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Bandits with concave rewards and convex knapsacks

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In this paper, we consider a very general model for exploration-exploitation tradeoff which allows arbitrary concave rewards and convex constraints on the decisions across time, in addition to the customary limitation on the time horizon. This model subsumes the classic multi-armed bandit (MAB) model, and the Bandits with Knapsacks (BwK) model of Badanidiyuru et al. [2013]. We also consider an extension of this model to allow linear contexts, similar to the linear contextual extension of the MAB model. We demonstrate that a natural and simple extension of the UCB family of algorithms for MAB provides a polynomial time algorithm that has near-optimal regret guarantees for this substantially more general model, and matches the bounds provided by Badanidiyuru et al. [2013] for the special case of BwK, which is quite surprising. We also provide computationally more efficient algorithms by establishing interesting connections between this problem and other well studied problems/algorithms such as the Blackwell approachability problem, online convex optimization, and the Frank-Wolfe technique for convex optimization.

We give examples of several concrete applications, where this more general model of bandits allows for richer and/or more efficient formulations of the problem.

1. INTRODUCTION

Multi-armed bandit (henceforth, MAB) is a classic model for handling exploration-exploitation tradeoff inherent in many sequential decision making problems. MAB algorithms have found a wide variety of applications in clinical trials, web search, internet advertising, multi-agent systems, queuing and scheduling etc. The classic MAB framework however only handles "local" constraints and "local" rewards: the constraint is only on the decision in each step and the total reward is necessarily a summation of the rewards in each step. (The only constraint allowed on decisions accross time is a bound on the number of trials.) For many real world problems there are multiple complex constraints on resources that are consumed during the entire decision process. Further, in some applications it may be desirable to evaluate the solution not simply by the sum of rewards obtained at individual time steps, but by a more complex utility function. We illustrate several such example scenarios in our Applications section (Section 3). This paper, in succession to the recent results by Badanidiyuru et al. [2013], extends the MAB framework to handle very general "global" constraints and rewards.

Badanidiyuru et al. [2013] took the first step in this direction by successfully extending the MAB model to include linear knapsack constraints on the resources consumed over time. In their model, which they call Bandits with Knapsacks (BwK), decision at any time t results in a reward and a d-dimensional resource consumption vector, and there is a pre-specified budget representing the maximum amount of each resource that can be consumed in time t. Badanidiyuru et al. [2013] combine techniques from UCB family of algorithms for MAB, and techniques from online learning algorithms in a non-trivial manner to provide an algorithm with near-optimal regret guarantees for this problem.

In this paper, we introduce a substantial generalization of the BwK setting, to include arbitrary concave rewards and arbitrary convex constraints. In our vector-valued bandit model, decision at any time t results in the observation of a d-dimensional vector $\boldsymbol{v}_t$. There is a prespecified convex set S and a prespecified concave obective function f, and the goal is that the average of the observed vectors in time T belongs to the specified convex set while maximizing the concave objective. This is essentially the most general convex optimization problem. We refer to this model as "Bandits with Convex knapsacks and concave Rewards" (henceforth, BwCR). We also consider an ex-

tension of BwCR to allow contexts, simiar to the linear contextual bandits extesion of MAB [Chu et al. 2011]. BwCR subsumes BwK as a special case when the convex set is simply given by the knapsack constraints, and the objective function is linear. We discuss applications in several domains such as sensor measurements, network routing, crowdsourcing, pay-per-click advertising, which substantially benefit from the more general BwCR framework – either by admitting richer models, or by more efficient formulation of existing models.

Another important contribution of this paper is to demonstrate that a conceptually simple and natural extension of the UCB family of algorithms for MAB [Auer et al. 2002; Auer 2003] provides near-optimal regret bounds for this substantially more general BwCR setting, and even for the contextual version of BwCR. Even in the special case of BwK, this natural extension of UCB algorithm achieves regret bounds matching the problem-dependent lower (and upper) bounds provided by Badanidiyuru et al. [2013]. This is quite surprising and is in contrast to the discussion in Badanidiyuru et al. [2013], where the need for special techniques for this problem was emphasized, in order to achieve sublinear regret.

However, this natural extension of the UCB algorithm for BwCR, even though polynomial-time implementable (as we show in this paper), may not be very computationally efficient. For example, our UCB algorithm for the special case of BwK requires solving an LP with m variables and d constraints at every time step. In general, we show that one would require solving a convex optimization problem by ellipsoid method at every time step, for which computing separating hyperplanes itself needs another application of the ellipsoid algorithm.

Our final contribution is giving computationally more efficient algorithms by establishing (sometimes surprising) connections between the BwCR problem and other well studied problems/algorithms such as the Blackwell approachability problem [Blackwell 1956], online convex optimization [Zinkevich 2003], and the Frank-Wolfe (projection-free) algorithm for convex optimization [Frank and Wolfe 1956]. We provide two efficient algorithms, a "primal" algorithm based on the Frank-Wolfe algorithm and a "dual" algorithm based on the reduction of Blackwell approachability to online convex optimization [Abernethy et al. 2011]. One may be faster than the other depending on the properties of the objective function f and convex set S. As an aside, the primal algorithm establishes a connection between Blackwell's algorithm for the approachability problem and the Frank-Wolf algorithm. The dual algorithm turns out to be almost identical to the primal-dual algorithm (PD-BwK) of Badanidiyuru et al. [2013] for the special case of BwK problem.

2. PRELIMINARIES AND MAIN RESULTS

2.1. Bandit with knapsacks (BwK)

The following problem was called Bandit with Knapsacks (BwK) by Badanidiyuru et al. [2013]. There is a fixed and known finite set of m arms (possible actions), available to the learner, henceforth called the algorithm. There are d resources and finite time-horizon T, where T is known to the algorithm. In each time step t, the algorithm plays an arm i_t of the m arms, receives reward $r_t \in [0, 1]$, and consumes amount $c_{t,j} \in [0, 1]$ of each resource j. The reward r_t and consumption $c_t \in \mathbb{R}^d$ are revealed to the algorithm after choosing arm i_t. The rewards and costs in every round are generated i.i.d. from some unknown fixed underlying distribution. More precisely, there is some fixed but unknown $\mu \in \mathbb{R}^m, C \in \mathbb{R}^{d \times m}$ such that

$$\mathbb{E}[r_t | i_t] = \mu_{i_t}, \quad \mathbb{E}[c_{t,j}(t) | i_t] = C_{j,i_t}.$$

In the beginning of every time step t, the algorithm needs to pick i_t, using only the history of plays and outcomes until time step $t - 1$. There is a hard constraint of B_j

on the resource consumption of every j. The algorithm stops at the earliest time τ when one or more of the constraints is violated, i.e. if $\sum_{t=1}^{\tau} c_{t,j}(t) > B_j$ for some j, or if the time horizon ends, i.e. $\tau > T$. Its total reward is given by the sum of rewards in all rounds preceding τ, i.e. $\sum_{t=1}^{\tau-1} r_t$. The goal of the algorithm is to maximize the expected total reward. The values of B_j are known to the algorithm, and without loss of generality we can assume $B_j = B = \min_j B_j$ for all j. (Multiply each $c_{t,j}$ by B/B_j.)

Regret and Benchmark. Regret is defined as the difference in the total reward obtained by the algorithm and OPT, where OPT denotes the total expected reward for the optimal dynamic policy.

$$\text{regret}(T) = \text{OPT} - \sum_{1 \leq t < \tau} r_t. \tag{1}$$

For any $\boldsymbol{\mu}, C$, let $\text{LP}(\boldsymbol{\mu}, C)$ denote the value of the following linear program.

$$\begin{aligned} \max_{\boldsymbol{p}} \quad & \boldsymbol{\mu} \cdot \boldsymbol{p} \\ \text{s.t.} \quad & C\boldsymbol{p} \preceq \tfrac{B}{T}\mathbf{1}, \\ & \boldsymbol{p} \in \Delta_m \end{aligned} \tag{2}$$

where Δ_m denotes the m-dimensional simplex, i.e., $\Delta_m = \{\boldsymbol{p} : \sum_{i=1}^{m} p_i = 1, p_i \geq 0, i = 1, \ldots, m\}$, and, $\preceq, \succeq$ denote component-wise $\leq$ and $\geq$ respectively. It is easy to show that $\text{LP}(\boldsymbol{\mu}, C) \geq \frac{\text{OPT}}{T}$. (For example, see Devanur et al. [2011], or Lemma 3.1 of Badanidiyuru et al. [2013].) Hence $T \cdot \text{LP}(\boldsymbol{\mu}, C)$ is commonly used in place of OPT in the analysis of regret.

2.2. Bandits with concave rewards and convex knapsacks (BwCR)

In this paper we consider a substantial generalization of BwK, to include arbitrary concave rewards and arbitrary convex constraints. This is essentially the most general convex optimization problem. We consider the problem with only convex constraints (BwC), and the problem with only concave rewards (BwR) as special cases.

In the Bandits with concave rewards and convex knapsacks (BwCR) setting, on playing an arm i_t at time t, we observe a vector $\boldsymbol{v}_t \in [0,1]^d$ generated independent of the previous observations, from a fixed but unknown distribution such that $\mathbb{E}[\boldsymbol{v}_t | i_t] = \boldsymbol{V}_{i_t}$, where $\boldsymbol{V} \in [0,1]^{d \times m}$. We are given a convex set S, and a concave objective function $f : [0,1]^d \to [0,1]$. We further make the following assumption regarding Lipschitz continuity of f.

ASSUMPTION 1. *Assume that function f is L-lipschitz with respect to norm $|| \cdot ||$, i.e., $f(\boldsymbol{x}) - f(\boldsymbol{y}) \leq L||\boldsymbol{x} - \boldsymbol{y}||$. Since f is concave, this is equivalent to the condition that for all $\boldsymbol{x}$ in the domain of f, and all supergradients $\boldsymbol{g} \in \partial f(\boldsymbol{x})$, we have that $||\boldsymbol{g}||_* \leq L$, where $|| \cdot ||_*$ is the dual norm (refer to Lemma 2.6 in [Shalev-Shwartz 2012]).*

The goal is to make the average of the observed vectors $\frac{1}{T}\sum_t \boldsymbol{v}_t$ be contained in the set S, and at the same time maximize $f(\frac{1}{T}\sum_t \boldsymbol{v}_t)$. Let OPT_f denote the expected value of the optimal dynamic solution to this problem. Then, the following lemma provides a benchmark for defining regret. The proof follows simply from concavity of f, and is provided in Appendix A.

LEMMA 2.1. *There exists a distribution $\boldsymbol{p}^* \in \Delta_m$, such that $\boldsymbol{V}\boldsymbol{p}^* \in S$, and $f(\boldsymbol{V}\boldsymbol{p}^*) \geq OPT_f$.*

We minimize two kinds of regret: regret in objective and regret in constraints. The (average) regret in objective is defined as

$$\text{avg-regret}_1(T) := \text{OPT}_f - f(\tfrac{1}{T}\textstyle\sum_{t=1}^{T} \boldsymbol{v}_t) \leq f(\boldsymbol{V}\boldsymbol{p}^*) - f(\tfrac{1}{T}\textstyle\sum_{t=1}^{T} \boldsymbol{v}_t). \tag{3}$$

And, (average) regret in constraints is the distance of average observed vector from S,

$$\text{avg-regret}_2(T) := d(\tfrac{1}{T}\textstyle\sum_{t=1}^{T} \boldsymbol{v}_t, S), \tag{4}$$

where the distance function $d(\boldsymbol{x}, S)$ is defined as $||\boldsymbol{x} - \pi_S(\boldsymbol{x})||$, $\pi_S(\boldsymbol{x})$ is the projection of $\boldsymbol{x}$ on S, and $|| \cdot ||$ denotes an L_q norm.

Below, we describe some special cases and extensions of this setting.

Hard constraints. In some applications, the constraints involved are hard constraints, that is, it is desirable that they are satisfied with high probability even if at a cost of higher regret in the objective. Therefore, we may want to tradeoff the regret in distance from S for possibly more regret in objective f. While this may not be always doable, under following conditions a simple modification of our algorithm can achieve this: the set S and function f are such that it is easy to define and use a shrunken set S^ϵ for any $\epsilon \in [0, 1]$, defined as a subset of S such that points within a distance of ϵ from this set lie in S. And, S^ϵ contains at least one good point $\boldsymbol{V}\boldsymbol{p}$ with objective function value within $K\epsilon$ of the optimal value. More precisely,

$$d(\boldsymbol{x}, S^\epsilon) \le \epsilon \Rightarrow \boldsymbol{x} \in S, \qquad \text{and}$$
$$\exists \boldsymbol{p} \in \Delta_m : \boldsymbol{V}\boldsymbol{p} \in S^\epsilon, f(\boldsymbol{V}\boldsymbol{p}) \ge f(\boldsymbol{V}\boldsymbol{p}^*) - K\epsilon, \tag{5}$$

for some $K \ge 0$. A special case is when S is a downward closed set, f is linear, and distance is L_∞ distance. In this case, we can define $S^\epsilon = \{\boldsymbol{x}(1 - \epsilon), \forall \boldsymbol{x} \in S\}$, for which $\boldsymbol{V}\boldsymbol{p}^*(1 - \epsilon) \in S^\epsilon$ and $f(\boldsymbol{V}\boldsymbol{p}^*(1 - \epsilon)) \ge (1 - \epsilon)f(\boldsymbol{V}\boldsymbol{p})$.

In our algorithms, we will be able to simply substitute S^ϵ for S to achieve the desired tradeoff. This observation will be useful for BwK problem, which involves hard (downward closed) resource consumption constraints – the algorithm needs to abort when the resource constraints are violated.

Linear contextual version of BwCR. We also consider an extension of our techniques to the linear contextual version of the BwCR problem, which can be derived from the linear contextual bandits problem [Auer et al. 2002; Chu et al. 2011]. In this setting, every arm i and component j is associated with a context vector $\boldsymbol{b}_{ji}$, which is known to the algorithm. There is an unknown n-dimensional weight vector $\boldsymbol{w}_j$ for every component j, such that $\boldsymbol{V}_{ji} = \boldsymbol{b}_{ji} \cdot \boldsymbol{w}_j$. Note that effectively, the d n-dimensional weight vectors are the unknown parameters to be learned in this problem, where n could be much smaller than the number of arms m. Algorithms for contextual bandits are expected to take advantage of this structure of the problem to produce low regret guarantees even when the number of arms is large.

In a more general setting, the context vector for arm i could even change with time (but are provided to the algorithm before taking the decision at time t), however that can be handled with only notational changes to our solution, and for simplicity of illustration, we will restrict to static contexts in the main body of this paper.

