

Graph Theory

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September 25, 2023

Contents

Chapter 1

The Basics

1.1 Graphs

Definition 1.1. A graph is a pair $G = (V, E)$ of sets with $E \subseteq V^2$.

- (a) The order of G is $|G|$ or $|V|$. The size of G is $\|G\|$ or $|E|$.
- (b) $v \in V$ is incident on $e \in E$ if $v \in e$, in which case, we say e is an edge at v .
- (c) e and f are adjacent if they share a vertex.
- (d) The coloring number, $\chi(G)$ is the smallest number of colors required to color each vertex so that no adjacent vertices are colored the same.
- (e) G is a complete graph if all vertices are pairwise adjacent. Let K^n be the complete graph on n vertices.
- (f) Pairwise non-adjacent vertices are called independent. A set of independent vertices is a stable set. $\alpha(G)$ is the size of largest stable set.
- (g) $G' \subseteq G$ if $V' \subseteq V$ and $E' \subseteq E$. Then G' is called a subgraph of G .
- (h) $G' \subseteq G$ and G' contains all edges $xy \in E$ with $x, y \in V'$, then G' is the subgraph induced by V' . Denote it as $G' = G[V']$.
- (i) An induced subgraph that is complete is a clique.
- (j) $\omega(G)$ is the size of the largest clique of G .
- (k) The complement of G is $\overline{G} = (V, \overline{E})$.
- (l) A path is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$, $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$. Or for shorthand, $P = x_0x_1 \cdots x_k$. If $P = x_0 \cdots x_{k-1}$ is a path with $k \geq 3$, then we call $C = P + x_{k-1}x_0$ a cycle.
- (m) For $G = (V, E)$ and $G' = (V', E')$, $G' \cong G$ if there exists a bijection $\phi : V \rightarrow V'$ with $xy \in E \iff \phi(x)\phi(y) \in E'$.

(n) G is edge maximal with respect to a property if G has the property but $G + uv$ does not for any $uv \notin E$.

(o) $N(v)$ is the neighbor set of v . $N(U)$ is the set of neighbors of vertices in $V \setminus U$.

(p) $d_G(v) = d(v)$ is the number of neighbors of v (when G is simple).

(q) $\delta(G) = \min\{d(v) | v \in V\}$. $\Delta(G) = \max\{d(v) | v \in V\}$.

(r) When all vertices have the same degree k , G is k -regular.

(s) The average degree $d(G) = \frac{\sum_{v \in V} d(v)}{|V|}$.

(t) G is perfect if and only if it contains no odd hole or antihole if and only if $\chi(G) = \omega(G)$.

Definition 1.2. The line graph $L(G)$ of $G = (V, E)$ is the graph on V with

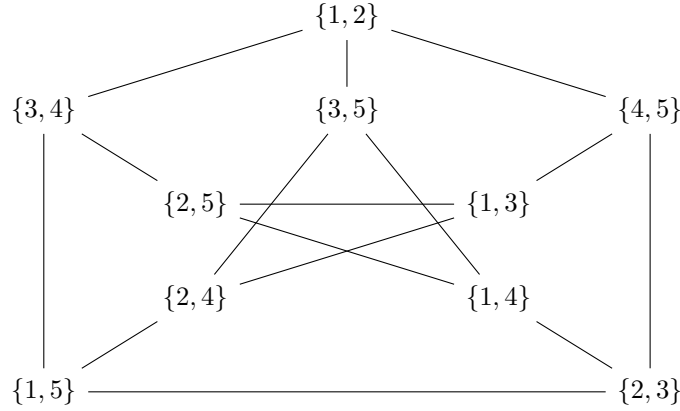
(a)

$$V(L(G)) = E.$$

(b) $ef \in E(L(G))$ if and only if e and f are adjacent in G .

Remark. The line graph of G represents adjacencies between edges.

Example 1.3. The $\overline{L(K^5)}$, i.e., Peterson graph is as follows.



Since $\chi(\overline{L(K^5)}) = 3 \geq 2 = \omega(\overline{L(K^5)})$, Peterson graph is not perfect.

1.2 The degree of vertex

Theorem 1.4. Every simple finite graph with at least one edge has a nonempty subgraph H with

$$\delta(H) > \frac{1}{2}d(H) \geq \frac{1}{2}d(G),$$

i.e.,

$$\delta(H) > \epsilon(H) \geq \epsilon(G).$$

Proof. Start with G and remove a vertex at a time, obtaining

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_i \supseteq \cdots \supseteq H.$$

Specifically, if $v_i \in V(G_i)$ with

$$d(v_i) \leq \epsilon(G_i),$$

then

$$G_{i+1} = G_i - v_i.$$

Otherwise, let $H = G_i$ and stop.

Claim 1. We do stop since G is finite.

Claim 2. The average degree is non-decreasing.

Let

$$G_i = (V, E).$$

Then

$$\epsilon(G_{i+1}) = \frac{|E| - d(v_i)}{|V| - 1} \geq \frac{|E| - \epsilon(G_i)}{|V| - 1} = \frac{|E| - \frac{|E|}{|V|}}{|V| - 1} = \frac{|E|}{|V|} = \epsilon(G_i).$$

Claim 3. $H \neq \emptyset$.

Suppose not. Then let $H = G_k$, then $G_{k-1} = K^1$ but then $\epsilon(K^1) = 0$.

But $\epsilon(G) > 0$ since we have at least one edge, contradicting Claim 2. □

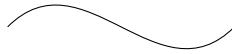
1.3 Path and Cycles

Definition 1.5. A path of length k is a graph $P = (V, E)$ with $V = \{x_0, x_1, \dots, x_k\}$ and $E = (x_0x_1, x_1x_2, \dots, x_{k-1}x_k)$, where x_i 's are all distinct. So The length is the number of edges. Sometimes we denote a path as a sequence of vertices

$$x_0x_1 \cdots x_k.$$

P^k is a path of length k . $P^0 = K^1$.

Definition 1.6. xPy : x and y are two intermediate points in the path P .



$$x_0 \text{ --- } x_1 \text{ --- } \cdots \text{ --- } x_{k-1} \text{ --- } x_k$$

$$P^0 = x_1Px_{k-1}.$$

Definition 1.7. Let $G = (V, E)$. In a path $x_0x_1 \cdots x_k$, if $x_0, x_k \in A$ but $x_1, \dots, x_{k-1} \notin A$, then $P = x_0x_1 \cdots x_k$ is an A -path.

Definition 1.8. Two u - v paths are independent (or internally disjoint) if they have only u, v in common.

Definition 1.9. A walk is a sequence $W = (v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k)$, where

$$e_i = v_{i-1}v_i, \forall 1 \leq i \leq k.$$

The length is the number of edges v_0 - v_k walk.

If $v_0 = v_k$, it is a closed walk.

Definition 1.10. A trial is a walk with no repeated edges.

Remark. A path is a walk with no repeated vertices.

Theorem 1.11. Let G be a graph.

- (a) Every u - v walk ($u \neq v$) contains a u - v path.
- (b) Every closed u - v walk contains a cycle.
- (c) Every closed walk with an odd number of edges contains an odd cycle.

Proof. Let

$$w = (u = v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k = v).$$

Let w' be a subsequence that is itself an u - v walk and is shortest possible.

Suppose w' is not a u - v path.

Then \exists a repeated vertex, say $v_j = v_l$ with $j < l$.

But then

$$(v_0 = u, e_1, v_1, \dots, v_j = v_l, e_l, \dots, e_k, v_k = v)$$

is a shorter subsequence that is also a walk. □

Definition 1.12. (a) The girth $g(G)$ is the length of a shortest cycle.

(b) The circumference of G is length of a longest cycle.

(c) $d(u, v)$ is length of shortest u - v path.

(d)

$$\text{diam}(G) = \max_{u, v \in V} d(u, v).$$

(e) The eccentricity

$$e(v) = \max_{u \in V} d(u, v).$$

(f) A vertex with the smallest eccentricity is central. The radius of G is $e(z)$, where z is central.

$$\text{rad}(G) = \min_{v \in V} e(v) = \min_{v \in V} \max_{u \in V} d(u, v).$$

Remark.

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G).$$

Theorem 1.13. Every graph G contains (provided that $\delta(G) \geq 2$.)

- (a) a path of length $\delta(G)$ and

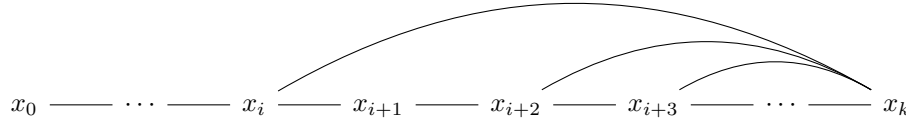
(b) a cycle of length at least $\delta(G) + 1$.

Proof. Let x_0, \dots, x_k be a longest path in G . Then all the neighbours of x_i lie on this path, otherwise, if w is a neighbor that is not in the path, then x_0, \dots, x_k, w is a longer path, a contradiction. Hence $d \geq d(x_k) \geq \delta(G)$.

Let

$$i = \min\{0 \leq i < k \mid x_i x_k \in E(G)\}.$$

Then $x_i \cdots x_k x_i$ is a cycle of length at least $\delta(G) + 1$.



□

Theorem 1.14. Every graph G containing a cycle satisfies $g(G) \leq 2 \text{diam}(G) + 1$.

Proof. Let C be a shortest cycle in G .

If $g(G) \geq 2 \text{diam}(G) + 2$, then C has two vertices whose distance in C is at least $\text{diam}(G) + 1$.

In G , these vertices have a lesser distance; any shortest path P between them is therefore not a subgraph of C .

Thus, P contains a C -path xPy .

Together with the shorter of the two $x-y$ paths in C , this path xPy forms a shorter cycle than C , a contradiction. □

1.4 Connectivity

Definition 1.15. Let $G = (V, E)$ be nonempty. G is connected if \exists a $u-v$ path for each $u, v \in V$. $U \subseteq V$ is connected if $G[U]$ is connected.

Theorem 1.16. G is connected, then vertices of G can be ordered as v_1, \dots, v_k so that each $G_i = [v_1, \dots, v_i]$ is connected for $i = 1, \dots, n$.

Proof. Pick any vertex as v_1 and assume inductively that we have picked v_1, \dots, v_j with G_j connected for $j = 1, \dots, i$.

Let $v \in G \setminus G_i$. Since G is connected, $\exists v_1-v$ path P in G .

Let v_{i+1} be the first vertex on P that is not in G_i .

Clearly, G_{i+1} is connected. □

Definition 1.17. The maximal connected subgraphs of G are its components.

Definition 1.18. Let $X \subseteq V \cup E$ and we call X a separating set if $G - X$ is disconnected.

If X is a separating set with $X \subseteq V$, we call X a separator.

Remark. Clearly, the components are induced subgraphs, and their vertex sets partition V . Since connected graphs are non-empty, the empty graph has no components.

Definition 1.19. Let $k \in \mathbb{N}_0$. G is k -connected if $|G| > k$ and $G - X$ is connected for all $X \subseteq V$ with $|X| < k$.

The connectivity $\kappa(G)$ is the largest k for which G is k -connected.

Remark. $\kappa(G) = 0$ if and only if G is disconnected or a K^1 .

Example 1.20. K^5 is 0-connected since it is connected.

K^5 is 1-connected since K^4 is connected.

K^5 is 2-connected since K^3 is connected.

K^5 is 3-connected since K^2 is connected.

K^5 is 4-connected since K^1 is connected.

K^5 is not 5-connected since $|K^1| = 5$.

So $\kappa(G) = 4$.

Since if a graph G is k -connected, then $|G| > k$,

$$\kappa(K^n) = n - 1, \forall n \in \mathbb{Z}^{\geq 1}.$$

Theorem 1.21. The smallest separator of G , X has $|X| = \kappa(G)$.

Definition 1.22. If $|G| > 1$ and $F \subseteq E$ with $G - F$ connected for all $F \subseteq E$ with $|F| < l$, then G is l -edge connected.

$\lambda(G)$ is the largest l for which G is l -edge connected.

Theorem 1.23. If G is non-trivial,

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Proof. The second inequality follows from the fact that all edges incident with a fixed vertex separate G .

To prove the first, let F be a set of $\lambda(G)$ edges such that $G - F$ is disconnected, i.e., F is a smallest separating set of edges.

We just need to show

$$\kappa(G) \leq |F|.$$

Idea is to construct a set $X \subseteq V$ that is a separator having $|X| \leq |F|$.

(a) Suppose first that G has a vertex that is not incident with an edge in F .

Let C be the component of $G - F$ containing v .

Then the vertices of C that are incident with an edge in F separate v from $G - C$.

Since no edge in F has both ends in C by the minimality of F , there are at most $|F|$ such vertices, giving

$$\kappa(G) \leq |F|.$$

(b) Suppose now that every vertex is incident with an edge in F .

Let v be any vertex, and let C be the component of $G - F$ containing v .

Then the neighbors w of v with $vw \notin F$ lie in C and are incident with distinct edges in F by the minimality of F , giving $d_G(v) \leq |F|, \forall v \in V$.

As $N_G(v)$ separates v from any other vertices in G , this yields $\kappa(G) \leq |F|$, unless there are no other vertices, i.e., unless $\{v\} \cup N(v) = V$.

But v was an arbitrary vertex. So we may assume that G is complete, giving $\kappa(G) = \lambda(G) = |G| - 1$.

□

1.5 Trees and forests

Definition 1.24. An acyclic graph is a forest.
A tree is a connected acyclic graph.

Example 1.25. List all trees on 6 vertices.
We have 6 trees.

Remark (Cayley's formula). The number of trees on n labeled vertices is n^{n-2} , $\forall n \in \mathbb{Z}^{\geq 0}$.
The formula equivalently counts the number of spanning trees of a complete graph with labeled vertices.

The number of unlabeled trees on n vertices: generating functions.

Theorem 1.26. *TFAE.*

- (a) T is a tree.
- (b) $\exists!$ u - v path in T for every $u, v \in V(T)$.
- (c) T is minimally connected.
- (d) T is maximally acyclic.

Proof. “(i) \Rightarrow (ii)”. Suppose there exists two distinct u - v paths in T for some $u, v \in T$.
Say

$$\begin{aligned} P_1 &= u = x_0 \cdots x_l = v, \\ P_2 &= u = y_0 \cdots y_k = v. \end{aligned}$$

But then

$$x_0 \cdots x_l y_k \cdots y_0$$

is a walk beginning and ending at u .

Hence it contains a cycle, a contradiction.

“(i) \Rightarrow (iii)”. Suppose T is not minimally connected.

Then for some edge uv , $T - uv$ is connected and hence contains a u - v path P .

But then $uPvu$ is a cycle.

“(i) \Rightarrow (iv)”. Suppose T is not maximally acyclic.

Then for some edge uv with $u \not\sim v$, we can connect u and v such that $T + uv$ is acyclic.

Let P be the unique uv path in T before adding new edge.

Then $uPvu$ is a cycle.

Others will be left as an exercise. □

Definition 1.27. A special vertex T is called a root.

A vertex of T other than the root, of degree 1 is called a leaf.

Theorem 1.28. *Every nontrivial tree contains a leaf.*

Proof. Let P be a longest path.

Let $P = x_0 \cdots x_k$. Then x_k is a leaf. □

Corollary 1.29. The vertices of a tree can be listed $v_0 \cdots v_n$ so that v_i has a unique neighbor in $\{v_0, \dots, v_{i-1}\}$, $\forall 1 \leq i \leq n$.

Proof. For any connected graph, by previous theorem, there exists an ordering $\{v_0, \dots, v_n\}$ so that for $1 \leq i \leq n$, $[v_0, \dots, v_i]$ is connected.

Assume inductively $[v_0, \dots, v_i]$ is a tree.

Claim the only new edge results $v_i v_{i+1}$ when we add v_{i+1} . □

Corollary 1.30. Let G be acyclic. Then G is a tree if and only if $\|G\| = n - 1$.

Proof. “ \Rightarrow ”. Induction on i shows that the subgraph spanned by the first i vertices in previous corollary has $i - 1$ edges.

“ \Leftarrow ”. Let G be any connected graph with n vertices and $n - 1$ edges.

Let G' be a spanning tree in G .

Since G' has $n - 1$ edges by the first implication, it follows $G = G'$. □

Theorem 1.31. A graph T with $|T| = n$ is a tree if and only if any 2 of the following hold.

(a) T is a cyclic.

(b) T is connected.

(c) $\|T\| = n - 1$.

Corollary 1.32. Let T be any tree of order n and let G be any graph with $\delta(G) = n - 1$. Then G contains a tree isomorphic to T as a subgraph.

Proof. List the tree $v_0 \cdots v_n$.

