Graph Theory

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Contents

Chapter 1

The Basics

1.1 Graphs

Definition 1.1. A graph is a pair $G = (V, E)$ of sets with $E \subseteq V^2$.

(a) The order of G is $|G|$ or $|V|$. The size of G is $\|G\|$ or $|E|$.

(b) $v \in V$ is incident on $e \in E$ if $v \in e$, in which case, we say e is an edge at v.

(c) e and f are adjacent if they share a vertex.

(d) The coloring number, $\chi(G)$ is the smallest number of colors required to color each vertex so that no adjacent vertices are colored the same.

(e) G is a complete graph if all vertices are pairwise adjacent. Let K^n be the complete graph on n vertices.

(f) Pairwise non-adjacent vertices are called independent. A set of independent vertices is a stable set. $\alpha(G)$ is the size of largest stable set.

(g) $G' \subseteq G$ if $V' \subseteq V$ and $E' \subseteq E$. Then G' is called a subgraph of G.

(h) $G' \subseteq G$ and G' contains all edges $xy \in E$ with $x, y \in V'$, then G' is the subgraph induced by V'. Denote it as $G' = G[V']$.

(i) An induced subgraph that is complete is a clique.

(j) $\omega(G)$ is the size of the largest clique of G.

(k) The complement of G is $\overline{G} = (V, \overline{E})$.

(l) A path is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$, $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$. Or for shorthand, $P = x_0x_1 \cdots x_k$. If $P = x_0 \cdots x_{k-1}$ is a path with $k \geqslant 3$, then we call $C = P + x_{k-1}x_0$ a cycle.

(m) For $G = (V, E)$ and $G' = (V', E')$, $G' \cong G$ if there exists a bijection $\phi : V \to V'$ with $xy \in E \Longleftrightarrow \phi(x)\phi(y) \in E'$.

(n) G is edge maximal with respect to a property if G has the property but $G + uv$ does not for any $uv \notin E$.

(o) $N(v)$ is the neighbor set of v. $N(U)$ is the set of neighbors of vertices in $V \smallsetminus U$.

(p) $d_G(v) = d(v)$ is the number of neighbors of v(when G is simple).

(q) $\delta(G) = \min\{d(v)|v \in V\}$. $\Delta(G) = \max\{d(v)|v \in V\}$.

(r) When all vertices have the same degree k, G is k-regular.

(s) The average degree $d(G) = \frac{\sum_{v \in V} d(v)}{V}$.

(t) G is perfect if and only if it contains no odd hole or antihole if and only if $\chi(G) = \omega(G)$.

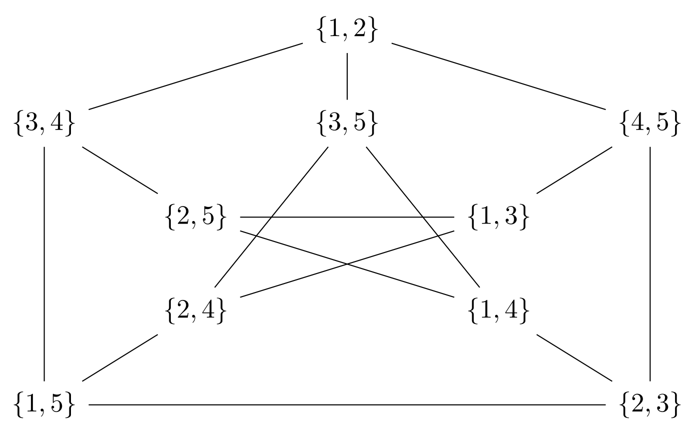
Definition 1.2. The line graph $L(G)$ of $G = (V, E)$ is the graph on V with

(a)
$$V(L(G)) = E.$$

(b) $ef \in E(L(G))$ if and only if e and f are adjacent in G.

Remark. The line graph of G represents adjacencies between edges.

Example 1.3. The $\overline{L(K^5)}$, i.e., Peterson graph is as follows.



Since $\chi(\overline{L(K^5)}) = 3 \geqslant 2 = \omega(\overline{L(K^5)})$, Peterson graph is not perfect.

1.2 The degree of vertex

Theorem 1.4. *Every simple finite graph with at least one edge has a nonempty subgraph H with*

$$\delta(H) > \frac{1}{2}d(H) \geqslant \frac{1}{2}d(G),$$

i.e.,

$$\delta(H) > \epsilon(H) \geqslant \epsilon(G).$$

Proof. Start with G and remove a vertex at a time, obtaining

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_i \supseteq \cdots \supseteq H.$$

Specifically, if $v_i \in V(G_i)$ with

$$d(v_i) \leqslant \epsilon(G_i),$$

then

$$G_{i+1} = G_i - v_i.$$

Otherwise, let $H = G_i$ and stop.
Claim 1. We do stop since G is finite.
Claim 2. The average degree is non-decreasing.
Let

$$G_i = (V, E).$$

Then

$$\epsilon(G_{i+1}) = \frac{|E| - d(v_i)}{|V| - 1} \geqslant \frac{|E| - \epsilon(G_i)}{|V| - 1} = \frac{|E| - \frac{|E|}{|V|}}{|V| - 1} = \frac{|E|}{|V|} = \epsilon(G_i).$$

Claim 3. $H \neq \emptyset$.
Suppose not. Then let $H = G_k$, then $G_{k-1} = K^1$ but then $\epsilon(K^1) = 0$.
But $\epsilon(G) > 0$ since we have at least one edge, contradicting Claim 2. □

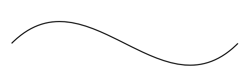
1.3 Path and Cycles

Definition 1.5. A path of length k is a graph $P = (V, E)$ with $V = \{x_0, x_1, \ldots, x_k\}$ and $E = (x_0x_1, x_1x_2, \ldots, x_{k-1}x_k)$, where x_i's are all distinct. So The length is the number of edges.
Sometimes we denote a path as a sequence of vertices

$$x_0x_1 \cdots x_k.$$

P^k is a path of length k. $P^0 = K^1$.

Definition 1.6. xPy: x and y are two intermediate points in the path P.



$$x_0 \;\text{———}\; x_1 \;\text{———}\; \cdots \;\text{———}\; x_{k-1} \;\text{———}\; x_k$$

$$P^0 = x_1 P x_{k-1}.$$

Definition 1.7. Let $G = (V, E)$. In a path $x_0x_1 \cdots x_k$, if $x_0, x_k \in A$ but $x_1, \ldots, x_{k-1} \notin A$, then $P = x_0x_1 \cdots x_k$ is an A-path.

Definition 1.8. Two u-v paths are independent (or internally disjoint) if they have only u, v in common.

Definition 1.9. A walk is a sequence $W = (v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k)$, where

$$e_i = v_{i-1}v_i, \forall 1 \leqslant i \leqslant k.$$

The length is the number of edges v_0-v_k walk.
If $v_0 = v_k$, it is a closed walk.

Definition 1.10. A trial is a walk with no repeated edges.

Remark. A path is a walk with no repeated vertices.

Theorem 1.11. *Let G be a graph.*

(a) Every u-v walk ($u \neq v$) contains a u-v path.

(b) Every closed u-v walk contains a cycle.

(c) Every closed walk with an odd number of edges contains an odd cycle.

Proof. Let

$$w = (u = v_0, e_1, v_1, e_2, v_2, \dots, e_k, v_k = v).$$

Let w' be a subsequence that is itself an u-v walk and is shortest possible.
Suppose w' is not a u-v path.
Then $\exists$a repeated vertex, say $v_j = v_l$ with $j < l$.
But them

$$(v_0 = u, e_1, v_1, \dots, v_j = v_l, e_l, \dots, e_k, v_k = v)$$

is a shorter subsequence that is also a walk. □

Definition 1.12. (a) The girth $g(G)$ is the length of a shortest cycle.

(b) The circumference of G is length of a longest cycle.

(c) $d(u, v)$ is length of shortest u-v path.

(d)

$$\operatorname{diam}(G) = \max_{u,v \in V} d(u, v).$$

(e) The eccentricity

$$e(v) = \max_{u \in V} d(u, v).$$

(f) A vertex with the smallest eccentricity is central. The radius of G is $e(z)$, where z is central.

$$\operatorname{rad}(G) = \min_{v \in V} e(v) = \min_{v \in V} \max_{u \in V} d(u, v).$$

Remark.

$$\operatorname{rad}(G) \leqslant \operatorname{diam}(G) \leqslant 2 \operatorname{rad}(G).$$

Theorem 1.13. *Every graph G contains (provided that $\delta(G) \geqslant 2$.)*

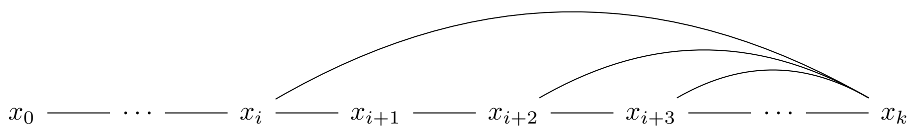
(a) a path of length $\delta(G)$ and

(b) a cycle of length at least $\delta(G)+1$.

Proof. Let $x_0,\dots,x_k$ be a longest path in G. Then all the neighbours of x_i lie on this path, otherwise, if w is a neighbor that is not in the path, then $x_0,\dots,x_k,w$ is a longer path, a contradiction. Hence $d \geqslant d(x_k) \geqslant \delta(G)$.
Let

$$i = \min\{0 \leqslant i < k \mid x_i x_k \in E(G)\}.$$

Then $x_i \cdots x_k x_i$ is a cycle of length at least $\delta(G)+1$.



□

Theorem 1.14. *Every graph G containing a cycle satisfies $g(G) \leqslant 2\operatorname{diam}(G)+1$.*

Proof. Let C be a shortest cycle in G.
If $g(G) \geqslant 2\operatorname{diam}(G)+2$, then C has two vertices whose distance in C is at least $\operatorname{diam}(G)+1$.
In G, these vertices have a lesser distance; any shortest path P between them is therefore not a subgraph of C.
Thus, P contains a C-path xPy.
Together with the shorter of the two x-y paths in C, this path xPy forms a shorter cycle than C, a contradiction. □

1.4 Connectivity

Definition 1.15. Let $G=(V,E)$ be nonempty. G is connected if $\exists$a u-v path for each $u,v\in V$.
$U \leqslant V$ is connected if $G[U]$ is connected.

Theorem 1.16. *G is connected, then vertices of G can be ordered as $v_1,\dots,v_k$ so that each $G_i = [v_1,\dots,v_i]$ is connected for $i=1,\dots,n$.*

Proof. Pick any vertex as v_1 and assume inductively that we have picked $v_1,\dots,v_j$ with G_j connected for $j=1,\dots,i$.
Let $v\in G\setminus G_i$. Since G is connected, $\exists v_1$-v path P in G.
Let v_{i+1} be the first vertex on P that is not in G.
Clearly, G_{i+1} is connected. □

Definition 1.17. The maximal connected subgraphs of G are its components.

Definition 1.18. Let $X\subseteq V\cup E$ and we call X a separating set if $G-X$ is disconneted.
If X is a separating set with $X\subseteq V$, we call X a separator.

Remark. Clearly, the components are induced subgraphs, and their vertex sets partition V.
Since connected graphs are non-empty, the empty graph has no components.

Definition 1.19. Let $k \in \mathbb{N}_0$. G is k-connected if $|G| > k$ and $G - X$ is connected for all $X \subseteq V$ with $|X| < k$.
The connectivity $\kappa(G)$ is the largest k for which G is k-connected.

Remark. $\kappa(G) = 0$ if and only if G is disconnected or a K^1.

Example 1.20. K^5 is 0-connected since it is connected.
K^5 is 1-connected since K^4 is connected.
K^5 is 2-connected since K^3 is connected.
K^5 is 3-connected since K^2 is connected.
K^5 is 4-connected since K^1 is connected.
K^5 is not 5-connected since $\left|K^1\right| = 5$.
So $\kappa(G) = 4$.
Since if a graph G is k-connected, then $|G| > k$,

$$\kappa(K^n) = n - 1, \forall n \in \mathbb{Z}^{\geqslant 1}.$$

Theorem 1.21. *The smallest separator of G, X has $|X| = \kappa(G)$.*

Definition 1.22. If $|G| > 1$ and $F \subseteq E$ with $G - F$ connected for all $F \subseteq E$ with $|F| < l$, then G is l-edge connected.
$\lambda(G)$ is the largest l for which G is l-edge connected.

Theorem 1.23. *If G is non-trivial,*

$$\kappa(G) \leqslant \lambda(G) \leqslant \delta(G).$$

Proof. The second inequality follows from the fact that all edges incident with a fixed vertex separate G.
To prove the first, let F be a set of $\lambda(G)$ edges such that $G - F$ is disconnected, i.e., F is a smallest separating set of edges.
We just need to show

$$\kappa(G) \leqslant |F|.$$

Idea is to construct a set $X \subseteq V$ that is a separator having $|X| \leqslant |F|$.

(a) Suppose first that G has a vertex that is not incident with an edge in F.
Let C be the component of $G - F$ containing v.
Then the vertices of C that are incident with an edge in F separate v from $G - C$.
Since no edge in F has both ends in C by the minimality of F, there are at most $|F|$ such vertices, giving

$$\kappa(G) \leqslant |F|.$$

(b) Suppose now that every vertex is incident with an edge in F.
Let v be any vertex, and let C be the component of $G - F$ containing v.
Then the neighbors w of v with $vw \notin F$ lie in C and are incident with distinct edges in F by the minimality of F, giving $d_G(v) \leqslant |F|, \forall v \in V$.
As $N_G(v)$ separates v from any other vertices in G, this yields $\kappa(G) \leqslant |F|$, unless there are no other vertices, i.e., unless $\{v\} \cup N(v) = V$.
But v was an arbitrary vertex. So we may assume that G is complete, giving $\kappa(G) = \lambda(G) = |G| - 1$.

□

1.5 Trees and forests

Definition 1.24. An acyclic graph is a forest.
A tree is a connected acyclic graph.

Example 1.25. List all tress on 6 vertices.
We have 6 tress.

Remark (Cayley's formula)**.** The number of trees on n labeled vertices is $n^{n-2}, \forall n \in \mathbb{Z}^{\geqslant 0}$.
The formula equivalently counts the number of spanning trees of a complete graph with labeled vertices.
The number of unlabeled trees on n vertices: generating functions.

Theorem 1.26. *TFAE.*

(a) T is a tree.

(b) $\exists!$ u-v path in T for every $u, v \in V(T)$.

(c) T is minimally connected.

(d) T is maximally acyclic.