BwK, BwR, and BwC as special cases. Observe that BwCR subsumes the BwK problem, on defining objective function $f(\boldsymbol{x}) = x_1$, and $S := \{\boldsymbol{x} : \boldsymbol{x}_{-1} \le \frac{B}{T}\boldsymbol{1}\}$. We define Bandits with concave Rewards (BwR) as a special case of BwCR when there are no constraints, i.e., the set $S = \mathbb{R}^n$. And, Bandits with Convex knapsacks (BwC) as the special case when the goal is only to satisfy the constraints, i.e. there is no objective function f. The average regret for BwR in time T is $\text{avg-regret}_1(T)$, and for BwC it is $\text{avg-regret}_2(T)$.

2.3. Summary of Results

Our main result is that a natural extension of UCB algorithm (Algorithm 1) for BwCR achieves bounds of

$$O(L||\mathbf{1}_d||\sqrt{\tfrac{m}{T}\ln(\tfrac{mTd}{\delta})}), \text{ and } O(||\mathbf{1}_d||\sqrt{\tfrac{m}{T}\ln(\tfrac{mTd}{\delta})}),$$

with probability $1 - \delta$, on the average regret in the objective (avg-regret$_1(T)$) and distance from constraint set (avg-regret$_2(T)$), respectively. Here $||\mathbf{1}_d||$ denotes the norm of d-dimensional vector of all 1's, with respect to the norm used in the Lipschitz condition of f, and, in defining the distance from set S, respectively.

We extend our results to the linear contextual version of BwCR, and provide an algorithm with average regret bounds of

$$O(Ln||\mathbf{1}_d||\sqrt{\tfrac{1}{T}\ln(\tfrac{Td}{\delta})}), \text{ and } O(n||\mathbf{1}_d||\sqrt{\tfrac{1}{T}\ln(\tfrac{Td}{\delta})}),$$

respectively, when contexts are of dimension n. Note that these regret bounds do not depend on the number of arms m, which is crucial when number of arms is large, possibly infinite.

Note that BwCR subsumes the MAB problem, and the contextual version of BwCR subsumes the linear contextual bandits problem, with $d = 1, L = 1$ and $S = \mathbb{R}^n$. And, our regret bounds for these problems match the lower bounds provided in Bubeck and Cesa-Bianchi [2012] (Section 3.3) and Dani et al. [2008], respectively, within logarithmic factors. A more refined problem-dependent lower bound (and matching upper bound) for the special case of BwK was provided in [Badanidiyuru et al. 2013]. We show that our UCB algorithm when specialized to this case (Algorithm 2) achieves a regret bound of

$$\mathbf{regret}(T) = O\left(\sqrt{\log(\tfrac{mdT}{\delta})}(\mathbf{OPT}\sqrt{\tfrac{m}{B}} + \sqrt{m\mathbf{OPT}} + m\sqrt{\log(\tfrac{mTd}{\delta})})\right),$$

which matches the bounds of [Badanidiyuru et al. 2013]. Thus, our UCB based algorithms provide near-optimal regret bounds. Precise statements of these results appear as Theorem 4.1 and Theorem 4.2.

Section 5 and 6 are devoted to developing a general framework for converting the UCB algorithm to fast algorithms. We provide algorithms BwC and BwR for which the arm selection problem at time t is simply of the form:

$$i_t = \arg\max_{i=1,\dots,m} \omega_{t,i}.$$

where $\omega_{t,i}$ for every i, can be computed using history until time $t - 1$ in $O(d)$ time. These fast algorithms can be viewed as approximate primal and dual implementations of the UCB algorithm, and come with a cost of increased regret, but we show that the regret increases by only constant factors. The derivation of these fast algorithms from UCB also provides interesting insights into connections between this problem, the Blackwell approachability problem, and the Frank-Wolfe projection technique for convex optimization, which may be of independent interest.

2.4. Related Work

The BwCR problem, as defined in the previous section, is closely related to the stochastic multi-armed bandits (MAB) problem, to the generalized secretary problems under stochastic assumption, and to the Blackwell approachability problem. As we mentioned in the introduction, the major difference between the classic MAB model and settings like BwCR (or BwK) is that the latter allow for "global" constraints – constraints on decisions accross time. The only global constraint allowed in the classic MAB model is the time horizon T.

Generalized secretary problems under i.i.d. distribution include online stochastic packing and covering problems (e.g., [Devanur et al. 2011], [Feldman et al. 2010]).

These problems involve "global" packing or covering constraints on decisions over time, as we have in BwCR. However, a major difference between the secretary problems and a bandit setting like BwCR is that in secretary problems, *before* taking the decision at time t the algorithm knows how much the reward or consumption (or in general v_t) will be for every possible decision. On the other hand, in the BwCR setting, v_t is revealed *after* the algorithm chooses the arm to play at time t. One of the ideas in this paper is to estimate the observations at time t by UCB estimates computed using only history til time $t-1$, and before choosing the arm i_t. This effectively reduces the problem to secretary problem, with error in the UCB estimates to account for in regret bounds.

Blackwell approachability problem considers a two player vector-valued game with a bi-affine payoff function, $r(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{p}^T M \boldsymbol{q}$. Further, it is assumed that for all $\boldsymbol{q}$, there exists a $\boldsymbol{p}$ such that $r(\boldsymbol{p}, \boldsymbol{q}) \in S$. The row player's goal is to direct the payoff vector to some convex set S. The Bandit with convex knapsacks (BwC) problem is closely related to the Blackwell approachability problem. The row player is the online algorithm and the column player is nature. However, in this case the nature always produces its outcome using a *fixed* (but unknown) mixed strategy (distribution) $\boldsymbol{q}^*$. Also, this means a weaker assumption should suffice: there exists a $\boldsymbol{p}^*$ for this particular $\boldsymbol{q}^*$, such that $r(\boldsymbol{p}^*, \boldsymbol{q}^*) \in S$ (stated as the assumption $\exists \boldsymbol{p}^*, \boldsymbol{V}\boldsymbol{p}^* \in S$). The bigger difference algorithmically is that there is nothing to statistically estimate in the Blackwell approachability problem, the only unknown is the column player strategy which may change every time. On the other hand, esitmating the expected consumption is inherently the core part of any algorithm for BwC.

Due to these differences, algorithms for none of these related problems directly solve the BwCR problem. Nonetheless, the similarities suffice to inspire many of the ideas for computationally efficient algorithms that we present in this paper.

The work closest to our work is that of Badanidiyuru et al. [2013] on the BwK problem. We successfully generalize their setting to include arbitrary convex constraints and concave objectives, as well as linear contexts. Additionally, we demonstrate that a simple and natural extension of UCB algorithm suffices to obtain optimal regret for BwCR which subsumes BwK, and provide generalized techniques for deriving multiple efficient implementations of this algorithm – one of which reduces to an algorithm similar to the PD-BwK algorithm of Badanidiyuru et al. [2013] for the speical case of BwK.

2.5. Fenchel duality

Fenchel duality will be used throughout the paper, below we provide some background on this useful mathematical concept. We define the Fenchel conjugate of f as

$$f^*(\boldsymbol{\theta}) := \max_{\boldsymbol{y} \in [0,1]^d} \{\boldsymbol{y} \cdot \boldsymbol{\theta} + f(\boldsymbol{y})\}$$

Suppose that f is a concave function defined on $[0,1]^d$, and as in Assumption 1, at every point $\boldsymbol{x}$, every supergradient $\boldsymbol{g}_x$ of f has bounded dual norm $||\boldsymbol{g}_x||_* \leq L$. Then, the following dual relationship is known between f and f^*. A proof is provided in Appendix A for completeness.

LEMMA 2.2. $f(\boldsymbol{z}) = \min_{||\boldsymbol{\theta}||_* \leq L} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \boldsymbol{z}$.

A special case is when $f(\boldsymbol{x}) = -d(\boldsymbol{x}, S)$ for some convex set S. This function is 1-Lipschitz with respect to norm $||\cdot||$ used in the definition of distance. In this case, $f^*(\boldsymbol{\theta}) = h_S(\boldsymbol{\theta}) := \max_{\boldsymbol{y} \in S} \boldsymbol{\theta} \cdot \boldsymbol{y}$, and Lemma 2.2 specializes to

$$d(\boldsymbol{x}, S) = \max_{||\boldsymbol{\theta}||_* \leq 1} \boldsymbol{\theta} \cdot \boldsymbol{x} - h_S(\boldsymbol{\theta}).$$

The derivation of this equality also appears in Abernethy et al. [2011].

2.6. Notations

We use bold alphabets or bold greek letters for vectors, and bold capital letters for matrices. Most matrices used in this paper will be $d \times m$ dimensional, and for a matrix $\boldsymbol{A}$, A_{ji} denotes its ji^{th} element, $\boldsymbol{A}_i$ denotes its ith column vector, and $\boldsymbol{A}_j$ its jth row vector. For matrices which represent time dependent estimates, we use $\boldsymbol{A}_t$ for the matrix at time t, and $\boldsymbol{A}_{t,i}$, $\boldsymbol{A}_{t,j}$ and $A_{t,ji}$ for its ith column, j^{th} row, and ji component, respectively. For two vectors $\boldsymbol{x}, \boldsymbol{y}$, $\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the inner product.

3. APPLICATIONS

Below, we demonstrate that BwCR setting and its extension to contextual bandits allows us to effectively handle much richer and complex models in applications like sensor networks, crowdsourcing, pay-per-click advertising etc., than those permitted by multi-armed bandits (MAB), or bandits with knapsacks (BwK) formulations. While some of these simply cannot be formulated in the MAB or BwK frameworks, others would require an exponential blowup of dimensions to convert the convex constraints to linear knapsack or covering constraints.

Sensor networks, network routing. Consider a sensor network with m sensors, each sensor i covering a subset A_i of N points, where $N >> m$, and N could even be exponential compared to m. Taking a reading from any sensor costs energy. Also, a sensor measurement may fail with probability q_i. The aim is to take atmost T measurements such that each point has at least b successful readings. We are given that there exists a strategy for selecting the sensors, so that in expectation these covering constraints can be satisfied. A strategy corresponds to a distribution $\boldsymbol{p} \in \Delta_m$ such that you measure sensor i with probability p_i. We are given that

$$\exists \boldsymbol{p}^* \in \Delta_m, T \sum_{i:k \in A_i} p_i^* q_i \geq b, \forall k = 1, \ldots, N.$$

We can model this as BwC by having $\boldsymbol{v}_t \in \{0,1\}^m$ (i.e., $d = m$), where on playing arm i_t, $\boldsymbol{v}_{t,i_t}$ denotes whether the sensor i_t was successfully measured or not: $\boldsymbol{v}_{t,i_t} = \boldsymbol{e}_{i_t}$ with probability q_{i_t}, and $\boldsymbol{0}$ otherwise, and $\mathbb{E}[\boldsymbol{v}_t|i_t] = \boldsymbol{V}_{i_t}$ where $\boldsymbol{V}$ is am $m \times m$ diagonal matrix with $\boldsymbol{V}_{ii} = q_i$. Define S as

$$S = \{\boldsymbol{x} \in [0,1]^m : \sum_{i:k \in A_i} x_i \geq \tfrac{b}{T}, k = 1, \ldots, N\}.$$

Note that S is an m-dimensional convex set. Then, we wish to achieve $\frac{1}{T}\sum_{t=1}^T \boldsymbol{v}_t \in S$. And, from above there exists $\boldsymbol{p}^* \in \Delta_m$ such that $\boldsymbol{V}\boldsymbol{p}^* \in S$. Our algorithms we will obtain $O(||\boldsymbol{1}_m||\sqrt{\frac{m}{T}\log(\frac{mT}{\delta})})$ regret as per the results metioned above ($d = m$).

Note that if we try to frame this problem in terms of linear covering constraints, we need to make $\boldsymbol{v}_t$ to be N dimensional (i.e, $d := N$), where $N >> m$. On playing arm i, $\boldsymbol{v}_t = \boldsymbol{e}_{A_i}$ with probability p_i. Then, the constraints can be written as linear constraints $\sum_t \boldsymbol{v}_{t,j} \geq b, j = 1, \ldots, N$. However, in that case, $d = N$ will result in a $||\boldsymbol{1}_N||\sqrt{\log(N)}$ term in the regret bound, which can be exponentially worse than $||\boldsymbol{1}_m||\sqrt{\log(m)}$.

Similar applications include *crowdsourcing* a survey or data collection task, where workers are sensors each covering his/her (overlapping) neighborhood, and *network monitoring*, where monitors located at some nodes of the network are sensors, each covering a subset of the entire network.

Another similar application is *network routing*, where routing requests are arriving online. There is a small number (d) of request types, and the hidden parameters to learn are expected usage for each type of request. But, there is a capacity constraint on each of the $N >> d$ edges. Then, modeling it as BwK would get an $||\boldsymbol{1}_N||\sqrt{\log(N)}$ term in the regret bound instead of $||\boldsymbol{1}_d||\sqrt{\log(d)}$.

Pay-per-click advertising. Pay-per-click advertising is one of the most touted applications for MAB, where explore-exploit tradeoff is observed in ad click-through rate (CTR) predictions. Our BwCR formulation with its contextual extension can considerably enrich the MAB formulations of this problem. Contexts are considered central to the effective use of bandit techniques in this problem – the CTR for an ad impression depends on the (query, ad) combination, and there are millions of these combinations, thus millions of arms. Contextual setting allows a compact representation of these arms as n-dimensional feature (context) vectors, and aims at learning the best weight vector that maps features to CTR.

BwCR allows using the contextual setting along with multiple complex constraints on the decision process over time. In addition to simple budget constraints for every advertiser/campaign, we can efficiently represent budget constraints on family of overlapping subset of those, without blowing up the dimension d, as explained in some of our earlier applications.

The ability to maximize a concave reward function is also very useful for such applications. Although in most models of pay-per-click advertising the reward is some simple linear function, the reality is more complex. A typical consideration is that advertisers (in dislay advertising) desire a certain mixture of different demographics such as equal number of men and women, or equal number of clicks from different cities. These are not hard constraints – the closer to the ideal mixture, the better it is. This is naturally modeled as a concave reward function of the vector of the number of clicks of each type the advertiser recieves.

Further, we can now admit more nuanced risk-sensitive constraints. This includes convex risk functions on budget expenditure or on distance from the target click or revenue performance.