Induction. $[v_0]$ is in G . Assume $[v_0, \dots, v_i]$ is a subgraph of G .

WTS

$$[v_0, \dots, v_i, v_{i+1}] \subseteq G.$$

□

1.6 Bipartite

Definition 1.33. A graph $G = (V, E)$ is r -partite if there exists an r -partition of V so that every edge of G has ends in distinct partite class.

If $r = 2$, G is called bipartite.

Definition 1.34. If G and G' are disjoint, then $G * G'$ is obtained by taking the disjoint union of G and G' and joining every vertex in $V(G)$ with every vertex in $v(G')$ with an edge.

Example 1.35. $P^1 * P^2$.

Definition 1.36.

Definition 1.37. An r -partite graph in which every two vertices from different partition classes are adjacent is called complete.

The complete r -partite graph $\overline{K^{n_1}} * \cdots * \overline{K^{n_r}}$ is written as

$$K_{n_1 \cdots n_k}.$$

Example 1.38. $K_{1,5}$ is a star.

Theorem 1.39. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. “ \Rightarrow ”. Let $G = (V, E)$ be bipartite with $V = V_1 \cup V_2$.

Suppose G contains an odd cycle $v_0 \cdots v_k$ with k even.

Wlog, let $v_0 \in V_1$, so

$$v_1 \in V_2, v_2 \in V_1, \dots, v_k \in V_1.$$

But $v_0 \sim v_k$, a contradiction.

“ \Leftarrow ”. Suppose G contains no odd cycle.

Fix $v_0 \in G$. Let

$$V_1 = \{v \in V(G) \mid d(v_0, v) \text{ is odd}\}.$$

$$V_2 = \{v \in V(G) \mid d(v_0, v) \text{ is even}\}.$$

If $u \in V_1$ and $w \in V_1$ and $u \sim w$, then we have an odd cycle.

If $u \in V_2$ and $w \in V_2$ and $u \sim w$, then we have an odd cycle. □

1.7 Contraction and minors

Definition 1.40. Let $G = (V, E)$ and $e \in E$ so that $\{e\}$ is not a separating set, i.e., e is not a bridge or cut edge. Then $G - e$ is the graph obtained from G by removing e .

Definition 1.41. An edge contraction $G \setminus e$ is obtained by removing an edge from a graph while simultaneously merging the two vertices that it previously joined and removing any resulting loops on multiple edges.

Definition 1.42. Any graph obtained from G by a series of deletions and contractions is called a minor of G .

Note we define the deletion of a cut edge to be the contraction of that edge.

To undo a deletion, we add the edge back.

Definition 1.43. Let X be a fixed graph. Replacing the vertices x of X with disjoint connected graphs G_x and replacing the edges xy of X with non-empty sets of $G_x - G_y$ edges, yields a graph that we shall call an IX , where $G_x - G_y$ is the set of all edges with an end in G_x and the other in G_y .

More formally, a graph G is an IX if its vertex set admits a partition $\{V_x \mid x \in V(X)\}$ into connected subsets V_x such that distinct vertices $x, y \in X$ are adjacent in X if and only if G contains a $V_x - V_y$ edge.

Definition 1.44. If a graph G contains an IX as a subgraph, then X is a minor of G .

Example 1.45. Peterson has a K^5 minor.

Definition 1.46. A subdividing of X , informally, any graph obtained from X by ‘subdividing’ some or all its edges by drawing new vertices on those edges. In other words, replace some edges of X with new paths between their ends, so that none of these paths has an inner vertex in $V(X)$. When G is a subdivision of X , we also say that G is a TX .

The original vertices of X are the branch vertices of the TX and its new vertices are called subdividing vertices.

Note that subdividing vertices have degree 2 while branch vertices retain their degree from X .

Definition 1.47. If a graph G contains a TX as a subgraph, then X is a topological minor of G .

1.8 Euler tours

Definition 1.48. Let $G = (V, E)$ be connected, simple and finite.

An Euler tour of G is a closed walk that uses each edge exactly once.

A graph is Eulerian if it contains an Euler tour.

Theorem 1.49. A connected graph G is Eulerian if and only if $\forall v \in V$, $d_G(v)$ is even.

Proof. “ \Rightarrow ”. Let W be an Euler tour. Then

$$d_W(v) = d_G(v).$$

Since $d_W(v)$ is even, $d_G(v)$ is even.

“ \Leftarrow ”. Let W be a longest walk that uses each edge at most once.

Claim W is closed.

Else $d_W(v)$ is odd for the last vertex u in W . But then W is not largest possible.

Claim $\forall u, v \in W$, the edge $uv \in W$, provided $uv \in E$. Else W is not longest possible.

Claim $\forall v \in V, v \in W$.

Suppose not. Then $v \in V$ but $v \notin W$.

Wlog, $v \sim u$ with $u \in W$. $uWuv$ is a longer walk. □

1.9 Some linear algebra

Definition 1.50. Let $G = (V, E)$ with $V = \{c_1, \dots, c_n\}$ and $E = \{e_1, \dots, e_n\}$. Associated any $U \subseteq V$ a vector $X_U \in \mathbb{F}_2^n$ with

$$X_U(v) = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, for $F \subseteq E$, $X_F \in \mathbb{F}_2^m$.

Remark. Add two vectors in \mathbb{F}_2^m means taking the symmetric differences.

We abuse notation slightly and refer to X_U as U and X_F as F .

Definition 1.51. Let $\mathcal{C}(G)$ be the subspaces of \mathbb{F}_2^m , spanned by the cycles of G . We call it the cycle space.

Definition 1.52. $F \subseteq E$ is a cut if V has a partition $\{V_1, V_2\}$ so that every edge $f \in F$ has one end in V_1 and one end in V_2 .

A minimal cut is a bond.

Definition 1.53. Let $\mathcal{C}^*(G)$ be the subspace of \mathbb{F}_2^n generated by all the bonds.

Special case of a bond: $V_1 = 1$ or $|V_2| = 1$, say $V_1 = \{v\}$, then the cut F is denoted as $E(v)$.

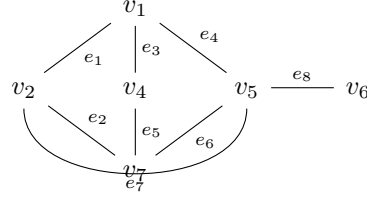
Theorem 1.54. Let $\{V_1, V_2\}$ partition V . Let F be corresponding cut. Then in \mathbb{F}_2^n

$$F = \sum_{v \in V_1} E(v).$$

Proof. Every edge in the sum appears twice if both ends are in V_1 and once if exactly one end is in V_1 . □

Lemma 1.55. $\{E(v)|v \in V\}$ generates $\mathcal{C}^*(G)$.

Example 1.56. Consider the following graph.



The vertex-edge incident matrix is

$$M = \begin{matrix} & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{matrix} E(v_1) \\ E(v_2) \\ E(v_3) \\ E(v_4) \\ E(v_5) \\ E(v_6) \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Definition 1.57. A tree T is spanning tree of G if

- (a) T is a subgraph of G .
- (b) $V(T) = V(G)$.

Theorem 1.58. The rank of the incident matrix is $n - 1$.

Proof. Find $n - 1$ linearly independent columns, equivalently, an spanning tree T in G . Then $|T| = n - 1$. \square

Theorem 1.59. Let M be the incident matrix. Then for any set of $(n - 1)$ linearly independent columns of M , the edges corresponding to these columns make up a spanning tree of G .
The columns corresponding to any tree are linearly indep.
The fundamental cycle are minimally linearly dependent.

Proof. $\{v|E(v)\}$ generates $\mathcal{C}^*(G)$. \square

Corollary 1.60.

$$\dim(\mathcal{C}^*(G)) = n - 1.$$

Definition 1.61. For $F, F' \in \mathbb{F}_2^m$, the inner product is

$$\langle F, F' \rangle = \sum_{i=1}^m F(e_i)F'(e_i) \in \mathbb{F}_2.$$

Theorem 1.62. The inner product is zero if and only if F and F' have an even number of edges in common.

Example 1.63. Let

$$F = (1, 0, 0, 0, 0, 1),$$

$$F' = (1, 1, 0, 0, 0, 1),$$

Then

$$\langle F, F' \rangle = 1 + 0 + 0 + 0 + 0 + 1 = 0.$$

Definition 1.64. For any subspace \mathfrak{F} of \mathbb{F}_2^m , we define

$$\mathfrak{F}^\perp = \{D \in \mathbb{F}_2^m \mid \langle F, D \rangle = 0, \forall F \in \mathfrak{F}\}.$$

Lemma 1.65. Every cut $C \in \mathcal{C}$ is a (possibly empty) disjoint union of edge of cycles in G .

Proof. □

Theorem 1.66.

$$\mathcal{C} = \mathcal{C}^{*\perp} \text{ and } \mathcal{C}^* = \mathcal{C}^\perp,$$

i.e.,

$$\mathcal{C} \oplus \mathcal{C}^* = \mathbb{F}_2^m.$$

Proof. Let $C \in \mathcal{C}(G)$ and $D \in \mathcal{C}^*(D)$.

Then C intersects D an even number of times.

So

$$\mathcal{C} \subseteq \mathcal{C}^{*\perp} \text{ and } \mathcal{C}^* \subseteq \mathcal{C}^\perp.$$

Exercise. □

Corollary 1.67.

$$\dim(\mathcal{C}(G)) = m - n + 1.$$

1.9.1 Basis

Theorem 1.68. (a) A basis for the cycle space \mathcal{C} is obtained as follows: for any spanning tree T of G , each out of tree edge ij creates a unique cycle if edge ij is concatenated to the unique in-tree ji path and there are exactly $m - n + 1$ such cycles.

The basis obtained in this way is called a fundamental cycle basis.

(b) Let T be a spanning tree. For every edge $f \in T$, the forest $T - f$ has exactly two component. The set $D_f \subseteq E$ of edges of G between these components is a bond in G , the fundamental cut of f with respect to T .

Then a fundamental cut of G with respect to T form a basis of $\mathcal{C}^*(G)$.

Theorem 1.69.

$$\text{Ker}(M) = \mathcal{C}(G).$$

$$\text{Im}(M^T) = \mathcal{C}^*(G).$$

Example 1.70. If we put M into standard form, we'd get

$$[I_{n-1} \mid A],$$

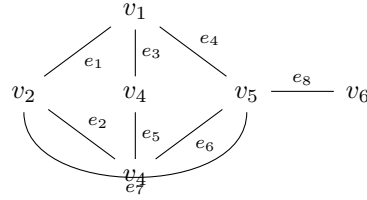
where A is $(n-1) \times (m-n+1)$ matrix.

Then the matrix

$$[A^T \mid I_{m-n+1}]$$

generates $C^*(G)$.

Example 1.71. Consider the following graph.



The vertex-edge incident matrix is

$$M = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \begin{matrix} E(v_1) \\ E(v_2) \\ E(v_3) \\ E(v_4) \\ E(v_5) \\ E(v_6) \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

Let $\{e_1, e_2, e_3, e_4, e_8\}$ be a spanning tree.

Then

$$[I_{n-1} \mid A] = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_8 & e_5 & e_6 & e_7 \\ \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_8 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{matrix}.$$

Add e_5 , there is a cycle $\{e_1, e_2, e_3, e_5\}$, which is a min. dependent set and also a fundamental cycle. So the e_5 column has a column vector $(1, 1, 1, 0, 0)^T$.

$a_{ij} = 1$ if and only if e_i is used in the fundamental cycle associated with e_j .

Note

$$[A^T \mid I_{m-n+1}] = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_8 & e_5 & e_6 & e_7 \\ \begin{matrix} e_5 \\ e_6 \\ e_7 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}.$$

Note every row is a fundamental cycle.

Remark. Let the edges of T be the basic elements and non-basic elements is called non-tree edges. Fundamental cycle is the unique cycle containing exactly one non-tree edge.

Theorem 1.72. Any collection of edges that induces a subgraph H with $d_H(v)$ even for all $v \in V(H)$ is a disjoint union of cycles.

Chapter 2

Matching Covering and Packing

Definition 2.1. A matching M in a simple graph $G = (V, E)$ is a set of independent edges. These vertices incident with the edges of a matching M are said to be saturated by M , the others are unsaturated.

Definition 2.2. A perfect matching in a graph is a matching that saturated every vertex, that is, a matching of size exactly $\frac{n}{2}$.

Remark. A perfect matching can only occur in a graph with evenly many vertices.

Remark. maximum: largest possible.

maximal: whether it can be extended by simply adding an edge.

Example 2.3. In P^3 ,

$$a \text{ --- } b \text{ --- } c \text{ --- } d$$

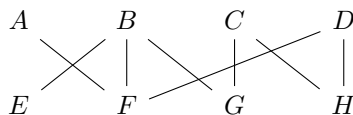
$\{ab, cd\}$ is maximal matching and a maximum matching ($\lfloor \frac{4}{2} \rfloor = 1$ since it contains 2 edges in 4 vertices.).

$\{bc\}$ is a maximal matching but not a maximum matching.

Definition 2.4. An M -alternating path is a path that alternates between edges in $E \setminus M$ and edges in M (in order).

Definition 2.5. An M -augmenting path $P = (v_1, \dots, v_k)$ is an M -alternating path s.t. $v_1, v_k \notin V(M)$

Example 2.6. Let $M = \{BF, CG\}$.



$EBFD$ and $AFBGCH$ are M alternating paths.

Lemma 2.7. Let M_1 and M_2 be matching of G . The degree of every vertex in $[M_1 \Delta M_2]$ is 1 or 2, Hence, $[M_1 \Delta M_2]$ is the disjoint union of paths and cycles.
Furthermore, each such cycle or path alternates in edges in M_1 and M_2 .

Theorem 2.8 (Berge). *A matching M in G is maximum if and only if G does not contain an M -augmenting path.*

Proof. “ \Rightarrow ”. By contrapositive.

“ \Leftarrow ”. Again, by contrapositive. Suppose M is not maximum.

Let M' be a larger matching.

Then $M' \Delta M$ is a collection of paths and even cycles that alternate between M and M' .

At least one such path begins at M' and ends at M' . But this is an M -augmenting path. \square

2.1 Matching, vertex covering in bipartite graph

Let $G = (V, E)$ be bipartite with $V = \{A, B\}$.

Definition 2.9. A vertex cover U is a subseq of V s.t. for all edges e , there is a vertex, say $u \in U$ with u incident with e .

Remark. For now, an alternating path w.r.t. a matching M begins at an unsaturated vertex in A , and contains, alternately edges from $E \setminus M$ and from M .

An alternating path that ends in an unmatched vertex of B is called an augmenting path.

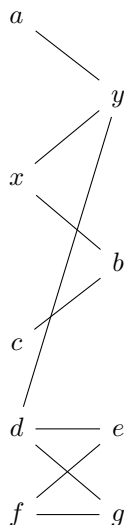
Definition 2.10. $\tau(G)$ = size of a smallest vertex cover.

$\nu(G)$ = size of a maximum matching.

Theorem 2.11 (König).

$$\tau(G) = \nu(G).$$

Proof. Consider



with $M = \{xy, cd, de, fg\}$ being maximum. So a is the only unsaturated vertex. Clearly,

$$\tau(G) \geq \nu(G).$$

Let M be a maximum matching.

Construct a vertex cover U as follows.

For each matching edge $xy \in M$ with $x \in A$ and $y \in B$, do the following: if y is reachable via an alternating path, then put y into U , otherwise, put x into U .

Claim: every edge is incident with a vertex in U .

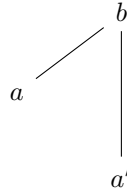
Let $ab \in E$ with $a \in A$ and $b \in B$.

If $ab \in M$, done.

Suppose $ab \notin M$.

Case 1: a is unsaturated. Then b is saturated. Else M is not maximum.

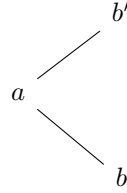
Say $a'b \in M$.



Then a, ab, b is an alternating path ending at b .

So b is reachable from a and then $b \in U$.

Case 2: a is saturated.



Say $ab' \in M$.

If $a \in U$, we are done.

Else $b' \in U$ and so b' is reachable via an alternating path P .

Let

$$P' = \begin{cases} Pb & \text{if } b \in P \\ Pb'ab & \text{if } b \notin P \end{cases}.$$

Then b must be reachable and so $b \in U$. □

Definition 2.12.

$$N(S) = \{u \in N(s) \text{ for all } s \in S\}.$$

Theorem 2.13. A necessary (marriage) condition for a matching saturating A is

$$|S| \leq |N(S)|, \forall S \subseteq A.$$

Theorem 2.14 (Hall 1935). *A bipartite graph $G = (V, E)$ with $V = \{A, B\}$ has a matching saturating A if and only if*

$$|S| \leq |N(S)|, \forall S \subseteq A.$$

Proof. “ \Rightarrow ”. By the marriage condition.