Proof. "(i)⇒(ii)". Suppose there exists two distinct u-v paths in T for some $u, v \in T$.
Say

$$P_1 = u = x_0 \cdots x_l = v,$$

$$P_2 = u = y_0 \cdots y_k = v.$$

But then

$$x_0 \cdots x_l y_k \cdots y_0$$

is a walk beginging and ending at u.
Hence it contains a cycle, a contradiction.
"(i)⇒(iii)". Suppose T is not minimally connected.
Then for some edge uv, $T - uv$ is connected and hence contains a u-v path P.
But then $uPvu$ is a cycle.
"(i)⇒(iv)". Suppose T is not maximally acyclic.
Then for some edge uv with $u \not\sim v$, we can connect u and v such that $T + uv$ is acylic.
Let P be the unique uv path in T before adding new edge.
Then $uPvu$ is a cycle.
Others will be left as a exercise. □

Definition 1.27. A special vertex T is called a root.
A vertex of T other than the root, of degree 1 is called a leaf.

Theorem 1.28. *Every nontrivial tree contains a leaf.*

Proof. Let P be a longest path.
Let $P = x_0 \cdots x_k$. Then x_k is a leaf. □

Corollary 1.29. The vertices of a tree can be listed $v_0 \cdots v_n$ so that v_i has a unique neighbor in $\{v_0, \ldots, v_{i-1}\}, \forall 1 \leqslant i \leqslant n$.

Proof. For any connected graph, by previous theorem, there exists an ordering $\{v_0, \ldots, v_n\}$ so that for $1 \leqslant i \leqslant n$, $[v_0, \ldots, v_i]$ is connected.
Assume inductively $[v_0, \ldots, v_i]$ is a tree.
Claim the only new edge results $v_i v_{i+1}$ when we add v_{i+1}. □

Corollary 1.30. Let G be acyclic. Then G is a tree if and only if $\|G\| = n - 1$.

Proof. "$\Rightarrow$". Induction on i shows that the subgraph spanned by the first i vertices in previous corollary has $i - 1$ edges.
"$\Leftarrow$". Let G be any connected graph with n vertices and $n - 1$ edges.
Let G' be a spanning tree in G.
Since G' has $n - 1$ edges by the first implication, it follows $G = G'$. □

Theorem 1.31. *A graph T with $|T| = n$ is a tree if and only if any 2 of the following hold.*

(a) T is a cyclic.

(b) T is connected.

(c) $\|T\| = n - 1$.

Corollary 1.32. Let T be any tree of order n and let G be any graph with $\delta(G) = n - 1$. Then G contains a tree isomorphic to T as a subgraph.

Proof. List the tree $v_0 \cdots v_n$.
Induction. $[v_0]$ is in G. Assume $[v_0, \ldots, v_i]$ is a subgraph of G.
WTS

$$[v_0, \ldots, v_i, v_{i+1}] \subseteq G.$$

□

1.6 Bipartite

Definition 1.33. A graph $G = (V, E)$ is r-partite if there exists an r-partition of V so that every edge of G has ends in distinct partite class.
If $r = 2$, G is called bipartite.

Definition 1.34. If G and G' are disjoint, then $G * G'$ is obtained by taking the disjoint union of G and G' and joining every vertex in $V(G)$ with every vertex in $v(G')$ with an edge.

Example 1.35. $P^1 * P^2$.

Definition 1.36.

Definition 1.37. An r-partite graph in which every two vertices from different partition classes are adjacent is called complete.
The complete r-partite graph $\overline{K^{n_1}} * \cdots * \overline{K^{n_r}}$ is written as

$$K_{n_1 \cdots n_k}.$$

Example 1.38. $K_{1,5}$ is a star.

Theorem 1.39. *A graph is bipartite if and only if it contains no odd cycles.*

Proof. "$\Rightarrow$". Let $G = (V, E)$ be bipartite with $V = V_1 \cup V_2$.
Suppose G contains and odd cycle $v_0 \cdots v_k$ with k even.
Wlog, let $v_0 \in V_1$, so

$$v_1 \in V_2,\ v_2 \in V_1, \ldots, v_k \in V_1.$$

But $v_0 \sim v_k$, a contradiction.
"$\Leftarrow$". Suppose G contains no odd cycle.
Fix $v_0 \in G$. Let

$$V_1 = \{v \in V(G) | d(v_0, v) \text{ is odd}\}.$$
$$V_2 = \{v \in V(G) | d(v_0, v) \text{ is even}\}.$$

If $u \in V_1$ and $w \in V_1$ and $u \sim w$, then we have an odd cycle.
If $u \in V_2$ and $w \in V_2$ and $u \sim w$, then we have an odd cycle. □

1.7 Contraction and minors

Definition 1.40. Let $G = (V, E)$ and $e \in E$ so that $\{e\}$ is not a separating set, i.e., e is not a brige or cut edge. Then $G - e$ is the graph obtained from G by removing e.

Definition 1.41. An edge contraction $G \setminus e$ is obtained by removes an edge from a graph while simultaneously merging the two vertices that it previous joined and removing any resulting loops on multiple edges.

Definition 1.42. Any graph obtained from G by a series of deletions and contractions is called a minors of G.
Note we define the deletion of a cut edge to be the contraction of that edge.
To undo a deletion, we add the edge back.

Definition 1.43. Let X be a fixed graph. Replacing the vertices x of X with disjoint connected graphs G_x and replacing the edges xy of X with non-empty sets of $G_x - G_y$ edeges, yields a graph that we shall call an IX, where $G_x - G_y$ is the set of all edges with an end in G_x and the other in G_y.
More formally, a graph G is an IX if its vertex set admits a partition $\{V_x | x \in V(X)\}$ into connected subsets V_x such that distinct vertices $x, y \in X$ are adjacent in X if and only if G contains a $V_x - V_y$ edge.

Definition 1.44. If a graph G contains an IX as a subgraph, then X is a minor of G.

Example 1.45. Peterson has a K^5 minor.

Definition 1.46. A subdividing of X, informally, any graph obtained from X by 'subdividing' some or all its edges by drawing new vertices on those edges. In other words, replace some edges of X with new paths between their ends, so that none of these paths has an inner vertex in $V(X)$.
When G is a subdivision of X, we also say that G is a TX.
The original vertices of X are the branch vertices of the TX and its new vertices are called subdividing vertices.
Note that subdividing vertices have degree 2 while branch vertices retain their degree from X.

Definition 1.47. If a graph G contains a TX as a subgraph, then X is a topological minor of Y.

1.8 Euler tours

Definition 1.48. Let $G = (V, E)$ be connected, simple and finite.
An Euler tour is G is a closed walk that uses each edge exactly once.
A graph is Eulerian if it contains an Euler tour.

Theorem 1.49. *A connected graph G is Eulerian if and only if $\forall v \in V$, $d_G(v)$ is even.*

Proof. "$\Rightarrow$". Let W be an Euler tour. Then

$$d_W(v) = d_G(v).$$

Since $d_W(v)$ is even, $d_G(v)$ is even.
"$\Leftarrow$". Let W be a longest walk that uses each edge at most once.
Claim W is closed.
Else $d_W(G)$ is odd for the last vertex u in W. But then W is not largest possible.
Claim $\forall u, v \in W$, the edge $uv \in W$, provided $uv \in E$. Else W is not longest possible.
Claim $\forall v \in V, v \in W$.
Suppose not. Then $v \in V$ but $v \notin W$.
Wlog, $v \sim u$ with $u \in W$. $uWuv$ is a longer walk. □

1.9 Some linear algebra

Definition 1.50. Let $G = (V, E)$ with $V = \{c_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_n\}$. Associated any $U \subseteq V$ a vector $X_U \in \mathbb{F}_2^n$ with

$$X_U(v) = \begin{cases} 1 & \text{if } v \in U \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, for $F \subseteq E$, $X_F \in \mathbb{F}_2^m$.

Remark. Add two vectors in $\mathbb{F}_2^m$ means taking the symmetric differences.
We abuse notation slightly and refer to X_U as U and X_F as F.

Definition 1.51. Let $\mathcal{C}(G)$ be the subspaces of $\mathbb{F}_2^m$, spanned by the cycles of G. We call it the cycle space.

Definition 1.52. $F \subseteq E$ is a cut if V has a partition $\{V_1, V_2\}$ so that every edege $f \in F$ has one end in V_1 and one end in V_2.
A minimal cut is a bond.

Definition 1.53. Let $\mathcal{C}^*(G)$ be the subspace of $\mathbb{F}_2^n$ generated by all the bonds.
Special case of a bond: $V_1 = 1$ or $|V_2| = 1$, say $V_1 = \{v\}$, then the cut F is denoted as $E(v)$.

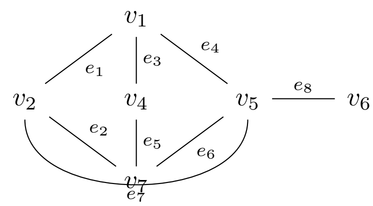
Theorem 1.54. *Let $\{V_1, V_2\}$ partition V. Let F be corresponding cut. Then in $\mathbb{F}_2^n$*

$$F = \sum_{v \in V_1} E(v).$$

Proof. Every edge in the sum appears twice if both ends are in V_1 and once if exactly one end is in V_1. □

Lemma 1.55. $\{E(V)|v \in V\}$ generates $\mathcal{C}^*(G)$.

Example 1.56. Consider the following graph.



The vertex-edge incident matrix is

$$M = \begin{array}{c} \\ E(v_1) \\ E(v_2) \\ E(v_3) \\ E(v_4) \\ E(v_5) \\ E(v_6) \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Definition 1.57. A tree T is spanning tree of G if

(a) T is a subgraph of G.

(b) $V(T) = V(G)$.

Theorem 1.58. *The rank of the incident matrix is $n-1$.*

Proof. Find $n-1$ linearly independent columns, equivalently, an spanning tree T in G. Then $|T| = n-1$. □

Theorem 1.59. *Let M be the incident matrix. Then for any set of $(n-1)$ linearly independent columns of M, the edges corresponding to these columns make up a spanning tree of G.*
The columns corresponding to any tree are linearly indep.
The fundamental cycle are minimally linearly dependent.

Proof. $\{v|E(v)\}$ generates $\mathcal{C}^*(G)$. □

Corollary 1.60.

$$\dim(\mathcal{C}^*(G)) = n-1.$$

Definition 1.61. For $F, F' \in \mathbb{F}_2^m$, the inner product is

$$\langle F, F' \rangle = \sum_{i=1}^{m} F(e_i)F'(e_i) \in \mathbb{F}_2.$$

Theorem 1.62. *The inner product is zero if and only if F and F' have an even number of edges in common.*

Example 1.63. Let

$$F = (1, 0, 0, 0, 0, 1),$$

$$F' = (1, 1, 0, 0, 0, 1),$$

Then

$$\langle F, F' \rangle = 1 + 0 + 0 + 0 + 0 + 1 = 0.$$

Definition 1.64. For any subspace $\mathfrak{F}$ os $\mathbb{F}_2^m$, we define

$$\mathfrak{F}^{\perp} = \{D \in \mathbb{F}_2^m | \langle F, D \rangle = 0, \forall F \in \mathfrak{F}\}.$$

Lemma 1.65. Every cut $C \in \mathcal{C}$ is a (possibly empty) disjoint union of edge of cycles in G.

Proof. □

Theorem 1.66.

$$\mathcal{C} = \mathcal{C}^{*\perp} \text{ and } \mathcal{C}^* = \mathcal{C}^{\perp},$$

i.e.,

$$C \oplus C^* = \mathbb{F}_2^m.$$

Proof. Let $C \in \mathcal{C}(G)$ and $D \in \mathcal{C}^*(D)$.
Then C intersects D an even number of times.
So

$$\mathcal{C} \subseteq \mathcal{C}^{*\perp} \text{ and } \mathcal{C}^* \subseteq \mathcal{C}^{\perp}.$$

Exercise. □

Corollary 1.67.

$$\dim(\mathcal{C}(G)) = m - n + 1.$$

1.9.1 Basis

Theorem 1.68. *(a) A basis for the cycle space $\mathcal{C}$ is obtained as follows: for any spanning tree T of G, each out of tree edge ij creates a unique cycle if edge ij is concatednated to the unique in-tree ji path and there are exactly $m - n + 1$ such cycles.*
The basis obtained in this way is called a fundamental cycle basis.

(b) Let T be a spanning tree. For every edge $f \in T$, the forest $T - f$ has exactly two component. The set $D_f \subseteq E$ of edges of G between these components is a bond in G, the fundamental cut of f with respect to T.
Then a fundamental cut of G with respect to T form a basis of $\mathcal{C}^*(G)$.

Theorem 1.69.

$$\mathrm{Ker}(M) = \mathcal{C}(G).$$

$$\mathrm{Im}(M^T) = \mathcal{C}^*(G).$$

Example 1.70. If we put M into standard form, we'd get

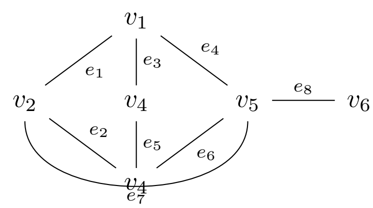
$$[I_{n-1} \mid A],$$

where A is $(n-1)\times(m-n+1)$ matrix.
Then the matrix

$$[A^T \mid I_{m-n+1}]$$

generates $C^*(G)$.

Example 1.71. Consider the following graph.



The vertex-edge incident matrix is

$$M = \begin{array}{c} \\ E(v_1) \\ E(v_2) \\ E(v_3) \\ E(v_4) \\ E(v_5) \\ E(v_6) \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Let $\{e_1, e_2, e_3, e_4, e_8\}$ be a spanning tree.
Then

$$[I_{n-1} \mid A] = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_8 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Add e_5, there is a cycle $\{e_1, e_2, e_3, e_5\}$, which is a min. dependent set and also a fundamental cycle.
So the e_5 column has a column vector $(1,1,1,0,0)^T$.
$a_{ij} = 1$ if and only if e_i is used in the fundamental cycle associated with e_j.
Note

$$[A^T | I_{m-n+1}] = \begin{array}{c} \\ e_5 \\ e_6 \\ e_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_8 & e_5 & e_6 & e_7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Note every row is a fundamental cycle.

Remark. Let the edges of T be the basic elements and non-basic elements is called non-tree edges. Fundamental cycle is the unique cycle containing exactly one non-tree edge.

Theorem 1.72. *Any collection of edges that induces a subgraph H with $d_H(v)$ even for all $v \in V(H)$ is a disjoint union of cycles.*

Chapter 2

Matching Covering and Packing

Definition 2.1. A matching M in a simple graph $G = (V, E)$ is a set of independent edges. These vertices incident with the edges of a matching M are said to be saturated by M, the others are unsaturated.