4. UCB FAMILY OF ALGORITHMS

In this section, we present algorithms derived from the UCB family of algorithms [Auer et al. 2002] for the multi-armed bandit problems. We demonstrate that simple extensions of UCB algorithm provide near-optimal regret bounds for BwCR and all its extensions introduced earlier. In particular, our UCB algorithm will match the optimal regret bound provided by Badanidiyuru et al. [2013] for the special case of BwK.

We start with some background on the UCB algorithm for classic multi-armed bandit problem. In the classic multi-armed bandit problem there are m arms and on playing an arm i_t at time t, a reward r_t is generated i.i.d. with fixed but unknown mean μ_{i_t}. The objective is to choose arms in an online manner in order to minimize regret defined as $\sum_{t=1}^{T}(\mu_{i^*} - r_t)$, where $i^* = \arg\max_i \mu_i$.

UCB algorithm for multi-armed bandits was introduced in Auer et al. [2002]. The basic idea behind this family of algorithms is to use the observations from the past plays of each arm i at time t to construct estimates ($\text{UCB}_{t,i}$) for the mean reward μ_i. These estimates are constructed to satisfy the following key properties.

(1) The estimate $\text{UCB}_{t,i}$ for every arm is guaranteed to be larger than its mean reward with high probability, i.e., it is an *Upper Confidence Bound* on the mean reward.

$$\text{UCB}_{t,i} \geq \mu_i, \forall i, t$$

(2) As an arm is played more and more, its estimate should approach the actual mean reward, so that with high probability, the total difference between estimated and actual reward for the *played arms* can be bounded as

$$|\sum_{t=1}^{T}(\text{UCB}_{t,i_t} - r_t)| \leq \tilde{O}(\sqrt{mT}).$$

This holds irrespective of how the arm i_t is chosen.

At time t, the UCB algorithm simply plays the best arm according to the current estimates, i.e., the arm with the highest value of $\text{UCB}_{t,i}$.

$$i_t = \arg\max_i \text{UCB}_{t,i}.$$

Then, a corollary of the first property above, and the choice of arm made by algorithm, is that with high probability,

$$\text{UCB}_{t,i_t} \geq \mu_{i^*}.$$

Using above observations, it is straightforward to bound the regret of this algorithm in time T.

$$\text{regret}(T) = \sum_{t=1}^{T}(\mu_{i^*} - r_t) \leq \sum_{t=1}^{T}(\text{UCB}_{t,i_t} - r_t) \leq \tilde{O}(\sqrt{mT}).$$

In our UCB based algorithms, we use this same basic idea for algorithm design and regret analysis.

4.1. Bandits with concave rewards and convex knapsacks (BwCR)

Since, our observation vector cannot be interpreted as cost or reward, we construct both lower and upper confidence bounds, and consider the range of estimates defined by these. More precisely, for every arm i and component j, we construct two estimates $\text{LCB}_{t,ji}(\boldsymbol{V})$ and $\text{UCB}_{t,ji}(\boldsymbol{V})$ at time t, using the past observations. The estimates for each component are constructed in a manner similar to the estimates used in the UCB algorithm for classic MAB, and satisfy the following generalization of the properties mentioned above.

(1) The mean for every arm i and component j is guaranteed to lie in the range defined by its estimates $\text{LCB}_{t,ji}(\boldsymbol{V})$ and $\text{UCB}_{t,ji}(\boldsymbol{V})$ with high probability. That is,

$$\boldsymbol{V} \in \mathcal{H}_t, \text{ where,} \tag{6}$$

$$\mathcal{H}_t := \{\tilde{\boldsymbol{V}} : \tilde{V}_{ji} \in [\text{LCB}_{t,ji}(\boldsymbol{V}), \text{UCB}_{t,ji}(\boldsymbol{V})], j = 1, \ldots, d, i = 1, \ldots, m\}. \tag{7}$$

(2) Let arm i is played with probability $p_{t,i}$ at time t. Then, the total difference between estimated and actual observations for the *played arms* can be bounded as

$$||\sum_{t=1}^{T}(\tilde{\boldsymbol{V}}_t \boldsymbol{p}_t - \boldsymbol{v}_t)|| \leq \mathcal{Q}(T), \tag{8}$$

for any $\{\tilde{\boldsymbol{V}}_t\}_{t=1}^{T}$ such that $\tilde{\boldsymbol{V}}_t \in \mathcal{H}_t$. Here, $\mathcal{Q}(T)$ is typically $\tilde{O}(||\mathbf{1}_d||\sqrt{mT})$.

A direct generalization of Property (2) from the MAB analysis mentioned before would have been a bound on $||\sum_{t=1}^{T}(\tilde{\boldsymbol{V}}_{t,i_t} - \boldsymbol{v}_t)||$. However, since we will choose a distribution $\boldsymbol{p}_t$ over arms at time t and sample i_t from this distribution, the form of bound in (8) is more useful, and a straightforward extension. A specialized expression for $\mathcal{Q}(T)$ in terms of problem specific parameters will be obtained in the specific case of BwK. As before, these are purely properties of the constructed estimates, and hold irrespective of how the choice of $\boldsymbol{p}_t$ is made by an algorithm.

At time t, our UCB algorithm plays the best arm (or, best distribution over arms) according to the best estimates in set $\mathcal{H}_t$.

ALGORITHM 1: UCB Algorithm for BwCR

    for all $t = 1, 2, \ldots, T$ **do**

$$\boldsymbol{p}_t = \begin{array}{c} \arg\max\limits_{\boldsymbol{p} \in \Delta_m} \max\limits_{\tilde{\boldsymbol{U}} \in \mathcal{H}_t} f(\tilde{\boldsymbol{U}}\boldsymbol{p}) \\ \text{s.t. } \min_{\tilde{\boldsymbol{V}} \in \mathcal{H}_t} d(\tilde{\boldsymbol{V}}\boldsymbol{p}, S) \leq 0 \end{array} \tag{9}$$

    If no feasible solution is found to the above problem, set $\boldsymbol{p}_t$ arbitrarily.
    Play arm i with probability $p_{t,i}$.
    end for

Observe that when $f(\cdot)$ is a monotone non-decreasing function as in the classic MAB problem (where $f(x) = x$), the inner maximizer in objective of (9) will be simply $\tilde{U}_t = \mathrm{UCB}_t(V)$, and therefore, for classic MAB problem this algorithm reduces to the UCB algorithm.

Let $\tilde{U}_t, \tilde{V}_t$ denote the inner maximizer and the inner minimizer in the problem (9). Then, a corollary of the first property above (refer to Equation (6)) is that with high probability,

$$f(\tilde{U}_t p_t) \geq f(V p^*), \quad \tilde{V}_t p_t \in S. \tag{10}$$

This is because the conditions $V \in \mathcal{H}_t$ and $V p^* \in S$ imply that $(p, \tilde{V}, \tilde{U}) = (p^*, V, V)$ forms a feasible solution for problem (9) at time t.

Using these observations, it is easy to bound the regret of this algorithm in time T. With high probability,

$$\mathrm{avg\text{-}regret}_1(T) \leq f(V p^*) - f(\tfrac{1}{T}\sum_{t=1}^T v_t) \leq f(\tilde{U}_t p_t) - f(\tfrac{1}{T}\sum_{t=1}^T v_t) \leq \tfrac{L}{T}\mathcal{Q}(T),$$

$$\mathrm{avg\text{-}regret}_2(T) = d(\tfrac{1}{T}\sum_{t=1}^T v_t, S) \leq d(\tfrac{1}{T}\sum_{t=1}^T v_t, \tfrac{1}{T}\sum_{t=1}^T \tilde{V}_t p_t) \leq \tfrac{1}{T}\mathcal{Q}(T), \tag{11}$$

where $\mathcal{Q}(T) = \tilde{O}(\|\mathbf{1}_d\|\sqrt{mT})$. Below is a precise statement for the regret bound.

THEOREM 4.1. *With probability $1 - \delta$, the regret of Algorithm 1 is bounded as*

$$\mathit{avg\text{-}regret}_1(T) = O(L\|\mathbf{1}_d\|\sqrt{\tfrac{\gamma m}{T}}), \quad \mathit{avg\text{-}regret}_2(T) = O(\|\mathbf{1}_d\|\sqrt{\tfrac{\gamma m}{T}})$$

where $\gamma = O(\log(\tfrac{mTd}{\delta}))$, $\mathbf{1}_d$ is the d dimensional vector of all 1's.

The detailed proof with exact expressions for $\mathrm{UCB}_t(V), \mathrm{LCB}_t(V)$ is in Appendix B.1.

4.1.1. Extensions.

Linear Contextual Bandits.. It is straightforward to extend Algorithm 1 to linear contextual bandits, using existing work on UCB family of algorithms for this problem. Using techniques in Abbasi-yadkori et al. [2012]; Auer [2003], instead of the hypercube $\mathcal{H}_t$ at time t, one can obtain an ellipsoid such that the weight vector w_j is guaranteed to lie in this ellipsoid, for every component j. Then, simply substituting $\mathcal{H}_t$ with these ellipsoids in Algorithm 1 will provide an algorithm for the linear contextual version of BwCR with regret bounds

$$\mathrm{avg\text{-}regret}_1(T) = O(Ln\|\mathbf{1}_d\|\sqrt{\tfrac{\gamma}{T}}), \quad \mathrm{avg\text{-}regret}_2(T) = O(n\|\mathbf{1}_d\|\sqrt{\tfrac{\gamma}{T}}),$$

with probability $1 - \delta$. Here $\gamma = O(\log(\tfrac{mTd}{\delta}))$. Further details are in Appendix B.2.

Hard constraints. In this case, a shrunket set S^ϵ can be used instead of S in Algorithm 1 (refer to Section 2.2 for definition of S^ϵ), with ϵ set to be an upper bound on $\mathrm{avg\text{-}regret}_2(T)$. For example, ϵ can be set as $\|\mathbf{1}_d\|\sqrt{\tfrac{\gamma m}{T}}$ using results in Theorem 4.1. Then, at the end of time horizon, with probability $1 - \delta$, the algorithm will satisfy,

$$d(\tfrac{1}{T}\sum_t v_t, S^\epsilon) \leq \epsilon \Rightarrow \tfrac{1}{T}\sum_t v_t \in S, \text{ and } \mathrm{avg\text{-}regret}_1(T) = O(L\|\mathbf{1}_d\|\sqrt{\tfrac{\gamma m}{T}} + K\epsilon).$$

4.2. Bandit with knapsacks (BwK)

This is a special case of BwCR with $v_t = \{r_t; c_t\}$, $f(x) = x_1$, and $S = \{x : x_{-1} \preceq \tfrac{B}{T}\mathbf{1}\}$. Then, the problem (9) in Algorithm 1 reduces to the following LP.

$$\begin{aligned} \max_{p \in \Delta_m} \quad & \mathrm{UCB}_t(\mu) \cdot p \\ \text{s.t.} \quad & \mathrm{LCB}_t(C)p \preceq \tfrac{B}{T}\mathbf{1}, \end{aligned} \tag{12}$$

where $\mathrm{UCB}_t(\mu) \in [0,1]^m$ denotes the UCB estimate constructed for μ and $\mathrm{LCB}_t(C) \in [0,1]^{d \times m}$ denotes the LCB estimate for C. Above is same as $\mathrm{LP}(\mathrm{UCB}_t(\mu), \mathrm{LCB}_t(C))$ (refer to Equation (2)).

Since this problem requires hard constraints on resource consumption, we would like to tradeoff the regret in constraint satisfaction for some more regret in reward. As discussed in Section 4.1.1, one way to achieve this is to use a shrunken constraint set. For any $\boldsymbol{\mu}, C$, we define $\mathrm{LP}(\boldsymbol{\mu}, C, \epsilon)$ by tightneing the constraints in $\mathrm{LP}(\boldsymbol{\mu}, C)$ by a $1 - \epsilon$ factor, i.e. replacing B by $(1 - \epsilon)B$. Then, at time t, the algorithms simply solves $\mathrm{LP}(\mathrm{UCB}_t(\boldsymbol{\mu}), \mathrm{LCB}_t(C), \epsilon)$ instead of $\mathrm{LP}(\mathrm{UCB}_t(\boldsymbol{\mu}), \mathrm{LCB}_t(C))$.

ALGORITHM 2: UCB algorithm for BwK

---

for all $t = 1, 2, \ldots, T$ **do**
    Exit if any resource consumption is more than B.
    Solve $\mathrm{LP}(\mathrm{UCB}_t(\boldsymbol{\mu}), \mathrm{LCB}_t(C), \epsilon)$, and let $\boldsymbol{p}_t$ denote the solution for this linear program.
    Play arm i with probability $p_{t,i}$.
end for

THEOREM 4.2. *For the BwK problem, with probability $1 - \delta$, the regret of Algorithm 2 with $\epsilon = \sqrt{\frac{\gamma m}{B}} + \log(T)\frac{\gamma m}{B}$, $\gamma = \log(\frac{mTd}{\delta})$, is bounded as*

$$\textbf{regret}(T) = O\left(\sqrt{\log(\tfrac{mdT}{\delta})}(\textbf{OPT}\sqrt{\tfrac{m}{B}} + \sqrt{m\textbf{OPT}} + m\sqrt{\log(\tfrac{mTd}{\delta})})\right).$$

PROOF. We use the same estimates for each component as in the previous section, to construct $\mathrm{UCB}_t(\boldsymbol{\mu})$ and $\mathrm{LCB}_t(C)$. We show that these UCB and LCB estimates satisfy the following more specialized versions of the properties given by Equation (6) and (8). With probability $1 - (mTd)e^{-\Omega(\gamma)}$,

$$(1) \qquad\qquad \mathrm{UCB}_t(\boldsymbol{\mu}) \succeq \boldsymbol{\mu}, \mathrm{LCB}_t(C) \preceq C. \qquad\qquad (13)$$

$$(2) \qquad \begin{aligned} \sum_{t=1}^{T}(\mathrm{UCB}_t(\boldsymbol{\mu}) \cdot \boldsymbol{p}_t - r_t)| &\leq O(\sqrt{\gamma m \left(\sum_t r_t\right)} + \gamma m), \\ |\sum_{t=1}^{T}(\mathrm{LCB}_t(C)\boldsymbol{p}_t - \boldsymbol{c}_t)| &\preceq \epsilon B \mathbf{1}. \end{aligned} \qquad (14)$$

Proof of the second property is similar to Lemma 7.4 of [Badanidiyuru et al. 2013], and is provided in Appendix B.3 for completeness.