“ \Leftarrow ”. Assume G contains no A matching. Then

$$\nu(G) < |A|.$$

Let U be a minimum vertex cover, say $U = A_1 \cup B_1$.

By König theorem,

$$|A_1| + |B_1| = |U| = \tau(G) = \nu(G) < |A|.$$

Then

$$|B_1| < |A| - |A_1| = |A \setminus A_1|.$$

Notice that there are no edges between $A \setminus A_1$ and $B \setminus B_1$.

So

$$N(A \setminus A_1) \subseteq B_1.$$

Thus,

$$|N(A \setminus A_1)| \leq |B_1| < |A \setminus A_1|,$$

which is contradicted by the assumption. □

Definition 2.15. A k -regular spanning subgraph is called a k -factor.

Corollary 2.16. 1-factor: a matching that saturates all vertices (perfect).

A subgraph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of V .

Corollary 2.17. Every k -regular bipartite graph has a 1-factor.

(Or every regular bipartite graph has a perfect matching.)

Proof. Let $G = (V, E)$ be k -regular with $V = \{A, B\}$.

Since $k|A| = k|B|$,

$$|A| = |B|.$$

Let $S \subseteq A$.

Then S is joined to $N(S)$ by a total of $k|S|$ edges.

These are among the $k|N(S)|$ edges of G incident with $N(S)$.

So

$$k|S| \leq k|N(S)|.$$

Then

$$|S| \leq |N(S)|.$$

So Hall's condition is satisfied.

Thus, G has a matching saturating A and so has a 1-factor. □

Definition 2.18. For $X \subseteq A$,

$$\text{def}_G(X) = |X| - |N(X)|.$$

We have

$$\text{def}(G) = \max_{X \subseteq A} \text{def}_G(X).$$

Theorem 2.19 (Refinement of Hall's theorem). *Let $G = (V, E)$ with $V = \{A, B\}$, then*

$$\nu(G) = |A| - \text{def}(G).$$

Proof. Let $d = \text{def}(G)$ and $\nu = \nu(G)$.

Clearly,

$$\nu \leq |A| - d.$$

Construct G' :

$$\begin{cases} \text{add } b_1, \dots, b_d \text{ to } B \\ \text{add edges } ab : \forall a \in A \end{cases}.$$

By Halls' theorem, G' has a matching M of A .

Note M use precisely edges in $E(G) \setminus E(G')$. □

2.2 Matching in general graphs

Definition 2.20. Let \mathcal{C}_G be the set of its components.

Definition 2.21. Let $q(G)$ be the number of components of G of odd order.

Theorem 2.22. *The necessary condition for the existence of a 1-factor (Tutte's condition) is:*

$$q(G - S) \leq |S|, \forall S \subseteq V(G).$$

Theorem 2.23 (Tutte). *A graph G has a 1-factor if and only if*

$$q(G - S) \leq |S|, \forall S \subseteq V(G).$$

Proof. Say G satisfies Tutte condition but has no 1-factor.

In fact, let G be edge maximal w.r.t. these property.

Let

$$K = \{v \in V : u \sim v, \forall u \neq v\}.$$

Claim: Every component of $G - K$ is a complete graph.

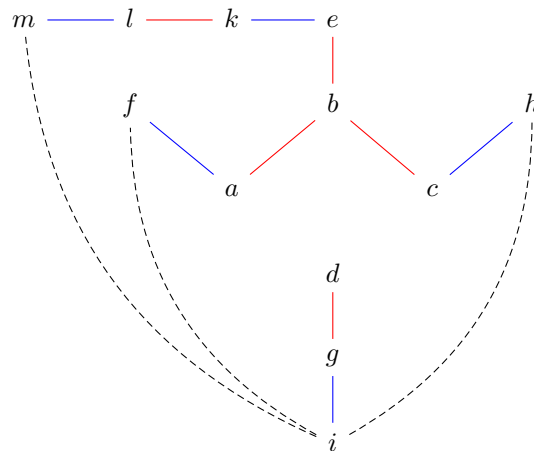
Suppose not. Then $\exists a, b, c \in V - K$ with

$$a \sim b, b \sim c, \text{ but } a \not\sim c.$$

Then since $b \notin K$, $\exists d \in V$ such that

$$d \not\sim b.$$

By edge maximality, there exists a matching M_1 , saturating all vertices except a and c , and a matching M_2 saturating all vertices except b and d .



Consider $M_1 \Delta M_2$: alternating cycles and paths.

Then we construct an augmenting path P : start d , alternating between edges in M_1 and edges in M_2 .

(a) P ends at b . But then P is an M_2 -augmenting path, a contradiction since M_2 is maximum.

(b) P ends at a or c . Consider Pab . Then Pab is an M_2 -augmenting path, a contradiction.

So every component of $G - K$ is a complete graph.

Thus, we have a 1-factor, a contradiction. \square

Corollary 2.24 (Peterson 1891). Every cubic bridgeless graph has a 1-factor.

Proof. We show that every graph satisfies Tutte's condition.

Let $S \subseteq V$.

Consider an odd component C of $G - S$.

Then $\partial(C)$, coboundary of C , is the set of all edges in G with exact one end in C .

Note

$$3|C| = \sum_{v \in C} d(v) = 2|E(C)| + |\partial(C)|.$$

So $|\partial(C)|$ is odd.

Since G is bridgeless,

$$|\partial(C)| \geq 3.$$

This is true for each odd component.

So with $\bar{S} = V - S$,

$$|\partial(\bar{S})| \geq 3 \cdot q(G - S).$$

Also,

$$|\partial(\bar{S})| = |\partial(S)| \leq 3|S|.$$

So

$$3|S| \geq |\partial(\bar{S})| \geq 3q(G - S).$$

Thus,

$$|S| \geq q(G - S).$$

□

2.3 Complementary

Definition 2.25. G is factor critical if it has no 1-factor but $G - u$ has a 1-factor for any $u \in V$.

Definition 2.26. A near factor is a matching in which only 1 vertex is unsaturated.

Definition 2.27. A vertex v is essential if every maximum matching covers v

Lemma 2.28. If G is connected and $\nu(G - u) = \nu(G), \forall u \in V$, then G is factor critical.

Proof. Let G be connected with $\nu(G) = \nu(G - u), \forall u \in V$.

So G has no 1-factor.

It suffices to show no maximum matching leaves two distinct vertices unmatched.

Suppose we have a maximum matching M s.t. x and y are unmatched and $d(x, y)$ is as smallest as possible.

Clearly,

$$d(x, y) \geq 2.$$

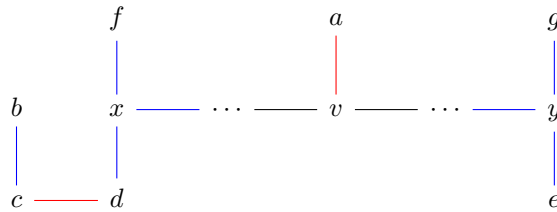
Let P be a shortest x - y path. Then there is a vertex v that is in the interior of P .

By the minimality of d , v is matched by M .

Since $\nu(G - v) = \nu(G)$, v is inessential. (All vertices of G is inessential.)

Then there exists a maximum matching M' missing v .

By the minimality of d , x, y is matched by M' .



In above graph, red edges are in M and blue edges are in M' and black edges are neither in M nor in M' .

In $M \Delta M'$, since each path alternates in edges in M_1 and M_2 , the paths in it starting at x and y are distinct.

Let Q be the path in $M \Delta M'$ starting at x , wlog, Q does not end at v .

Then $Q \Delta M'$ is a maximum matching avoiding x and v .

□

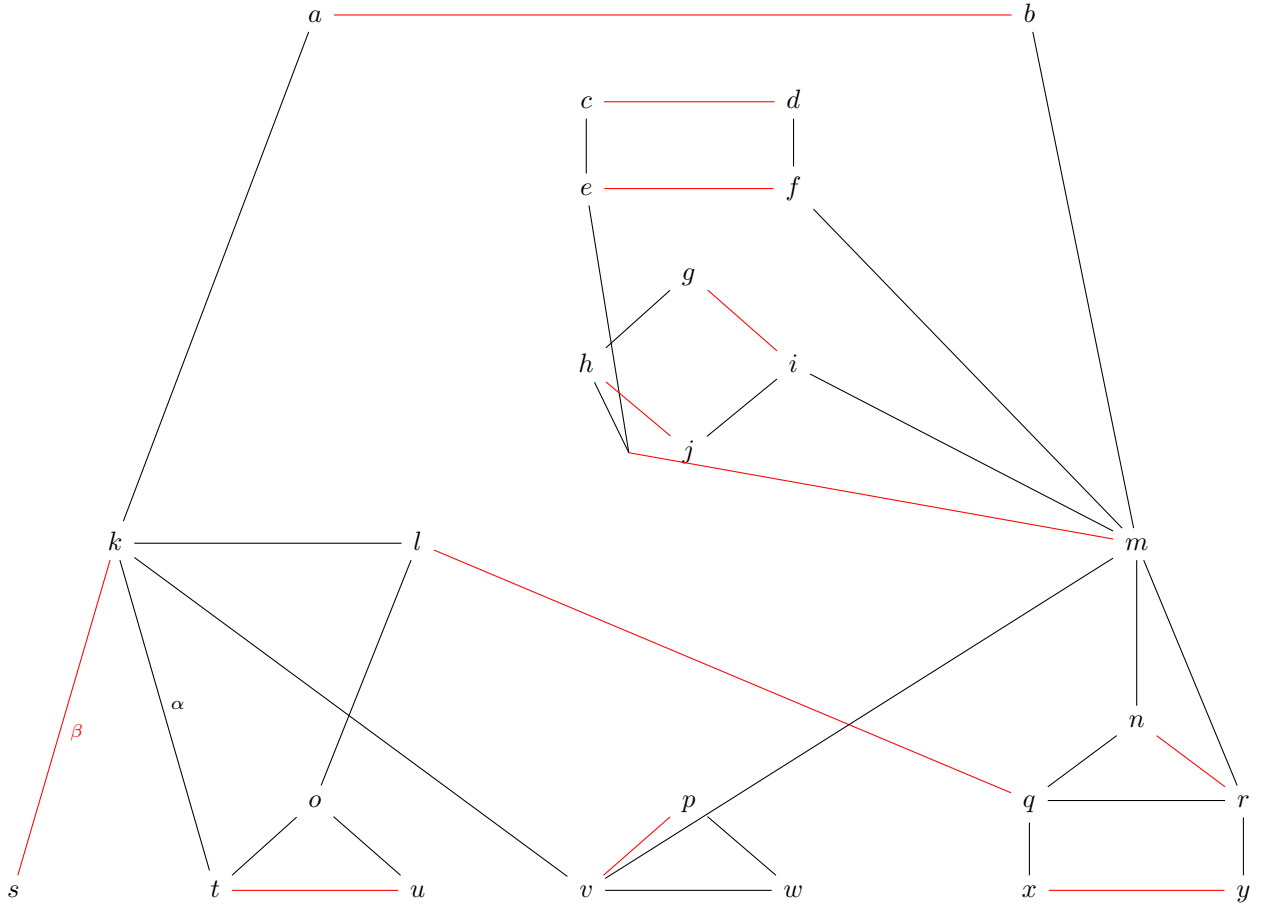
Definition 2.29. Let $G = (V, E)$ be a graph with no 1-factor. Define

$$D(G) = \{v \in V : v \text{ is an inessential vertex}\},$$

$$A(G) = \{v \in V \setminus D(G) : v \in N(D(G))\},$$

$$C(G) = V \setminus \{D(G) \cup A(G)\}.$$

Theorem 2.30. *Consider*



Since $D(G)$ is in the bottom, $A(G)$ is in the middle and then $C(G)$ is the left ones.

Lemma 2.31 (Stability lemma). Let $G = (V, E)$ be a graph with no 1-factor. Then $\forall u \in A$, we have

$$D(G - u) = D(G),$$

$$A(G - u) = A(G) - u,$$

$$C(G - u) = C(G).$$

Proof. Claim

$$\nu(G - u) = \nu(G) - 1, \forall u \in A.$$

Since $u \in A$ is essential, no matching of $G - u$ has cardinality $\nu(G)$.

So

$$\nu(G - u) < \nu(G).$$

Furthermore, let M be a maximum matching of G , then

$$|M| = \nu(G),$$

and $u \in A$ is saturated by M , say by $\alpha \in M$.

Then $M - \alpha$ is a matching of $G - u$ and $|M - \alpha| = \nu(G) - 1$.

So

$$\nu(G - u) \geq |M - \alpha| = \nu(G) - 1.$$

Thus,

$$\nu(G - u) = \nu(G) - 1.$$

Claim

$$D(G) \subseteq D(G - u), \forall u \in A.$$

Let

$$o \in D(G).$$

Let M_o be a maximum matching of G leaving o unmatched.

Then

$$|M_o| = \nu(G).$$

Let $\beta \in M_o$ be incident with $u \in A$.

Then $M_o - \beta$ is a matching of $G - k$ of size $|M_o - \beta| = \nu(G) - 1$, and hence, by previous claim, a maximum matching of $G - u$ leaving o unmatched.

So

$$o \in D(G - u).$$

Next,

$$D(G - u) \subseteq D(G).$$

Choose $v \in D(G - u)$. Let M' be a maximum matching of $G - u$ missing v .

Let $w \in D(G)$ with $w \sim u$ and let M be a maximum matching of G missing w .

We need to construct a maximum matching of G missing v , (this would imply that $v \in D(G - u)$ as required.)

If M misses v , then we are done. So assume not.

Then v is matched by M .

Let P be the path of $M \Delta M'$ starting at v .

Case 1: P_1 ends with an edge of M' . Then $M \Delta P$ is a matching in G missing v , and it is the same cardinality as M , hence maximum. So we are done.

Case 2: P ends with an edge of M . Consider $M' \Delta P$. It is maximum. Hence it must match u . So M ends at u . But then $M \Delta (P + uw)$ is a maximum matching avoiding v as required. \square

Corollary 2.32. Let $G = (V, E)$ be no 1-factor.

(a) Let M be a maximum matching in G , let $u \in A(G)$ and let f be the unique edge in M incident with u . Then $M - f$ is a maximum matching of $G - u$.

(b) Let M be a maximum matching in G . Then if f is an edge of M with one end in $A(G)$, then the other end of f is necessarily in $D(G)$.

Theorem 2.33 (Edmond's Gallai's structure theorem). *Let $G = (V, E)$ be a graph with no 1-factor and D, A, C be defined before. Then*

(a) *Every component of $[D]$ is factor critical (odd).*

(b) *Every component of $[C]$ has a 1-factor (even).*

(c) *Define a bipartite graph $\{A, B\}$, where $A = A(G)$ and a vertex of B is a component of $[D]$, with an edge if and only if a is adjacent to at least one vertex in B .*

Hall's condition holds with a surplus,

$$|N(X)| \geq |X| + 1, \forall X \subseteq A.$$

(d) *Let M be a maximum matching of G . Then M contains a near-factor of each component of $[D]$.*

A 1-factor of each component of $[C]$ and vertices in A are matched to vertices in distinct component of $[D]$.

(e)

$$\nu(G) = \frac{1}{2}(|V| - q(G - A) + |A|).$$

Proof. Delete vertex of A one at a time.

$$D(G - A) = D(G),$$

$$A(G - A) = \emptyset,$$

$$C(G - A) = C(G).$$

(a) Since any matching M of G saturating A , $M \cap E(G - A)$ has cardinality $\nu(G) - |A|$ and is a maximum matching of $G - A$.

Use Gallai's lemma, it is enough to show

$$\nu(G_i - v) = \nu(G_i), \forall v \in V(G_i),$$

where G_i is a component of $[D]$.

So let

$$v \in V(G_i).$$

Let M_v be a maximum matching of G , leaving v unsaturated.

Remove all edges of M_v incident with A and then the part left has cardinality $\nu(G) - |A|$ and is a maximum matching of $G - A$.

Since the component of $[D]$ are disjoint, **restricting** $M_v - E[A]$ is a maximum matching of G_i avoiding v .

So

$$\nu(G_i) = \nu(G_i - v).$$

By Gallai's lemma, G_i is factor critical.

(b) Note that $[C]$ has 1-factor (start with a maximum matching M of G and remove all edges incident with A .) (Again we used consequence (2) above).

(d) (Key point: every vertex $k \in A$ is saturated by any maximum matching M of G , say $\beta \in M$ is incident with k and the other end of β must be in D . Else remove k and β to get a maximum matching of $G - k$?)