Definition 2.2. A perfect matching in a graph is a matching that saturated every vertex, that is, a matching of size exactly $\frac{n}{2}$.

Remark. A perfect matching can only occur in a graph with evenly many vertices.

Remark. maximum: largest possible.
maximal: whether it can be extended by simply adding an edge.

Example 2.3. In P^3,

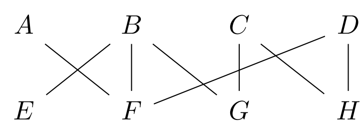
$$a \text{ --- } b \text{ --- } c \text{ --- } d$$

$\{ab, cd\}$ is maximal matching and a maximum matching ($\left\lfloor \frac{4}{2} \right\rfloor = 1$ since it contains 2 edges in 4 vertices.).
$\{bc\}$ is a maximal mathching but not a maximum matching.

Definition 2.4. An M-alternating path is a path that alternates between edges in $E \setminus M$ and edges in M (in order).

Definition 2.5. An M-augmenting path $P = (v_1, \ldots, v_k)$ is an M-alternating path s.t. $v_1, v_k \notin V(M)$

Example 2.6. Let $M = \{BF, CG\}$.



$EBFD$ and $AFBGCH$ are M alternating paths.

Lemma 2.7. Let M_1 and M_2 be matching of G. The degree of every vertex in $[M_1 \Delta M_2]$ is 1 or 2, Hence, $[M_1 \Delta M_2]$ is the disjoint union of paths and cycles.
Furthermore, each such cycle or path alternates in edges in M_1 and M_2.

Theorem 2.8 (Berge)**.** *A matching M in G is maximum if and only if G does not contain an M-augmenting path.*

Proof. "$\Rightarrow$". By contrapositive.
"$\Leftarrow$". Again, by contrapositive. Suppose M is not maximum.
Let M' be a larger matching.
Then $M' \Delta M$ is a collection of paths and even cycles that alternate between M and M'.
At least one such path begins at M' and ends at M'. But this is an M-augmenting path. $\square$

2.1 Matching, vertex covering in bipartite graph

Let $G = (V, E)$ be bipartite with $V = \{A, B\}$.

Definition 2.9. A vertex cover U is a subseteq of V s.t. for all eges e, there is a vertex, say $u \in U$ with u incident with e.

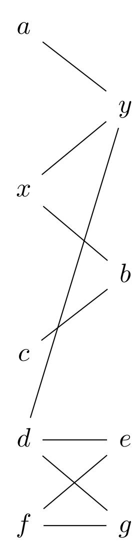
Remark. For now, an alternating path w.r.t. a matching M begins at an unsaturated vertex in A, and contains, alternately edges from $E \setminus M$ and from M.
An alternating path that ends in an unmatched vertex of B is called an augmenting path.

Definition 2.10. $\tau(G) =$ size of a smallest vertex cover.
$\nu(G) =$ size of a maximum matching.

Theorem 2.11 (König)**.**

$$\tau(G) = \nu(G).$$

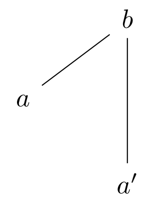
Proof. Consider



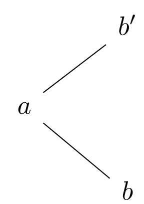
with $M = \{xy, cd, de, fg\}$ being maximum. So a is the only unsaturated vertex.
Clearly,

$$\tau(G) \geqslant \nu(G).$$

Let M be a maximum matching.
Construct a vectex cover U as follows.
For each matching edge $xy \in M$ with $x \in A$ and $y \in B$, do the following: if y is reachable via an alternating path, then put y into U, otherwise, put x into U.
Claim: every edge is incident with a vertex in U.
Let $ab \in E$ with $a \in A$ and $b \in B$.
If $ab \in M$, done.
Suppose $ab \notin M$.
Case 1: a is unsaturated. Then b is saturated. Else M is not maximum.
Say $a'b \in M$.



Then a, ab, b is an alternating path ending at b.
So b is reachable from a and then $b \in U$.
Case 2: a is saturated.



Say $ab' \in M$.
If $a \in U$, we are done.
Else $b' \in U$ and so b' is reachable via an alternating path P.
Let

$$P' = \begin{cases} Pb & \text{if } b \in P \\ Pb'ab & \text{if } b \notin P \end{cases}.$$

Then b must be reachable and so $b \in U$. □

Definition 2.12.

$$N(S) = \{u \in N(s) \text{ for all } s \in S\}.$$

Theorem 2.13. *A necessary (marriage) condition for a matching saturating A is*

$$|S| \leqslant |N(s)|, \forall S \subseteq A.$$

Theorem 2.14 (Hall 1935)**.** *A bipartite graph $G = (V, E)$ with $V = \{A, B\}$ has a matching saturating A if and only if*

$$|S| \leqslant |N(S)|, \forall S \subseteq A.$$

Proof. "$\Rightarrow$". By the marriage condition.
"$\Leftarrow$". Assume G contains no A matching. Then

$$\nu(G) < |A|.$$

Let U be a minimum vertex cover, say $U = A_1 \cup B_1$.
By König theorem,

$$|A_1| + |B_1| = |U| = \tau(G) = \nu(G) < |A|.$$

Then

$$|B_1| < |A| - |A_1| = |A \smallsetminus A_1|.$$

Notice that there are no edges between $A \smallsetminus A_1$ and $B \smallsetminus B_1$.
So

$$N(A \smallsetminus A_1) \subseteq B_1.$$

Thus,

$$|N(A \smallsetminus A_1)| \leqslant |B_1| < |A \smallsetminus A_1|,$$

which is contradicted by the assumption. □

Definition 2.15. A k-regular spanning subgraph is called a k-factor.

Corollary 2.16. 1-factor: a matching that saturates all vertices (perfect).
A subgraph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of V.

Corollary 2.17. Every k-regular bipartite graph has a 1-factor.
(Or every regular bipartite graph has a perfect matching.)

Proof. Let $G = (V, E)$ be k-regular with $V = \{A, B\}$.
Since $k|A| = k|B|$,

$$|A| = |B|.$$

Let $S \subseteq A$.
Then S is joined to $N(S)$ by a total of $k|S|$ edges.
These are among the $k|N(S)|$ edges of G incident with $N(S)$.
So

$$k|S| \leqslant k|N(S)|.$$

Then

$$|S| \leqslant |N(S)|.$$

So Hall's condition is satisfied.
Thus, G has a matching saturating A and so has a 1-factor. □

Definition 2.18. For $X \subseteq A$,

$$\mathrm{def}_G(X) = |X| - |N(X)|.$$

We have

$$\mathrm{def}(G) = \max_{X \subseteq A} \mathrm{def}_G(X).$$

Theorem 2.19 (Refinement of Hall's theorem)**.** *Let* $G = (V, E)$ *with* $V = \{A, B\}$*, then*

$$\nu(G) = |A| - \operatorname{def}(G).$$

Proof. Let $d = \operatorname{def}(G)$ and $\nu = \nu(G)$.
Clearly,

$$\nu \leqslant |A| - d.$$

Construct G':

$$\begin{cases} \text{add } b_1, \ldots, b_d \text{ to } B \\ \text{add edges } ab : \forall\, a \in A \end{cases}.$$

By Halls' theorem, G' has a matching M of A.
Note M use prevcisely edges in $E(G) \setminus E(G')$. □

2.2 Matching in general graphs

Definition 2.20. Let $\mathcal{C}_G$ be the set if its components.

Definition 2.21. Let $q(G)$ be the number of components of G of odd order.

Theorem 2.22. *The necessary condition for the existence of a 1-factor (Tutte's condition) is:*

$$q(G - S) \leqslant |S|, \forall S \subseteq V(G).$$

Theorem 2.23 (Tutte)**.** *A graph* G *has a 1-factor if and only if*

$$q(G - S) \leqslant |S|, \forall S \subseteq V(G).$$

Proof. Say G satisfies Tutte condition but has no 1-factor.
In fact, let G be edge maximal w.r.t. these property.
Let

$$K = \{v \in V : u \sim v, \forall u \neq v\}.$$

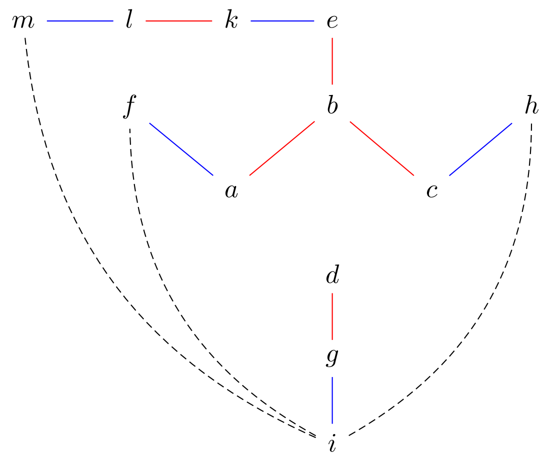
Claim: Every component of $G - K$ is a complete graph.
Suppose not. Then $\exists a, b, c \in V - K$ with

$$a \sim b, b \sim c, \text{ but } a \not\sim c.$$

Then since $b \notin K$, $\exists d \in V$ such that

$$d \not\sim b.$$

By edge maximality, there exists a matching M_1, saturating all vertices except a and c, and a matching M_2 saturating all vertices except b and d.



Consider $M_1 \Delta M_2$: alternating cycles and paths.
Then we construct an augmenting path P: start d, alternating between edges in M_1 and edges in M_2.

(a) P ends at b. But then P is an M_2-augmenting path, a contradiction since M_2 is maximum.

(b) P ends at a or c. Consider Pab. Then Pab is an M_2-augmenting path, a contradiction.

So every component of $G - K$ is a complete graph.
Thus, we have a 1-factor, a contradiction. □

Corollary 2.24 (Peterson 1891)**.** Every cubic bridgeless graph has a 1-factor.

Proof. We show that every graph satisfies Tutte's condition.
Let $S \subseteq V$.
Consider an odd component C of $G - S$.
Then $\partial(C)$, coboundary of C, is the set of all edges in G with exact one end in C.
Note

$$3|C| = \sum_{v \in C} d(v) = 2|E(C)| + |\partial(C)|.$$

So $|\partial(C)|$ is odd.
Since G is bridgeless,

$$|\partial(C)| \geqslant 3.$$

This is true for each odd component.
So with $\overline{S} = V - S$,

$$\left|\partial(\overline{S})\right| \geqslant 3 \cdot q(G - S).$$

Also,

$$\left|\partial(\overline{S})\right| = |\partial(S)| \leqslant 3|S|.$$

So

$$3|S| \geqslant \left|\partial(\overline{S})\right| \geqslant 3q(G - S).$$

Thus,

$$|S| \geqslant q(G - S).$$

$\square$

2.3 Complementary

Definition 2.25. G is factor critial if it has no 1-factor but $G - u$ has a 1-factor for any $u \in V$.

Definition 2.26. A near factor is a matching in which only 1 vertex is unsaturated.

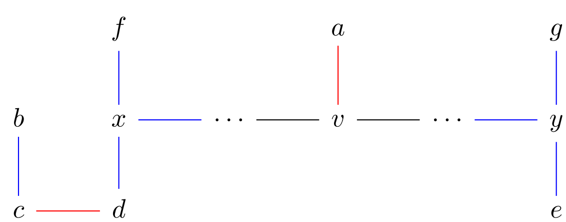
Definition 2.27. A vertex v is essential if every maximum matching covers v

Lemma 2.28. If G is connected and $\nu(G - u) = \nu(G), \forall u \in V$, then G is factor critical.

Proof. Let G be connected with $\nu(G) = \nu(G - u), \forall u \in V$.
So G has no 1-factor.
It suffices to show no maximum matching leaves two distinct vertices unmatched.
Suppose we have a maximum matching M s.t. x and y are unmatched and $d(x, y)$ is as smallest as possible.
Clearly,

$$d(x, y) \geqslant 2.$$

Let P be a shorstest x-y path. Then there is a vertex v that is in the interior of P.
By the minimality of d, v is matched by M.
Since $\nu(G - v) = \nu(G)$, v is inessential. (All vertices of G is inessential.)
Then there exists a maximum matching M' missing v.
By the minimality of d, x, y is matched by M'.



In above graph, red ededges are in M and blue edges are in M' and black edges are neither in M nor in M'.
In $M \Delta M'$, since each path alternates in edges in M_1 and M_2, the paths in it starting at x and y are distinct.
Le Q be the path in $M \Delta M'$ starting at x, wlog, Q does not end at v?
Then $Q \Delta M'$ is a maximum matching avoiding x and v? $\square$

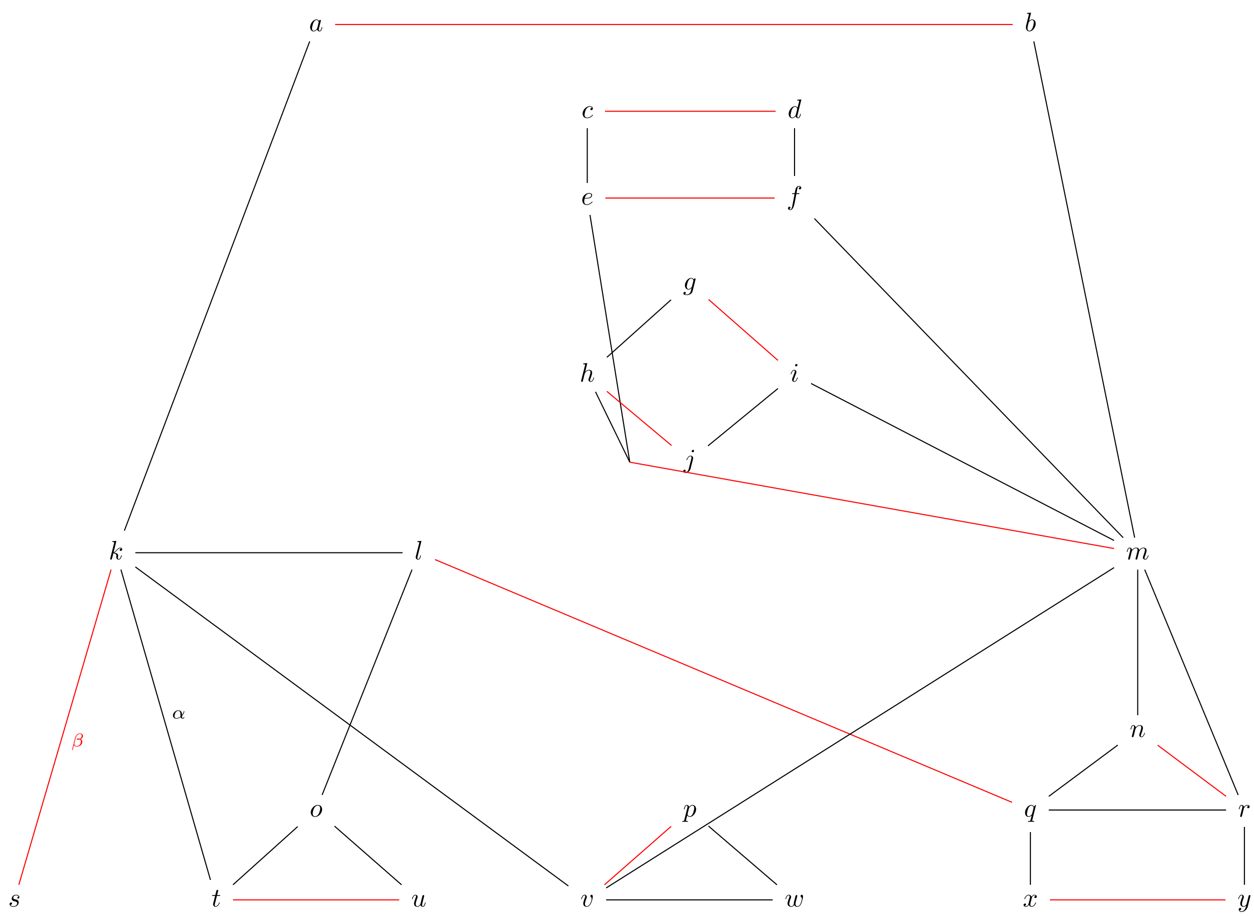
Definition 2.29. Let $G = (V, E)$ be a graph with no 1-factor.Define

$$D(G) = \{v \in V : v \text{ is an inessential vertex}\},$$

$$A(G) = \{v \in V \setminus D(G) : v \in N(D(G))\},$$

$$C(G) = V \setminus \{D(G) \cap A(G)\}.$$

Theorem 2.30. *Consider*



Since $D(G)$ is in the bottom, $A(G)$ is in the middle and then $C(G)$ is the left ones.

