Generalized $\mathbb{N}$-weighted simplicial complex

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The Stanley-Reisner correspondence uses simplicial complexes to study square-free monomial ideals. In order to use similar techniques to study certain non-square-free monomial ideals, in my dissertation, I define a weighted version which I call a *generalized* $\mathbb{N}$*-weighted simplicial complex*.

A *generalized* $\mathbb{N}$*-weighted simplicial complex* on V is a non-empty collection Υ of sets of form (V', δ') with $V' \subseteq V$ and $\delta' : V' \to \mathbb{N}$ that is closed under $\preccurlyeq$: For subsets $V_1', V_2' \subseteq V'$ and functions $\delta_1' : V_1' \to \mathbb{N}$ and $\delta_2' : V_2' \to \mathbb{N}$, we write $(V_1', \delta_1') \preccurlyeq (V_2', \delta_2')$ if $V_1' \subseteq V_2'$ and $\delta_1' \leq \delta_2'|_{V_2'}$.

For a subset $V' \subseteq V$ and a function $\delta' : V' \to \mathbb{N}$, we notationally set

$$(V', \delta') = \{v_i^{\delta'(v_i)} \mid v_i \in V'\},$$

where $\delta'(v_i)$ is the *weight* of v_i.

For instance,

$$\Upsilon = \{\{v_2^2, v_3\}, \{v_2, v_3^2\}, \{v_2, v_3\}, \{v_2^2\}, \{v_3^2\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\}$$

is a generalized $\mathbb{N}$-weighted simplicial complex.

Associated to Υ, we define *weighted Stanley-Reisner ideal* J_Υ by

$$\begin{aligned} J_\Upsilon &= \left(\underline{X}^{(V',\delta')} \mid V' \subseteq V,\ \delta' : V' \to \mathbb{N} \text{ and } (V', \delta') \notin \Upsilon\right)R \\ &= \left(\prod_{v_i \in V'} X_i^{\delta'(v_i)} \mid V' \subseteq V,\ \delta' : V' \to \mathbb{N} \text{ and } (V', \delta') \notin \Upsilon\right)R. \end{aligned}$$

If $\delta' : V' \to \mathbb{N}$ is the constant function 1 for all $(V', \delta') \in \Upsilon$, then Υ becomes a usual simplicial complex Δ, and this recovers the definition of the original notion of a Stanley-Reisner ideal.

For example, given the generalized $\mathbb{N}$-weighted simplicial complex

$$\Upsilon = \{\{v_2^2, v_3\}, \{v_2, v_3^2\}, \{v_2, v_3\}, \{v_2^2\}, \{v_3^2\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\},$$

we have

$$J_\Upsilon = (X_2^2 X_3^2, X_1 X_2, X_1 X_3)R.$$

Let $I \subseteq R$ be a monomial ideal [3], for $j = 1, \ldots, d$ assume M_j is the maximal exponent of X_j on the minimal generators of I. We define

$$\Upsilon(I) = \{(V', \delta') \text{ with } V' \subseteq V,\ \delta' : V' \to \mathbb{N} \text{ and } \delta'(v_i) \leq M_i \forall v_i \in V' \mid \underline{X}^{(V',\delta')} \notin I\},$$

which is then a generalized $\mathbb{N}$-weighted simplicial complex. The next result from my dissertation is a weighted version of the Stanley-Reisner correspondence.

Theorem 0.1. *(a) Let $I \subseteq R$ be a monomial ideal [3], for $j = 1, \ldots, d$ assume M_j is the maximal exponent of X_j on the minimal generators of I. Then*

$$J_{\Upsilon(I)} = I.$$

(b) Let Υ be a generalized $\mathbb{N}$-weighted simplicial complex such that for $i = 1, \ldots, d$ we have $M_i^\Upsilon := \max\{\delta'(v_i) \mid v_i \in V'$ and $(V', \delta') \in \Upsilon\} < \infty$. Then

$$\Upsilon(J_\Upsilon) = \Upsilon.$$

For instance, with $J_\Upsilon = (X_2^2 X_3^2, X_1 X_2, X_1 X_3)R$, we have

$$\begin{aligned} \Upsilon(J_\Upsilon) &= \{(V', \delta') \text{ with } V' \subseteq V \text{ and } \delta' : V' \to \mathbb{N} \mid \underline{X}^{(V',\delta')} \notin J_\Upsilon\} \\ &= \{\{v_2^2, v_3\}, \{v_2, v_3^2\}, \{v_2, v_3\}, \{v_2^2\}, \{v_3^2\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\} \\ &= \Upsilon. \end{aligned}$$