From (a) and (b), it follows that a maximum matching in $G - A$ consists of a 1-factor of $[C]$ and a near factor of each component of $[D]$, i.e., we can do better than this, so this must be largest possible. We also know that removing all edges incident with A from any maximum matching of G results in a maximum matching of $G - A$, and hence leaves exactly 1 vertex unsaturated in each component G_i of $G - A$ in D .

(e) Clearly now.

(c) Let $C \subseteq A$. Let $u \in X$ and let $u \sim v$ with $v \in b$, where b is some component of $[D]$. Let M be a maximum matching of G avoiding v . By (d), the rest of the vertices in b are matched to vertices also in b . Hence no vertex in X is matched to a vertex of b . It follows that each of the $|X|$ vertices in X is matched to a distinct component other than b in $[D]$. These $|X|$ distinct components, together with b form our requisite set of size at least $|X| + 1$ elements in B in the neighbor set of X .

□

Chapter 3

Connectivity

3.1 2-connected graphs and subgraphs

Definition 3.1. A cut vertex is one that separates of two other vertices.

Definition 3.2. G is 2-connected if it contains at least 3 vertices and has no cut vertex.

Definition 3.3. Ear decomposition is a simple recursive procedure for generating any 2-connected graph starting with a cycle.

Definition 3.4. An F -path is also called an ear of F in G .

Theorem 3.5. Let F be a nontrivial subgraph of a 2-connected graph G . Then F has an ear in G .

Proof. Case 1: F spans G . Then $\exists e \in E(G - F)$. Then e is an ear.

Case 2: F is not spanning.

Since G is connected, $\exists xy \in E(G)$ with $x \in V(F)$ and $y \in V(G - F)$.

Since G is 2-connected, there is a $(y, F - x)$ -path Q in $G - x$.

So $P = xyQ$ is an ear in F . □

Theorem 3.6. Let F be an 2-connected subgraph of G . Let P be an ear of F . Then $F \cup P$ is 2-connected.

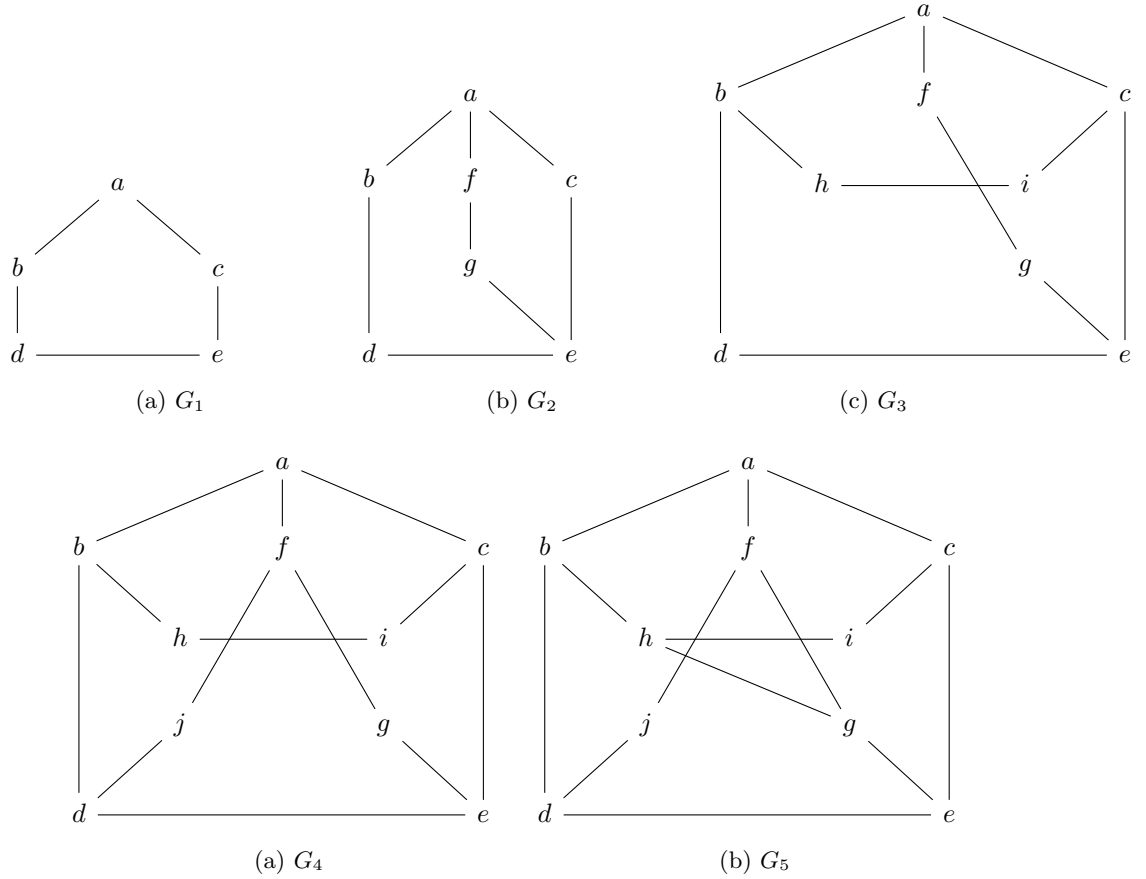
Proof. Exercise. □

Definition 3.7. A nested sequence of graphs is a (finite) sequence (G_0, \dots, G_k) with $G_i \subsetneq G_{i+1}$ for $0 \leq i \leq k - 1$.

Definition 3.8. An ear decomposition of 2-connected graph is a nested sequence (G_0, \dots, G_k) of a 2-connected so that

- (a) G_0 is a cycle;
- (b) $G_{i+1} = G_i \cup P_i$, where P_i is an ear of G_i in G , $0 \leq i \leq k - 1$.

Example 3.9. Consider



Lemma 3.10. Every 2-connected graph G has a cycle.

Proof. Since G is 2-connected, G is connected.

Suppose G is acyclic, then G is a tree.

So G contains a leaf x .

Let y be the unique vertex adjacent to x , then y is a cut vertex, a contradiction. \square

Lemma 3.11. G is 2-connected if and only if it has an ear decomposition.

Proof. “ \Leftarrow ”. Induction on the number of ears.

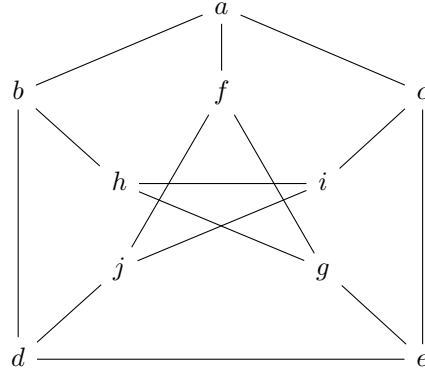
G_0 is a cycle and 2-connected.

Then inductively apply previous theorem that the union of 2-connected graph and an ear is still 2-connected.

“ \Rightarrow ”. Use previous lemma and the following theorem. \square

Theorem 3.12. Let F be a nontrivial proper subgraph of a 2-connected graph G . Then F has an ear in G .

Definition 3.13. A block of a graph G is a maximal connected subgraph without a cut vertex.

Figure 3.3: G_6

Remark. Types of blocks.

- (a) maximal 2-connected graph.
- (b) a bridge.
- (c) an isolated vertex.
- (d) If different blocks overlap, then they overlap in one vertex (a cut vertex).
- (e) Every edge lies in a unique block.
- (f) G is the union of its blocks.

Definition 3.14. A bond is a minimal cut. Assume G is cut into two parts A and B , then either A or B is connected.

Theorem 3.15. If F is a cut with $xy \in F$, then F is a bond if and only if it is a minimal intersection set of all x - y path.

Lemma 3.16. (a) cycles of G are cycles of the blocks.

(b) bonds of G are bonds of the blocks.

Proof. (a) A cycle is 2-connected. So it must be part of some maximal 2-connected subgraphs.

(b) Let F be a bond of G , let $xy \in F$. So F separates x and y in G .

Let B be the block containing xy , by the maximality of B , G contains no B -path.

Hence B contains all x - y paths (or use previous theorem).

So $F \cap E(B)$ separates x and y in B .

Thus, F is also a bond in B .

□

Lemma 3.17. For distinct edges e and f of G , TFAE.

- (a) e and f belong to the same block;

(b) e and f belong to the same cycle;

(c) e and f belong to the same bond;

Proof. “(i) \Rightarrow (ii)”. Let e and f be in the same block B .

Let B be 2-connected.

Claim in any 2-connected graph, any two edges are in the same cycle.

It suffices to show that for any two distinct pairs $\{u_1, u_2\}$ and $\{v_1, v_2\}$ of vertices, there are two disjoint paths.

Since B is 2-connected, it has an ear decomposition $\{B_0, \dots, B_k\}$.



If $k = 0$, then B is a cycle, done .

Then induct on k .

Let

$$0 \leq i \leq k - 1.$$

Case 1: both $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are in B_i . True by inductive assumption.

Case 2: both are in P , which is an ear of B_i , since G_i is connected, they are in the same cycle.

Case 3: one pair is in B_i and the other is in P_k .

Use induction with the pairs $\{u_1, u_2\}$ in G_i and $\{u, v\}$ in G_i (By symmetry).

“(iii) \Rightarrow (ii)”. Let e and f be in the same cycle C .

Removing e and f from C leaving two paths P_1 and P_2 .

Grow P_1 and P_2 into a partition V_1, V_2 of G so that $e, f \in V_1 - V_2$, so that $[V_2]$ is connected?

Then edges between V_1 and V_2 form a bond of G ?

Let V_2 be the connected component of $G - P_1$ containing P_2 and V_2 ?

Let

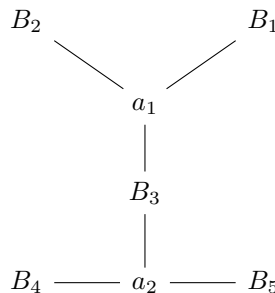
$$V_1 = V(G) - V_2.$$

“(iii) \Rightarrow (i)”. Assume e and f are in the same bond of G .

That bond is also a bond of some block B of G .

Then B contains e and f . □

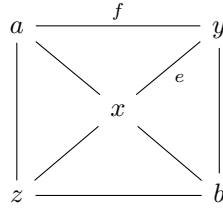
Definition 3.18. Bipartite $\{A, B\}$, where A is the set of cut vertices and B is the set of blocks. $a \sim B$ in this block graph if $a \in B$.



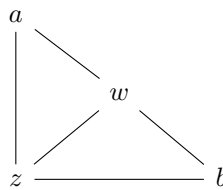
Definition 3.19. The block graph of a connected graph is a tree.

3.2 The structure of 3-connected graphs

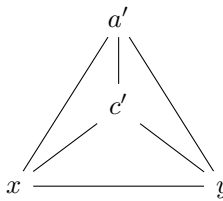
Example 3.20. Consider G



Then G/e is



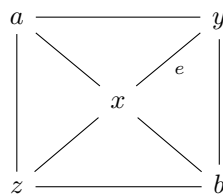
So it is not 3-connected since it contains a 2-vertex cut $\{x, c\}$.
 G/f is



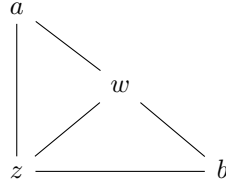
It is 3-connected.

Lemma 3.21. Let G be a 3-connected with $|G| \geq 5$ and let $e = xy \in E(G)$ s.t. G/e is not 3-connected. Then $\exists z \in V$ such that $\{x, y, z\}$ is a 3-vertex cut of G .

Proof. Let G be

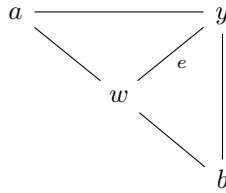


Let $\{z, w\}$ be a 2-vertex cut of G/e .



Both z and w cannot be the result of contracting e , say z is that vertex.
Set

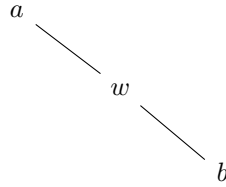
$$F := G - z.$$



(Since G is 3-connected, F is 2-connected.)

However,

$$F/e = (G - z)/e = G/e - z.$$



Note $G/e - z$ has a 1-vertex cut $\{w\}$.

Hence w must be the result of contracting e .

Thus, $\{x, y, z\}$ is a 3-vertex cut of G .

(z is not the resulting of contracting xy .) □

Lemma 3.22. If G is 2-connected and $\{x, y\}$ is a 2-vertex cut of G with $x \sim y$ and C is any component of $G - \{x, y\}$, then $H = [V(C) \cup \{x, y\}]$ is also 2-connected.

C

D



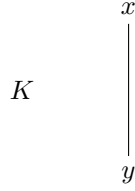
E

F

Proof. Suppose not. Then there is a cut vertex $u \in V(H)$.

Case 1. $u = x$ or y . Wlog, let $u = x$.

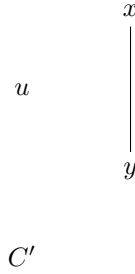
Then G looks like



So there is no y - C edges?

Then $G - x$ contains C as a component, a contradiction?

Case 2. $u \in C$. Let C' be component of $H - \{u\}$.



Then v is a cut vertex of G , a contradiction. \square

Theorem 3.23 (Thomason 1981). *Let G be a 3-connected graph with at least 5 vertices. Then G contains an edge such that G/e is 3-connected.*

Proof. Suppose not.

Then for any edge $e = xy$ of G , G/e is not 3-connected.

By previous lemma, $\exists z \in V$ associated with xy such that $\{z, x, y\}$ is a 3-vertex cut of G .

Choose e and z such that $G - \{x, y, z\}$ has a component F with as many vertices as possible.

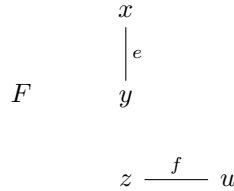
Consider $G - z$.

Since G is 3-connected, $G - z$ is 2-connected.

Also $G - z$ has the 2-vertex cut $\{x, y\}$.

Hence $H = [V(F) \cup \{x, y\}]$ is 2-connected by previous lemma.

Let u be a neighbor of z in a component of $G - \{x, y, z\}$, other than F .



Since $f = zu \in E(G)$, by our assumption, $\exists v \in V$ such that $\{z, u, v\}$ is a 3-vertex cut of G .

Since H is 2-connected, $H - v$ is connected and is thus contained in component of $G - \{z, u, v\}$.

But the order of $H - v$ is larger than $|F|$, which is contradicted by the maximality of F . \square

3.3 Menger's theorem

Theorem 3.24 (Menger 1927). *Let $G = (V, E)$ and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the largest collection of disjoint A - B path in G .*

Proof. Let

$$k = \kappa(G, A, B) = \text{minimum number of vertices separating } A \text{ from } B.$$

Clearly, the cardinality of the largest collection of vertex disjoint A - B path $\leq k$.

Induct on $\|G\|$.

If $\|G\| = 0$, the only A - B paths are the singletons $|A \cap B|$, which is the largest number of disjoint A - B path.

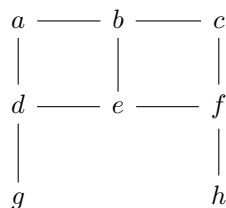
Also, the smallest separating set is $A \cap B$.

Assume

$$\|G\| \geq 1.$$

Then there exists

$$e = xy \in E.$$



with $A = \{b, c, e, f\}$ and $B = \{a, b, e, d\}$.

Inductively, assume statement holds for graphs of smallest size.

Suppose G has no k disjoint A - B paths, then neither does G/e .

Let v_e be the contracted vertex. Replace A with A' and B with B' .

Put v_e into A' if $\{x, y\} \cap A \neq \emptyset$. Put v_e into B' if $\{x, y\} \cap B \neq \emptyset$.

By the induction hypothesis, G/e contains an A - B separator Y of fewer than k vertices.

Note

$$v_e \in Y,$$

otherwise, $Y \subseteq V$ would be an A - B separator.

Hence $X := (Y - \{v_e\}) \cup \{xy\}$ is an A - B separator in G of cardinality k .

Let

$$k = \kappa(G, A, B) \text{ and } p = \text{maximum number of } A\text{-}B \text{ disjoint paths in } G;$$

$$k' = \kappa(G, A', B') \text{ and } p' = \text{maximum number of } A'\text{-}B' \text{ disjoint paths in } G.$$

Then

$$p' \leq p, \quad p < k \text{ and } p' = k'.$$

Also,

$$k' = k \text{ or } k - 1.$$

So

$$p = k - 1,$$

$$p' = k - 1 = k'.$$

Consider $G - e$.