Lemma 2.31 (Stability lemma)**.** Let $G = (V, E)$ be a graph with no 1-factor. Then $\forall u \in A$, we have

$$D(G - u) = D(G),$$

$$A(G - u) = A(G) - u,$$

$$C(G - u) = C(G).$$

Proof. Claim

$$\nu(G - u) = \nu(G) - 1, \forall u \in A.$$

Since $u \in A$ is essential, no matching of $G - u$ has cardinality $\nu(G)$.
So

$$\nu(G - u) < \nu(G).$$

Furthermore, let M be a maximum matching of G, then

$$|M| = \nu(G),$$

and $u \in A$ is saturated by M, say by $\alpha \in M$.
Then $M - \alpha$ is a matching of $G - u$ and $|M - \alpha| = \nu(G) - 1$.
So

$$\nu(G - u) \geqslant |M - \alpha| = \nu(G) - 1.$$

Thus,

$$\nu(G - u) = \nu(G) - 1.$$

Claim

$$D(G) \subseteq D(G - u), \forall u \in A.$$

Let

$$o \in D(G).$$

Let M_o be a maximum matching of G leaving o unmatched.
Then

$$|M_o| = \nu(G).$$

Let $\beta \in M_o$ be incident with $u \in A$.
Then $M_o - \beta$ is a matching of $G - k$ of size $|M_o - \beta| = \nu(G) - 1$, and hence, by previous claim, a maximum matching of $G - u$ leaving o unmatched.
So

$$o \in D(G - u).$$

Next,

$$D(G - u) \subseteq D(G).$$

Choose $v \in D(G - u)$. Let M' be a maximum matching of $G - u$ missing v.
Let $w \in D(G)$ with $w \sim u$ and let M be a maximum matching of G missing w.
We need to construct a maximum matching of G missing v, (this would imply that $v \in D(G - u)$ as required.)
If M misses v, then we are done. So assume not.
Then v is matched by M.
Let P be the path of $M \Delta M'$ starting at v.
Case 1: P_1 ends with an edge of M'. Then $M \Delta P$ is a matching in G missing v, and it is the same cardinality as M, hence maximum. So we are done.
Case 2: P ends with an edge of M. Consider $M' \Delta P$. It is maximum. Hence it must match u. So M ends at u. But then $M \Delta (P + uw)$ is a maximum matching avoiding v as required. □

Corollary 2.32. Let $G = (V, E)$ be no 1-factor.

(a) Let M be a maximum matching in G, let $u \in A(G)$ and let f be the unique edge in M incident with u. Then $M - f$ is a maximum matching of $G - u$.

(b) Let M be a maximum matching in G. Then if f is an edge of M with one end in $A(G)$, then the other end of f is necessarily in $D(G)$.

Theorem 2.33 (Edmond's Gallai's structure theorem)**.** *Let $G = (V, E)$ be a graph with no 1-factor and D, A, C be defined before. Then*

(a) Every component of $[D]$ is factor critical (odd).

(b) Every component of $[C]$ has a 1-factor (even).

(c) Define a bipartite graph $\{A, B\}$, where $A = A(G)$ and a vertex of B is a component of $[D]$, with ab an edge if and only if a is adjacent to at least one vertex in B.
Hall's condition holds with a surplus,

$$|N(X)| \geqslant |X| + 1, \forall X \subseteq A.$$

(d) Let M be a maximum matching of G. Then M contains a near-factor of each component of $[D]$.
A 1-factor of each component of $[C]$ and vertices in A are matched to vertices in distinct component of $[D]$.

(e)

$$\nu(G) = \frac{1}{2}(|V| - q(G - A) + |A|).$$

Proof. Delete vertex of A one at a time.

$$D(G - A) = D(G),$$

$$A(G - A) = \emptyset,$$

$$C(G - A) = C(G).$$

(a) Since any matching M of G saturating A, $M \cap E(G - A)$ has cardinality $\nu(G) - |A|$ and is a maximum matching of $G - A$.
Use Gallai's lemma, it is enough to show

$$\nu(G_i - v) = \nu(G_i), \forall v \in V(G_i),$$

where G_i is a component of $[D]$.
So let

$$v \in V(G_i).$$

Let M_v be a maximum matching of G, leaving v unsaturated.
Remove all edges of M_v incident with A and then the part left has cardinality $\nu(G) - |A|$ and is a maximum matching of $G - A$.
Since the component of $[D]$ are disjoint, **restricting** $M_v - E[A]$ is a maximum matching of G_i avoiding v.
So

$$\nu(G_i) = \nu(G_i - v).$$

By Gallai's lemma, G_i is factor critical.

(b) Note that $[C]$ has 1-factor (start with a maximum matching M of G and remove all edges incident with A.) (Again we used consequence (2) above).

(d) (Key point: every vertex $k \in A$ is saturated by any maximum matching M of G, say $\beta \in M$ is incident with k and the other end of β must be in D. Else remove k and β to get a maximum matching of $G - k$?)
From (a) and (b), it follows that a maximum matching in $G - A$ consists of a 1-factor of $[C]$ and a near factor of each component of $[D]$, i.e., we can do better than this, so this must be largest possible. We also know that removing all edges incident with A from any maximum matching of G results in a maximum matching of $G - A$, and hence leaves exactly 1 vertex unsaturated in each component G_i of $G - A$ in D.

(e) Clearly now.

(c) Let $C \subseteq A$. Let $u \in X$ and let $u \sim v$ with $v \in b$, where b is some component of $[D]$. Let M be a maximum matching of G avoiding v. By (d), the rest of the vertices in b are matched to vertices also in b. Hence no vertex in X is matched to a vertex of b. It follows that each of the $|X|$ vertices in X is matched to a distinct component other than b in $[D]$. These $|X|$ distinct components, together with b form our requisite set of size at least $|X| + 1$ elements in B in the neighbor set of X.

$\square$

Chapter 3

Connectivity

3.1 2-connected graphs and subgraphs

Definition 3.1. A cut vertex is one that separates of two other vertices.

Definition 3.2. G is 2-connected if it contains at least 3 vertices and has no cut vertex.

Definition 3.3. Ear decomposition is a simple recursive procedure for generating any 2-connected graph starting with a cycle.

Definition 3.4. An F-path is also called an ear of F in G.

Theorem 3.5. *Let F be a nontrivial subgraph of a 2-connected graph G. Then F has an ear in G.*

Proof. Case 1: F spans G. Then $\exists e \in E(G-F)$. Then e is an ear.
Case 2: F is not spanning.
Since G is connected, $\exists xy \in E(G)$ with $x \in V(F)$ and $y \in V(G-F)$.
Since G is 2-connected, there is a $(y, F-x)$-path Q in $G-x$.
So $P = xyQ$ is an ear in F. $\square$

Theorem 3.6. *Let F be an 2-connected subgraph of G. Let P be an ear of F. Then $F \cup P$ is 2-connected.*

Proof. Exercise. $\square$

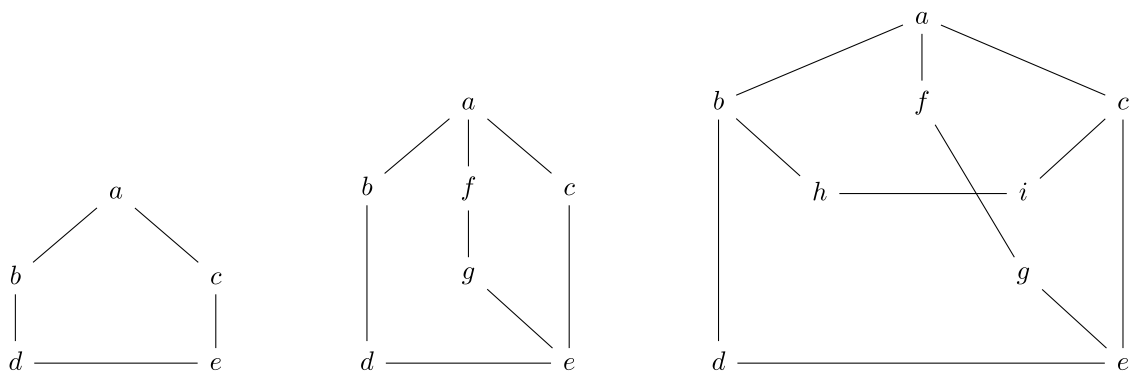
Definition 3.7. A nested sequence of graphs is a (finite) sequence $(G_0, \ldots, G_k)$ with $G_i \subsetneq G_{i+1}$ for $0 \leqslant i \leqslant k-1$.

Definition 3.8. An ear decomposition of 2-connected graph is a nested sequence $(G_0, \ldots, G_k)$ of a 2-connected so that

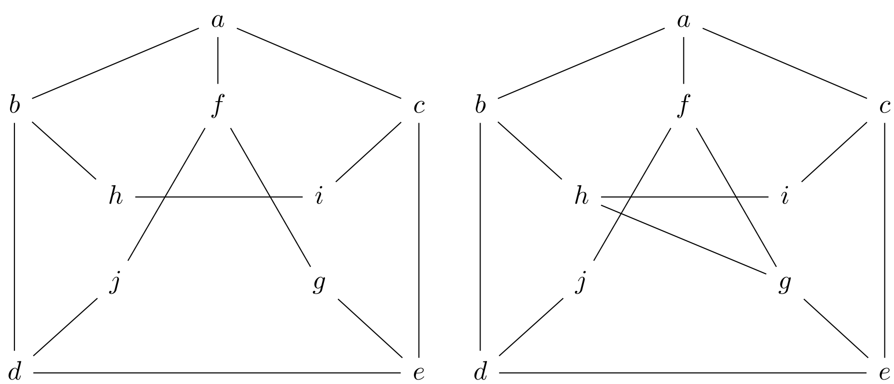
(a) G_0 is a cycle;

(b) $G_{i+1} = G_i \cup P_i$, where P_i is an ear of G_i in G, $0 \leqslant i \leqslant k-1$.

Example 3.9. Consider



(a) G_1 (b) G_2 (c) G_3



(a) G_4 (b) G_5

Lemma 3.10. Every 2-connected graph G has a cycle.

Proof. Since G is 2-connected, G is connected.
Suppose G is acyclic, then G is a tree.
So G contains a leaf x.
Let y be the unique vertex adjoint to x, then y is a cut vertex, a contradiction. □

Lemma 3.11. G is 2-connected if and only if it has an ear decomposition.

Proof. "$\Leftarrow$". Induction on the number of ears.
G_0 is a cycle and 2-conncetd.
Then inductively apply previous theorem that the union of 2-connected graph and an ear is still 2-connected.
"$\Rightarrow$". Use previous lemma and the following theorem. □

Theorem 3.12. *Let F be a nontrivial proper subgraph of a 2-connected graph G. Then F has an ear in G.*

Definition 3.13. A block of a graph G is a maximal connected subgraph without a cut vertex.

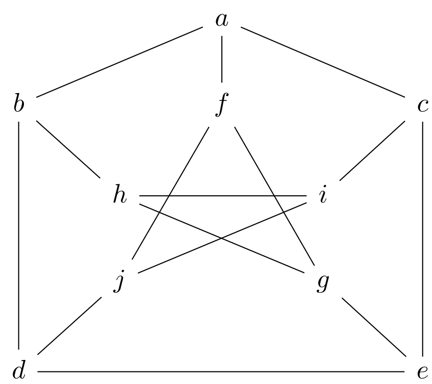


Figure 3.3: G_6

Remark. Types of blocks.

(a) maximal 2-connected graph.

(b) a bridge.

(c) an isolated vertex.

(d) If different blocks overlap, then they overlap in one vertex (a cut vertex).

(e) Every edge lies in a unique block.

(f) G is the union of its blocks.

Definition 3.14. A bond is a minimal cut. Assume G is cut into two parts A and B, then either A or B is connected.

Theorem 3.15. *If F is a cut with $xy \in F$, then F is a bond if and only if it is a minimal intersection set of all x-y path.*

Lemma 3.16. (a) cycles of G are cycles of the blocks.

(b) bonds of G are bonds of the blocks.

Proof. (a) A cycle is 2-connected. So it must be part of some maximal 2-connected subgraphs.

(b) Let F be a bond of G, let $xy \in F$. So F separates x and y in G.
Let B be the block containing xy, by the maximality of B, G contains no B-path.
Hence B contains all x-y paths (or use previous theorem).
So $F \cap E(B)$ separates x and y in B.
Thus, F is also a bond in B.

□

Lemma 3.17. For distinct edges e and f of G, TFAE.