Then, similar to (10), following is a corollary of the first property and the choice made by the algorithm at time step t.

$$\sum_{t=1}^{T} \mathrm{UCB}_t(\boldsymbol{\mu}) \cdot \boldsymbol{p}_t = \mathrm{LP}(\mathrm{UCB}_t(\boldsymbol{\mu}), \mathrm{LCB}_t(C), \epsilon) \geq \mathrm{LP}(\boldsymbol{\mu}, C, \epsilon) \geq (1 - \epsilon)\mathrm{OPT},$$
$$\sum_{t=1}^{T} \mathrm{LCB}_t(C)\boldsymbol{p}_t \preceq (1 - \epsilon)B\mathbf{1}. \qquad (15)$$

Then, using the second property above and (15), $\sum_{t=1}^{T} \boldsymbol{c}_t \leq B\mathbf{1}$, and the algorithm will not terminate before time T. This means that the total reward for the algorithm will be given by $\mathrm{ALGO} = \sum_{t=1}^{T} r_t$. Also, using the second property,

$$\mathrm{ALGO} = \sum_{t=1}^{T} r_t \geq (1 - \epsilon)\mathrm{OPT} - O(\sqrt{\gamma m\, \mathrm{ALGO}}) - O(\gamma m)$$

Therefore, either $\mathrm{ALGO} \geq \mathrm{OPT}$ or

$$\mathrm{ALGO} \geq (1 - \epsilon)\mathrm{OPT} - O(\sqrt{\gamma m\mathrm{OPT}}) - O(\gamma m).$$

Now, assuming $m\gamma \leq O(B)$, [1] $\epsilon \mathrm{OPT} = O(\mathrm{OPT}\sqrt{\frac{m\gamma}{B}})$. Therefore,

$$\textbf{regret}(T) = \mathrm{OPT} - \mathrm{ALGO} \leq O\left(\mathrm{OPT}\sqrt{\tfrac{\gamma m}{B}} + \sqrt{\gamma m\mathrm{OPT}} + \gamma m\right).$$

---

[1] This assumption was also made in [Badanidiyuru et al. 2013]

Then, substituting $\gamma = \Theta(\log(\frac{mTd}{\delta}))$, we get the desired result. □

4.3. Implementability

Next, we investigate whether our UCB algorithm is efficiently implementable. For the special case of BwK problem, this reduces to Algorithm 2 which only requires solving an LP at every step. However, the poynomial-time implementability of Algorithm 1 is not so obvious. Below, we prove that the problem (9) required to be solved in every time step t is in fact a convex optimization problem, with separating hyperplanes computable in polynomial time. Thus, this problem can be solved by ellipsoid method, and every step of Algorithm 2 can be implemented in polynomial time.

LEMMA 4.3. *The functions* $\psi(\boldsymbol{p}) := \max_{\tilde{\boldsymbol{U}} \in \mathcal{H}_t} f(\tilde{\boldsymbol{U}}\boldsymbol{p})$, *and* $g(\boldsymbol{p}) = \min_{\tilde{\boldsymbol{V}} \in \mathcal{H}_t} d(\tilde{\boldsymbol{V}}\boldsymbol{p}, S)$
are concave and convex functions respectively, and the subgradients for these functions at any given point can be computed in polynomial time using ellipsoid method for convex optimization.

The proof of above lemma is provided in Appendix B.1.

5. COMPUTATIONALLY EFFICIENT ALGORITHMS FOR BWC AND BWR

In the UCB algorithm for BwCR, at every time step t, we need to solve the optimization problem (9). Even though this can be done in polynomial time (Lemma 4.3), this is an expensive step. It requires solving a convex optimization problem in $\boldsymbol{p}$ (possibly using ellipsoid method), for which computing the separating hyperplane at any point itself requires solving a convex optimization problem (again, possibly using ellipsoid method). For practical reasons, it is desirable to have a faster algorithm. In this section, we present alternate algorithms that are very efficient computationally at the expense of a slight increase in regret. The regret bounds remain the same in the $O(\cdot)$ notation and the increase is only in the constants. We present two such algorithms, a "primal" algorithm based on the Frank-Wolfe algorithm [Frank and Wolfe 1956] and a "dual" algorithm based on the reduction of the Blackwell approachability problem to online convex optimization (OCO) in Abernethy et al. [2011]. In this section, for simplicity of illustration, we consider only the BwC and BwR problems, i.e., the problem with only constraint set S, and the problem with only the objective function f, respectively. In Section 6 we show that one could use any combination of these algorithms, or the UCB algorithm, for each of BwC and BwR to get an algorithm for BwCR.

The basic idea is to replace the convex optimization problem with its "linearization", which turns out to be a problem of optimizing a linear function over the unit simplex, and hence very easy to solve. For the BwC problem, the convex optimization problem (9) specializes to finding a $\boldsymbol{p}_t$ such that $\tilde{\boldsymbol{V}}\boldsymbol{p}_t \in S$ for some $\tilde{\boldsymbol{V}} \in \mathcal{H}_t$. In our "linearized" version, instead of this, we will only need to find a $\boldsymbol{p}_t$ such that $\tilde{\boldsymbol{V}}\boldsymbol{p}_t$ is in a halfspace containing the set S. A half space that contains S and is tangential to S is given by a vector $\boldsymbol{\theta}$; such a halfspace is $H_S(\boldsymbol{\theta}) := \{\boldsymbol{x} : \boldsymbol{\theta} \cdot \boldsymbol{x} \le h_S(\boldsymbol{\theta})\}$, where $h_S(\boldsymbol{\theta}) := \max_{\boldsymbol{s} \in S} \boldsymbol{\theta} \cdot \boldsymbol{s}$. Now given a $\boldsymbol{\theta}_t$ in time step t, a point in $H_S(\boldsymbol{\theta}_t)$ can be found by simply minimizing $\boldsymbol{\theta}_t \cdot \boldsymbol{x}$, which is a linear function. This is exactly what the algorithm does, at each time step t, it picks a vector $\boldsymbol{\theta}_t$ and sets

$$(\boldsymbol{p}_t, \tilde{\boldsymbol{V}}_t) = \arg \min_{\boldsymbol{p} \in \Delta_m} \min_{\tilde{\boldsymbol{V}} \in H_t} \boldsymbol{\theta}_t \cdot (\tilde{\boldsymbol{V}}\boldsymbol{p}). \tag{16}$$

The inner minimization is actually trivial and the optimal solution is at a vertex of $\mathcal{H}_t$, independent of the value of $\boldsymbol{p}$, i.e., $\tilde{\boldsymbol{V}}_t = \boldsymbol{Z}_t(\boldsymbol{\theta}_t)$, where

$$\boldsymbol{Z}_t(\boldsymbol{\theta})_{ji} := \begin{cases} \text{UCB}_{t,ji}(\boldsymbol{V}), & \theta_j \le 0, \\ \text{LCB}_{t,ji}(\boldsymbol{V}), & \theta_j > 0 \end{cases}, \tag{17}$$

for $j = 1, \ldots, d, i = 1, \ldots, m$.

With this observation, the outer minimization is also quite simple, since it optimizes a linear function over the unit simplex and the optimal solution occurs at one of the vertices. It is solved by setting $\boldsymbol{p}_t = \mathbf{e}_{i_t}$, where

$$i_t = \arg \min_{i \in \{1, \ldots, m\}} \boldsymbol{\theta}_t \cdot \tilde{\boldsymbol{V}}_{t,i}. \tag{18}$$

Hence given $\boldsymbol{\theta}_t$, the procedure for picking the arm i_t is quite simple.

A generalization of this idea is used for BwR: instead of optimizing f, we optimize a linear function that is tangential to f. A linear function that is tangential to f at a point $\boldsymbol{y}$ (and is an upper bound on f since f is concave) is

$$l_f(\boldsymbol{x}; \boldsymbol{y}) := f(\boldsymbol{y}) + \nabla f(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq f(\boldsymbol{x}) \ \forall \boldsymbol{x}, \boldsymbol{y}.$$

Then, instead of maximizing $f(\tilde{\boldsymbol{V}}\boldsymbol{p})$ as in (9), we maximize $l_f(\tilde{\boldsymbol{V}}\boldsymbol{p}; \boldsymbol{y}_t)$ over $\tilde{\boldsymbol{V}} \in \mathcal{H}_t$ and $\boldsymbol{p} \in \Delta_m$, for some $\boldsymbol{y}_t$. The latter is equivalent to minimizing $\boldsymbol{x} \cdot \boldsymbol{\theta}_t$ where $\boldsymbol{\theta}_t = -\nabla f(\boldsymbol{y}_t)$, therefore $\boldsymbol{p}_t$ is still set as per (16) (which reduces to the simple rule in (18)).

We introduce some notation here, let $\boldsymbol{x}_t := \tilde{\boldsymbol{V}}_t \boldsymbol{p}_t$, $\boldsymbol{x}^* := \boldsymbol{V} \boldsymbol{p}^*$, $\bar{\boldsymbol{x}}_{1:t} := \frac{1}{T} \sum_{s=1}^t \boldsymbol{x}_s$ and $\bar{\boldsymbol{v}}_{1:t} := \frac{1}{T} \sum_{s=1}^t \boldsymbol{v}_s$.

The regret bound for the UCB algorithm followed rather straight-forwardly from the two properties (6) and (8), but the regret bounds for these algorithms will not be as easy. For one, we no longer have (10), instead we have the corresponding relations for l_f and H_S respectively:

$$\begin{aligned} l_f(\boldsymbol{x}_t; \boldsymbol{y}_t) &\geq l_f(\boldsymbol{x}^*; \boldsymbol{y}_t) \geq f(\boldsymbol{x}^*), \\ \boldsymbol{x}_t &\in H_S(\boldsymbol{\theta}_t). \end{aligned} \tag{19}$$

Since we don't have that $f(\boldsymbol{x}_t) \geq f(\boldsymbol{x}^*)$ (or $\boldsymbol{x}_t \in S$), the main task is to bound $f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T})$ (and $d(\bar{\boldsymbol{x}}_{1:T}, S)$), and these will be extra terms in the regret bound. In particular, (11) is replaced by

$$\begin{aligned} \text{avg-regret}_1(T) &\leq f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{v}}_{1:T}) \leq f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T}) + f(\bar{\boldsymbol{x}}_{1:T}) - f(\bar{\boldsymbol{v}}_{1:T}) \\ &\leq f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T}) + \frac{L}{T}\mathcal{Q}(T). \end{aligned} \tag{20}$$

$$\text{avg-regret}_2(T) = d(\bar{\boldsymbol{v}}_{1:T}, S) \leq d(\bar{\boldsymbol{v}}_{1:T}, \bar{\boldsymbol{x}}_{1:T}) + d(\bar{\boldsymbol{x}}_{1:T}, S) \leq \frac{1}{T}\mathcal{Q}(T) + d(\bar{\boldsymbol{x}}_{1:T}, S).$$

The bounds on $f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T})$ and $d(\bar{\boldsymbol{x}}_{1:T}, S)$ will depend on the choice of $\boldsymbol{\theta}_t$s. Each of the two algorithms we present provides a specific method for choosing $\boldsymbol{\theta}_t$s to achieve desired regret bounds.

5.1. The dual algorithm

This algorithm is inspired by the reduction of the Blackwell approachability problem to online convex optimization (OCO) in Abernethy et al. [2011]. It is also related to the fast algorithms to solve covering/packing LPs using multiplicative weight update [Devanur et al. 2011] and the algorithm of Badanidiyuru et al. [2013]. In fact, we give a *reduction* to OCO; any algorithm for OCO can then be used.

In OCO, the algorithm has to pick a vector, say $\boldsymbol{\theta}_t$ in each time step t. (The domain of $\boldsymbol{\theta}_t$ is such that $||\boldsymbol{\theta}_t||_* \leq L$ for our purpose here, where L is the Lipschitz constant of f, and $L = 1$ for distance function.) Once $\boldsymbol{\theta}_t$ is picked the algorithm observes a convex *loss* function, g_t, and the process repeats. The objective is to minimize regret defined as

$$\mathcal{R}^c(T) := \sum_{t=1}^T g_t(\boldsymbol{\theta}_t) - \min_{||\boldsymbol{\theta}||_* \leq L} \sum_{t=1}^T g_t(\boldsymbol{\theta}).$$

Recall from our discussion earlier, in each step t, the algorithm sets $\boldsymbol{p}_t$ as per (16) for some $\boldsymbol{\theta}_t$. The choice of $\boldsymbol{\theta}_t$ is via a reduction to OCO: we define a convex function g_{t-1}

based on the history upto time $t - 1$ which is then fed as input to the OCO algorithm, whose output $\boldsymbol{\theta}_t$ is used in picking $\boldsymbol{p}_t$. We first define g_t for the BwR problem; the g_t for BwC is obtained as a special case with $f(\boldsymbol{x}) = -d(\boldsymbol{x}, S)$. Define

$$g_t(\boldsymbol{\theta}) := f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \boldsymbol{x}_t,$$

where f^* is the Fenchel conjugate of f,(see Section 2.5 for the definition), and $\boldsymbol{x}_t = \tilde{\boldsymbol{V}}_t \boldsymbol{p}_t$.

ALGORITHM 3: Fenchel dual based algorithm for BwR

Inititalize $\boldsymbol{\theta}_1$.
for all $t = 1, 2, \ldots, T$ **do**
    Set $(\boldsymbol{p}_t, \tilde{\boldsymbol{V}}_t) = \arg\min_{\boldsymbol{p} \in \Delta_m, \tilde{\boldsymbol{V}} \in \mathcal{H}_t} \boldsymbol{\theta}_t \cdot (\tilde{\boldsymbol{V}} \boldsymbol{p})$.
    Play arm i with probability $p_{t,i}$.
    Choose $\boldsymbol{\theta}_{t+1}$ by doing an OCO update for the convex function $g_t(\boldsymbol{\theta}) = f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot (\tilde{\boldsymbol{V}}_t \boldsymbol{p}_t)$.
end for

The following geometric intuition for the Fenchel conjugate is useful in the analysis: if $\boldsymbol{y}_t = \arg\max_{\boldsymbol{y}} \{\boldsymbol{y} \cdot \boldsymbol{\theta}_t + f(\boldsymbol{y})\}$ then $-\boldsymbol{\theta}_t \in \nabla f(\boldsymbol{y}_t)$ and $f^*(\boldsymbol{\theta}_t) = \boldsymbol{y}_t \cdot \boldsymbol{\theta}_t + f(\boldsymbol{y}_t) = l_f(\boldsymbol{0}; \boldsymbol{y}_t)$, i.e., $f^*(\boldsymbol{\theta}_t)$ is the y-intercept of $l_f(\boldsymbol{x}; \boldsymbol{y}_t)$. We can therefore rewrite $l_f(\boldsymbol{x}; \boldsymbol{y}_t)$ in terms of f^* as follows

$$l_f(\boldsymbol{x}; \boldsymbol{y}_t) = f^*(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t \cdot \boldsymbol{x}.$$

With this and (19), we have

$$g_t(\boldsymbol{\theta}_t) = f^*(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t \cdot \boldsymbol{x}_t = l_f(\boldsymbol{x}_t; \boldsymbol{y}_t) \geq f(\boldsymbol{x}^*).$$

The above inequality states that the optimum of BwR is bounded above by what the algorithm gets for OCO. We next show that the optimum value for OCO is equal to what the algorithm of BwR gets, so there is a flip. This will produce bound on $f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T})$ in terms of $\mathcal{R}^c(T)$.