Since $x, y \in X$, every A - X separator in $G - e$ is also an A - B separator in G and hence contains at least k vertices.

So by induction there are k disjoint A - X paths in $G - e$, and similarly there are k disjoint X - B paths in $G - e$.

As X separates A from B , these two path systems do not meet outside X , and can thus be combined to k disjoint A - B paths. \square

Remark. We have the following stronger statement.

If P is any set of fewer than k disjoint A - B paths in G , then G contains a set of $|P| + 1$ disjoint A - B paths exceeding P .

Corollary 3.25 (König theorem). Let $G = (V, E)$ be a bipartite with bipartition $\{A, B\}$. Every A - B path is an edge in G . Every vertex cover is an A - B separating set.

Definition 3.26. Let $G = (V, E)$. If $a \in V$ and $B \subseteq V$ with $a \notin B$, then an a - B fan is a collection of paths with pairwise intersection at a .

Corollary 3.27 (To Menger). For $B \subseteq V$ and $a \in V \setminus B$, the size of a smallest a - B separation not containing a is equal to the maximum number of paths in an a - B fan.

Proof. Apply Menger to $G - a$ with $A = N_G(a)$. \square

Corollary 3.28. Let a, b (s, t) be two distinct vertices of $G = (V, E)$.

If $ab \notin E$, then the minimum number of vertices not containing $\{a, b\}$ separating a from b in G ($\kappa(a, b)$) is equal to the maximum number of independent (internally disjoint) a - b paths in G ($\lambda(a, b)$).

Corollary 3.29 (Edge a - b version). The minimum number of edges separating a from b ($\kappa'(a, b)$) is equal to the maximum number of edge disjoint a - b paths in G ($\lambda'(a, b)$).

Proof. Apply Menger's a - b version of the line graph of G . \square

Theorem 3.30 (Menger's global version). (a) A simple graph is k -connected if and only if it contains k independent paths between any 2 distinct vertices.

(b) A simple graph is k -edge-connected if and only if it contains k edge-disjoint paths between any 2 distinct vertices.

Proof. (a) " \Leftarrow ". Say G contains k -independent paths between 2 distinct vertices.

Then $|G| > k$.

Furthermore, G cannot be separated by fewer than k vertices.

Hence G is k -connected.

" \Rightarrow ". Assume G is k -connected.

Then $|G| > k$ and any separating set has size at least k .

Assume $\exists a, b \in V$ s.t. there are at most $k - 1$ independent paths between a and b .

If $ab \notin E$, by previous corollary, the minimum number of vertices separating a from b is at most

$k - 1$, which is contradicted by that G is k -connected.
So

$$ab \notin E.$$

Set

$$G' = G - ab.$$

Since ab is a - b path, which must be independent of any other a - b paths, G' contains at most $k - 2$ independent a - b paths.

Then G' has an a - b separator X with at most $k - 2$ vertices.

Since $|G'| > k$, $|G'| \geq k$.

Also,

$$|X| \leq k - 2.$$

So $\exists v \in V$ such that $v \notin X \cup \{a, b\}$ in G' .

It must be the case that in G' either X separates a from v or X separates b from v , wlog, say a .

But then $X \cup \{b\}$ is a set of at most $k - 1$ vertices separating v from a in G .

Thus, G is not k -connected, a contradiction.

□

Chapter 4

Planar Graphs

Remark (Problem). Given distinct vertices x_1, \dots, x_k and y_1, \dots, y_k , find k independent paths P_1, \dots, P_k , where P_i is an x_i - y_i path, called an x - y linkage. This is a NP-hard problem even if $k = 2$.

4.1 Topological prerequisites

Definition 4.1. A topology is a collection of subsets called open sets of a ground set X that is closed under arbitrary union and finite intersection. X is called a topological space.

Example 4.2. The smallest topology on X is

$$\{\emptyset, X\}.$$

Example 4.3. In discrete topology, every subset is open.

Example 4.4. In metric space, open sets are generated by open sets.

Definition 4.5. A function between two topological spaces is continuous if the preimage of every open set is open.

Definition 4.6. A homeomorphism is a continuous bijection between two topological spaces for which the inverse function is continuous.

Example 4.7.

$$(\mathbb{R}, d_{\text{disc}}) \xrightarrow{\text{id}} (\mathbb{R}, |\cdot|),$$

is bijection continuous but not a homeomorphism since the inverse

$$(\mathbb{R}, |\cdot|) \xrightarrow{\text{id}} (\mathbb{R}, d_{\text{disc}}),$$

is not continuous since the open set $\{x\}$ in $(\mathbb{R}, d_{\text{disc}})$ is not open in $(\mathbb{R}, |\cdot|)$ (closed).

Lemma 4.8. A continuous bijective map is a homeomorphism if and only if the image of every open is open.

Definition 4.9. A set is closed if it is the complement of an open set.

Remark. In a metric space, closed sets contain all limit points.

Definition 4.10. A set is compact if every open cover has an finite subcover.

Remark. In \mathbb{R} , closed and bounded sets are compact.

Remark. Topological studies properties of objects that does not change under homeomorphism.

Example 4.11. $[0, 1]$ is homeomorphic to a polygonal arc in \mathbb{R}^2 .

Remark. Topological graph theory was studied first to address 4-color theorem.

Remark. Two homeomorphic spaces share the same topological properties. For example, if one of them is compact, then the other is as well; if one of them is connected, then the other is as well; if one of them is Hausdorff, then the other is as well; their homotopy and homology groups will coincide.

Definition 4.12. In \mathbb{R}^2 , a set S is open if $\forall x \in S, \exists r > 0$ such that the open disk $B_r(x) \subseteq S$, where $B_r(x)$ is called a neighborhood of x .

Definition 4.13. A straight line segment in \mathbb{R}^2 between p and q is of the form

$$\{p + \lambda(p - q) : 0 \leq \lambda \leq 1\}.$$

Definition 4.14. A polygonal arc P is a set $A \subseteq \mathbb{R}^2$ and is a union of finitely many line segment and is homeomorphic to $[0, 1]$ in \mathbb{R}^1 . The images of 0 and 1, say x and y are called the ends of P . Say P links x and y , define

$$\overset{o}{P} = P \setminus \{x, y\}.$$

Definition 4.15. A polygon is a subset of \mathbb{R}^2 , which is the union of finitely many straight line segment and is homeomorphic to the unit cycle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Definition 4.16. A bond of a polygonal arc or a polygon P is a point in P where line segments meet. Note there are just finitely many bonds.

Theorem 4.17. *Complement of finite union of (polygon) arcs is open.*

Definition 4.18. Let $\Omega \subseteq \mathbb{R}^2$ be an open set. Define $x \sim y$ if $x, y \in \Omega$ and there is a polygonal arc $A \subseteq \Omega$ having ends x and y . Note “ \sim ” is an equivalent relation and equivalence classes are called arcwise connected components of Ω , or region of Ω .

Definition 4.19. If $x \sim y$, for any $x, y \in \Omega$, we say that Ω is arcwise connected.

Definition 4.20. If $X \subseteq \mathbb{R}^2$ is closed, we call an arcwise connected component of $\mathbb{R}^2 - X$ a face of X .

Definition 4.21. The frontier or (boundary) of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points in \mathbb{R}^2 such that every neighbor of y meets both X and $\mathbb{R}^2 - X$.

Theorem 4.22. *If X is open, frontier of X is in $\mathbb{R}^2 - X$.*

Theorem 4.23 (Jordan curve theorem for polygon). *Every polygon $P \subseteq \mathbb{R}^2$ has exactly two faces of which exactly one is bounded. The boundary of each of the two faces is P .*

Proof. Let $x \in \mathbb{R}^2 - P$ and L be a half line starting at x and containing no bonds of P .

Let

$$\pi(x, L) = |L \cap P| \pmod{2}.$$

Check if L_1 and L_2 are two such lines starting at x , then

$$\pi(x, L_1) = \pi(x, L_2).$$

Call this number $\pi(x)$.

Check π is a continuous function.

Then π is constant on each arcwise connected component of $\mathbb{R}^2 - P$.

Choose two points x_1 and x_2 close to each other but on opposite side of a line segment of P .

Then

$$\pi(x_1) \neq \pi(x_2).$$

So P has at least two faces.

Suppose P has at least 3 faces.

Choose x_1, x_2, x_3 on each faces.

Let x be on the boundary of P (but not a bound).

So x is on a line segment S .

Pick O a small open neighborhood of x with

$$O \cap P = O \cap S.$$

For each of x_1, x_2, x_3 , shoot a half line towards P but not on the way.

Travel on the line segment along lots of neighbors of P to O from there.

Going backwards, we get a polygonal arc from O to x_1 .

So each of x_1, x_2, x_3 can be reached from a point in O by a polygonal arc not intersecting P .

But $O - P$ has at most two arcwise connected components.

So by PHP and the def. of face, at least two of x_1, x_2, x_3 are in the same region of $\mathbb{R}^2 \setminus P$.

Hence P has at most 2 faces.

Furthermore, every point of $O \cap S$ belongs to the boundary of both faces.

Also, since x is arbitrary, P is the boundary of both faces.

Check one region is unbounded. □

Lemma 4.24. Let P_1, P_2, P_3 be 3 (polygonal) arcs between the same two end points and are otherwise disjoint. Then $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly 3 regions with

(a) frontier $P_1 \cup P_2, P_2 \cup P_3$ and $P_1 \cup P_3$.

(b) If P is an arc between a point in $\overset{\circ}{P}_1$ and $\overset{\circ}{P}_3$ whose intersection lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ that contains P_2 , then $\overset{\circ}{P} \cap \overset{\circ}{P}_2 \neq \emptyset$.

Proof. (Sketch) $\overset{\circ}{P}_i$ is entirely contained in one of the 2 faces in $\mathbb{R}^2 \setminus \{P_j \cup P_k\}$.

(1) follows from PJCT, too.

(2) P_2 separates one of the two regions defined by $P_1 \cup P_3$ into two parts.

Consider Pab stated. a is in one of these regions, the one bounded by $P_1 \cup P_2$ and b is in the one bounded by $P_2 \cup P_3$. Let c be the first point on P that is in both. Then $c \in P_2$. □

Definition 4.25. A closed set X separates an open region O if $O \setminus X$ has more than 1 region.

4.2 Drawing graphs

Definition 4.26. A drawing of a graph $G = (V, E)$ is a function f that maps each $v \in V$ to $f(v) \in \mathbb{R}^2$. f maps each edge $e = uv \in E$ to $f(e)$, a polygonal arc, with ends $f(u)$ and $f(v)$.

Definition 4.27. A point in $f(e) \cap f(e')$ other than the common ends is a crossing.

Remark (Perturbation assumption for planar graph). • The interior of an edge contains no vertex and no point of any other edge.

- If 2 edges cross more than once, we can reduce the number of crossings.
- No pair of edges is parallel.

Definition 4.28. A graph is *planar* if it has a drawing with no crossings. Such a drawing is a plane embedding of G . A *plane graph* is a particular drawing of a planar graph with no crossing.

Definition 4.29. Let G be a planar and consider a **plane drawing** of G . The (open) regions of $\mathbb{R}^2 \setminus G$ are the faces of G .

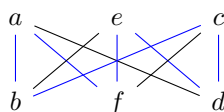
Remark (Fact). • If G is finite and so bounded, then we can construct a big disk containing all of G and so G has only one unbounded face.

- The faces of G are pairwise disjoint.
- The points p, q not on an edge of a plane graph are in the same face if and only if there exists a p - q arc crossing no edges of G .

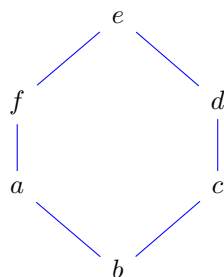
Definition 4.30. A chord of a cycle C is an edge e joining two vertices on C but with $e \notin C$.

Theorem 4.31. Neither K^5 nor $K_{3,3}$ are planar.

Proof. Consider a drawing of $G = K^5$ or $K_{3,3}$ in the plane. Let C be a spanning cycle in $G = K_{3,3}$.



Then we can draw C as a polygon:



By PJCJ, $\mathbb{R}^2 - C$ has exactly 2 faces.

Let e be a chord of C , then by definition, e is entirely contained in one of these two faces.

We will say that two chords of C conflict if their endpoints on C occur in alternating order, for example, the chords fc and eb conflict.

Conflicting chords must be drawn in different faces. But $K_{3,3}$ has 3 pairwise conflicting chords and $\mathbb{R} \setminus C$ has only 2 faces, so $K_{3,3}$ cannot be drawn in the plane.

A similar argument holds for K_5 . □

The following Lemmas are used for proving Kuratowski's theorem.

Lemma 4.32. Let G be a planar graph and E be the edge set of a face F of G . Then there is an embedding in which F is the unbounded face.

Lemma 4.33. Every minimal nonplanar graph is 2-connected.

Proof. Let G be minimal nonplanar.

Suppose G were not connected, then one of the component would be a nonplanar, which is contradicted by the minimality and so G is connected.

Suppose v were a cut vertex and let C_1, \dots, C_k be the components of $G - v$ with $k \geq 2$.

For $i = 1, \dots, k$, let H_i be the subgraph of G induced by $C_i \cup \{v\}$.

By the minimality of G , each H_i is planar for $i = 1, \dots, k$.

Squeeze each to fit an angle less than $\frac{360^\circ}{k}$ at v and merge.

But then G is planar, a contradiction and so G is 2-connected. □

Definition 4.34. A minimal nonplanar graph is a nonplanar graph for which every proper subgraph is planar.

Lemma 4.35. Let G be minimal nonplanar and has a separator S of size 2, say $S = \{x, y\}$.

Let C_1 be one component of $G - \{x, y\}$ and let $C_2 = G - \{x, y\} - C_1$.

Let G_i be the subgraph of G induced by $C_i \cup \{x, y\}$ for $i = 1, 2$.

Note

$$V(G_1) \cap V(G_2) = \{x, y\}.$$

Define for $i = 1, 2$,

$$H_i = G_i \cup xy.$$

Then at least one of H_1, H_2 is nonplanar, otherwise we could glue H_1 and H_2 at xy and remove xy to obtain a planar graph G , a contradiction.

Definition 4.36. A Kuratowski graph is a subdivision of K^5 or $K_{3,3}$.

Lemma 4.37. A minimal nonplanar graph with no Kuratowski subgraph is 3-connected.

Proof. Assume G is minimal non-planar. Then G is 2-connected by previous Lemma.

Suppose G were not 3-connected.

Then by last Lemma, H_1 or H_2 defined in last Lemma is nonplanar, say H_1 .

Since H_1 has fewer edges than G , H_1 must contain a Kuratowski subgraph.

Replace xy with an x - y path using only edges in H_2 and this gives a Kuratowski subgraph of G , a contradiction. □

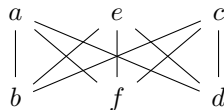
Lemma 4.38. A 3-connected graph with at least 5 vertices has an edge whose contraction leaves the graph 3-connected.

Lemma 4.39. If G/e has a Kuratowski subgraph, then G also does.

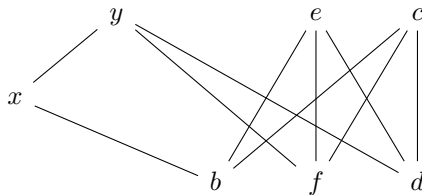
Proof. Let H be the Kuratowski subgraph of $G' = G/e$.

Let $e = xy$ and z be the vertex resulting from contracting the edge e .

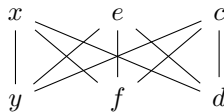
Case 1: z is a nonbranching vertex of H . Uncontracted to get a Kuratowski subgraph of G , for example z is on ab .



Case 2: If when we uncontract z (inflation), at least one of the vertices $\{x, y\}$ has degree 2 in the subgraph of G induced by $(V(H) - z) \cup \{x, y\}$. Still, we have a Kuratowski subgraph after expanding z .



Case 3: x and y have degree greater than 2 in this same subgraph, i.e., $\deg_H(z) = 4$.



□

Remark. Sometimes, the contrapositive statement is more useful.

Theorem 4.40 (Kuratowski, 1930). G is planar if and only if G contains no subdivision of K^5 or $K_{3,3}$ (no Kuratowski subgraph).

Proof. The goal is to show

- (a) Show that a minimal nonplanar graph with no Kuratowski subgraph must be 3-connected.
- (b) Prove that a 3-connected graph with no Kuratowski subgraph must in fact be planar.

□

Remark (Fact). • Subdividing edges does not affect planarity.

- Deletion and contraction preserve planarity.
- So it makes sense to seek minimal non-planar graphs with respect to these operations.