(a) e and f belong to the same block;

(b) e and f belong to the same cycle;

(c) e and f belong to the same bond;

Proof. "(i)⇒(ii)". Let e and f be in the same block B.
Let B be 2-connected.
Claim in any 2-connected graph, any two edges are in the same cycle.
It suffices to show that for any two distinct pairs $\{u_1, u_2\}$ and $\{v_1, v_2\}$ of vertices, there are two disjoint paths.
Since B is 2-connected, it has an ear decomposition $\{B_0, \ldots, B_k\}$.

If $k = 0$, then B is a cycle, done .
Then induct on k.
Let

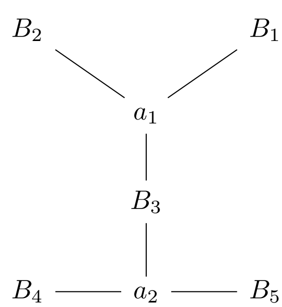
$$0 \leqslant i \leqslant k - 1.$$

Case 1: both $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are in B_i. True by inductive assumption.
Case 2: both are in P, which is an ear of B_i, since G_i is connected, they are in the same cycle.
Case 3: one pair is in B_i and the other is in P_k.
Use induction with the pairs $\{u_1, u_2\}$ in G_i and $\{u, v\}$ in G_i (By symmetry).
"(iii)⇒(ii)". Let e and f be in the same cycle C.
Removing e and f from C leaving two paths P_1 and P_2.
Grow P_1 and P_2 into a partition V_1, V_2 of G so that $e, f \in V_1 - V_2$, so that $[V_2]$ is connected?
Then edges between V_1 and V_2 form a bond of G?
Le V_2 be the connected componet of $G - P_1$ containing P_2 and V_2?
Let

$$V_1 = V(G) - V_2.$$

"(iii)⇒(i)". Assume e and f are in the same bond of G.
That bond is also a bond of some block B of G.
Then B contains e and f. □

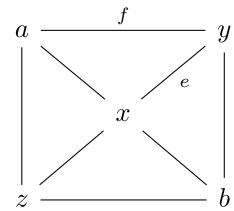
Definition 3.18. Bipartite $\{A, B\}$, where A is the set of cut vertices and B is the set of blocks. $a \sim B$ in this block graph if $a \in B$.



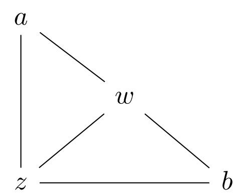
Definition 3.19. The block graph of a connected graph is a tree.

3.2 The structure of 3-connected graphs

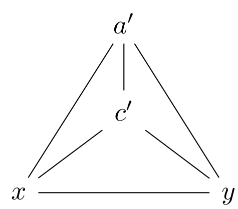
Example 3.20. Consider G



Then G/e is



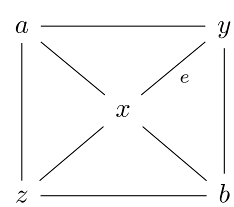
So it is not 3-connected since it contains a 2-vertex cut $\{x, c\}$.
G/f is



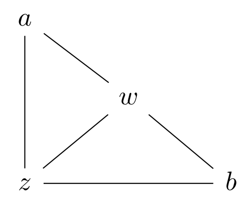
It is 3-connected.

Lemma 3.21. Let G be a 3-conneted with $|G| \geqslant 5$ and let $e = xy \in E(G)$ s.t. G/e is not 3-connected. Then $\exists\, z \in V$ such that $\{x, y, z\}$ is a 3-vertex cut of G.

Proof. Let G be

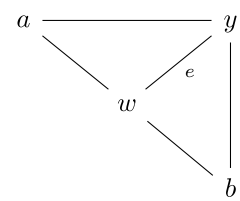


Let $\{z, w\}$ be a 2-vertex cut of G/e.



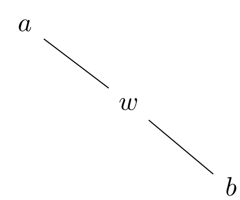
Both z and w cannot be the result of contracting e, say z is that vertex.
Set

$$F := G - z.$$



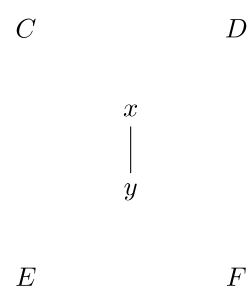
(Since G is 3-connected, F is 2-connected.)
However,

$$F/e = (G - z)/e = G/e - z.$$

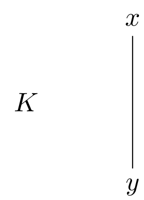


Note $G/e - z$ has a 1-vertex cut $\{w\}$.
Hence w must be the result of contracting e.
Thus, $\{x, y, z\}$ is a 3-vertex cut of G.
(z is not the resulting of contracting xy.) □

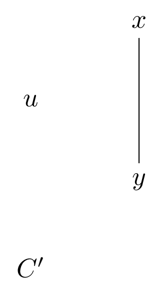
Lemma 3.22. If G is 2-connected and $\{x, y\}$ is a 2-vertex cut of G with $x \sim y$ and C is any component of $G - \{x, y\}$, then $H = [V(C) \cup \{x, y\}]$ is also 2-connected.



Proof. Suppose not. Then there is a cut vertex $u \in V(H)$.
Case 1. $u = x$ or y. Wlog, let $u = x$.
Then G looks like



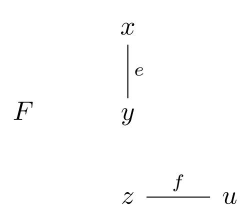
So there is no y-C edges?
Then $G - x$ contains C as a component, a contradiction?
Case 2. $u \in C$. Let C' be component of $H - \{u\}$.



Then v is a cut vertex of G, a contradiction. □

Theorem 3.23 (Thomason 1981)**.** *Let G be a 3-connected graph with at least 5 vertices. Then G contains an edge such that G/e is 3-connected.*

Proof. Suppose not.
Then for any edge $e = xy$ of G, G/e is not 3-connected.
By previous lemma, $\exists z \in V$ associated with xy such that $\{z, x, y\}$ is a 3-vertex cut of G.
Choose e and z such that $G - \{x, y, z\}$ has a component F with as many vertices as possible.
Consider $G - z$.
Since G is 3-conneted, $G - z$ is 2-connected.
Also $G - z$ has the 2-vertex cut $\{x, y\}$.
Hence $H = [V(F) \cup \{x, y\}]$ is 2-connected by previous lemma.
Let u be a neighbor of z in a component of $G - \{x, y, z\}$, other than F.



Since $f = zu \in E(G)$, by our assumption, $\exists v \in V$ such that $\{z, u, v\}$ is a 3-vertex cut of G.
Since H is 2-connected, $H - v$ is connected and is thus contained in component of $G - \{z, u, v\}$.
But the order of $H - v$ is larger than $|F|$, which is contradicted by the maximality of F. □

3.3 Menger's theorem

Theorem 3.24 (Menger 1927)**.** *Let $G = (V, E)$ and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the largest collection of disjoint A-B path in G.*

Proof. Let

$$k = \kappa(G, A, B) = \text{minimum number of vertices separating } A \text{ from } B.$$

Clearly, the cardinality of the largest collection of vertex disjoint A-B path $\leqslant k$.
Induct on $\|G\|$.
If $\|G\| = 0$, the only A-B paths are the singletons $|A \cap B|$, which is the largest number of disjoint A-B path.
Also, the smallest separating set is $A \cap B$.
Assume

$$\|G\| \geqslant 1.$$

Then there exists

$$e = xy \in E.$$

$$\begin{array}{ccccc} a & \text{——} & b & \text{——} & c \\ | & & | & & | \\ d & \text{——} & e & \text{——} & f \\ | & & & & | \\ g & & & & h \end{array}$$

with $A = \{b, c, e, f\}$ and $B = \{a, b, e, d\}$.
Inductively, assume statement holds for graphs of smallest size.
Suppose G has no k disjoint A-B paths, then neither does G/e.
Let v_e be the contracted vertex. Replace A with A' and B with B'.
Put v_e into A' if $\{x, y\} \cap A \neq \emptyset$. Put V_e into B' if $\{x, y\} \cap B \neq \emptyset$.
By the induction hypothesis, G/e contains an A-B speparator Y of fewer than k vertices.
Note

$$v_e \in Y,$$

otherwise, $Y \subseteq V$ would be an A-B separater.
Hence $X := (Y - \{v_e\}) \cup \{xy\}$ is an A-B separators in G of cardinality k.
Let

$$k = \kappa(G, A, B) \text{ and } p = \text{maximum number of } A\text{-}B \text{ disjoint paths in } G;$$

$$k' = \kappa(G, A', B') \text{ and } p' = \text{maximum number of } A'\text{-}B' \text{ disjoint paths in } G.$$

Then

$$p' \leqslant p, \ p < k \text{ and } p' = k'.$$

Also,

$$k' = k \text{ or } k - 1.$$

So

$$p = k - 1,$$

$$p' = k - 1 = k'.$$

Consider $G - e$.
Since $x, y \in X$, every A-X separator in $G - e$ is also an A-B separator in G and hence contains at least k vertices.
So by induction there are k disjoint A-X paths in $G - e$, and similarly there are k disjoint X-B paths in $G - e$.
As X separates A from B, these two path systems do not meet outside X, and can thus be combined to k disjoint A-B paths. □

Remark. We have the following stronger statement.
If P is any set of fewer than k disjoint A-B paths in G, then G contains a set of $|P| + 1$ disjoint A-B paths exceeding P.

Corollary 3.25 (König theorem)**.** Let $G = (V, E)$ be a bipartite with bipartition $\{A, B\}$. Every A-B path is an edge in G. Every vertex cover is an A-B separating set.

Definition 3.26. Let $G = (V, E)$. If $a \in V$ and $B \subseteq V$ with $a \notin B$, then an a-B fan is a collection of paths with pairwise intersection at a.

Corollary 3.27 (To Menger)**.** For $B \subseteq V$ and $a \in V \setminus B$, the size of a smallest a-B separation not containing a is equal to the maximum number of paths in an a-B fan.

Proof. Apply Menger to G-a with $A = N_G(a)$. □

Corollary 3.28. Let a, b (s, t) be two distinct vertices of $G = (V, E)$.
If $ab \notin E$, then the minimum number of vertices not containing $\{a, b\}$ separating a from b in G $(\kappa(a, b))$ is equal to the maximum number of independent (internally disjoint) a-b paths in G $(\lambda(a, b))$.

Corollary 3.29 (Edge a-b version)**.** The minimum number of edges separating a from b $(\kappa'(a, b))$ is equal to the maximum number of edge disjoint a-b paths in G $(\lambda'(a, b))$.

Proof. Apply Menger' a-b version of the line graph of G. □

Theorem 3.30 (Menger's global version)**.** *(a) A simple graph is k-connected if and only if it contains k independent paths between any 2 distinct vertices.*

(b) A simple graph is k-edge-connected if and only if it contains k edge-disjoint paths between any 2 distinct vertices.

Proof. (a) "$\Leftarrow$". Say G contains k-independent paths between 2 distinct vertices.
Then $|G| > k$.
Furthermore, G connot be separated by fewer than k vertices.
Hence G is k-connected.
"$\Rightarrow$". Assume G is k-connected.
Then $|G| > k$ and any separating set has size at least k.
Assume $\exists a, b \in V$ s.t. there are at most $k - 1$ independent paths between a and b.
If $ab \notin E$, by previous corollary, the minimum number of vertices separating a from b is at most

$k-1$, which is contradicted by that G is k-connected.
So
$$ab \notin E.$$
Set
$$G' = G - ab.$$
Since ab is a-b path, which must be independent of any other a-b paths, G' contains at most $k-2$ independent a-b paths.
Then G' has an a-b sepatators X with at most $k-2$ vertices.
Since $|G'| > k$, $|G'| \geqslant k$.
Also,
$$|X| \leqslant k-2.$$
So $\exists v \in V$ such that $v \notin X \cup \{a, b\}$ in G'.
It must be the case that in G' either X separates a from v or X separates b from v, wlog, say a.
But then $X \cup \{b\}$ is a set of at most $k-1$ vertices separating v from a in G.
Thus, G is not k-connected, a contradiction.

$\square$

Chapter 4

Planar Graphs

Remark (Problem)**.** Given distinct vertices $x_1, \ldots, x_k$ and$y_1, \ldots, y_k$, find k independent paths $P_1, \ldots, P_k$, where P_i is an x_i-y_i path, called an x-y linkage. This is a NP-hard problem even if $k = 2$.

4.1 Topological prerequisites

Definition 4.1. A topological is a collection of subsets called open sets of a ground set X that is closed under arbitrary union and finite intersection. X is called a topological space.

Example 4.2. The smallest topology on X is

$$\{\emptyset, X\}.$$

Example 4.3. In discrete topology, every subset is open.

Example 4.4. In metric space, open sets are generated by open sets.

Definition 4.5. A function between two topological spaces is continuous if the preimage of every open set is open.

Definition 4.6. A homeomorphism is a continous bijection between two topological spaces for which the inverse function is continuous.

Example 4.7.

$$(\mathbb{R}, d_{\text{disc}}) \xrightarrow{\text{id}} (\mathbb{R}, |\cdot|),$$

is bijection continuous but not a homeomorphism since the inverse

$$(\mathbb{R}, |\cdot|) \xrightarrow{\text{id}} (\mathbb{R}, d_{\text{disc}}),$$

is not continuous since the open set $\{x\}$ in $(\mathbb{R}, d_{\text{disc}})$ is not open in $(\mathbb{R}, |\cdot|)$ (closed).

Lemma 4.8. A continuous bijective map is a homeomorphism if and only if the image of every open is open.

Definition 4.9. A set is closed if it is the complement of an open set.

Remark. In a metric space, closed sets contain all limit points.

Definition 4.10. A set is compact if every open cover has an finite subcover.

Remark. In $\mathbb{R}$, closed and bounded sets are compact.

Remark. Topological studies properties of objects that does not change under homeomorphism.

Example 4.11. $[0,1]$ is homeomorphic to a polygonal arc in $\mathbb{R}^2$.

Remark. Topological graph theory was studied first to address 4-color theorem.

Remark. Two homeomorphic spaces share the same topological properties. For example, if one of them is compact, then the other is as well; if one of them is connected, then the other is as well; if one of them is Hausdorff, then the other is as well; their homotopy and homology groups will coincide.

Definition 4.12. In $\mathbb{R}^2$, a set S is open if$\forall x \in S, \exists r > 0$ such that the open disk $B_r(x) \subseteq S$, where $B_r(x)$ is called a neighborhood of x.