Note that for a fixed $\boldsymbol{\theta}$, g_t's differ only in the linear term, so the average of g_t's for all t is equal to $f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \bar{\boldsymbol{x}}_{1:T}$. Then, minimizing this over all $\boldsymbol{\theta}$ gives $f(\bar{\boldsymbol{x}}_{1:T})$, by Lemma 2.2.

$$\min_{||\boldsymbol{\theta}||_* \leq L} \tfrac{1}{T} \sum_t g_t(\boldsymbol{\theta}) = \min_{||\boldsymbol{\theta}||_* \leq L} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \bar{\boldsymbol{x}}_{1:T} = f(\bar{\boldsymbol{x}}_{1:T}).$$

These two observations together give

$$f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T}) \leq \tfrac{1}{T} \sum_t g_t(\boldsymbol{\theta}_t) - \min_{||\boldsymbol{\theta}||_* \leq L} \tfrac{1}{T} \sum_t g_t(\boldsymbol{\theta}) = \tfrac{1}{T} \mathcal{R}^c(T).$$

Algorithm for BwC. The algorithm for BwC is obtained by letting $f(\boldsymbol{x}) = -d(\boldsymbol{x}, S)$. Note that for this function $L = 1$. Also, it can be shown that $f^*(\boldsymbol{\theta}) = h_S(\boldsymbol{\theta})$, therefore $g_t(\boldsymbol{\theta}) := h_S(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \boldsymbol{x}_t$. And, using the same calculations as in above, we will get

$$d(\bar{\boldsymbol{x}}_{1:T}, S) \leq \tfrac{1}{T} \mathcal{R}^c(T).$$

This and (20) imply the following theorem.

THEOREM 5.1. *With probability $1 - \delta$, the regret of Algorithm 3 is bounded as*

$$avg\text{-}regret_1(T) = O\left(L||\boldsymbol{1}_d|| \sqrt{\tfrac{\gamma m}{T}} + \tfrac{\mathcal{R}^c(T)}{T}\right) \text{ for BwR, and}$$

$$avg\text{-}regret_2(T) = O\left(||\boldsymbol{1}_d|| \sqrt{\tfrac{\gamma m}{T}} + \tfrac{\mathcal{R}^c(T)}{T}\right) \text{ for BwC,}$$

when used with $f(\boldsymbol{x}) = -d(\boldsymbol{x}, S)$. Here $\gamma = \log(\tfrac{mTd}{\delta})$, and $\mathcal{R}^c(T)$ is the regret for the OCO method used.

In case of Eucledian norm, online gradient descent (OGD) can be used to get $\mathcal{R}^c(T) = \tilde{O}(GD\sqrt{T})$, where G is an upper bound on Eucledian norm of subgradient of g_t, and D is an upper bound on Eucledian norm of $\boldsymbol{\theta}$ (refer to Zinkevich [2003], and Corollary 2.7 in Shalev-Shwartz [2012]). For our purpose, $G \leq \sqrt{d}$ and $D \leq L$. For other norms FoRel algorithm with appropriate regularization may provide better guarantees. For example, when $||\cdot||$ is L_∞ norm (i.e., $||\cdot||_*$ is L_1), we can use FoRel algorithm with Entropic regularization (essentially a generalization of the Hedge algorithm [Freund and Schapire 1995]), to obtain an improved bound of $O(L\sqrt{T\log(d)})$ on $\mathcal{R}^c(T)$ (refer to Corollary 2.14 in Shalev-Shwartz [2012]).

Implementability. OCO algorithms like online gradient descent require gradient computation. In this case, we need to compute the gradient of the dual f^* (that is why we call it the dual algorithm) which can be computed as $\arg\max_{\boldsymbol{y}}\{\boldsymbol{\theta}\cdot\boldsymbol{y} + f(\boldsymbol{y})\}$. for a given $\boldsymbol{\theta}$.

5.2. The primal algorithm

The algorithm presented in Section 5.1 required computing the gradient of the Fenchel dual f^* which may be computationally expensive in some cases. Here we present a primal algorithm (for BwR) that requires computing the gradient of f in each step, based on the Frank-Wolfe algorithm [Frank and Wolfe 1956]. A caveat is that this requires a stronger assumption on f, that f is smooth in the following sense.

ASSUMPTION 2. *We call concave function $f(\cdot)$ to be β-smooth if*

$$f(\boldsymbol{z} + \alpha(\boldsymbol{y} - \boldsymbol{z})) \geq f(\boldsymbol{z}) + \alpha\nabla f(\boldsymbol{z})\cdot(\boldsymbol{y} - \boldsymbol{z}) - \frac{\beta}{2}\alpha^2, \qquad (21)$$

for all $\boldsymbol{y}, \boldsymbol{z} \in [0,1]^d$ and $\alpha \in [0,1]$. If f is such that the gradient *of f is Lipshitz continuous (with respect to any L_q norm) with a constant G, then $\beta \leq Gd$.*

Note that the distance function ($f(\boldsymbol{z}) = -d(\boldsymbol{z}, S)$) does not satisfy this assumption.

Like the Fenchel dual based algorithm, in each step, this algorithm too picks a $\boldsymbol{\theta}_t$ and sets $\boldsymbol{p}_t$ according to (16). The difference is that $\boldsymbol{\theta}_t$ is now simply $-\nabla f(\bar{\boldsymbol{x}}_{1:t-1})$!

ALGORITHM 4: Frank-Wolfe based primal algorithm for BwR

   for all $t = 1, 2, \ldots, T$ **do**
$$(\boldsymbol{p}_t, \tilde{\boldsymbol{V}}_t) = \arg\max_{\boldsymbol{p}\in\Delta_m}\max_{\tilde{\boldsymbol{V}}\in\mathcal{H}_t}(\tilde{\boldsymbol{V}}\boldsymbol{p})\cdot\nabla f(\bar{\boldsymbol{x}}_{1:t-1}),$$
     where $\boldsymbol{x}_t = \tilde{\boldsymbol{V}}_t\boldsymbol{p}_t$. Play arm i with probability $p_{t,i}$.
   end for

THEOREM 5.2. *With probability $1 - \delta$, the regret of Algorithm 4 for BwR problem with β-smooth function f (Assumption 2), is bounded as*

$$\text{avg-regret}_1(T) = O(L||\boldsymbol{1}_d||\sqrt{\tfrac{\gamma m}{T}} + \tfrac{\beta\log(T)}{T}).$$

PROOF. Using (20), proving this regret bound essentially means bounding $f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T})$. This quantity can be bounded by $\frac{\beta\log(2T)}{2T}$ using techniques similar to those used in the analysis of Frank-Wolfe algorithm for convex optimization [Frank and Wolfe 1956]. The complete proof is provided in Appendix C. □

5.3. Smooth approximation of Non-smooth f

Assumption 2 may be stronger than Assumption 1. For instance, for distance function ($f(\boldsymbol{z}) = -d(\boldsymbol{z}, S)$) Assumption 1 is satisfied with $L = 1$, but not Assumption 2. In this section, we show how to use the technique of [Nesterov 2005] to convert a non-smooth f that only satisfies Assumption 1 into one that satisfies Assumption 2. For

simpicity, we assume $|| \cdot ||$ to be Eucledian norm in this section. Interestingly, for the smooth approximation of distance function, this algorithm will have essentially the same structure as the (primal) algorithm for the Blackwell approachability problem, thus drawing a connection between two well known algorithms.

THEOREM 5.3. *[Nesterov 2005] Define*

$$\hat{f}_\eta(\boldsymbol{z}) := \min_{||\boldsymbol{\theta}|| \leq L} \{ f^*(\boldsymbol{\theta}) + \tfrac{\eta}{2L} \boldsymbol{\theta} \cdot \boldsymbol{\theta} - \boldsymbol{\theta} \cdot \boldsymbol{z} \}. \tag{22}$$

Then, $\hat{f}_\eta$ is concave, differentiable, and $\frac{dL}{\eta}$-smooth. Further, $\hat{f}_\eta - \frac{\eta}{2}L \leq f \leq \hat{f}_\eta$.

Now, if we run Algorithm 4 on $\hat{f}_\eta$, with $\eta = \sqrt{\frac{d}{T} \log(2T)}$, we get that $f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:T}) \leq \frac{\beta \log(2T)}{2T} + \frac{\eta}{2}L \leq \frac{L}{2}\sqrt{\frac{d \log(2T)}{T}}$. The algorithm and regret bound for BwC can be obtained similarly by using this smooth approximation for distance function, i.e., for $f(\boldsymbol{z}) = -d(\boldsymbol{z}, S)$. We thus obtain the following theorem.

THEOREM 5.4. *With probability $1 - \delta$, the regret of Algorithm 4 when used with smooth approximation $\hat{f}_\eta(\boldsymbol{z})$ of function $f(\boldsymbol{z})$ (or, $-\hat{d}_\eta(\boldsymbol{z}, S)$ of function $-d(\boldsymbol{z}, S)$), is bounded as*

$$\textit{avg-regret}_1(T) = O(L||\mathbf{1}_d||\sqrt{\tfrac{\gamma m}{T}} + L\sqrt{\tfrac{d \log(T)}{T}}) \textit{ for BwR, and}$$

$$\textit{avg-regret}_2(T) = O(||\mathbf{1}_d||\sqrt{\tfrac{\gamma m}{T}} + \sqrt{\tfrac{d \log(T)}{T}}) \textit{ for BwC.}$$

For the distance function, this smooth approximation has some nice characteristics.

LEMMA 5.5. *For the distance function $d(\boldsymbol{z}, S)$, (22) provides smooth approximation $\hat{d}_\eta(\boldsymbol{z}, S) = \max_{||\boldsymbol{\theta}|| \leq 1} \boldsymbol{\theta} \cdot \boldsymbol{z} - h_S(\boldsymbol{\theta}) - \frac{\eta}{2} \boldsymbol{\theta} \cdot \boldsymbol{\theta}$, and, the gradient of this function is given by*

$$\nabla \hat{d}_\eta(\boldsymbol{z}) = \begin{cases} \frac{\boldsymbol{z} - \pi_S(\boldsymbol{z})}{||\boldsymbol{z} - \pi_S(\boldsymbol{z})||} & \textit{if } ||\boldsymbol{z} - \pi_S(\boldsymbol{z})|| \geq \eta \\ \frac{\boldsymbol{z} - \pi_S(\boldsymbol{z})}{\eta} & \textit{if } 0 < ||\boldsymbol{z} - \pi_S(\boldsymbol{z})|| < \eta \ , \\ \mathbf{0} & \textit{if } \boldsymbol{z} \in S \end{cases}$$

where $\pi_S(\boldsymbol{z})$ denotes the projection of $\boldsymbol{z}$ on S.

The proof of the above lemma along with a proof of Theorem 5.3 is in Appendix C. Note that for Algorithm 4 only the direction of the gradient of f matters, and in this case the direction of gradient of $f = -\hat{d}_\eta$ at $\boldsymbol{z}$ is $-(\boldsymbol{z} - \pi_S(\boldsymbol{z}))$ for all $\boldsymbol{z} \notin S$. For $\boldsymbol{z} \in S$, the gradient is $\mathbf{0}$, which means it does not really matter what $\boldsymbol{p}$ is picked. Therefore, Algorithm 4 reduces to the following.

ALGORITHM 5: Frank-Wolfe based primal algorithm for BwC

> **for all** $t = 1, 2, \ldots, T$ **do**
>   If $\bar{\boldsymbol{x}}_{1:t-1} \in S$, set $\boldsymbol{p}_t$ arbitrarily.
>   If $\bar{\boldsymbol{x}}_{1:t-1} \notin S$, find projection $\pi_S(\bar{\boldsymbol{x}}_{1:t-1})$ of this point on S. And compute
> $(\boldsymbol{p}_t, \tilde{\boldsymbol{V}}_t) = \arg\min_{\boldsymbol{p} \in \Delta_m} \min_{\tilde{\boldsymbol{V}} \in \mathcal{H}_t} (\tilde{\boldsymbol{V}}\boldsymbol{p}) \cdot (\bar{\boldsymbol{x}}_{1:t-1} - \pi_S(\bar{\boldsymbol{x}}_{1:t-1})),$
>   Play arm i with probability $p_{t,i}$.
> **end for**

Algorithm 5 has the same structure as the Blackwell's algorithm for the approachability problem [Blackwell 1956], which asks to play anything at time t if $\bar{\boldsymbol{x}}_{1:t-1}$ is in S. Otherwise, find a point $\boldsymbol{x}_t$ such that $\boldsymbol{x}_t - \bar{\boldsymbol{x}}_{1:t-1}$ makes a negative angle with $(\bar{\boldsymbol{x}}_{1:t-1} - \pi_S(\bar{\boldsymbol{x}}_{1:t-1}))$. We have $\boldsymbol{x}_t = \tilde{\boldsymbol{V}}_t \boldsymbol{p}_t$. However, the proof of convergence of Blackwell's algorithm as given in [Blackwell 1956] seems to be different from the proof derived here, via the smooth approximation and Frank-Wolfe type analysis. This gives

an interesting connection between well known algorithms, Blackwell's algorithm for the approachability problem and Frank-Wolfe algorithm for convex optimization, via Nesterov's method of smooth approximations!!

Implementability. The algorithm with smooth approximation needs to compute the gradient of $\hat{f}_\eta$ in each step and in general there is no easy method to compute this, except in some special cases like the distance function discussed above. Alternatively, one could use the smooth approximation $\hat{f}_\eta(z) = \mathbb{E}_{u \in \mathbb{B}}[f(z + \delta u)]$ given by [Flaxman et al. 2005], which has slightly worse smoothness coefficient but has easy-to-compute gradient by sampling.

6. COMPUTATIONALLY EFFICIENT ALGORITHMS FOR BWCR

Any combination of the primal and dual approaches mentioned in the previous sections can be used to get an efficient algorithm for the BwCR problem. Using the observations in Equation (16) and (17), we obtain an algorithm with the following structure.