Theorem 4.41 (Wagner, 1937). *G is planar if and only if it has no subgraph contractible to K^5 or $K_{3,3}$.*

Remark (Fact). A graph contains K^5 or $K_{3,3}$ as a minor if and only if it contains K^5 or $K_{3,3}$ as a topological minor.

Theorem 4.42 (Fary's Theorem, 1948). *Every finite planar graph has an embedding in which all edges are straight line segments.*

Remark (Recall). An embedding is a drawing of the graph in the plane.

Remark (Fact). If each face boundary is convex, we say the representation is convex.

Definition 4.43. A set A is convex if for any $x, y \in A$ and $\forall 0 \leq \lambda \leq 1$,

$$(1 - \lambda)x + \lambda y \in A.$$

Definition 4.44. A convex embedding of G is a planar embedding in which each inner face is convex.

Theorem 4.45 (Tutte, 1969, 1963). *Every 3-connected planar graph has a convex embedding in the plane.*

Remark (Fact). $K_{2,n}$ for $n \geq 4$ has no convex representation.

Theorem 4.46 (Tutte). *If G is 3-connected with no Kuratowski subgraph, then G has a convex embedding in the plane.*

Proof. Induction on $|G|$.

If $|G| \leq 4$, then $G = K^4$ and K^4 has a convex embedding.

Assume $|G| \geq 5$. Assume the statement holds for all graphs with fewer vertices.

By previous Lemma, there exists $e = xy$ with G/e 3-connected.

Let z be the vertex resulting from contracting e .

Previous lemma implies that G/e has no Kuratowski subgraph.

So by inductive hypothesis, there exists a convex embedding of $G' = G/e$.

Consider removing all edges in G' incident with z .

The resulting graph has a face containing z .

A cycle of $G' - z$ bounds the face.

There exists straight line segments from z to each of its neighbors on C .

Some connect x to C . Some connect y to C .

Let x_1, \dots, x_k be the neighbors of x in order on C .

Case 1: All neighbors of y lie between x_i and x_{i+1} for some $1 \leq i \leq k-1$ or between x_1 and x_k .

Case 2: Consider subcases (2a) and (2b). We claim that both of these subcases allow us to conclude that we have a Kuratowski subgraph.

(2a) y shares 3 neighbors with x . Then we have a K^5 subdivision.

(2b) y has two neighbors u and v in C (breaking C into two segments) and x has two neighbors u' and v' that are in different segments of C .

Then we have a $K_{3,3}$ subdivision.

□

Remark (Interesting Fact). excluded minors characterization for our planar graphs: K^4 and $K_{2,3}$.

Theorem 4.47.

$$2\|G\| = \sum_i l(F_i).$$

Lemma 4.48 (Euler's formula, 1258). If G is planar and connected with n vertices, m edges and l faces, then

$$n - m + l = 2.$$

Corollary 4.49. A simple 2-connected planar graph has at most $3|G| - 6$ edges.

Proof. Let G has n vertices, m edges and l faces.

Since G is simple and 2-connected, every face has length at least 3.

So

$$2m = \sum_i l(F_i) \geq 3l.$$

Also, by Euler's formula,

$$3m = 3n + 3l - 6 \leq 3n + 2m - 6.$$

So

$$m \leq 3n - 6.$$

□

Remark (Exercise). Use this to show K^5 is not planar.

Corollary 4.50. Let G be a planar, simple and 2-connected. Then the average degree of G

$$d(G) = \frac{\sum_v d(v)}{n} = \frac{2\|G\|}{n} = \frac{2m}{n} \leq \frac{6n - 12}{n} = 6 - \frac{12}{n} < 6.$$

We conclude that every simple 2-connected planar graph has a vertex of degree ≤ 5 .

Chapter 5

Coloring

Remark. How many colors do we need to color the countries of a map in such a way that adjacent countries are colored differently? How many days have to be scheduled for committee meetings of a parliament if every committee intends to meet for one day and some members of parliament serve on several committees? How can we find a school timetable of minimum total length, based on the information of how often each teacher has to teach each class?

Definition 5.1. A (vertex) coloring of a graph is an assignment of colors to vertices. Specifically, let $G = (V, E)$. Let S be the set of colors and be finite. A vertex coloring of G is a map

$$c : V \rightarrow S.$$

A coloring is proper if $c(v) \neq c(u)$ when $v \sim u$.

Definition 5.2. An edge coloring of $G = (V, E)$ is map $c : E \rightarrow S$ with $c(e) \neq c(f)$ for any adjacent edges e, f .

Remark. Clearly, every edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice versa; in particular,

$$\chi'(G) = \chi(L(G)).$$

Remark. Often $S = \{1, \dots, k\}$. If there is a coloring using only elements in $[k]$, we say G is k -colorable and the associated coloring is a k -coloring.

Definition 5.3. Let $\chi(G)$ be the chromatic number of G , which is the smallest integer k so that G is k -colorable.

Definition 5.4. Let $\chi'(G)$ be the edge chromatic number of G or called chromatic index of G , which is the smallest integer k so that G is k -edge-colorable.

Remark. If $\chi(G) \leq k$, we say G is k -colorable. If $\chi(G) = k$, we say G is k -chromatic.

Remark. Note that a k -coloring is nothing but a vertex partition into k independent sets, now called color classes. The non-trivial 2-colorable graphs, for example, are precisely the bipartite graphs.

Remark. How many colors are needed to color the regions of a planar graph? Equivalent to the vertex coloring problem of the dual. Find $\chi(G^*)$.

Theorem 5.5 (4 color theorem). *For any planar graph G , $\chi(G^*) = 4$.*

Proof. • 1976, Appel, Haken

• 1997 Robertson, Sanders, Seymour, Thomas

• 1879 Kempe

•

Kempe's ideal helped prove a weaker theorem, Heawood 1890. □

Theorem 5.6. *For any planar graph G , $\chi(G) \leq 5$, i.e., every planar graph is 5-colorable.*

Proof. Let G be planar graph. Use induction on $|G|$.

If $|G| \leq 5$, done.

Let $n = |G| \geq 6$ and $m = \|G\|$.

Assume that any planar graph with less than n vertices is 5-colorable.

Let v be a vertex with $d(v) \leq 5$ and $H := G - v$.

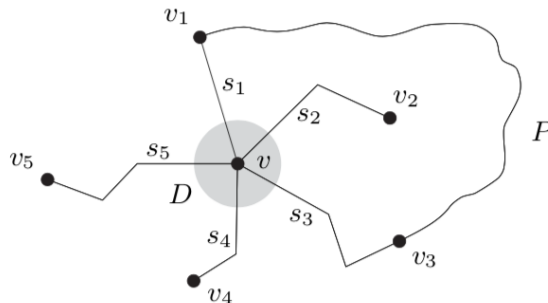
By inductive hypothesis, H has a coloring $c : V(H) \rightarrow \{1, 2, 3, 4, 5\}$.

If c uses at most 4 colors for the neighbors of v , we can extend it to a 5-coloring for the neighbors of v and done

Assume, therefore, that v has exactly 5 neighbors $\{v_1, \dots, v_5\}$ and let $c(v_i) = i$ for $i = 1, \dots, 5$. Let D be an open disc around v , so small that it meets only those five straight edge segments of G that contain v .

Let us enumerate these segments according to their cyclic position in D as s_1, \dots, s_5 .

Let vv_i be the edge containing s_i for $i = 1, \dots, 5$.



We first show every v_1 - v_3 path $P \subseteq H - \{v_2, v_4\}$ separates v_2 from v_4 in H .

Clearly, this is the case if and only if the cycle $C := vv_1Pv_3v$ separates v_2 from v_4 in G .

We prove this by showing that v_2 and v_4 lie in different faces of C .

Let x_2 be an inner point of s_2 in D and x_4 be an inner point of s_4 in D .

Then in $D \setminus (s_1 \cup s_3) \subseteq \mathbb{R}^2 \setminus C$, every point can be linked by a polygonal arc to x_2 or to x_4 .

This implies x_2 and x_4 (and hence also v_2 and v_4) lie in different faces of C , otherwise, D would meet only one of the two faces of C , which would contradict the fact that v lies on the frontier of

both these faces since by Jordan Curve Theorem for Polygons, any neighbor sets of a point in the boundary will meet two faces of a polygon.

Let H_{ij} be the subgraph of H induced by vertices colored i or j for $i, j \in \{1, 2, 3, 4, 5\}$.

We may assume that the component C_1 containing v_1 of $H_{1,3}$ also contains v_3 . Indeed, if we interchange the colors 1 and 3 at all the vertices of C_1 , we obtain another 5-coloring of H ; if $v_3 \notin C_1$, then v_1 and v_3 are both colored 3 in this new coloring, and we may assign remaining color 1 to v and done.

So H_{13} contains a v_1 - v_3 path $P \in H_{13}$.

As shown above, P separates v_2 from v_4 in H .

Since $P \cap H_{2,4} = \emptyset$, v_2 and v_4 lie in different components of $H_{2,4}$.

In the component containing v_2 , we now interchange the colors 2 and 4, thus recoloring v_2 with color 4.

Now v no longer has a neighbor colored 2 and we may give it this color. \square

Theorem 5.7. Every graph G with m edges satisfies $\chi(G) \leq 1/2 + \sqrt{2m + 1/4}$

Proof. Let c be a vertex coloring of G with $k = \chi(G)$ colors.

Then G has at least one edge between any two color classes: if not, we could have used the same color for both classes.

So letting $m = \|G\|$, we have $m \geq \binom{k}{2} = \frac{k(k-1)}{2}$, i.e., $2m \geq k(k-1) = (k-1/2)^2 - 1/4$, i.e., $k \leq 1/2 + \sqrt{2m + 1/4}$. \square

Theorem 5.8 (Another easy bound).

$$\chi(G) \leq \Delta + 1,$$

where $\Delta = \Delta(G) = \max_{v \in V(G)} d(v)$.

Proof. We can establish this bound algorithmically.

Greedy method: list the vertices of G in any order v_1, \dots, v_n .

Color v_1 with 1 and at step i , color v_i with the smallest color (positive integer) not used so far by any neighbor of v_i among v_1, \dots, v_{i-1} .

In this way, we never use more than $\Delta(G) + 1$ colors. \square

Remark. Can we do better? and how can we make our algorithm better with the same idea? Consider C_n with n odd and for any n ,

$$\Delta(K_n) = n - 1.$$

When we come to color the vertex v_i in the above algorithm, we only need a supply of $d_{G[v_1, \dots, v_i]}(v_i) + 1$ rather than $d_G(v_i)$ colors to proceed and the algorithm ignores any neighbors v_j of v_i with $j > i$. Hence in most graphs, there will be scope for an improvement of the $\Delta + 1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbors are ignored) and vertices of small degree last.

Definition 5.9. The last number k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbors is called the coloring number $\text{col}(G)$ of G .

Proposition 5.10.

$$\text{col}(G) = \max_{H \subseteq G} \delta(H) + 1.$$

Proof. The enumeration we just discussed shows that $\text{col}(G) \leq \max_{H \subseteq G} \delta(H) + 1$. But for $H \subseteq G$, clearly $\text{col}(G) \geq \text{col}(H) \geq \delta(H) + 1$. \square

Theorem 5.11. *Every graph satisfies*

$$\chi(G) \leq 1 + \max_{H \subseteq G} \{\delta(H)\} = \text{col}(G).$$

Proof. Since the ‘back-degree’ of the last vertex in any enumeration of H is just its ordinary degree in H , which is at least $\delta(H)$. \square

Remark. It is tight for G not regular.

Corollary 5.12. Every k -chromatic graph G has a k -chromatic subgraph with minimum degree at least $\chi(G) - 1$.

Proof. Given G with $\chi(G) = k$, let $H \subseteq G$ be minimal with $\chi(H) = k$. If H had a vertex v of degree $d_H(v) \leq k - 2$, we could extend a $(k - 1)$ -coloring of $H - v$ to one of H , contradicting the choice of H . \square

Remark. What can we say when G is regular?
If $G = C_n$ with n odd or K^n for any $n \in \mathbb{N}$, then

$$\chi(G) = \Delta + 1.$$

Remark. For G connected and not regular, $\chi(G) \leq \Delta$.

Theorem 5.13 (Brooks 1941). *If G is connected and neither an odd cycle nor a complete graph, then*

$$\chi(G) \leq \Delta.$$

Proof. Induction on $|G|$.

If $\Delta(G) \leq 2$, then G is a path or a cycle, and the assertion is trivial.

Assume $\Delta(G) \geq 3$ and that the assertion holds for graphs of smaller order.

Suppose $\chi(G) > \Delta(G)$.

Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leq \Delta(G)$.

Also, every component H' of H satisfies $\chi(H') \leq \Delta(H') \leq \Delta(G)$ unless H' is complete or an odd cycle, in which case since every vertex of H' has maximum degree in H' and one such vertex is also adjacent to v in G , we have

$$\chi(H') = \Delta(H') + 1 \leq \Delta(G).$$

Since H can be $\Delta(G)$ -colored but G cannot, we have the following:

Every $\Delta(G)$ -coloring of H uses all the colors $1, \dots, \Delta$ on the neighbors of v ; in particular, $d(v) = \Delta(G)$. (1)

Given any Δ -coloring of H , let us denote the neighbor of v colored i by v_i for any $i = 1, \dots, \Delta$.

For all $i \neq j$, let $H_{i,j}$ denote the subgraph of H spanned by all the vertices colored i or j .

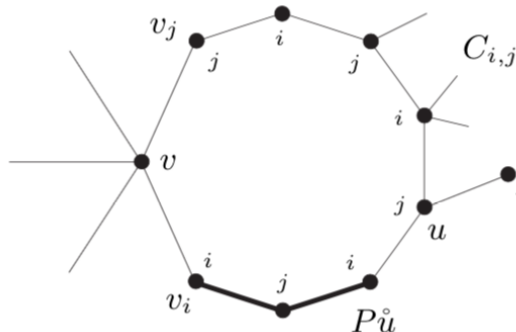
For all $i \neq j$, the vertices v_i and v_j lie in a common component $C_{i,j}$ of $H_{i,j}$. (2)

Otherwise we could interchange the colors i and j in one of those components; then v_i and v_j would be colored the same, contrary to (1).

$C_{i,j}$ is always a v_i - v_j path. (3)

Indeed, let P be a v_i - v_j path in $C_{i,j}$. Since $\Delta(H') + 1 \leq \Delta(G)$, $d_H(v_i) \leq \Delta - 1$ and then the

neighbors of v_i have pairwise different colors: otherwise we could recolor v_i (interchange the color i and the color of its neighbor at all vertices of H), contrary to (1). Hence the neighbor of v_i on $P \in C_{i,j}$ is its only neighbor in $C_{i,j}$, and similarly for v_j . Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically colored neighbors in H ; let u (clearly not v_i or v_j) be the first such vertex on P . Since at least 3 neighbors of u have the same color, at most $\Delta(G) - 2$ colors are used on the neighbors of u and so we may recolor u . But this makes Pu into a component of $H_{i,j}$, contradicting (2).



For distinct i, j, k , the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i . (4)

For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbors colored j and two colored k , so we may recolor u . In the new coloring, v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).

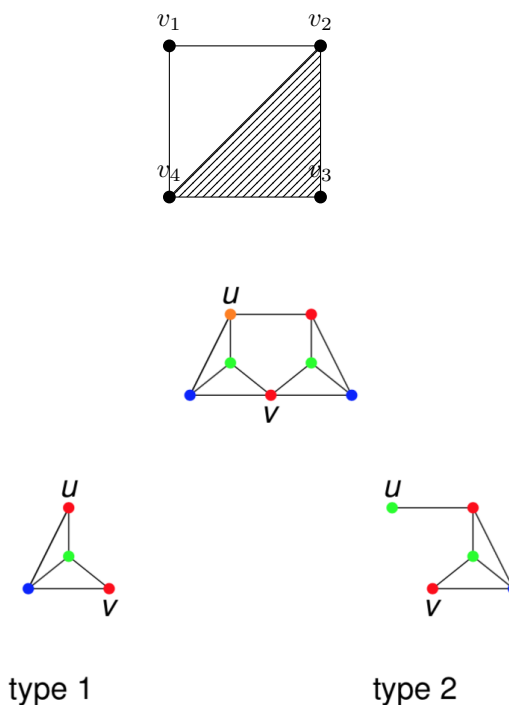
The proof of the theorem now follows easily. If the neighbors of v are pairwise adjacent, then each has $\Delta(G)$ neighbors in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta(G)}$. As G is complete, there is nothing to show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \dots, v_{\Delta(G)}$ derive their names from fixed Δ -coloring c of H . Let $u \neq v_2$ be the neighbor of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colors 1 and 3 in $C_{1,3}$, we obtain a new coloring c' of H ; let $v'_i, H'_{i,j}, C'_{i,j}$ etc. be defined with respect to c' in the obvious way. As a neighbor of $v_1 = v'_3$, our vertex u now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for c , however, the path $v_1 C_{1,2}$ retained its original coloring, so $u \in v_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for c' . \square

Theorem 5.14 (Erdős 1959, 1961). *For every positive integer k , there exists a graph G having girth $g(G) > k$ and chromatic number $\chi(G) > k$.*

Definition 5.15. A k -chromatic graph G is critically k -chromatic or k -critical if $\chi(G - v) < k$ for every $v \in V(G)$. (Obviously, $\chi(G - v) = k - 1$ for any $v \in V(G)$.)