Definition 4.13. A straight line segment in $\mathbb{R}^2$ between p and q is of the form

$$\{p + \lambda(p - q) : 0 \leqslant \lambda \leqslant 1\}.$$

Definition 4.14. A polygonal arc P is a set $A \subseteq \mathbb{R}^2$ and is a union of finitely many line segment and is homeomorphic to $[0,1]$ in $\mathbb{R}^1$. The images of 0 and 1, say x and y are called the ends of P. Say P links x and y, define

$$\overset{o}{P} = P \setminus \{x, y\}.$$

Definition 4.15. A polygon is a subset of $\mathbb{R}^2$, which is the union of finitely many straight line segment and is homeomorphic to the unit cycle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Definition 4.16. A bond of a polygonal arc or a polygon P is a point in P where line segments meet. Note there are just finitely many bonds.

Theorem 4.17. *Complement of finite union of (polygon) arcs is open.*

Definition 4.18. Let $\Omega \subseteq \mathbb{R}^2$ be an open set. Define $x \sim y$ if $x, y \in \Omega$ and there is a polygonal arc $A \subseteq \Omega$ having ends x and y. Note "$\sim$" is an equivalent relation and equivalence classes are called arcwise connected components of Ω, or region of Ω.

Definition 4.19. If $x \sim y$, for any $x, y \in \Omega$, we say that Ω is arcwise connected.

Definition 4.20. If $X \subseteq \mathbb{R}^2$ is closed, we call an arcwise connected component of $\mathbb{R}^2 - X$ a face of X.

Definition 4.21. The frontier or (boundary) of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points in $\mathbb{R}^2$ such that every neighbor of y meets both X and $\mathbb{R}^2 - X$.

Theorem 4.22. *If X is open, frontier of X is in $\mathbb{R}^2 - X$.*

Theorem 4.23 (Jordan curve theorem for polygon). *Every polygon $P \subseteq \mathbb{R}^2$ has exactly two faces of which exactly one is bounded. The boundary of each of the two faces is P.*

Proof. Let $x \in \mathbb{R}^2 - P$ and L be an half line starting at x and containing no bonds of P.
Let

$$\pi(x, L) = |L \cap P| \pmod 2.$$

Check if L_1 and L_2 are two such lines starting at x, then

$$\pi(x, L_1) = \pi(x, L_2).$$

Call this number $\pi(x)$.
Check π is a continuous function.
Then π is constant on each arcwise connected component of $\mathbb{R}^2 - P$.
Choose two points x_1 and x_2 close to each other but on opposite side of a line segment of P.
Then

$$\pi(x_1) \neq \pi(x_2).$$

So P has at least two faces.
Suppose P has at least 3 faces.
Choose x_1, x_2, x_3 on each faces.
Let x be on the boundary of P (but not a bound).
So x is on a line segment S.
Pick O a small open neighborhood of x with

$$O \cap P = O \cap S.$$

For each of x_1, x_2, x_3, shoot a half line towards P but not on the way.
Travel on the line segment along lots of neighbors of P to O from there.
Going backwards, we get a polygonal arc from O to x_1.
So each of x_1, x_2, x_3 can be reached from a point in O by a polygonal arc not intersecting P.
But $O - P$ has at most two arcwise connected components.
So by PHP and the def. of face, at least two of x_1, x_2, x_3 are in the same region of $\mathbb{R}^2 \setminus P$.
Hence P has at most 2 faces.
Furthermore, every point of $O \cap S$ belongs to the boundary of both faces.
Also, since x is arbitrary, P is the boundary of both faces.
Check one region is unbounded. $\square$

Lemma 4.24. Let P_1, P_2, P_3 be 3 (polygonal) arcs between the same two end points and are otherwise disjoint. Then $\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly 3 regions with

(a) fontier $P_1 \cup P_2, P_2 \cup P_3$ and $P_1 \cup P_3$.

(b) If P is an arc between a point in $\overset{o}{P}_1$ and $\overset{o}{P}_3$ whose intersection lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ that contains P_2, then $\overset{o}{P} \cap \overset{o}{P}_2 \neq \emptyset$.

Proof. (Sketch) $\overset{o}{P}_i$ is entirely contained in one of the 2 faces in $\mathbb{R}^2 \setminus \{P_j \cup P_k\}$.
(1) follows from PJCT, too.
(2) P_2 separates one of the two regions defined by $P_1 \cup P_3$ into two parts.
Consider Pab stated. a is in one of these regions, the one bounded by $P_1 \cup P_2$ and b is in the one bounded by $P_2 \cup P_3$. Let c be the first point on P that is in both. Then $c \in P_2$. $\square$

Definition 4.25. A closed set X separates an open region O if $O \setminus X$ has more than 1 region.

4.2 Drawing graphs

Definition 4.26. A drawing of a graph $G = (V, E)$ is a function f that maps each $v \in V$ to $f(v) \in \mathbb{R}^2$. f maps each edge $e = uv \in E$ to $f(e)$, a polygonal arc, with ends $f(u)$ and $f(v)$.

Definition 4.27. A point in $f(e) \cap f(e')$ other than the common ends is a crossing.

Remark (Perturbation assumption for planar graph)**.**

- The interior of an edge contains no vertex and no point of any other edge.
- If 2 edges cross more than once, we can reduce the number of crossings.
- No pair of edges is parallel.

Definition 4.28. A graph is *planar* if it **has** a drawing with no crossings. Such a drawing is a plane embedding of G. A *plane graph* is a particular drawing of a planar graph with no crossing.

Definition 4.29. Let G be a planar and consider a **plane drawing** of G. The (open) regions of $\mathbb{R}^2 \setminus G$ are the faces of G.

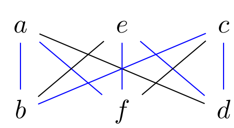
Remark (Fact)**.**

- If G is finite and so bounded, then we can constrcut a big desk containing all of G and so G has only one unbounded face.
- The faces of G are pairwise disjoint.
- The points p, q not on an edges of a plane graph are in the same face if and only if there exists a p-q arc crossing no edges of G.

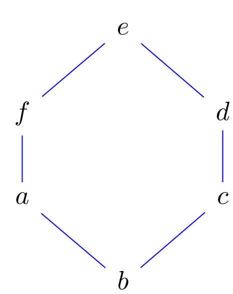
Definition 4.30. A chord of a cycle C is an edge e joining two vertices on C but with $e \notin C$.

Theorem 4.31. *Neither K^5 nor $K_{3,3}$ are planar.*

Proof. Consider a drawing of $G = K^5$ or $K_{3,3}$ in the plane.
Let C be a spanning cycle in $G = K_{3,3}$.



Then we can draw C as a polygon:



By PJCJ, $\mathbb{R}^2 - C$ has exactly 2 faces.
Let e be a chord of C, then by definition, $\overset{o}{e}$ is entirely contained in one of these two faces.
We will say that two chords of C conflict if their endpoints on C occur in alternating order, for example, the chords fc and eb conflict.
Conflicting chords must be drawn in different faces. But $K_{3,3}$ has 3 pairwise conflicting chords and $\mathbb{R} \setminus C$ has only 2 faces, so $K_{3,3}$ cannot be drawn in the plane.
A similar argument holds for K_5. $\square$

The following Lemmas are used for proving Kuratowski's theorem.

Lemma 4.32. Let G be a planar graph and E be the edge set of a face F of G. Then there is an embedding in which F is the unbounded face.

Lemma 4.33. Every minimal nonplanar graph is 2-connected.

Proof. Let G be minimal nonplanar.
Suppose G were not connected, then one of the component would be a nonplanar, which is contradicted by the minimality and so G is connected.
Suppose v were a cut vertex and let $C_1, \dots, C_k$ be the components of $G - v$ with $k \geqslant 2$.
For $i = 1, \dots, k$, let H_i be the subgraph of G induced by $C_i \cup \{v\}$.
By the minimality of G, each H_i is planar for $i = 1, \dots, k$.
Squeeze each to fit an angle less than $\frac{360^o}{k}$ at v and merge.
But then G is planar, a contradition and so G is 2-connected. $\square$

Definition 4.34. A minimal nonplanar graph is a nonplanar graph for which every proper subgraph is planar.

Lemma 4.35. Let G be minimal nonplanar and has a separator S of size 2, say $S = \{x, y\}$.
Let C_1 be one component of $G - \{x, y\}$ and let $C_2 = G - \{x, y\} - C_1$.
Let G_i be the subgraph of G induced by $C_i \cup \{x, y\}$ for $i = 1, 2$.
Note

$$V(G_1) \cap V(G_2) = \{x, y\}.$$

Define for $i = 1, 2$,

$$H_i = G_i \cup xy.$$

Then at least one of H_1, H_2 is nonplanar, otherwise we could glue H_1 and H_2 at xy and remove xy to obtain a planar graph G, a contradiction.

Definition 4.36. A Kuratowski graph is a subdivision of K^5 or $K_{3,3}$.

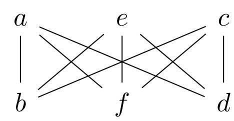
Lemma 4.37. A minimal nonplanar graph with no Kuratowski subgraph is 3-connected.

Proof. Assume G is minimal non-planar. Then G is 2-connected by previous Lemma.
Suppose G were not 3-connected.
Then by last Lemma, H_1 or H_2 defined in last Lemma is nonplanar, say H_1.
Since H_1 has fewer edges than G, H_1 must contain a Kuratowski subgraph.
Replace xy with an x-y path using only edges in H_2 and this gives a Kuratowski subgraph of G, a contradiction. $\square$

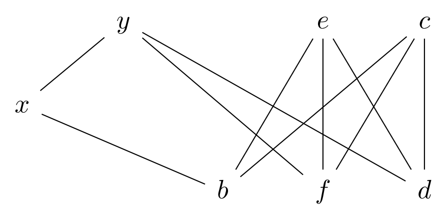
Lemma 4.38. A 3-connected graph with at least 5 vertices has an edge whose contraction leaves the graph 3-connected.

Lemma 4.39. If G/e has a Kuratowski subgraph, then G also does.

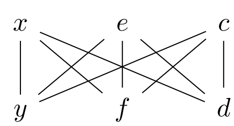
Proof. Let H be the Kuratowski subgraph of $G' = G/e$.
Let $e = xy$ and z be the vertex resulting from contracting the edge e.
Case 1: z is a nonbranching vertex of H. Uncontracted to get a Kuratowski subgraph of G, for example z is on ab.



Case 2: If when we uncontract z (inflation), at least one of the vertices $\{x, y\}$ has degree 2 in the subgraph of G induced by $(V(H) - z) \cup \{x, y\}$. Still, we have a Kuratowski subgraph after expanding z.



Case 3: x and y have degree greater than 2 in this same subgraph, i.e., $\deg_H(z) = 4$.



□

Remark. Sometimes, the contrapositive statement is more useful.

Theorem 4.40 (Kuratowski, 1930)**.** *G is planar if and only if G contains no subdivision of K^5 or $K_{3,3}$ (no Kuratowski subgraph).*

Proof. The goal is to show

(a) Show that a minimal nonplanar graph with no Kuratowski subgraph must be 3-connected.

(b) Prove that a 3-connected graph with no Kuratowski subgraph must in fact be planar.

□

Remark (Fact)**.**

- Subdividing edges does not affect planarity.
- Deletion and contraction preserve planarity.
- So it makes sense to seek minimal non-planar graphs with respect to these operations.

Theorem 4.41 (Wagner,1937)**.** *G is planar if and only if it has no subgraph contractible to K^5 or $K_{3,3}$.*

Remark (Fact)**.** A graph contains K^5 or $K_{3,3}$ as a minor if and only if it contains K^5 or $K_{3,3}$ as a topological minor.

Theorem 4.42 (Fary's Theorem, 1948)**.** *Every finite planar graph has an embedding in which all edges are straight line segments.*

Remark (Recall)**.** An embedding is a drawing of the graph in the plane.

Remark (Fact)**.** If each face boundary is convex, we say the representation is convex.

Definition 4.43. A set A is convex if for any $x, y \in A$ and$\forall 0 \leqslant \lambda \leqslant 1$,

$$(1-\lambda)x + \lambda y \in A.$$

Definition 4.44. A convex embedding of G is a planar embedding in which each inner face is convex.

Theorem 4.45 (Tutte, 1969, 1963)**.** *Every 3-connected planar graph has a convex embedding in the plane.*

Remark (Fact)**.** $K_{2,n}$ for $n \geqslant 4$ has no convex representation.

Theorem 4.46 (Tutte)**.** *If G is 3-connected with no Kuratowski subgraph, then G has a convex embedding in the plane.*

Proof. Induction on $|G|$.
If $|G| \leqslant 4$, then $G = K^4$ and K^4 has a convex embedding.
Assume $|G| \geqslant 5$. Assume the statement holds for all graphs with fewer vertices.
By previous Lemma, there exists $e = xy$ with G/e 3-connected.
Let z be the vertex resulting from contracting e.
Previous lemma implies that G/e has no Kuratowski subgraph.
So by inductive hypothesis, there exists a convex embedding of $G' = G/e$.
Consider removing all edges in G' incident with z.
The resulting graph has a face containing z.
A cycle of $G' - z$ bounds the face.
There exists straight line segments from z to each of its neighbors on C.
Some connect x to C. Some connect y to C.
Let $x_1, \ldots, x_k$ be the neighbors of x in order on C.
Case 1: All neighbors of y lie between x_i and x_{i+1} for some $1 \leqslant i \leqslant k-1$ or between x_1 and x_k.
Case 2: Consider subcases (2a) and (2b). We claim that both of these subcases allow us to conclude that we have a Kuratowski subgraph.

(2a) y shares 3 neighbors with x. Then we have a K^5 subdivision.

(2b) y has two neighbors u and v in C (breaking C into two segments) and x has two neighbors u' and v' that are in different segments of C.
Then we have a $K_{3,3}$ subdivision.

□

Remark (Interesting Fact)**.** excluded minors characterization for our planar graphs: K^4 and $K_{2,3}$.

Theorem 4.47.

$$2\|G\| = \sum_i l(F_i).$$

Lemma 4.48 (Euler's formula,1258)**.** If G is planar and connected with n vertices, m edges and l faces, then

$$n - m + l = 2.$$

Corollary 4.49. A simple 2-connected planar graph has at most $3|G| - 6$ edges.

Proof. Let G has n vertices, m edges and l faces.
Since G is simple and 2-connected, every face has length at least 3.
So

$$2m = \sum_i l(F_i) \geqslant 3l.$$

Also, by Euler's formula,

$$3m = 3n + 3l - 6 \leqslant 3n + 2m - 6.$$

So

$$m \leqslant 3n - 6.$$

□

Remark (Exercise)**.** Use this to show K^5 is not planar.