ALGORITHM 6: Efficient algorithm for BwCR

Inititalize θ_1.
for all $t = 1, 2, \ldots, T$ **do**
$$p_t = \begin{array}{c} \arg\min_{p \in \Delta_m} \\ \text{s.t.} \end{array} \begin{array}{c} (\theta_t \cdot Z_t(\theta_t))p \\ (\phi_t \cdot Z_t(\phi_t))p \le h_S(\phi_t). \end{array} \tag{23}$$

Play arm i with probability $p_{t,i}$. Compute θ_{t+1}, ϕ_{t+1}.
end for

Here, $Z_t(\cdot)$ is a vertex of $\mathcal{H}_t$ as defined in (17). Now, either primal or dual approach can be used to update θ, irrespective of what approach is being used for updating ϕ, and vice-versa. The choice between primal and dual approach will depend on properties of f and S, e.g., whether it is easy to compute the gradient of f or its dual f^*. It is easy to derive regret bounds for this efficient algorithm for BwCR using results in the previous section.

THEOREM 6.1. *For Algorithm 6, avg-regret$_1$(T) is given by Theorem 5.1, Theorem 5.2, or Theorem 5.4, respectively, depending on whether the dual, primal, or primal approach with smooth approximation is used for updating θ. And, avg-regret$_2$(T) is given by Theorem 5.1 or Theorem 5.4, respectively, depending on whether the dual or primal approach is used for updating ϕ.*

One can also substitute the constraint or objective in (23) by the corresponding expression in Equation (9) of the UCB algorithm, if efficiency is not as much of a concern as regret for either constraint or objective.

Implementability. Every step t of Algorithm 6 requires solving a linear optmization problem over simplex with one additional linear constraint. This is a major improvement in efficiency over Algorithm 1, which required solving a difficult convex optimization problem over domain $\{p\tilde{V} : \tilde{V} \in \mathcal{H}_t, p \in \Delta_m\}$.

Also, Algorithm 6 is particularly simple to implement when the given application allows *not* playing *any* arm at a given time step, i.e. relaxing the constraint $\sum_i p_i = 1$ to $\sum_i p_i \le 1$. This is true in many applications, for example, an advertiser is allowed to not participate in a given auction. In particular, in BwK, the algorithm can abort at any time step, effectively chosing not to play any arm in the remaining time steps. In such an application, if further $h_S(\phi_t) \ge 0$, $\phi_t^T Z_t(\phi_t)_i > 0, \forall i$, (23) is a special case of fractional knapsack problem, and the greedy optimal solution in this case reduces to simply choosing the arm i that minimizes $\frac{\theta_t^T Z_t(\theta_t)_i}{\phi_t^T Z_t(\phi_t)_i}$, and playing it with probabil-

ity p, where $p \in [0,1]$ is the highest value satisfying $\phi_t^T \mathbf{Z}_t(\phi_t)_i p \leq h_S(\phi_t)$. Even if $\phi_t^T \mathbf{Z}_t(\phi_t)_i \leq 0$ for some arms i, some simple tweaks to this greedy choice work.

In the special case of BwK, it is not difficult to compute that $-\boldsymbol{\theta}_t^T \mathbf{Z}_t(\boldsymbol{\theta}_t)_i = \text{UCB}_{t,i}(\boldsymbol{\mu})$, $\phi_t^T \mathbf{Z}_t(\phi_t)_i = \phi_t^T \text{LCB}_{t,i}(\boldsymbol{C})$, and $h_S(\phi_t) = \frac{B}{T}$ for all t, so that the above greedy rule simply becomes that of selecting arm

$$i_t = \arg\max_i \frac{\text{UCB}_{t,i}(\boldsymbol{\mu})}{\phi_t^T \text{LCB}_{t,i}(\boldsymbol{C})},$$

and playing it with largest probability p such that $\phi_t^T \text{LCB}_{t,i}(\boldsymbol{C})p \leq \frac{B}{T}$. This is remarkably similar to the PD-BwK algorithm of Badanidiyuru et al. [2013], except that their algorithm plays this greedy choice with probability 1 and aborts when any constraint is violated.

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A. PRELIMINARIES

PROOF OF LEMMA 2.1. For a random instance of the problem, let $\tilde{p}_i$ denote the empirical probability of playing arm i in the *optimal instance specific solution in hindsight*, and $\boldsymbol{v}_t$ denote the observation vector at time t. Then, it must be true that $\frac{1}{T}\sum_t \boldsymbol{v}_t \in S$. Let $\boldsymbol{p}^* = \mathbb{E}[\tilde{\boldsymbol{p}}]$. Then,

$$\mathbb{E}[\frac{1}{T}\textstyle\sum_t \boldsymbol{v}_t] = \frac{1}{T}\mathbb{E}[\textstyle\sum_t \mathbb{E}[\boldsymbol{v}_t|i_t]] = \mathbb{E}[\boldsymbol{V}\tilde{\boldsymbol{p}}_t] = \boldsymbol{V}\boldsymbol{p}^*.$$

So that, due to convexity of S, $\frac{1}{T}\sum_t \boldsymbol{v}_t \in S$ implies that $\boldsymbol{V}\boldsymbol{p}^* \in S$. And, by concavity of f,

$$\text{OPT}_f \le \mathbb{E}[f(\frac{1}{T}\sum_t \boldsymbol{v}_t)] \le f(\mathbb{E}[\frac{1}{T}\sum_t \boldsymbol{v}_t]) = f(\boldsymbol{V}\boldsymbol{p}^*).$$

□

PROOF OF LEMMA 2.2.

$$\min_{||\boldsymbol{\theta}||_* \le L} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \boldsymbol{z} = \min_{||\boldsymbol{\theta}||_* \le L} \max_{\boldsymbol{y}} \{\boldsymbol{y} \cdot \boldsymbol{\theta} + f(\boldsymbol{y}) - \boldsymbol{\theta} \cdot \boldsymbol{z}\}$$
$$= \max_{\boldsymbol{y}} \min_{||\boldsymbol{\theta}||_* \le L} \{\boldsymbol{y} \cdot \theta + f(\boldsymbol{y}) - \boldsymbol{\theta} \cdot \boldsymbol{z}\}.$$

The last equality uses minmax theorem. Now, by our assumption, for any $\boldsymbol{z}$, there exists a vector $||\boldsymbol{g}_z||_* \le L$ which is a supergradient of f at $\boldsymbol{z}$, i.e.,

$$f(\boldsymbol{y}) - f(\boldsymbol{z}) \le \boldsymbol{g}_z \cdot (\boldsymbol{y} - \boldsymbol{z}), \forall \boldsymbol{y}.$$

Therefore, for all $\boldsymbol{y}$,

$$\min_{||\theta|| \le L} \{\boldsymbol{y} \cdot \theta + f(\boldsymbol{y}) - \theta \cdot \boldsymbol{z}\} \le (-\boldsymbol{g}_z) \cdot \boldsymbol{y} + f(\boldsymbol{y}) - (-\boldsymbol{g}) \cdot \boldsymbol{z} \le f(\boldsymbol{z}),$$

with equality achieved for $\boldsymbol{y} = \boldsymbol{z}$. □

B. UCB FAMILY OF ALGORITHMS

We will use the following concentration theorem.

LEMMA B.1. *[Kleinberg et al. 2008; Babaioff et al. 2012; Badanidiyuru et al. 2013] Consider some distribution with values in $[0, 1]$, and expectation ν. Let $\hat{\nu}$ be the average of N independent samples from this distribution. Then, with probability at least $1 - e^{-\Omega(\gamma)}$, for all $\gamma > 0$,*

$$|\hat{\nu} - \nu| \le \text{rad}(\hat{\nu}, N) \le 3\text{rad}(\nu, N), \tag{24}$$

where $\text{rad}(\nu, N) = \sqrt{\frac{\gamma \nu}{N}} + \frac{\gamma}{N}$. More generally this result holds if $X_1, \dots, X_N \in [0, 1]$ are random variables, $N\hat{\nu} = \sum_{t=1}^N X_t$, and $N\nu = \sum_{t=1}^N \mathbb{E}[X_t|X_1, \dots, X_{t-1}]$.

LEMMA B.2. *[Badanidiyuru et al. 2013]*
For any two vectors $\boldsymbol{a}, \boldsymbol{n} \in \mathbb{R}_+^m$,

$$\sum_{j=1}^m \text{rad}(a_j, n_j)n_j \le \sqrt{\gamma m (\boldsymbol{a} \cdot \boldsymbol{n})} + \gamma m.$$

B.1. BwCR

LEMMA B.3. *Define empirical average $\widehat{V}_{t,ji}$ for each arm i and component j at time t as*

$$\widehat{V}_{t,ji} = \frac{\sum_{s<t:i_s=i} v_{t,j}}{k_{t,i} + 1}, \tag{25}$$

where $k_{t,i}$ is the number of plays of arm i before time t. Then, $\widehat{V}_{t,ji}$ is close to the actual mean V_{ji}: for every i, j, t, with probability $1 - e^{-\Omega(\gamma)}$,

$$|\widehat{V}_{t,ji} - V_{t,ji}| \leq 2\text{rad}(\widehat{V}_{t,ji}, k_{t,i} + 1).$$

PROOF. This proof follows from application of Lemma B.1. We apply Lemma B.1 to $v_{1,j}, \ldots, v_{T,j}$, for each j, using $\mathbb{E}[v_{t,j}|i_t] = V_{t,ji_t}$, to get that with probability at least $1 - e^{-\Omega(\gamma)}$,

$$
\begin{aligned}
|\widehat{V}_{t,ji} - V_{t,ji}| &\leq \frac{k_{t,i}}{k_{t,i} + 1} \cdot \text{rad}(\widehat{V}_{t,ji}, k_{t,i}) + \frac{V_{ji}}{k_{t,i} + 1} \\
&\leq \text{rad}(\widehat{V}_{t,ji}, k_{t,i} + 1) + \frac{V_{ji}}{k_{t,i} + 1} \\
&\leq 2\text{rad}(\widehat{V}_{t,ji}, k_{t,i} + 1).
\end{aligned}
$$

□

PROOF OF THEOREM 4.1. We use the following estimates

$$
\begin{aligned}
\text{UCB}_{t,ji}(\boldsymbol{V}) &= \min\{1, \widehat{V}_{t,ji} + 2\text{rad}(\widehat{V}_{t,ji}, k_{t,i} + 1)\}, \\
\text{LCB}_{t,ji}(\boldsymbol{V}) &= \max\{0, \widehat{V}_{t,ji} - 2\text{rad}(\widehat{V}_{t,ji}, k_{t,i} + 1)\},
\end{aligned}
\tag{26}
$$

for $i = 1, \ldots, m, j = 1, \ldots, d, t = 1, \ldots, T$. Here $\text{rad}(\nu, N) = \sqrt{\frac{\gamma\nu}{N}} + \frac{\gamma}{N}$, $k_{t,i}$ is the number of plays of arm i before time t, and $\widehat{V}_{t,ji}$ is the empirical average as defined in Equation (25). These estimates are similar to those used in literature on UCB algorithm for classic MAB and to those used in [Badanidiyuru et al. 2013].

Then, using concentration Lemma B.1, we will prove that the properties in Equation (6) and (8) hold with probability $1 - (mTd)e^{-\Omega(\gamma)}$, and with $\mathcal{Q}(T) = O(\|\mathbf{1}_d\|\sqrt{\gamma mT})$ where $\|\mathbf{1}_d\|$ denotes the norm of d dimensional vector of all 1s. Theorem 4.1 will then follow from the calculations in Equation (11).

Property (1) stated as Equation (6) is obtained as a corollary of Lemma B.3 by taking a union bound for all i, j, t. With probability $1 - (mTd)e^{-\Omega(\gamma)}$,

$$\text{UCB}_{t,ji}(\boldsymbol{V}) \geq V_{ji} \geq \text{LCB}_{t,ji}(\boldsymbol{V}), \forall i, j, t.$$

Next, we prove Property (2) stated in Equation (8). Given that arm i was played with probability $p_{t,i}$ at time t, for any $\{\tilde{\boldsymbol{V}}_t\}_{t=1}^T$ such that $\tilde{\boldsymbol{V}}_t \in H_t$ for all t, we will show that with probability $1 - (mTd)e^{-\Omega(\gamma)}$,

$$\|\sum_{t=1}^T (\tilde{\boldsymbol{V}}_t \boldsymbol{p}_t - \boldsymbol{v}_t)\| = O(\|\mathbf{1}_d\|\sqrt{\gamma mT}).$$

We use the observation that $\mathbb{E}[\boldsymbol{v}_t|i_t] = \boldsymbol{V}_{i_t}$. Then, using concentration Lemma B.1, with probability $1 - de^{-\Omega(\gamma)}$

$$|\sum_{t=1}^T (V_{ji_t} - v_{t,j})| \leq 3\text{rad}(\frac{1}{T}\sum_{t=1}^T V_{ji_t}, T) = O(\sqrt{\gamma T}), \tag{27}$$

for all $j = 1, \ldots, d$. Therefore, it remains to bound $\sum_t (\tilde{\boldsymbol{V}}_t \boldsymbol{p}_t - \boldsymbol{V}_{i_t})$. Again, since $\mathbb{E}[\tilde{\boldsymbol{V}}_{t,i_t}|\boldsymbol{p}_t, \tilde{\boldsymbol{V}}_t] = \tilde{\boldsymbol{V}}_t \boldsymbol{p}_t$, we can obtain, using Lemma B.1,

$$|\sum_{t=1}^T (\tilde{V}_{t,ji_t} - \tilde{\boldsymbol{V}}_{t,j}\boldsymbol{p}_t)| \leq 3\text{rad}(\frac{1}{T}\sum_{t=1}^T \tilde{\boldsymbol{V}}_{t,j}\boldsymbol{p}_t, T) = O(\sqrt{\gamma T}), \tag{28}$$

for all j with probability $1 - de^{-\Omega(\gamma)}$. Now, it remains to bound $|\sum_{t=1}^{T}(\tilde{V}_{t,ji_t} - V_{ji_t})|$. Using Lemma B.3, with probability $1 - (mTd)e^{-\Omega(\gamma)}$, for all i, j, t,

$$|\widehat{V}_{t,ji} - V_{t,ji}| \leq 2\mathrm{rad}(\widehat{V}_{t,ji}, k_{t,i} + 1).$$

Applying this, along with the observation that for any $\tilde{V} \in \mathcal{H}_t$, $\mathrm{LCB}_{t,ij}(V) \leq \tilde{V}_{t,ji} \leq \mathrm{UCB}_{t,ji}(V)$, we get

$$
\begin{aligned}
|\textstyle\sum_{t=1}^{T}(\tilde{V}_{t,ji_t} - V_{ji_t})| &\leq \left(\sum_t 4\mathrm{rad}(\widehat{V}_{t,ji_t}, k_{t,i_t} + 1)\right) \\
&= \left(\sum_i \sum_{N=1}^{k_{T,i}+1} 4\mathrm{rad}(\widehat{V}_{N,ji}, N)\right) \\
&\leq 4\left(\sum_i (k_{T,i} + 1)\mathrm{rad}(1, k_{T,i} + 1)\right) \\
&\leq O(\sqrt{\gamma mT}). \qquad (29)
\end{aligned}
$$

where we used $\widehat{V}_{N,i}$ to denote the empirical average for i^{th} arm over its past $N-1$ plays. In the last inequality, we used Lemma B.2 along with the observation that $\sum_{i=1}^{m} k_{T,i} = T$. Equation (27), (28), and (29) together give

$$||\textstyle\sum_{t=1}^{T}\boldsymbol{v}_t - \boldsymbol{V}\boldsymbol{p}_t|| \leq O(||\boldsymbol{1}_d||\sqrt{\gamma mT}).$$

□

PROOF OF LEMMA 4.3. We use Fenchel duality to derive an equivalent expression for f: $f(\boldsymbol{x}) = \min_{||\boldsymbol{\theta}||_* \leq L} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot \boldsymbol{x}$ (refer to Section 2.5). Then,

$$\psi(\boldsymbol{p}) = \max_{\tilde{\boldsymbol{U}} \in \mathcal{H}_t} \min_{||\boldsymbol{\theta}||_* \leq L} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot (\tilde{\boldsymbol{U}}\boldsymbol{p}) = \min_{||\boldsymbol{\theta}||_* \leq L} f^*(\boldsymbol{\theta}) - \min_{\tilde{\boldsymbol{U}} \in \mathcal{H}_t} \boldsymbol{\theta} \cdot (\tilde{\boldsymbol{U}}\boldsymbol{p}),$$

by application of the minimax theorem.

