleveland} = 5 - 4 = 1.$

But $x_{Akron,Clevend} = 1, => \mu_{Cleveland} - \mu_{Akron} = 1 => 4 - 5 = 1$, which is not valid.

```
## Subtour constraint: Miller-Tucker-Zemlin formulation
t has a range (1,N).

**When $x_{ij} =0$**, $\mu_i - \mu_j \le n-1$, eg. Toledo->Cincy->Dayton->Cleveland->Arkon, $\mu_{Arkon}-\mu_{Toledo} \le 5-1 =4$, which is largest difference.
This always holds true.

**When $x_{ij} = 1$**, we have $\mu_i - \mu_j + nx_{ij}\le (t) - (t-1) +n*1 = n-1$,

Constraints:  $ u_i - u_j + n x_{ij} \leq n-1, \quad 2 \leq i \neq j \leq n, $
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Subtour constraint: Miller-Tucker-Zemlin formulation

t has a range (1,N).

When $x_{ij} = 0$, $\mu_i - \mu_j \leq n-1$, eg. Toledo->Cincy->Dayton->Cleveland->Arkon, $\mu_{Arkon} - \mu_{Toledo} \leq 5-1=4$, which is largest difference. This always holds true.

When $x_{ij} = 1$, we have $\boxed{\mu_i - \mu_j + nx_{ij} \leq (t) - (t-1) + n * 1 = n-1}$,

Constraints: $u_i - u_j + nx_{ij} \leq n-1, \quad 2 \leq i \neq j \leq n,$

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$$
\begin{aligned}
\min &\sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} c_{ij} x_{ij}, \\
\text{subject to} \\
&\sum_{i=1, i \neq j}^{n} x_{ij} = 1, \quad j = 1, 2, \ldots, n, \\
&\sum_{j=1, j \neq i}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n, \\
&u_i - u_j + nx_{ij} \leq n-1, \quad 2 \leq i \neq j \leq n, \\
&x_{ij} \in \{0, 1\} \quad i, j = 1, 2, \ldots, n, \quad i \neq j, \\
&u_i \in \mathbb{R}^+ \quad i = 1, 2, \ldots, n.
\end{aligned}
$$

$$
\begin{gathered}
\sum_i y_{ij} = 1, \;\; j = 0, 1, \ldots, n-1 \\
\sum_i \sum_j y_{ij} \leq |S| - 1 \;\; S \subset V, 2 \leq |S| \leq n-2 \\
y_{ij} \in \{0, 1\} \; \forall i, j \in E
\end{gathered}
$$

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