Corollary 4.50. Let G be a planar, simple and 2-connected. Then the average degree of G

$$d(G) = \frac{\sum_v d(v)}{n} = \frac{2\|G\|}{n} = \frac{2m}{n} \leqslant \frac{6n-12}{n} = 6 - \frac{12}{n} < 6.$$

We conclude that every simple 2-connected planar graph has a vertex of degree $\leqslant 5$.

Chapter 5

Coloring

Remark. How many colors do we need to color the countries of a map in such a way that adjacent countries are colored differently? How many days have to be scheduled for commitee meetings of a parliament if every committee intends to meet for one day and some members of parliament serve on serveral committees? How can we find a school timetable of minimum total length, based on the information of how often each teacher has to teach each class?

Definition 5.1. A (vertex) coloring of a graph is an assignment of colors to vertices. Specifically, let $G = (V, E)$. Let S be the set of colors and be finite. A vertex coloring of G is a map

$$c : V \to S.$$

A coloring is proper if $c(v) \neq c(u)$ when $v \sim u$.

Definition 5.2. An edge coloring of $G = (V, E)$ is map $c : E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f.

Remark. Clearly, every edge coloring of G is a vertex coloring of its line graph $L(G)$, and vice verce; in particular,

$$\chi'(G) = \chi(L(G)).$$

Remark. Often $S = \{1, \ldots, k\}$. If there is a coloring using only elements in $[k]$, we say G is k-colorable and the associated coloring is a k-coloring .

Definition 5.3. Let $\chi(G)$ be the chromatic number of G, which is the smallest integer k so that G is k-colorable.

Definition 5.4. Let $\chi'(G)$ be the edge chromatic number of G or called chromatic index of G, which is the smallest integer k so that G is k-edge-colorable.

Remark. If $\chi(G) \leqslant k$, we say G is k-colorable. If $\chi(G) = k$, we say G is k-chromatic.

Remark. Note that a k-coloring is nothing but a vertex partition into k independent sets, now called color classes. The non-trivial 2-colorable graphs, for example, are precisely the bipartite graphs.

Remark. How many colors are needed to color the regions of a planar graph? Equivalent to the vertex coloring problem of the dual. Find $\chi(G^*)$.

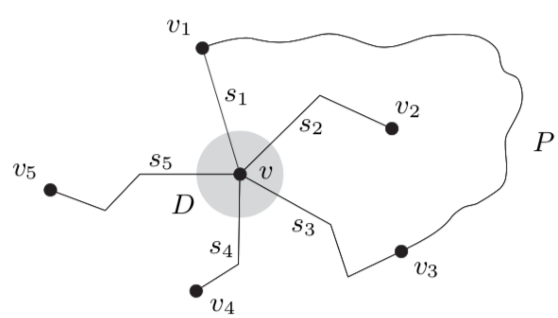
Theorem 5.5 (4 color theorem)**.** *For any planar graph G, $\chi(G^*) = 4$.*

Proof.

- 1976, Appel, Haken
- 1997 Robertson, Sanders, Seymour, Thomas
- 1879 Kempe
- Kempe's ideal helped prove a weaker theorem, Heawood 1890. □

Theorem 5.6. *For any planar graph G, $\chi(G) \leqslant 5$, i.e., every planar graph is 5-colorable.*

Proof. Let G be planar graph. Use induction on $|G|$.
If $|G| \leqslant 5$, done.
Let $n = |G| \geqslant 6$ and $m = \|G\|$.
Assume that any planar graph with less than n vertices is 5-colorable.
Let v be a vertex with $d(v) \leqslant 5$ and $H := G - v$.
By inductive hypothesis, H has a coloring $c : V(H) \to \{1, 2, , 3, 4, 5\}$.
If c uses at most 4 colors for the neighbors of v, we can extend it to a 5-coloring for the neighbors of v and done
Assume, therefore, that v has exactly 5 neighbors $\{v_1, \ldots, v_5\}$ and let $c(v_i) = i$ for $i = 1, \ldots, 5$. Let D be an open disc around c, so small that it meets only those five straight edge segments of G that contain v.
Let us enumerate these segments according to their cyclic position in D as $s_1, \ldots, s_5$.
Let vv_i be the edge containing s_i for $i = 1, \ldots, 5$.



We first show every v_1-v_3 path $P \subseteq H - \{v_2, v_4\}$ separates v_2 from v_4 in H.
Clearly, this is the case if and only if the cycle $C := vv_1Pv_3v$ separates v_2 from v_4 in G.
We prove this by showing that v_2 and v_4 lie in different faces of C.
Let x_2 be an inner point of s_2 in D and x_4 be an inner point of s_4 in D.
Then in $D \setminus (s_1 \cup s_3) \subseteq \mathbb{R}^2 \setminus C$, every point can be linked by a polygonal arc to x_2 or to x_4.
This implies x_2 and x_4 (and hence also v_2 and v_4) lie in different faces of C, otherwise, D would meet only one of the two faces of C, which would contradict the fact that v lies on the frontier of

both these faces since by Jordan Curve Theorem for Polygons, any neighbor sets of a point in the boundary will meet two faces of a polygon.
Let H_{ij} be the subgraph of H induced by vertices colored i or j for $i, j \in \{1, 2, 3, 4, 5\}$.
We may assume that the component C_1 containing v_1 of $H_{1,3}$ also contains v_3. Indeed, if we interchange the colors 1 and 3 at all the vertices of C_1, we obtain another 5-coloring of H; if $v_3 \notin C_1$, then v_1 and v_3 are both colored 3 in this new coloring, and we may assign remaining color 1 to v and done.
So H_{13} contains a v_1-v_3 path $P \in H_{13}$.
As shown above, P separates v_2 from v_4 in H.
Since $P \cap H_{2,4} = \emptyset$, v_2 and v_4 lie in different components of $H_{2,4}$.
In the component containing v_2, we now interchange the colors 2 and 4, thus recoloring v_2 with color 4.
Now v no longer has a neighbor colored 2 and we may give it this color. □

Theorem 5.7. *Every graph G with m edges satisfies* $\chi(G) \leqslant 1/2 + \sqrt{2m + 1/4}$

Proof. Let c be a vertex coloring of G with $k = \chi(G)$ colors.
Then G has at least one edge between any two color classes: if not, we could have used the same color for both classes.
So letting $m = \|G\|$, we have $m \geqslant \binom{k}{2} = \frac{k(k-1)}{2}$, i.e., $2m \geqslant k(k-1) = (k-1/2)^2 - 1/4$, i.e., $k \leqslant 1/2 + \sqrt{2m + 1/4}$. □

Theorem 5.8 (Another easy bound)**.**

$$\chi(G) \leqslant \Delta + 1,$$

where $\Delta = \Delta(G) = \max_{v \in V(G)} d(v)$.

Proof. We can establish this bound algorithmically.
Greedy method: list the vertices of G in any order $v_1, \ldots, v_n$.
Color v_1 with 1 and at step i, color v_i with the smallest color (positive integer) not used so far by any neighbor of v_i among $v_1, \ldots, v_{i-1}$.
In this way, we never use more than $\Delta(G) + 1$ colors. □

Remark. Can we do better? and how can we make our algorithm better with the same idea? Consider C_n with n odd and for any n,

$$\Delta(K_n) = n - 1.$$

When we come to color the vertex v_i in the above algorithm, we only need a supply of $d_{G[v_1,\ldots,v_i]}(v_i)+$ 1 rather then $d_G(v_i)$ colors to proceed and the algorithm ignores any neighbors v_j of v_i with $j > i$. Hence in most graphs, there will be scope for an improvement of the $\Delta + 1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbors are ignored) and vertices of small degree last.

Definition 5.9. The last number k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbors is called the coloring number $\mathrm{col}(G)$ of G.

Proposition 5.10.

$$\mathrm{col}(G) = \max_{H \subseteq G} \delta(H) + 1.$$

Proof. The enumeration we just discussed shows that $\mathrm{col}(G) \leqslant \max_{H\subseteq G} \delta(H) + 1$.
But for $H \subseteq G$, clearly $\mathrm{col}(G) \geqslant \mathrm{col}(H) \geqslant \delta(H) + 1$. □

Theorem 5.11. *Every graph satisfies*

$$\chi(G) \leqslant 1 + \max_{H\subseteq G}\{\delta(H)\} = \mathrm{col}(G).$$

Proof. Since the 'back-degree' of the last vertex in any enumeration of H is just its ordinary degree in H, which is at least $\delta(H)$. □

Remark. It is tight for G not regular.

Corollary 5.12. Every k-chromatic graph G has a k-chromatic subgraph with minimum degree at least $\chi(G) - 1$.

Proof. Given G with $\chi(G) = k$, let $H \subseteq G$ be minimal with $\chi(H) = k$. If H had a vertex v of degree $d_H(v) \leqslant k - 2$, we could extend a $(k-1)$-coloring of $H - v$ to one of H, contradicting the choice of H. □

Remark. What can we say when G is regular?
If $G = C_n$ with n odd or K^n for any $n \in \mathbb{N}$, then

$$\chi(G) = \Delta + 1.$$

Remark. For G connected and not regular, $\chi(G) \leqslant \Delta$.

Theorem 5.13 (Brooks 1941)**.** *If G is connected and neither an odd cycle or not a complete graph, then*

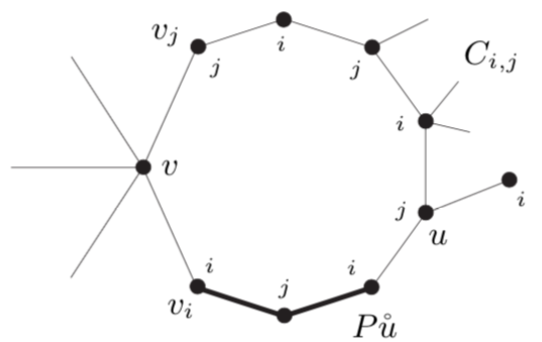
$$\chi(G) \leqslant \Delta.$$

Proof. Induction on $|G|$.
If $\Delta(G) \leqslant 2$, then G is a path or a cycle, and the assertion is trivial.
Assume $\Delta(G) \geqslant 3$ and that the assertion holds for graphs of smaller order.
Suppose $\chi(G) > \Delta(G)$.
Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leqslant \Delta(G)$.
Also, every component H' of H satisfies $\chi(H') \leqslant \Delta(H') \leqslant \Delta(G)$ unless H' is complete or an odd cycle, in which case since every vertex of H' has maximum degree in H' and one such vertex is also adjacent to v in G, we have

$$\chi(H') = \Delta(H') + 1 \leqslant \Delta(G).$$

Since H can be $\Delta(G)$-colored but G connot, we have the following:
Every $\Delta(G)$-coloring of H uses all the colors $1, \ldots, \Delta$ on the neighbors of v; in particular, $d(v) = \Delta(G)$. (1)
Given any Δ-coloring of H, let us denote the neighbor of v colored i by v_i for any $i = 1, \ldots, \Delta$.
For all $i \neq j$, let $H_{i,j}$ denote the subgraph of H spanned by all the vertices colored i or j.
For all $i \neq j$, the vertices v_i and v_j lie in a common component $C_{i,j}$ of $H_{i,j}$. (2)
Otherwise we could interchange the colors i and j in one of those components; then v_i and v_j would be colored the same, contrary to (1).
$C_{i,j}$ is always a v_i-v_j path. (3)
Indeed, let P be a v_i-v_j path in $C_{i,j}$. Since $\Delta(H') + 1 \leqslant \Delta(G)$, $d_H(v_i) \leqslant \Delta - 1$ and then the

neighbors of v_i have pairwise different colors: otherwise we could recolor v_i (interchange the color i and the color of its neighbor at all vertices of H), contrary to (1). Hence the neighbor of v_i on $P \in C_{i,j}$ is its only neighbor in $C_{i,j}$, and similarly for v_j. Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically colored neighbors in H; let u (clearly not v_i or v_j) be the first such vertex on P. Since at least 3 neighbors of u have the same color, at most $\Delta(G) - 2$ colors are used on the neighbors of u and so we may recolor u. But this makes $P\overset{o}{u}$ into a component of $H_{i,j}$, contradicting (2).



For distinct i, j, k, the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i. (4)
For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbors colored j and two colored k, so we may recolor u. In the new coloring, v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).
The proof of the theorem now follows easily. If the neighbors of v are pairwise adjacent, then each has $\Delta(G)$ neighbors in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta(G)}$. As G is complete, there is nothing to show. We may thus assume that $v_1v_2 \notin G$, where $v_1, \dots, v_{\Delta(G)}$ derive their names from fixed Δ-coloring c of H. Let $u \neq v_2$ be the neighbor of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colors 1 and 3 in $C_{1,3}$, we obtain a new coloring c' of H; let $v_i', H_{i,j}', C_{i,j}'$ etc. be defined with respect to c' in the obvious way. As a neighbor of $v_1 = v_3'$, our vertex u new lies in $C_{2,3}'$, since $c'(u) = c(u) = 2$. By (4) for c, however, the path $\overset{o}{v}_1C_{1,2}$ retained its original coloring, so $u \in \overset{o}{v}_1C_{1,2} \subseteq C_{1,2}'$. Hence $u \in C_{2,3}' \cap C_{1,2}'$, contradicting (4) for c'. $\square$

Theorem 5.14 (Erdös 1959, 1961)**.** *For every positive integer k, there exists a graph G having girth $g(G) > k$ and chromatic number $\chi(G) > k$.*

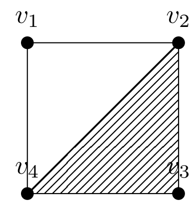
Definition 5.15. A k-chromatic graph G is critically k-chromatic or k-critical if $\chi(G - v) < k$ for every $v \in V(G)$. (Obviously, $\chi(G - v) = k - 1$ for any $v \in V(G)$.)

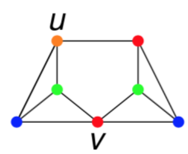
Theorem 5.16. *Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Let C_1 and C_2 be the components of $G - \{u, v\}$. For $i = 1, 2$, let $G_i = G[V(C_i) + \{u, v\}]$. Then*

(a) $G = G_1 \cup G_2$, and G_1 and G_2 are $(k-1)$-colorable.

(b) One of G_1 and G_2, say G_1 has $c(u) = c(v)$ in all $k-1$-colorings. G_2 has $c(u) \neq c(v)$ in all $k-1$-coloring.