Now, due to the structure of set $\mathcal{H}_t$, observe that for any given $\boldsymbol{\theta}$, a vertex $\boldsymbol{Z}_t(\boldsymbol{\theta})$ (as defined in Equation (17)) of $\mathcal{H}_t$ minimizes $\boldsymbol{\theta} \cdot \tilde{\boldsymbol{U}}$ componentwise. Therefore, irrespective of what $\boldsymbol{p}$ is,

$$\psi(\boldsymbol{p}) = \min_{||\boldsymbol{\theta}||_* \leq L} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot (\boldsymbol{Z}_t(\boldsymbol{\theta})\boldsymbol{p}),$$

which is a concave function, and a subgradient of this function at a point $\boldsymbol{p}$ is $-\boldsymbol{\theta}'^T \boldsymbol{Z}_t(\boldsymbol{\theta}')$, where $\boldsymbol{\theta}'$ is the minimizer of the above expression. The minimizer

$$\boldsymbol{\theta}' = \arg\min_{||\boldsymbol{\theta}||_* \leq L} \left(\max_{\tilde{\boldsymbol{U}} \in \mathcal{H}_t} f^*(\boldsymbol{\theta}) - \boldsymbol{\theta} \cdot (\tilde{\boldsymbol{U}}\boldsymbol{p})\right)$$

is computable (e.g., by ellipsoid method) because it minimizes a convex function in $\boldsymbol{\theta}$, with subgradient $\partial f^*(\boldsymbol{\theta}) - \boldsymbol{Z}_t(\boldsymbol{\theta})\boldsymbol{p}$ at point $\boldsymbol{\theta}$.

The same analysis can be applied for $g(\boldsymbol{p})$, by using $f(\boldsymbol{x}) = -d(\boldsymbol{x}, S)$. □

B.2. Linear contextual Bandits

It is straightforward to extend Algorithm 1 to linear contextual bandits, using existing work on UCB family of algorithms for this problem. Recall that in the contextual setting a n-dimensional context vector $\boldsymbol{b}_{ji}$ is associated with every arm i and component j, and there is an unknown weight vector $\boldsymbol{w}_j$ for every component j, such that

$V_{ji} = \boldsymbol{b}_{ji} \cdot \boldsymbol{w}_j$. Now, consider the following ellipsoid defined by inverse of Gram matrix at time t,

$$\mathcal{E}_j(t) = \{\boldsymbol{x} : (\boldsymbol{x} - \widehat{\boldsymbol{w}}_j(t))^T \boldsymbol{A}_j(t)(\boldsymbol{x} - \widehat{\boldsymbol{w}}_j(t)) \le n\},$$

where

$$\boldsymbol{A}_j(t) = \boldsymbol{I}_n + \sum_{s=1}^{t-1} \boldsymbol{b}_{ji_s}\boldsymbol{b}_{ji_s}^T, \text{ and } \widehat{\boldsymbol{w}}_j(t) = \boldsymbol{A}_j(t)^{-1}\sum_{s=1}^{t-1} \boldsymbol{b}_{ji_s} v_{s,j},$$

for $j = 1,\ldots,d$. Results from existing literature on linear contextual bandits [Abbasi-yadkori et al. 2012; Chu et al. 2011; Auer 2003] provide that with high probability, the actual weight vector $\boldsymbol{w}_j$ is guaranteed to lie in this elliposid, i.e.,

$$\boldsymbol{w}_j \in \mathcal{E}_j(t).$$

This allows us to define new estimate set $\mathcal{H}_t$ as

$$\mathcal{H}_t = \{\tilde{\boldsymbol{V}} : \tilde{V}_{ji} = \boldsymbol{b}_{ji} \cdot \tilde{\boldsymbol{w}}_j, \forall \tilde{\boldsymbol{w}}_j \in \mathcal{E}_j(t)\}.$$

Then, using results from the above-mentioned literature on linear contextual bandits, it is easy to show that the properties (1) and (2) in Equation (6) and (8) hold with high probability for this $\mathcal{H}_t$ with $\mathcal{Q}(T) = ||\boldsymbol{1}_d||n\sqrt{T\log(\frac{dT}{\delta})}$. Therefore, simply substituing this $\mathcal{H}_t$ in Algorithm 1 provides an algorithm for linear contextual version of BwCR , with regret bounds,

$$\text{avg-regret}_1(T) \le O(L||\boldsymbol{1}_d||n\sqrt{\tfrac{1}{T}\log(\tfrac{dT}{\delta})}), \text{ and, } \text{avg-regret}_2(T) \le O(||\boldsymbol{1}_d||n\sqrt{\tfrac{1}{T}\log(\tfrac{dT}{\delta})}).$$

B.3. BwK
Property (1) for BwK(stated in Equation (13)), is simply a special case of Property (1) for BwCR, which was proven in the previous subsection. The following two lemmas prove the Property (2) for BwK(stated as Equation (14)). The proofs are similar to the proof of Property (2) for BwCR illustrated in the previous section, except that a little more careful analysis is done to get the bounds in terms of problem dependent parameters B and OPT.

LEMMA B.4. *With probability* $1 - (mT)e^{-\Omega(\gamma)}$,

$$||\tfrac{1}{T}\textstyle\sum_{t=1}^T (r_t - \boldsymbol{UCB}_t(\boldsymbol{\mu}) \cdot \boldsymbol{p}_t)|| \le \sqrt{\gamma m(\textstyle\sum_t r_t)} + \gamma m.$$

PROOF. Similar to Equation (27) and Equation (28), we can apply the concentration bounds given by Lemma B.1 to get that with probability $1 - (mT)e^{-\Omega(\gamma)}$,

$$|\tfrac{1}{T}\textstyle\sum_{t=1}^T (r_t - \mu_{i_t})| \le 3\text{rad}(\tfrac{1}{T}\textstyle\sum_{t=1}^T \mu_{i_t}, T)$$
$$\le 3\text{rad}(\tfrac{1}{T}\textstyle\sum_{t=1}^T \text{UCB}_{t,i_t}(\boldsymbol{\mu}), T) \quad (30)$$
$$|\tfrac{1}{T}\textstyle\sum_{t=1}^T (\text{UCB}_t(\boldsymbol{\mu}) \cdot \boldsymbol{p}_t - \text{UCB}_{t,i_t}(\boldsymbol{\mu}))| \le \text{rad}(\tfrac{1}{T}\textstyle\sum_{t=1}^T \text{UCB}_{t,i_t}(\boldsymbol{\mu}), T) \quad (31)$$

Also, using Lemma B.3,

$$
\begin{aligned}
|\textstyle\sum_{t=1}^{T}(\mu_{i_t} - \mathrm{UCB}_{t,i_t}(\boldsymbol{\mu}))| &\leq 4\textstyle\sum_t \mathrm{rad}(\widehat{\mu}_{t,i_t}, k_{t,i_t}+1) \\
&\leq 12\textstyle\sum_t \mathrm{rad}(\mu_{i_t}, k_{t,i_t}+1) \\
&= 12\sum_i \sum_{N=1}^{k_{T,i}+1} \mathrm{rad}(\mu_i, N) \\
&\leq 12\sum_i (k_{T,i}+1)\mathrm{rad}(\mu_i, k_{T,i}+1) \\
&\leq 12\sqrt{\gamma m\left(\sum_i \mu_i(k_{T,i}+1)\right)} + 12\gamma m \\
(\text{using Lemma B.2}) \quad &\leq 12\sqrt{\gamma m\left(\sum_t \mu_{i_t}\right)} + 24\gamma m \\
&\leq 12\sqrt{\gamma m\left(\sum_t \mathrm{UCB}_{t,i_t}(\boldsymbol{\mu})\right)} + 24\gamma m \qquad (32)
\end{aligned}
$$

Let $A = \sum_{t=1}^{T} \mathrm{UCB}_{t,i_t}(\boldsymbol{\mu})$. Then, from (30) and (32), we have that for some constant α

$$
A - 2\sqrt{\alpha\gamma m A} \leq \sum_{t=1}^{T} r_t + O(\gamma m).
$$

which implies

$$
(\sqrt{A} - \sqrt{\alpha\gamma m})^2 \leq \sum_{t=1}^{T} r_t + O(\gamma m).
$$

Therefore,

$$
\sqrt{\sum_{t=1}^{T} \mathrm{UCB}_{t,i_t}(\boldsymbol{\mu})} \leq \sqrt{\sum_{t=1}^{T} r_t} + O(\sqrt{\gamma m}). \qquad (33)
$$

Substituting (33) in (30), (31), (32), we get

$$
|\textstyle\sum_{t=1}^{T}(r_t - \sum_{t=1}^{T}\mathrm{UCB}_t(\boldsymbol{\mu})\boldsymbol{p}_t)| \leq O(\sqrt{\gamma m(\sum_{t=1}^{T} r_t)} + \gamma m).
$$

□

LEMMA B.5. *With probability* $1 - (mTd)e^{-\Omega(\gamma)}$, *for all* $j = 1, \ldots, d$,

$$
|\textstyle\sum_{t=1}^{T}(c_{t,j} - \boldsymbol{LCB}_{t,j}(\boldsymbol{C})\boldsymbol{p}_t| \leq \sqrt{\gamma m B} + \gamma m.
$$

PROOF. Similar to Equation (27) and Equation (28), we can apply the concentration bounds given by Lemma B.1 to get that with probability $1 - (mTd)e^{-\Omega(\gamma)}$, for all j

$$|\frac{1}{T}\sum_{t=1}^{T}(c_{t,j} - C_{ji_t})| \leq 3\mathrm{rad}(\frac{1}{T}\sum_{t=1}^{T}C_{ji_t}, T) \tag{34}$$

$$|\frac{1}{T}\sum_{t=1}^{T}(\mathrm{LCB}_{t,j}(\boldsymbol{C})\boldsymbol{p}_t - \mathrm{LCB}_{t,ji_t}(\boldsymbol{C}))| \leq \mathrm{rad}(\frac{1}{T}\sum_{t=1}^{T}\mathrm{LCB}_{t,ji_t}(\boldsymbol{C}), T)$$

$$\leq \mathrm{rad}(\frac{1}{T}\sum_{t=1}^{T}C_{ji_t}, T) \tag{35}$$

Also, using Lemma B.3,

$$
\begin{aligned}
|\sum_{t=1}^{T}(C_{ji_t} - \mathrm{LCB}_{t,i_t}(\boldsymbol{C}))| &\leq 4\sum_{t}\mathrm{rad}(\widehat{C}_{t,ji_t}, k_{t,i_t} + 1) \\
&\leq 12\sum_{t}\mathrm{rad}(C_{ji_t}, k_{t,i_t} + 1) \\
&= 12\sum_{i}\sum_{N=1}^{k_{T,i}+1}\mathrm{rad}(C_{ji}, N) \\
&\leq 12\sum_{i}(k_{T,i} + 1)\mathrm{rad}(C_{ji}, k_{T,i} + 1) \\
&\leq 12\sqrt{\gamma m\left(\sum_{i}C_{ji}(k_{T,i} + 1)\right)} + 12\gamma m \\
&\leq 12\sqrt{\gamma m\left(\sum_{t}C_{ji_t}\right)} + 24\gamma m
\end{aligned}
\tag{36}
$$

Let $A = \sum_t C_{ji_t}$. Then, from (35) and (36), we have that for some constant α

$$A \leq \sum_{t}\mathrm{LCB}_t(\boldsymbol{C})\boldsymbol{p}_t + 2\sqrt{\alpha\gamma mA} + O(\gamma m) \leq B + 2\sqrt{\alpha\gamma mA} + O(\gamma m),$$

where we used that $\sum_{t=1}^{T}\mathrm{LCB}_t(\boldsymbol{C})\boldsymbol{p}_t \leq B$, which is a corollary of the choice of $\boldsymbol{p}_t$ made by the algorithm. Then,

$$(\sqrt{A} - \sqrt{\alpha\gamma m})^2 \leq B + O(\gamma m).$$

That is,

$$\sqrt{\sum_{t}C_{ji_t}} \leq \sqrt{B} + O(\sqrt{\gamma m}). \tag{37}$$

Substituting (37) in (34), (35), (36), we get

$$|\sum_{t=1}^{T}(\boldsymbol{c}_{t,j} - \sum_{t=1}^{T}\mathrm{LCB}_{t,j}(\boldsymbol{C})\boldsymbol{p}_t)| \leq O(\sqrt{\gamma mB} + \gamma m).$$