In my dissertation, I show that, as in the classical setting, these yield an irredundant irreducible decomposition of $J_{\Upsilon^{\mathrm{M}}}$:

Theorem 0.2. *Let Υ be a generalized $\mathbb{N}$-weighted simplicial complex such that for $i = 1, \ldots, d$ we have $M_i^{\Upsilon} := \max\{\delta'(v_i) \mid v_i \in V' \text{ and } (V', \delta') \in \Upsilon\} < \infty$. Then the ideal J_Υ has the following irreducible (primary) decomposition*

$$J_\Upsilon = \bigcap_{\textit{face } \rho \in \Upsilon} Q_\rho = \bigcap_{\rho \textit{ facet}} Q_\rho,$$

where $Q_\rho = (X^{\tau_\rho(v_i)} \mid v_i \in (V \smallsetminus V') \sqcup W_\rho)R$, $W_\rho = \{v_i \in V' \mid \delta'(v_i) < M_i^{\Upsilon}\}$ and $\tau_\rho : (V \smallsetminus V') \sqcup W_\rho \to \mathbb{N}$ is given by

$$\tau_\rho(v_i) = \begin{cases} 1 & \text{if } v_i \in V \smallsetminus V', \\ \delta'(v_i) + 1 & \text{if } v_i \in W_\rho. \end{cases}$$

The second intersection is irredundant.

For instance, with

$$\Upsilon = \{\{v_2^2, v_3\}, \{v_2, v_3^2\}, \{v_2, v_3\}, \{v_2^2\}, \{v_3^2\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\},$$

and $M_1^{\Upsilon} = 1$, $M_2^{\Upsilon} = 2$ and $M_3^{\Upsilon} = 2$, we have

$$\begin{aligned} J_\Upsilon &= \bigcap_{\text{face } \rho \in \Upsilon} Q_\rho \\ &= Q_{\{v_2^2, v_3\}} \cap Q_{\{v_2, v_3^2\}} \cap Q_{\{v_2, v_3\}} \cap Q_{\{v_2^2\}} \cap Q_{\{v_3^2\}} \cap Q_{\{v_1\}} \cap Q_{\{v_2\}} \cap Q_{\{v_3\}} \cap Q_\emptyset \\ &= (X_1, X_3^2)R \cap (X_1, X_2^2)R \cap (X_1, X_2^2, X_3^2)R \cap (X_1, X_3)R \cap (X_1, X_2)R \cap (X_2, X_3)R \\ &\quad \cap (X_1, X_3)R \cap (X_1, X_2)R \cap (X_1, X_2, X_3)R \\ &= (X_1, X_3^2)R \cap (X_1, X_2^2)R \cap (X_2, X_3)R \\ &= Q_{\{v_2^2, v_3\}} \cap Q_{\{v_2, v_3^2\}} \cap Q_{\{v_1\}} \\ &= \bigcap_{\text{facet } \rho \in \Upsilon} Q_\rho. \end{aligned}$$

Much of our interest in this area focuses on the edge ideal $I(G)$ of a graph G [1] in some weighted settings. For the classical construction, this is the same as the Stanley-Reisner ideal J_{Δ_G} of the independence complex Δ_G; this is the set of all independent subsets of V where $W \subseteq V$ is *independent* if no two vertices in W are adjacent in G. In my dissertation I introduce and investigate the following weighted version of this notion.

Let G_ω be an edge weighted graph on V. We say that an ordered pair (V', δ') with $V' \subseteq V$ and $\delta' : V' \to \mathbb{N}$ satisfying $\delta'(v_i) \leq \max\{\omega(v_i v_j) \mid v_i v_j \in E\}$ for any $v_i \in V'$ is a *weighted independent* set in G_ω if for any $v_i, v_j \in V'$, either $v_i v_j \notin E$, or $\min\{\delta'(v_i), \delta'(v_j)\} < \omega(v_i v_j)$. Let Υ_ω denote the set of weighted independent sets (V', δ') in G_ω. This is the *weighted independence complex* of G_ω.