Theorem 5.16. *Let G be a k -critical graph with a 2-vertex cut $\{u, v\}$. Let C_1 and C_2 be the components of $G - \{u, v\}$. For $i = 1, 2$, let $G_i = G[V(C_i) + \{u, v\}]$. Then*

- (a) $G = G_1 \cup G_2$, and G_1 and G_2 are $(k - 1)$ -colorable.
- (b) One of G_1 and G_2 , say G_1 has $c(u) = c(v)$ in all $k - 1$ -colorings. G_2 has $c(u) \neq c(v)$ in all $k - 1$ -coloring.
- (c) $H_1 := G_1 + e$, where $e = uv$ and $H_2 := (G_2 + e)/e$ are each k -critical.



Proof. (1)(2) Clearly, Each component of $G - \{u, v\}$ is $k - 1$ -colorable.

In fact, $\chi(G_1) = \chi(G_2) = k - 1$. (Hint: glue).

If there exists a $k - 1$ -coloring of G_1 and G_2 where the colors of u and v agree, then glue to get a $k - 1$ coloring of G , a contradiction.

(3) Adding uv forces chromatic number of G_1 up by 1 and similarly for G_2 .

Exer: show that the result is k -critical. □

Proposition 5.17. If G is k -critical, then G does not contain a cut set consisting of pairwise adjacent vertices.

Proof. Let S be a cut set. Let H_1, \dots, H_t be the components of $G - S$.

Since each $H_i \cup S$ is a proper subgraph, $H_i \cup S$ is $(k - 1)$ -colorable.

Suppose S is a clique, then one can permute the colors such that G is $(k - 1)$ -colorable, a contradiction.

So S is not a clique. □

Theorem 5.18 (Dirac.). Every graph G with $\chi(G) \geq 4$ contains a K^4 -subdivision.

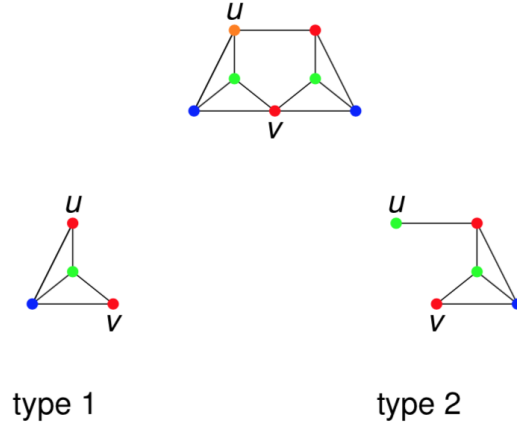
Proof. Induction on $n = |G|$.

If $n = 4$, then $G = K^4$.

Assume $n > 4$ with $\chi(G) \geq 4$ and we can let H be a 4-critical subgraph of G .

By previous proposition, H has no cut vertex.

Case 1: $\kappa(H) = 2$. Let $\{x, y\}$ be a cut vertex.
 $x \sim y$, let G_1 and G_2 be as in the lemma.
 By previous lemma, $\chi(G_1 + xy) = 4$.
 By induction, $H_1 = H_1 + xy$ has a K^4 -subdivision.



If necessary, remove xy from this subdivision and replace it with any x - y path in G_2 .

Case 2: H is 3-connected. Select a vertex $x \in V(G)$.

Since $H - x$ is 2-connected, it has a cycle C of length at least 3.

By the Fan version of Menger's theorem, there exists an $x, V(C)$ -Fan of size 3 in H .

So we have our K^4 subdivision. □

Remark (Question). Can the invariant χ have a direct structural effect on a graph in terms of forcing a specific substructure?

Hadwiger 1943, Famous conjecture: For every $r \in \mathbb{Z}^+$,

$$\chi(G) \geq r \implies G \geq K^r,$$

i.e., every graph G with $\chi(G) \geq 5$ has a K^5 minor.

$r = 4$:

$r = 5$:

$r = 6$:

$r \geq 7$:

5.1 k -edge coloring

Theorem 5.19 (König 1916). For a bipartite graph G , $\chi'(G) = \Delta(G)$.

Proof. Induction on $m = \|G\|$.

If $\|G\| = 0$. Done.

Assume $\|G\| \geq 1$ and the assertion holds for graphs with fewer edges.

Let $\Delta := \Delta(G)$.

Pick $xy \in E(G)$, by inductive hypothesis, there exists a coloring of the edge of $G - \{xy\}$ using the colors $\{1, \dots, \Delta\}$.

In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges.

So there exists $\alpha, \beta \in \{1, \dots, \Delta\}$ such that no edge in $N(x)$ is colored α and no edge in $N(y)$ is colored β .

If $\alpha = \beta$, we can color the edge xy with this color and are done, so assume $\alpha \neq \beta$.

In fact, y is incident with an α edge and x is incident with a β edge.

Let us extend this edge to a maximum walk W from x whose edges are colored β and α alternatively. Since no such walk contains a vertex twice, W exists and is a path. Moreover, W does not contain y : if it did, it would end in y on an α -edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G . We now recolor all the edges on W , swapping α with β . By the choice of α and the maximality of W , adjacent of $G - xy$ are still colored differently. We have thus found a Δ -edge-coloring of $G - xy$ in which neither x nor y is incident with a β -edge. Coloring xy with β , we extend this coloring to a Δ -edge-coloring of G . \square

Remark. If G is an odd cycle, it needs $\Delta + 1$ colors, so $\chi'(G) = \Delta + 1$.

Theorem 5.20 (Vizing 1964). *Every simple graph G satisfies $\Delta \leq \chi'(G) \leq \Delta + 1$.*

Proof. Induction on $\|G\|$.

If $\|G\| = 0$. Done.

Let $\Delta := \Delta(G) > 0$ and assume the assertion is true for all graphs with fewer edges.

Instead of ‘ $(\Delta + 1)$ -edge-coloring’ let us just say ‘coloring’.

Suppose there is not $\Delta + 1$ coloring of G .

Let $e = xy$ and color $G - xy$ with $\{0, 1, \dots, \Delta\}$.

A color is missing at x , wlog., let this missing color be 0.

There exists a missing color at y . Not 0, call it 1, this is a 1 edge at x , let xy be colored 1.

Something missing at y .

If this “, color is 0, else down-shifting, i.e., coloring xy_1 with 0 and xy_0 with 1.

So the missing color is neither 0 nor 1, wlog., let it be 2.

x is incident with a 2-edge, (else recolor xy with 2 and ‘downshift’ coloring xy_0 with 1).

Continue in this way.

But we have only $\Delta + 1$ colors.

At some point, the missing color has already been used-let k be the smallest index where this happens.

y_k is missing 0, (else coloring y_kx with 0 and down-shift from y_k .)

Let p_i =maximal and path of edges using 0 and i .

Case 1: p reaches y_i along a 0 edge. Then continues to x and stops. Down-shift from y and switch on P and coloring y_ix with 0.

Case 2: p doesn’t reach y_i but dows reach y_{i-1} .

So stop at $y_i - 1$ since no i at y_{i-1} .

Downshift from y_{i-1} , switch on P and color xy_{i-1} .

Case 3: P reaches neither y_i nor y_{i-1} .

So then P also avoids x (P can only arrive at x via i through y .)

Now down-shift from y_k , then switch on P and color xy_k with 0. \square

Definition 5.21. A lattice square is an $n \times n$ array with n different symbols such that no row or column has 2 of the same symbols.

Example 5.22. Consider lattice squares

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

Then AB is also a lattice square.

5.2 List coloring

Definition 5.23. Suppose we are given a graph $G = (V, E)$, and for each vertex of G , a list of colors permitted at that particular vertex: when can we color G so that each vertex receives a color from its list? More formally, let $(S_v)_{v \in V}$ be a family of sets. We call a vertex coloring c of G with $c(v) \in S_v$ for all $v \in V$ a coloring from the lists S_v . The graph G is called k -list-colorable, or k -choosable, if for every family $(S_v)_{v \in V}$ with $|S_v| = k$ for all v , there is a vertex coloring of G from the lists S_v . The least integer k for which G is k -choosable is the list-chromatic number or choice number $\text{ch}(G)$ of G .

Definition 5.24. The least integer k such that G has an edge coloring from any family of lists of size k is the list-chromatic index $\text{ch}'(G)$ of G ; formally, we just set $\text{ch}'(G) := \text{ch}(L(G))$.

Theorem 5.25 (Dinitz Conjecture, 1979). *Given an $n \times n$ square array and n^2 arbitrary sets A_{ij} with $1 \leq i, j \leq n$ and $|A_{ij}| = n$, it is always possible to pick $a_{ij} \in A_{ij}$ such that each row and each column has all n vertices distinct.*

Remark. Given $G = (V, E)$ and put a set of allowable colors S_v on each vertex v , can we properly color $V(G)$ so that every vertex gets a color from its list?

Lemma 5.26.

$$\text{ch}(G) \geq \chi(G).$$

Lemma 5.27. $\text{ch}(G) \geq \chi(G)$.

Remark. Nobody knows a case where $\text{ch}'(G) > \chi'(G)$.

Example 5.28. $L(K_{3,3})$

$$\begin{array}{ccc} 11' & 12' & 13' \\ 21' & 22' & 23' \\ 31' & 32' & 33' \end{array}$$

Theorem 5.29 (List coloring conjecture). $\text{ch}'(G) = \chi'(G)$ for all G .

Remark. Dinitz problem can be seen as a special case of LCC.

Same graph as above.

Every cell has set A_{ij} .

Define G : Let $V(G)$ be the cells (n^2 of them).

$(i, j) \sim (i, j')$ for all $j' \neq j$.

$(i, j) \sim (i', j)$ for all $i' \neq i$.

We want to show that G is n -choosable.

Note: G is the line graph of $K_{n,n}$.

So Dinitz conjecture $\iff \text{ch}'(K_{n,n}) = n$.

Also, recall that

$$\chi'(G) = \Delta(K_{n,n}) = n.$$

So Dinitz conjecture \iff LCC for $K_{n,n}$.

F.Galvin 94, LCC holds for all bipartite graphs, $\text{ch}(L(G)) = \chi(L(G))$ for all G .

An orientation of a graph means we put a direction on each edge. ij : $i \rightarrow j$.

Definition 5.30.

$$N^+(v) = \{w \in V(G) : v \rightarrow w\}.$$

$$d^+(v) = |N^+(v)|.$$

Definition 5.31. An independent set $U \subseteq V(D)$ is a kernel of D if for every $v \in D - U$, there is a $w \in U$ so that $v \rightarrow w$.

Definition 5.32 (Property X). D has this property, for every non-empty induced subgraph D' of D , D has a kernel.

Lemma 5.33. Let H be a graph and let $\{S_v\}$ be a collection of sets. If H has an orientation D so that

(a) $|S_v| \geq d^+(v)$ for any v ;

(b) D has property X .

Then H can be colored from the lists S_v .

Proof. Induction on $|H|$.

If $|H| = 0$, no color needed.

Induction step: let H be a graph with orientation D as stated.

Pick any color α , □

Definition 5.34. Let $U \subseteq D$, if for any $v \notin D - U$, there exists $w \in U$ with $v \rightarrow w$, then U is a kernel of D .

Lemma 5.35. Let H be a graph and $\{S_v\}_{v \in V}$ be a collection of sets. If H has a orientation D with

(a) $|S_v| \geq d^+(v)$ for any $v \in V$.

(b) Every nonempty induced subgraph D' at D has a kernel.

Then H can be colored from the lists (sets) $\{S_v\}_{v \in V}$.

Theorem 5.36 (Galvin 94). *LCC holds for all bipartite graph.*

List chromatic conjecture: $\text{ch}'(G) = \chi'(G)$ for any G .

Proof. Let G be bipartite with bipartition $\{x, y\}$ and let $\chi'(G) = k$.

We know that $\text{ch}'(G) \geq k$. We will show $\text{ch}'(G) \leq k$, i.e., we will show $L(G)$ is k -colorable.

Let c be a k -edge coloring of G with $c: E(G) \rightarrow [k]$.

We need an orientation D of the line graph of G satisfying

(a) $d^+(e) \leq k$ for any $e \in E(G)$.

(b) Every nonempty induced subgraph of D has a kernel.

Define D as follows. If e and e' meet at X and $c(e) < c(e')$, then $e' \rightarrow e$. If e and e' meet at Y and $c(e) < c(e')$, then $e \rightarrow e'$.

Let $c(e) = i$. For every $e' \in N^+(e)$ meeting e in X , $c(e') \in \{1, \dots, i-1\}$ and for every $e' \in N^+(e)$ meeting e in Y , $c(e') \in \{i+1, \dots, k\}$.

None of these can be the same. $d^+(v) = |N^+(v)| \leq k-1 < k$.

Let D' be a nonempty induced subgraph of D .

Interpret direction in D as a preference. $e <_v e'$ if $e \rightarrow e'$.

Let M be a stable matching in the graph $(X \cup Y, V(D'))$, then for every edge $e \in E(D') \setminus M$, there exists $f \in M$ such that they have a common vertex with $e <_v f$, i.e., for which $e \rightarrow f$, i.e., M is an required kernel. \square

Example 5.37. Graph.

Let $e \in E(G)$. Compute $d^+(e)$.

Chapter 6

Hamilton Cycles

Definition 6.1. When does a graph G contain a closed walk that contains every vertex of G exactly once? If $|G| \geq 3$, then any such walk is a cycle: a Hamilton cycle of G .

Definition 6.2. A Hamilton path in G is a path in G containing every vertex of G .

Definition 6.3. If G has a Hamilton cycle, it is called Hamiltonian. If G has a Hamilton path, it is called traceable.

Definition 6.4. Define the number of component of H as $c(H)$.

Remark. Look for some sufficient and necessary conditions.

Easy necessary conditions: $\delta(G) \geq 2$. If $G = K_{m,n}$, then $m = n$.

A necessary condition for Hamiltonicity is $c(G - S) \leq |S|$ for every separator S .

Remark. Consider this example. Not Hamiltonian.

Hint: remove the white vertex, then we left with 4 components.

Definition 6.5. A graph G is tough if $c(G - S) \leq |S|$ for every separator S .

Definition 6.6. For $t \in \mathbb{R}^{>0}$, G is t -tough if $c(G - S) \leq \frac{|S|}{t}$ for every separator S .

Remark (Conjecture 1973). There exists $t \in \mathbb{Z}^+$ so that every t -tough graph is Hamiltonian.

Theorem 6.7 (Dirac 1952). *Every graph with $n \geq 3$ vertices and $\delta(G) \geq n/2$ is Hamiltonian.*

Lemma 6.8. Let $G = (V, E)$ be simple. Let $u, v \in V$ and $u \not\sim v$. If $d(u) + d(v) \geq n$, then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Theorem 6.9. *Let $G = (V, E)$ be simple. Let $u, v \in V$. If $d(u) + d(v) \geq n$ for all $u \not\sim v$, then G is Hamiltonian.*

Theorem 6.10 (Bondy and Chóatal 1970). *A simple graph is Hamiltonian if and only if its closure is Hamiltonian.*

Theorem 6.11. *Every graph G with $|G| \geq 3$ and $\alpha(G) \leq \kappa(G)$ has a Hamilton cycle.*

Chapter 7

Extremal Graph Theory

How many edges can G of order n have and be triangle free?

Theorem 7.1 (Mantel 1907). *The maximal number of edges a simple triangle free graph G can have is $\left\lfloor \frac{n^2}{4} \right\rfloor$, where $n = |G|$.*

Proof. Idea:

(a) Show a simple triangle free graph G has $\|G\| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$, where $|G| = n$.

(b) Exhibit a triangle free graph G with $\|G\| = \left\lfloor \frac{n^2}{4} \right\rfloor$.

(a) Let G be a simple and triangle free.

Let $\Delta(G) = k$. Pick u with $\deg_G(u) = k$.

graph:

Since G is triangle free, $N(u)$ is an independent set.