□

C. FRANK-WOLFE

PROOF OF THEOREM 5.2. Let $\Delta_t := f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:t})$. We prove that $\Delta_t \leq \frac{\beta \log(2t)}{2t}$. (The base of the log is 2.) Once again, we use (19) for $t + 1$, with $\boldsymbol{y}_{t+1} = \bar{\boldsymbol{x}}_{1:t}$, and rearrange terms as follows:

$$\nabla f(\bar{\boldsymbol{x}}_{1:t}) \cdot (\boldsymbol{x}_{t+1} - \bar{\boldsymbol{x}}_{1:t}) \geq f(\boldsymbol{x}^*) - f(\bar{\boldsymbol{x}}_{1:t}). \tag{38}$$

In order to use (21), we rewrite $\bar{x}_{1:t+1} = \bar{x}_{1:t} + \frac{1}{t+1}(x_{t+1} - \bar{x}_{1:t})$. Using (21) first, followed by (38), gives us the following.

$$f(\bar{x}_{1:t+1}) \geq f(\bar{x}_{1:t}) + \frac{1}{t+1}\nabla f(\bar{x}_{1:t}) \cdot (x_{t+1} - \bar{x}_{1:t}) - \frac{\beta}{2(t+1)^2}$$

$$\geq f(\bar{x}_{1:t}) + \frac{1}{t+1}(f(x^*) - f(\bar{x}_{1:t})) - \frac{\beta}{2(t+1)^2}$$

With this we can bound Δ_{t+1} in terms of Δ_t.

$$\Delta_{t+1} \leq \Delta_t - \frac{1}{(t+1)}\Delta_t + \frac{\beta}{2(t+1)^2} = \frac{t}{(t+1)}\Delta_t + \frac{\beta}{2(t+1)^2} \qquad (39)$$

Recall that we wish to show that $\Delta_t \leq \beta\log(2t)/2t$. The rest of the proof is by induction on t. For the base case, we note that we can still use (39) with $t = 0$ and an arbitrary x_0 which is used to set p_1. This gives us that $\Delta_1 \leq \beta/2$. The inductive step for $t + 1$ follows from (39) and the inductive hypothesis for t if

$$\frac{t}{(t+1)} \cdot \frac{\beta\log(2t)}{2t} + \frac{\beta}{2(t+1)^2} \leq \frac{\beta\log(2(t+1))}{2(t+1)}$$

$$\Leftrightarrow \log(t) + \frac{1}{t+1} \leq \log(t+1)$$

$$\Leftrightarrow \frac{1}{t+1} \leq \log(1 + \tfrac{1}{t}).$$

The last inequality follows from the fact that for any $a > 0$, $\log(1+a) > \frac{a}{1+a}$, by setting $a = 1/t$. This completes the induction. Therefore, $\Delta_T = f(x^*) - f(\bar{x}_{1:T}) \leq \frac{\beta\log(2T)}{2T}$ and combined with (20), we get the desired theorem statement. $\square$

PROOF OF THEOREM 5.3 . We first show Lipshitz continuity of $\nabla\hat{f}_\eta$. Let x_1 and x_2 be any two points in the domain of f, then for $\ell = 1, 2$, $\nabla\hat{f}_\eta(x_\ell) = -\theta_\ell$ where

$$\theta_\ell = \arg\min_{||\theta|| \leq L} \{f^*(\theta) + \frac{\eta}{2L}||\theta||^2 - \theta \cdot x_\ell\}.$$

We use the following fact about convex functions: if y^* minimizes a convex function ψ over some domain and y is any other point in the domain then $\nabla\psi(y^*) \cdot (y - y^*) \geq 0$. Using this fact for $y^* = \theta_1$ and $y = \theta_2$, we get that

$$\left(\nabla f^*(\theta_1) + \frac{\eta}{L}\theta_1 - x_1\right) \cdot (\theta_2 - \theta_1) \geq 0. \qquad (40)$$

Using convexity of f^* and strong convexity of $||\cdot||^2$, we get that

$$f^*(\theta_2) \geq f^*(\theta_1) + \nabla f^*(\theta_1) \cdot (\theta_2 - \theta_1), \qquad (41)$$

$$\frac{\eta}{2L}||\theta_2||^2 \geq \frac{\eta}{2L}\left(||\theta_1||^2 + 2\theta_1 \cdot (\theta_2 - \theta_1) + ||\theta_2 - \theta_1||^2\right). \qquad (42)$$

Adding (40–42) we get that

$$-x_1 \cdot (\theta_2 - \theta_1) + f^*(\theta_2) + \frac{\eta}{2L}||\theta_2||^2 \geq f^*(\theta_1) + \frac{\eta}{2L}\left(||\theta_1||^2 + ||\theta_2 - \theta_1||^2\right).$$

Similarly, by switching x_1 and x_2, we get

$$-x_2 \cdot (\theta_1 - \theta_2) + f^*(\theta_1) + \frac{\eta}{2L}||\theta_1||^2 \geq f^*(\theta_2) + \frac{\eta}{2L}\left(||\theta_2||^2 + ||\theta_2 - \theta_1||^2\right).$$

Adding these two, we get

$$(\boldsymbol{x}_1 - \boldsymbol{x}_2) \cdot (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) \geq \frac{\eta}{L} ||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||^2.$$

By Caucy-Schwartz inequality, we have

$$\begin{aligned}
(\boldsymbol{x}_1 - \boldsymbol{x}_2) \cdot (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2) &\leq ||\boldsymbol{x}_1 - \boldsymbol{x}_2|| \cdot ||\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2|| \\
\therefore \frac{\eta}{L} ||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1||^2 &\leq ||\boldsymbol{x}_1 - \boldsymbol{x}_2|| \cdot ||\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2|| \\
\Rightarrow ||\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1|| &\leq \frac{L}{\eta} ||\boldsymbol{x}_1 - \boldsymbol{x}_2||.
\end{aligned}$$

This shows that the Lipschitz constant of $\nabla \hat{f}_\eta$ is L/η.

Then, we can show that $\hat{f}_\eta$ is $\frac{dL}{\eta}$ smooth as follows:

$$\begin{aligned}
\hat{f}_\eta(\boldsymbol{x} + \alpha(\boldsymbol{y} - \boldsymbol{x})) &= \hat{f}_\eta(\boldsymbol{x}) - \int_{w:0}^{\alpha} \nabla \hat{f}_\eta(\boldsymbol{x} + w(\boldsymbol{y} - \boldsymbol{x})) \cdot (\boldsymbol{y} - \boldsymbol{x}) dw \\
&= \hat{f}_\eta(\boldsymbol{x}) + \alpha \nabla \hat{f}_\eta(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) + \int_{w:0}^{\alpha} (\nabla \hat{f}_\eta(\boldsymbol{x} + w(\boldsymbol{y} - \boldsymbol{x})) - \nabla \hat{f}_\eta(\boldsymbol{x})) \cdot (\boldsymbol{y} - \boldsymbol{x}) dw
\end{aligned}$$

Then, using Lipschitz continuity of $\hat{f}_\eta$,

$$\begin{aligned}
\left| \int_{w:0}^{\alpha} (\nabla \hat{f}_\eta(\boldsymbol{x} + w(\boldsymbol{y} - \boldsymbol{x})) - \nabla \hat{f}_\eta(\boldsymbol{x})) \cdot (\boldsymbol{y} - \boldsymbol{x}) dw \right| &\leq \frac{L\alpha^2}{\eta} ||\boldsymbol{x} - \boldsymbol{y}|| \cdot ||\boldsymbol{x} - \boldsymbol{y}|| \int_0^{\alpha} (w) dw \\
&= \frac{L\alpha^2}{2\eta} ||\boldsymbol{x} - \boldsymbol{y}||^2 \\
&\leq \frac{dL}{\eta} \cdot \frac{\alpha^2}{2}
\end{aligned}$$

It remains to show that $\hat{f}_\eta - \frac{\eta L}{2} \leq f \leq \hat{f}_\eta$. This follows almost immediately from Lemma 2.2 and (22): the function inside the minimization for $\hat{f}_\eta$ is always larger than that of f, but not by more than $\frac{\eta L}{2}$. □

LEMMA C.1. $\nabla \hat{f}_\eta(\boldsymbol{z}) = -\boldsymbol{\theta}$ iff $\exists \boldsymbol{y}$ such that

(1) $-\boldsymbol{\theta}$ is a supergradient of f at $\boldsymbol{y}$. We denote this by $-\boldsymbol{\theta} \in \partial f(\boldsymbol{y})$, and
(2) $-\boldsymbol{\theta} = \alpha(\boldsymbol{y} - \boldsymbol{z})$ where $\alpha = \min\{L/\eta, L/||\boldsymbol{y} - \boldsymbol{z}||\}$.

PROOF. The gradient of $\hat{f}_\eta$ is equal to $-\boldsymbol{\theta}$ where $\boldsymbol{\theta}$ is the $\arg\min$ in (22), which is equivalent to

$$\min_{||\boldsymbol{\theta}|| \leq L} \max_{\boldsymbol{y}} \{f(\boldsymbol{y}) + \boldsymbol{\theta} \cdot \boldsymbol{y} + \frac{\eta}{2L} \boldsymbol{\theta} \cdot \boldsymbol{\theta} - \boldsymbol{\theta} \cdot \boldsymbol{z}\} = \max_{\boldsymbol{y}} \min_{||\boldsymbol{\theta}||_* \leq L} \{f(\boldsymbol{y}) + \boldsymbol{\theta} \cdot \boldsymbol{y} + \frac{\eta}{2L} \boldsymbol{\theta} \cdot \boldsymbol{\theta} - \boldsymbol{\theta} \cdot \boldsymbol{z}\},$$

by the min-max theorem. The two conditions in the hypothesis of the lemma are essentially the KKT conditions for the above. Given $\boldsymbol{\theta}$, it must be that $\boldsymbol{y}$ optimizes the inner maximization in the first form, which happens when $-\boldsymbol{\theta} \in \partial f(\boldsymbol{y})$. On the other hand, given $\boldsymbol{y}$, it must be that $\boldsymbol{\theta}$ optimizes the inner minimization in the second form. Note that due to the spherical symmetry of the domain of $\boldsymbol{\theta}$, the *direction* that minimizes is $\boldsymbol{z} - \boldsymbol{y}$. Therefore we may assume that $\boldsymbol{\theta} = \alpha(\boldsymbol{z} - \boldsymbol{y})$ for some $0 \leq \alpha \leq L/||\boldsymbol{z} - \boldsymbol{y}||$, since $||\boldsymbol{\theta}|| \leq L$. Given this, the inner minimization reduces to minimizing $\eta \alpha^2/(2L) - \alpha$, subject to the constraint on α above, the solution to which is $\alpha = \min\{L/\eta, L/||\boldsymbol{y} - \boldsymbol{z}||\}$. □

PROOF OF LEMMA 5.5. One can get a closed form expression for the subgradients of the distance function. Let $\pi_S(\boldsymbol{z})$ be the projection of $\boldsymbol{z}$ onto S for $\boldsymbol{z} \notin S$, and $\nu_S(\boldsymbol{z})$ be the set of unit normal vectors to S at $\boldsymbol{z}$, for a $\boldsymbol{z}$ that is on the boundary of S. We extend the defintion of $\nu_S(\boldsymbol{z})$ to $\boldsymbol{z} \notin S$ as

$$\nu_S(\boldsymbol{z}) := \frac{\boldsymbol{z} - \pi_S(\boldsymbol{z})}{||\boldsymbol{z} - \pi_S(\boldsymbol{z})||}.$$

Then, the set of subgradients of the distance function $\partial d(\boldsymbol{z}, S)$ is as follows.

$$\partial d(\boldsymbol{z}, S) = \begin{cases} \nu_S(\boldsymbol{z}) & \text{if } \boldsymbol{z} \notin S \\ \{\alpha\nu_S(\boldsymbol{z}), \text{ for all } \alpha \in [0, 1]\} & \text{if } \boldsymbol{z} \text{ is on the boundary of } S \\ \boldsymbol{0} & \text{if } \boldsymbol{z} \in \text{ interior of } S \end{cases}$$

Note that $d(\cdot, S)$ is non-smooth near the boundary of S. We now show how $\hat{d}_\eta(\cdot, S)$ becomes smooth, and give the stated closed form expression for $\nabla \hat{d}_\eta(\cdot, S)$.

We use Lemma C.1 for $f(\boldsymbol{z}) = -d(\boldsymbol{z}, S)$ to construct for each $\boldsymbol{z}$, a $\boldsymbol{y}$ that satisfies the two conditions in the lemma, and gives $\nabla \hat{f}_\eta(\boldsymbol{z}) = -\nabla \hat{d}_\eta(\boldsymbol{z}, S)$ as claimed. Note that $L = 1$ in this case.

Case 1: $||\boldsymbol{z} - \pi_S(\boldsymbol{z})|| \geq \eta$. Pick $\boldsymbol{y} = \pi_S(\boldsymbol{z})$. Note that $\nu_S(\boldsymbol{z}) \in \nu_S(\boldsymbol{y})$ therefore $-\nu_S(\boldsymbol{z}) \in \partial f(\boldsymbol{y})$, and the first condition in Lemma C.1 is satisfied. Since $||\boldsymbol{z}-\boldsymbol{y}|| \geq \eta$, $\alpha = 1/||\boldsymbol{z}-\boldsymbol{y}||$ and $\alpha(\boldsymbol{y} - \boldsymbol{z}) = -\nu_S(\boldsymbol{z})$, so the second condition in Lemma C.1 is satisfied.

Case 2: $0 < ||\boldsymbol{z} - \pi_S(\boldsymbol{z})|| < \eta$. Pick $\boldsymbol{y} = \pi_S(\boldsymbol{z})$. As in Case 1, $\nu_S(\boldsymbol{z}) \in \nu_S(\boldsymbol{y})$ therefore $-\nu_S(\boldsymbol{z})\frac{||\boldsymbol{z}-\pi_S(\boldsymbol{z})||}{\eta} \in \partial f(\boldsymbol{y})$, and the first condition in Lemma C.1 is satisfied. Since $||\boldsymbol{z} - \boldsymbol{y}|| < \eta$, $\alpha = 1/\eta$ and $\alpha(\boldsymbol{y} - \boldsymbol{z}) = \frac{\pi_S(\boldsymbol{z})-\boldsymbol{z}}{\eta}$ so the second condition in Lemma C.1 is satisfied.

Case 3: $\boldsymbol{z} \in S$. Pick $\boldsymbol{y} = \boldsymbol{z}$. Note that $\boldsymbol{0} \in \partial f(\boldsymbol{y})$, and the conditions in Lemma C.1 are satisfied trivially.

□