The weighted independent sets in G_ω from Example **??** are by definition

$$\Upsilon_\omega = \{\{v_1^2, v_3\}, \{v_1, v_3^2\}, \{v_1, v_3\}, \{v_2^2, v_3\}, \{v_2, v_3^2\}, \{v_2, v_3\}, \{v_1^2\}, \{v_2^2\}, \{v_3^2\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\}.$$

The weighted edge ideal $I(G_\omega)$ can be decomposed in terms of weighted vertex covers of G_ω by [4]. In my dissertation I give the following decomposition result for these ideals in terms of weighted independent sets.

Theorem 0.3. *We have $M_i^{\Upsilon_\omega} = \max\{\omega(v_i v_j) \mid v_i v_j \in E\}$ for $i = 1, \ldots, d$. The ideal J_{Υ_ω} has the following irreducible (primary) decompositions*

$$I(G_\omega) = J_{\Upsilon_\omega} = \bigcap_{\rho \in \Upsilon_\omega} Q_\rho = \bigcap_{\textit{maximal } \rho \in \Upsilon_\omega} Q_\rho.$$

The second intersection is irredundant.

For instance, in Example **??**, with $M_1^{\Upsilon_\omega} = 2$, $M_2^{\Upsilon_\omega} = 2$ and $M_3^{\Upsilon_\omega} = 2$, we have

$$\begin{aligned} J_{\Upsilon_\omega} &= \bigcap_{\text{face } \rho \in \Upsilon_\omega} Q_\rho \\ &= Q_{\{v_1^2, v_3\}} \cap Q_{\{v_1, v_3^2\}} \cap Q_{\{v_1, v_3\}} \cap Q_{\{v_2^2, v_3\}} \cap Q_{\{v_2, v_3^2\}} \cap Q_{\{v_2, v_3\}} \\ &\cap Q_{\{v_1^2\}} \cap Q_{\{v_2^2\}} \cap Q_{\{v_3^2\}} \cap Q_{\{v_1\}} \cap Q_{\{v_2\}} \cap Q_{\{v_3\}} \cap Q_\emptyset \\ &= (X_2, X_3^2)R \cap (X_1^2, X_2)R \cap (X_1^2, X_2, X_3^2)R \cap (X_1, X_3^2)R \cap (X_1, X_2^2)R \cap (X_1, X_2^2, X_3^2)R \\ &\cap (X_2, X_3)R \cap (X_1, X_3)R \cap (X_1, X_2)R \cap (X_2, X_3)R \cap (X_1, X_3)R \cap (X_1, X_2)R \cap (X_1, X_2, X_3)R \\ &= (X_2, X_3^2) \cap (X_1^2, X_2) \cap (X_1, X_3^2)R \cap (X_1, X_2^2)R \\ &= Q_{\{v_1^2, v_3\}} \cap Q_{\{v_1, v_3^2\}} \cap Q_{\{v_2^2, v_3\}} \cap Q_{\{v_2, v_3^2\}} \\ &= \bigcap_{\text{facet } \rho \in \Upsilon_\omega} Q_\rho. \end{aligned}$$

Let Υ be a generalized $\mathbb{N}$-weighted simplicial complex such that for $i = 1, \ldots, d$ we have $M_i^{\Upsilon} < \infty$. Define the *Alexander dual* $\Upsilon^{\vee}$ of Υ by

$$\Upsilon^{\vee} = \{\rho := (V', \delta') \text{ with } V' \subseteq V,\ \delta' : V' \to \mathbb{N} \mid ((V \smallsetminus V') \sqcup W_\rho, \varphi_\rho) \notin \Upsilon\},$$

where $\varphi_\rho : (V \smallsetminus V') \sqcup W_\rho \to \mathbb{N}$ is defined by

$$\varphi_\rho(v_i) = \begin{cases} M_i & \text{if } v_i \in V \smallsetminus V', \\ M_i - \delta'(v_i) & \text{if } v_i \in W_\rho. \end{cases}$$

Then $\Upsilon^{\vee}$ is a generalied $\mathbb{N}$-weighted simplicial complex.