So every edge is incident with at least one vertex in $V(G) - N(u)$

Hence

$$\|G\| \leq |G - N(u)| \cdot k = (n - k)k.$$

Therefore, $\|G\| \leq \max_k (n - k)k$, where the equality is attained for $n = 2k$ and $n(n - k) = \frac{n^2}{4}$.

(b) The graph we need is $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$.

□

Remark (Bipartite $K_{n,m}$). Multipartite graphs k -partite graph.

We denote a complete k -partite graph by K_{n_1, \dots, n_k} , where n_i is cardinality of the i^{th} part.

All edges between distinct parts,

$$K_r^l = K_{r, \dots, r},$$

where the number of r 's is l .

Definition 7.2. The Turan graph $T^r(n)$ is the unique n -vertex, complete r -partite simple graph whose partite sets dif and only ifer in cardinality by at most 1.

Example 7.3.

$$T^3(8) = K_{3,3,2}.$$

Proposition 7.4. Let $n, r \in \mathbb{N}$ and $n \geq r$ and choose l and $0 \leq j < r$ so that $n = rl + j$. Then the Turan graph $T^r(n)$ is defined as follows.

$$T^r(n) = K_{l, \dots, l, l+1, \dots, l+1},$$

where there are j $l+1$ and $r-j$ l .

Definition 7.5.

$$T^1(n) = \overline{K}_n,$$

which are n isolates.

Remark (Question). Given n, r , can we find an r -partite graph having more edges than $T^r(n)$?

Lemma 7.6. Among all n -vertex simple r -partite graphs, $T^r(n)$ has the maximum number of edges.

Proof. Say G is r -partite with $|G| = n$ and $\|G\| \geq \|T^r(n)\|$.

Then there are parts L and S with $|L| - |S| \geq 2$.

Pick $v \in L$ and move it to S . Then the number of edges changes by $|L| - |S| - 1 \geq 1$. □

Remark. Denote

$$\|T^r(n)\| = t_r(n).$$

Remark. Note that $T^2(n)$ is K^3 free.

In general, $T^{r-1}(n)$ is K^r free.

Each complete graph has at most 1 vertex in each part.

Remark. Is $t_r(n)$ best possible? Is it the largest size of a graph of order n having no K^r subgraph? Is $T^{r-1}(n)$ the only such graph?

That is, what is the largest size for a graph G of order n with $G \not\supseteq K^r$.

More generally, let H be a graph with $|H| < n$. What is the largest size for a graph G on n vertices having $G \not\supseteq H$? Such a graph is called extremal for n and H . Its size is $\text{ex}(n, H)$.

Remark (Question). Is $\text{ex}(n, K^r) = t_{r-1}(n)$ and is $T^{r-1}(n)$ the only graph that is extremal for n and K^r ?

Theorem 7.7 (Turan 1941). For all integers r, n with $r > 1$, every $G \not\supseteq K^r$ with n vertices and $\text{ex}(n, K^r)$ edges is $T^{r-1}(n)$.

Proof. Let $G \not\supseteq K^r$ of order n .

We will construct an $r-1$ partite graph H with $V(H) = V(G)$ and show that $\|G\| \leq \|H\|$.

Then the result will follow from the lemma ($\|H\| \leq t_{r-1}(n)$).

Induction on r .

$r = 2$. If $|G| = n$ and $G \not\supseteq K^2$, then $G = \overline{K}^n$.

So let $r \geq 3$.

Let $k = \Delta(G)$ and pick u with $d_G(u) = k$.

Let $G' = G[N_G(u)]$.

Since $G \not\supseteq K^r$, $G' \not\supseteq K^{r-1}$.

By induction, there exists an $(r-2)$ -partite graph H' with $V(H') = N_G(u)$ and $\|G'\| \leq \|H'\|$.

Construct H as follows.

\vdots

□

Remark. Uniquely so, $\|T^{r-1}(n)\| = t_{r-1}(n)$. This generates Mantel's Theorem.

Definition 7.8. For a graph H with $|H| \leq n$, $\text{ex}(n, H)$ is the largest number of edges of a graph G of order n , can have and still not contain a subgraph H . Such a graph G is called extremal in n and H .

Definition 7.9. Let $|G| = n$. Let density of a graph G be $\frac{\|G\|}{\binom{n}{2}}$, where $n = |G|$. If $\|G\|$ is of order n^2 , then G is dense. Otherwise, G is sparse.

Remark. Turan graphs are dense. Specifically,

$$t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1},$$

with equality when $r-1 \mid n$.

Proof. Hint: choose k and i so that $n = (r-1)k + i$, where $0 \leq i < r-1$.

When $i = 0$, the number of edges in $T^{r-1}(n)$ is

$$\binom{r-1}{2} k^2 = \frac{(r-1)(r-2)}{2} \frac{n^2}{(r-1)^2} = \frac{1}{2} n^2 \frac{r-2}{r-1}.$$

For $i \neq 0$, show that

$$t_{r-1}(n) = \frac{1}{2} \frac{r-2}{r-1} (n^2 - i^2) + \binom{i}{2} < \frac{1}{2} n^2 \frac{r-2}{r-1}.$$

□

Remark. What happens when we add edges to $T^{r-1}(n)$?

Surprising answer: Just a few more edges not only forces a K^r but forces many copies of K^r in the form of a subgraph $K_s^r = K_{s, \dots, s}$ for some s .

Any set of vertices with exactly one vertex in each part induces a K^r .

Specifically: fix $\epsilon \in \mathbb{R}^+$, fix $s \in \mathbb{Z}^+$, then there exists n_0 so that for any $n \geq n_0$, adding ϵn^2 edges to $T^{r-1}(n)$ forces a K_s^r .

Theorem 7.10 (Erdős Stone). *For all $r \geq 2$ and $s \geq 1$ and every $\epsilon \in \mathbb{R}^+$, there exists an integer n_0 so that every graph with $n \geq n_0$ vertices and at least $t_{r-1}(n) + \epsilon n^2$ edges contains K_s^r as a subgraph.*

Definition 7.11. Given a graph H with $|H| \leq n$, $h_n = \frac{\text{ex}(n, H)}{\binom{n}{2}}$, a critical number. This is maximum edge density that an n -vertex graph can have without containing H as a subgraph.

Remark. What happens to this critical number as $n \rightarrow \infty$. It converges to a number that depends only on $\chi(H)$.

Lemma 7.12.

$$\lim_{n \rightarrow \infty} \frac{t_{r-1}(n)}{\binom{n}{2}} = \frac{r-2}{r-1}.$$

Corollary 7.13. For every graph H with at least one edge,

$$\lim_{n \rightarrow \infty} \frac{\text{ex}(n, H)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

Proof. Let H be a graph with at least one edge.

Let $r := \chi(H)$.

- Note $H \not\subseteq T^{r-1}(n)$ for any $n \in \mathbb{N}$. Otherwise, H would be $(r-1)$ -colorable. Since $T^{r-1}(n)$ has no H -subgraph, $t_{r-1}(n) \leq \text{ex}(n, H)$
- Note $H \subseteq K_s^r$ for sufficiently large s . So $\text{ex}(n, H) \leq \text{ex}(n, K_s^r)$ for sufficiently large s .
- Fix such an s . By Erdős Stone, $\text{ex}(n, K_s^r) < t_{r-1}(n) + \epsilon n^2$ for n big enough. So

$$\begin{aligned} t_{r-1}(n) / \binom{n}{2} &\leq \text{ex}(n, H) / \binom{n}{2} \leq \text{ex}(n, K_s^r) / \binom{n}{2} < \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{\epsilon n^2}{\binom{n}{2}} \\ &= \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{2\epsilon n^2}{\binom{n}{2}} = \frac{t_{r-1}(n)}{\binom{n}{2}} + . \end{aligned}$$

□

Remark (Conjecture Hadwiger 1993). For every $r \in \mathbb{N}$ and every graph G , if $\chi(G) = r$, then $G \geq K^r$.

$r = 1, 2, 3, 4$ has been proved.

- $r = 1$: G contains a vertex.
- $r = 2$: G contains a edge.
- $r = 3$: G contains a cycle, which implies K^3 minor.
- $r = 4$: need a few work.

Proposition 7.14. A graph G with $|G| \geq 3$ is edge-maximal with no K^4 minor if and only if it can be considered by recursively pasting triangles.

(Note any subgraph has $2|G| - 3$ cycles.)

Proof. “ \Leftarrow ”. Exercise (For $|G| > 3$).

“ \Rightarrow ”. WTS if G is maximal with no K^4 , then G is triangle-pasted.

Induction on $|G|$.

If $|G| = 3$, done.

Let $|G| \geq 4$ and G is maximal with no K^4 minor but not triangle-pasted.

If G is not complete, done.

Let S be a separator with $|S| = \kappa(G)$.

Case 1: $\kappa(G) \geq 3$.

Graph

There exists P_1, P_2, P_3 , $G - \{v_1, v_2, v_3\}$ is connected.

There exists a shortest path P connected two of P_1, P_2, P_3 .

Graph. K^4 minor.

So $\kappa(G) \leq 2$.

Use fact K^4 minor $\cong TK^4$. (Lemma 4.4.4)

□

Corollary 7.15. Hadwiger holds for $r = 4$.

Graph.

$\chi(G) = \max(\chi(G_1), \chi(G_2))$.

Use induction on $|G|$ and Thm 7.3.1 to show all edge maximal graphs.

Chapter 8

Ramsey Theory for Graphs

Remark. We've seen that $\text{tr}(n)$ edges forces a K^r in G for $|G| = n$. What if we want to know how to force a K^r or a \overline{K}^r .

Theorem 8.1 (Ramsey 1930). *For every $r \in \mathbb{N}$, there exists $n \in \mathbb{N}$ so that if $|G| \geq n$, then G contains either K^r or \overline{K}^r as a subgraph.*

Remark. Trivial for $r \leq 1$.

Let $n = 2^{2^{r-3}}$ and $|G| = n$.

Define a sequence of subsets of $V(G)$ $V_1, \dots, V_{2^{r-2}}$ with $V_1 \supseteq V_2 \supseteq \dots \supseteq V_{2^{r-2}}$ and with $v_i \in V_i - V_{i-1}$ as follows: pick $V_1 \subseteq V(G)$ with $|V_1| = 2^{2^{r-3}}$ and let $v_1 \in V_1$.

Let $A = N(v_1) \cap V_1$ and $B = (V_1 - \{v_1\}) - A$.

Then A or B contains at least $2^{2^{r-4}}$ vertices.

Let V_2 be $2^{2^{r-4}}$ of the vertices in that set.

So either $v_1 \sim w$ for any $w \in V_2$ or $v_1 \not\sim w$ for any $w \in V_2$.

Pick v_2 arbitrary. Continue the process, $|V_3| = 2^{2^{r-5}}$. Pick v_3 .

So $V_i = 2^{2^{r-2-i}}$ and v_{i-1} is either adjacent to all vertices in V_i or $v_{i-1} \not\sim w$ for any $w \in V_i$.

Among the vertices $v_1, \dots, v_{2^{r-2}}$, at least $r - 1$ showed the same behavior when viewed as v_{i-1} when choosing V_i .

So this set of $r - 1$ vertices together with the last one either induces a K^r or a \overline{K}^r .

Definition 8.2. Define $R(r)$ to be the least number n so that $|G| \geq n$ so that $G \supseteq K^r$ or $G \supseteq \overline{K}^r$. We showed that $R(r) \leq 2^{2^{r-3}}$, can't say much more. We'll show that $R(r) \leq 2^{r/2}$ using probabilistic method.

Definition 8.3. Define $R(H_1, H_2)$ to be the least number n so that $|G| \geq n$ so that $G \supseteq H_1$ or $G \supseteq \overline{H_2}$.

Remark.

$$R(r) = R(K^r, K^r).$$

Remark. Trees-an exception-not so hard.

Theorem 8.4. *Let s, t be positive integers and let T be a tree of order s . Then $R(T, K^s) = (s - 1)(t - 1) + 1$.*

Proof. Prove part of this.

Consider the graph G build as the disjoint union of $s - 1$ copies of K^{t-1} .

Then $G \not\supseteq T$.

graph.

$s - 1$ of these because the largest component of G has order $t - 1$.

$G \not\supseteq \bar{K}^s$ (if and only if $\bar{G} \not\supseteq K^s$) because the largest independent set of G has cardinality $s - 1$.

So $R(T, K^s) > (s - 1)(t - 1)$.

To show $R(T, K^s) = (s - 1)(t - 1) + 1$, consider a graph G containing no \bar{K}^s , then show that $G \supseteq T$.

Hint: consider a proper coloring with $\chi(G)$ colors. \square

Chapter 9

Random Graph

Remark. Intuitively, we build a random graph G on n vertices by performing an experiment for each possible edge e in G . Fix $0 < p < 1$, let $P(e \in E(G)) = p$ and $P(e \notin E(G)) = 1 - p$.

Remark. A latter model by Erdős-Renyi, $G(n, m)$. Think of this as a process. Start with $G_{n,0}$ with no edges. At step we add 1 more edge so that all possible new edges are equally likely.

$$G_{n,0} \subseteq G_{n,1} \subseteq \cdots \subseteq G_{n,\binom{n}{2}}.$$

What kind of questions can we answer?

(a) Deterministic question.

- What is a better bound on $R(r)$? ($2^{r/2}$).
- What is a bound on the number of crossings in a graph with $\|G\| \geq 4|G|$?

(b) Erdős-Renyi.

How big should m be to ensure $G_{n,m}$ is Hamiltonian? Same question because $\Delta(G_{n,m}) = 2$?

Theorem 9.1 (Erdős 1947). *For every integer $k \geq 3$, $R(k) > 2^{k/2}$.*

Proof. For $k = 3$, the statement is $R(3) > 2^{3/2}$. $R(3) = 6 > 2^{3/2}$.

So let $k \geq 4$, let $k \leq 2^{k/2}$.

We will show there exists a graph of order n with no K^k or \overline{K}^k subgraph.

Take a random graph on n vertices $G(n, p)$.

Let $p = 1/2$. $P(\alpha(G) \geq k)$ and $P(\omega(G) \geq k)$ are each since $1/k! < 1/2^k$,

$$\begin{aligned} &\leq \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}} < \frac{n^k}{2^k} 2^{-\frac{1}{2}k(k-1)} \left(\frac{n(n-1) \cdots (n-k+1)}{k!} < \frac{n^k}{2^k} \right) \leq \frac{(2^{k/2})^k}{2} 2^{k(k-1)} \\ &= 2^{k^2/2 - k - k^2/2 + k/2} = 2^{-k/2} < 1/2. \end{aligned}$$

So $P(\alpha(G) \geq k)$ or $P(\omega(G) \geq k) < 1/2 + 1/2 = 1$.

Then the probability that a graph $G(n, p)$ has either a K^k or \overline{K}^k subgraph is less than 1.

So there exists a graph of order n having no K^k or \overline{K}^k subgraph.

Thus, $R(k) > 2^{k/2}$. □

Remark (Backgraph). We have

- Euler's formula: For planar graph, $n - m + l = 2$.
- For a planar graph, $m \leq 3n - 6$.
- We can embed any graph in the plane so that each crossing point is incident with at least 2 edges.
- Linearity of expectation $E(X + Y) = E(X) + E(Y)$.
- From any graph G , we can construct a new graph H : Assume G is embedded in plane. $V(H) = V(G) + \text{crossing points}$. $E(H) = \text{all pieces of the original edges}$. $N(H) = n + \text{cr}(G)$. $E(H) = m + 2 \text{cr}(G)$. So $m + \text{cr}(G) \leq 3(n + \text{cr}(G) - 6)$. Hence $\text{cr}(G) \geq m - 3n - 6$. Thus, $\text{cr}(G) - m - 3n \geq 6 > 0$.

Theorem 9.2. If G is simple with n vertices and m edges, where $m \geq 4n$, then $\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}$.

Proof. Let $0 < p < 1$. Start with a graph G drawn in the plane with $\text{cr}(G)$ crossings.

Generate G_p : Pick vertices independently with probability p and consider the resulting induced subgraph.

Let n_p be the number of G_p and m_p be the number of edges of G_p and X_p be the number of crossing points of G_p .

By previous result, $E(X_p - m_p + 3n_p) \geq 0$, $E(n_p) = pn$, $E(m_p) = p^2m$ and $E(X_p) = p^4 \text{cr}(G)$.

We get

$$0 \leq E(X_p) - E(m_p) + 3E(n_p),$$

i.e.,

$$0 \leq p^4 \text{cr}(G) - p^2m + 3pn,$$

i.e.,

$$\text{cr}(G) \geq \frac{p^2m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}.$$

Hence where we pick $p = \frac{4n}{m}$, plugging in it, we get

$$\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

□

9.1 Properties of almost all graphs