For instance, the dual of the weighted independence complex Υ_ω for Example **??** is

$$\Upsilon_\omega^{\vee} = \{\{v_1, v_3^2\}, \{v_2, v_3^2\}, \{v_1, v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3^2\}, \{v_3\}, \emptyset\}.$$

Then

$$J_{\Upsilon_\omega^{\vee}} = (X_1^2 X_2, X_1 X_2^2, X_1^2 X_3, X_2^2 X_3)R.$$

In addition, I introduce and investigate a variation of E. Miller's algebraic Alexander duality [2] in this context which is slightly better behaved in some respects.

Let $I \subseteq R$ be a monomial ideal, for $j = 1, \ldots, d$ assume M_j and m_j are the maximal and minimal exponent of X_j on the minimal generators of I, respectively. Write $I = \bigcap_{i=1}^{n} P(V_i', \delta_i')$ as a irredundant irreducible decomposition, where $P(V_i', \delta_i') = (X_{j_i}^{\delta_i'(v_{j_i})} \mid v_{j_i} \in V_i')R$. Define the *Alexander dual* $I^{\vee}$ of I by

$$I^{\vee} = (\underline{X}^{(V_i', \delta_i'')} \mid i = 1, \ldots, n),$$

where for $i = 1, \ldots, n$: $\delta_i''(v_j) = M_j + m_j - \delta_i'(v_j)$ for any $v_j \in V_i'$ and $\underline{X}^{(V_i', \delta_i'')} = \prod_{v_i \in V_i'} X_i^{\delta''(v_i)}$. If all the m_j's are 1 in the above definition, then it recovers the classical Alexander dual. In terms of this definition, I am working on proving that $I^{\vee\vee} = I$.

In some situations, E. Miller's definition works better. So for the rest of this section I'll use his duality definition for monomial ideals. I justify the name of the previous construction by showing in the next two results from my dissertation that Alexander duality commutes with the weighted Stanley-Reisner correspondence.

Theorem 0.4. *Let Υ be a generalized $\mathbb{N}$-weighted simplicial complex such that for $i = 1, \ldots, d$ we have $M_i^{\Upsilon} < \infty$. Then*

$$J_{\Upsilon}^{\vee} = J_{\Upsilon^{\vee}}.$$

For instance, consider the weighted independence complex Υ_ω for Example **??**. Since the irredundant irreducible decomposition of $I(G_\omega)$ is

$$I(G_\omega) = (X_1 X_2, X_2^2 X_3^2, X_1^2 X_3^2) = (X_1, X_2^2)R \cap (X_1^2, X_2)R \cap (X_1, X_3^2)R \cap (X_2, X_3^2)R,$$

we have

$$J_{\Upsilon_\omega}^{\vee} = I(G_\omega)^{\vee} = (X_1^2 X_2, X_1 X_2^2, X_1^2 X_3, X_2^2 X_3)R = J_{\Upsilon_\omega^{\vee}}.$$

Theorem 0.5. *If $I \subseteq R$ is a monomial ideal, then*

$$\Upsilon(I^{\vee}) = \Upsilon(I)^{\vee}.$$

For instance, consider the weighted edge ideal $I(G_\omega)$ for Example **??**. We have by Macaulay2

$$I(G_\omega)^{\vee} = (X_1^2 X_2, X_1 X_2^2, X_1^2 X_3, X_2^2 X_3).$$

Then

$$\Upsilon(I(G_\omega)^{\vee}) = \{\{v_1, v_2\}, \{v_1, v_3^2\}, \{v_1, v_3\}, \{v_2, v_3^2\}, \{v_2, v_3\}, \{v_3^2\}, \{v_1\}, \{v_2\}, \{v_3\}, \emptyset\}.$$

We can show that $\Upsilon(I(G_\omega)) = \Upsilon_\omega$ generally. In this particular example, we have

$$\begin{aligned} \Upsilon(I(G_\omega))^{\vee} &= \Upsilon_\omega^{\vee} \\ &= \{\{v_1, v_3^2\}, \{v_2, v_3^2\}, \{v_1, v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3^2\}, \{v_3\}, \emptyset\} \\ &= \Upsilon(I(G_\omega)^{\vee}). \end{aligned}$$

References

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