(c) $H_1 := G_1 + e$, where $e = uv$ and $H_2 := (G_2 + e)/e$ are each k-critical.





type 1 type 2

Proof. (1)(2) Clearly, Each component of $G - \{u, v\}$ is $k-1$-colorable.
In fact, $\chi(G_1) = \chi(G_2) = k - 1$. (Hint: glue).
If there exists a $k-1$-coloring of G_1 and G_2 where the colors of u and v agree, then glue to get a $k - 1$ coloring of G, a contradiction.

(3) Adding uv forces chromatic number of G_1 up by 1 and similarly for G_2.
Exer: show that the result is k-critical.

□

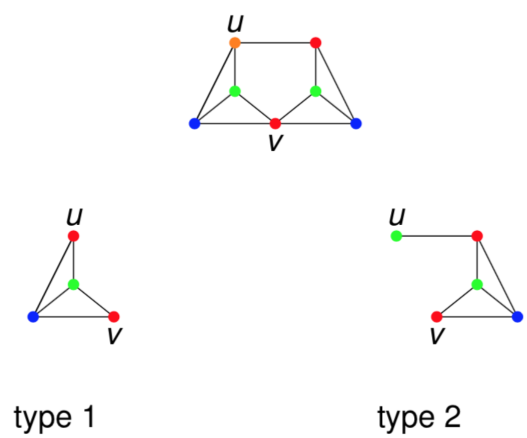
Proposition 5.17. If G is k-critical, then G does not contain a cut set consisting of pairwise adjacent vertices.

Proof. Let S be a cut set. Let $H_1, \ldots, H_t$ be the components of $G - S$.
Since each $H_i \cup S$ is a proper subgraph, $H_i \cup S$ is $(k-1)$-colorable.
Suppose S is a clique, then one can permute the colors such that G is $(k-1)$-colorable, a contradiction.
So S is not a clique. □

Theorem 5.18 (Dirac.)**.** *Every graph G with $\chi(G) \geqslant 4$ contains a K^4-subdivision.*

Proof. Induction on $n = |G|$.
If $n = 4$, then $G = K^4$.
Assume $n > 4$ with $\chi(G) \geqslant 4$ and we can let H be a 4-critical subgraph of G.
By previous proposition, H has no cut vertex.

Case 1: $\kappa(H) = 2$. Let $\{x, y\}$ be a cut vertex.
$x \sim y$, let G_1 and G_2 be as in the lemma.
By previous lemma, $\chi(G_1 + xy) = 4$.
By induction, $H_1 = H_1 + xy$ has a K^4-subdivision.



If necessary, remove xy from this subdivion and replace it with any x-y path in G_2.
Case 2: H is 3-connected. Select a vertex $x \in V(G)$.
Since $H - x$ is 2-connected, it has a cycle C of length at least 3.
By the Fan version of Menger's theorem, there exists an $x, V(C)$-Fan of size 3 in H.
So we have our K^4 subdivision. $\square$

Remark (Question)**.** Can the invarient x have a direct structural effect on a graph in terms of forcing a specific substructure?
Hadwiger 1943, Famous conjecture: For every $r \in \mathbb{Z}^+$,

$$\chi(G) \geqslant r \Longrightarrow G \geqslant K^r,$$

i.e., every graph G with $\chi(G) \geqslant 5$ has a K^5 minor.
$r = 4$:
$r = 5$:
$r = 6$:
$r \geqslant 7$:

5.1 k-edge coloring

Theorem 5.19 (kónig 1916)**.** *For a bipartite graph G, $\chi'(G) = \Delta(G)$.*

Proof. Induction on $m = \|G\|$.
If $\|G\| = 0$. Done.
Assume $\|G\| \geqslant 1$ and the assertion holds for graphs with fewer edges.

Let $\Delta := \Delta(G)$.
Pick $xy \in E(G)$, by inductive hypothesis, there exists a coloring of the edge of $G - \{xy\}$ using the colors $\{1, \ldots, \Delta\}$.
In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges.
So there exists $\alpha, \beta \in \{1, \ldots, \Delta\}$ such that no edge in $N(x)$ is colored α and no edge in $N(y)$ is colored β.
If $\alpha = \beta$, we can color the edge xy with this color and are done, so assume $\alpha \neq \beta$.
In fact, y is incident with an α edge and x is incident with a β edge.
Let us extend this edge to a maximum walk W from x whose edges are colored β and α alternatively. Since no such walk contains a vertex twice, W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G. We now recolor all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent of $G - xy$ are still colored differently. We have thus found a Δ-edge-coloring of $G - xy$ in which neither x nor y is incident with a β-edge. Coloring xy with β, we extend this coloring to a Δ-edge-coloring of G. □

Remark. If G is an odd cycle, it needs $\Delta + 1$ colors, so $\chi'(G) = \Delta + 1$.

Theorem 5.20 (Vizing 1964)**.** *Every simple graph G satisfies $\Delta \leqslant \chi'(G) \leqslant \Delta + 1$.*

Proof. Induction on $\|G\|$.
If $\|G\| = 0$. Done.
Let $\Delta := \Delta(G) > 0$ and assume the assertion is true for all graphs with fewer edges.
Instead of '$(\Delta + 1)$-edge-coloring' let us just say 'coloring'.
Suppose there is not $\Delta + 1$ coloring of G.
Let $e = xy$ and color $G - xy$ with $\{0, 1, \ldots, \Delta\}$.
A color is missing at x, wlog., let this missing color be 0.
There exists a missing color at y. Not 0, call it 1, this is a 1 edge at x, let xy be colored 1.
Something missing at y.
If this ", color is 0, else down-shifting, i.e., coloring xy_1 with 0 and xy_0 with 1.
So the missing color is neither 0 nor 1, wlog., let it be 2.
x is incident with a 2-edge, (else recolor xy with 2 and 'downshift' coloring xy_0 with 1).
Continue in this way.
But we have only $\Delta + 1$ colors.
At some point, the missing color has already been used-let k be the smallest index where this happens.
y_k is missing 0, (else coloring $y_k x$ with 0 and down-shift from y_k.)
Let p_i =maximal and path of edges using 0 and i.
Case 1: p reaches y_i along a 0 edge. Then continues to x and stops. Down-shift from y and switch on P and coloring $y_i x$ with 0.
Case 2: p doesn't reach y_i but dows reach y_{i-1}.
So stop at $y_i - 1$ since no i at y_{i-1}.
Downshift from y_{i-1}, switch on P and color xy_{i-1} .
Case 3: P reaches neither y_i nor y_{i-1}.
So then P also avoids x (P can only arrive at x via i through y.)
Now down-shift from y_k, then switch on P and color xy_k with 0. □

Definition 5.21. A lattice square is an $n \times n$ array with n different symbols such that no row or column has 2 of the same symbols.

Example 5.22. Consider lattice squares

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}.$$

Then AB is also a lattice square.

5.2 List coloring

Definition 5.23. Suppose we are given a graph $G = (V, E)$, and for each vertex of G, a list of colors permitted at that particular vertex: when can we color G so that each vertex receives a color from its list? More formally, let $(S_v)_{v \in V}$ be a family of sets. We call a vertex coloring c of G with $c(v) \in S_v$ for all $v \in V$ a coloring from the lists S_v. The graph G is called k-list-colorable, or k-choosable, if for every family $(S_v)_{v \in V}$ with $|S_v| = k$ for all v, there is a vertex coloring of G from the lists S_v. The least integer k for which G is k-choosable is the list-chromatic number or choice number $\mathrm{ch}(G)$ of G.

Definition 5.24. The least integer k such that G has an edge coloring from any family of lists of size k is the list-chromatic index $\mathrm{ch}'(G)$ of G; formally, we just set $\mathrm{ch}'(G) := \mathrm{ch}(L(G))$.

Theorem 5.25 (Dinitz Conjecture,1979)**.** *Given an $n \times n$ square array and n^2 arbitrary sets A_{ij} with $1 \leqslant i, j \leqslant n$ and $|A_{ij}| = n$, it is aways possible to pick $a_{ij} \in A_{ij}$ such that each row and each column has all n vertices distinct.*

Remark. Given $G = (V, E)$ and put a set of allowable colors S_v on each vertex v, can we properly color $V(G)$ so that every vertex gets a color from its list?

Lemma 5.26.

$$\mathrm{ch}(G) \geqslant \chi(G).$$

Lemma 5.27. $\mathrm{ch}(G) \geqslant \chi(G)$.

Remark. Nobody knows a case where $\mathrm{ch}'(G) > \chi'(G)$.

Example 5.28. $L(K_{3,3,})$

$$\begin{matrix} 11' & 12' & 13' \\ \\ 21' & 22' & 23' \\ \\ 31' & 32' & 33' \end{matrix}$$

Theorem 5.29 (List coloring conjecture)**.** $\mathrm{ch}'(G) = \chi'(G)$ *for all G.*

Remark. Dinitz problem can be seen as a special case of LCC.
Same graph as above.
Every ceil has set A_{ij}.
Define G: Let $V(G)$ be the cells (n^2 of them).
$(i,j) \sim (i,j')$ for all $j' \neq j$.
$(i,j) \sim (i',j)$ for all $i' \neq i$.
We want to show that G is n-choosable.
Note: G is the line graph of $K_{n,n}$.
So Dinitz conjecture $\iff \mathrm{ch}'(K_{n,n}) = n$.
Also, recall that

$$\chi'(G) = \Delta(K_{n,n}) = n.$$

So Dinitz conjecture $\iff$ LCC for $K_{n,n}$.
F.Galvin 94, LCC holds for all bipartite graphs, $\mathrm{ch}(L(G)) = \chi(L(G))$ for all G.
An orientation of a graph means we put a direction on each edge. ij: $i \to j$.

Definition 5.30.

$$N^+(v) = \{w \in V(G) : v \to w\}.$$

$$d^+(v) = \left|N^+(v)\right|.$$

Definition 5.31. An independent set $U \subseteq V(D)$ is a kernel of D if for every $v \in D - U$, there is a $w \in U$ so that $v \to w$.

Definition 5.32 (Property X)**.** D has this property, for every non-empty induced subgraph D' of D, D has a kernel.

Lemma 5.33. Let H be a graph and let $\{S_v\}$ be a collection of sets. If H has an orientation D so that

(a) $|S_v| \geqslant d^+(v)$ for any v;

(b) D has property X.

Then H can be colored from the lists S_v.

Proof. Induction on $|H|$.
If $|H| = 0$, no color needed.
Induction step: let H be a graph with orientation D as stated.
Pick any color α, □

Definition 5.34. Let $U \subseteq D$, if for any $v \notin D - U$, there exists $w \in U$ with $v \to w$, then U is a kernel of D.

Lemma 5.35. Let H be a graph and $\{S_v\}_{v \in V}$ be a collection of sets. If H has a orientation D with

(a) $|S_v| \geqslant d^+(v)$ for any $v \in V$.

(b) Every nonempty induced subgraph D' at D has a kernel.

Then H can be colored from the lists (sets) $\{S_v\}_{v\in V}$.

Theorem 5.36 (Galvin 94)**.** *LCC holds for all bipartite graph.*
List chromatic conjecture: $\mathrm{ch}'(G) = \chi'(G)$ *fo any* G.

Proof. Let G be bipartite with bipartition $\{x, y\}$ and let $\chi'(G) = k$.
We know that $\mathrm{ch}'(G) \geqslant k$. We will show $\mathrm{ch}'(G) \leqslant k$, i.e., we will show $L(G)$ is k-colorable.
Let c be a k-edge coloring of G with $c : E(G) \to [k]$.
We need an orientation D of the line graph of G satisfying

(a) $d^+(e) \leqslant k$ for any $e \in E(G)$.

(b) Every nonempty induced subgraph of D has a kernel.

Define D as follows. If e and e' meet at X and $c(e) < c(e')$, then $e' \to e$. If e and e' meet at Y and $c(e) < c(e')$, then $e \to e'$.
Let $c(e) = i$. For every $e' \in N^+(e)$ meeting e in X, $c(e') \in \{1, \ldots, i-1\}$ and for every $e' \in N^+(e)$ meeting e in Y, $c(e') \in \{i+1, \cdots .k\}$.
None of these can be the same. $d^+(v) = |N^+(v)| \leqslant k - 1 < k$.
Let D' be a nonempty induced subgraph of D.
Interpret direction in D as a preference. $e <_v e'$ if $e \to e'$.
Let M be a stable matching in the graph $(X \cup Y, V(D'))$, then for every edge $e \in E(D') \setminus M$, there exists $f \in M$ such that they have a common vertex with $e <_v f$, i.e., for which $e \to f$, i.e, M is an required kernel. ☐

Example 5.37. Graph.
Let $e \in E(G)$. Compute $d^+(e)$.

Chapter 6

Hamilton Cycles

Definition 6.1. When does a graph G contain a closed walk that contains every vertex of G exactly once? If $|G| \geqslant 3$, then any such walk is a cycle: a Hamilton cycle of G.

Definition 6.2. A Hamilton path in G is a path in G containing every vertex of G.

Definition 6.3. If G has a Hamilton cycle, it is called Hamiltonian. If G has a Hamilton path, it is called traceable.

Definition 6.4. Define the number of component of H as $c(H)$.

Remark. Look for some sufficient and necessary conditions.
Easy necessary conditions: $\delta(G) \geqslant 2$. If $G = K_{m,n}$, then $m = n$.
A necessary condition for Hamiltonicity is $c(G - S) \leqslant |S|$ for every separator S.

Remark. Consider this example. Not Hamiltonion.
Hint: remove the white vertex, then we left with 4 components.

Definition 6.5. A graph G is tough if $c(G - S) \leqslant |S|$ for every separator S.

Definition 6.6. For $t \in \mathbb{R}^{>0}$, G is t-tough if $c(G - S) \leqslant \frac{|S|}{t}$ for every separator S.

Remark (Conjecture 1973)**.** There exists $t \in \mathbb{Z}^{+}$ so that every t-tough graph is Hamiltonion.

Theorem 6.7 (Dirac 1952)**.** *Every graph with $n \geqslant 3$ vertices and $\delta(G) \geqslant n/2$ is Hamiltonion.*

Lemma 6.8. Let $G = (V, E)$ be simple. Let $u, v \in V$ and $u \nsim v$. If $d(u) + d(v) \geqslant n$, then G is Hamiltonion if and only if $G + uv$ is Hamiltonion.

Theorem 6.9. *Let $G = (V, E)$ be simple. Let $u, v \in V$. If $d(u) + d(v) \geqslant n$ for all $u \nsim v$, then G is Hamiltonion.*

Theorem 6.10 (Bondy and Chóatal 1970)**.** *A simple graph is Hamiltonion if and only if its closure is Hamiltonion.*

Theorem 6.11. *Every graph G with $|G| \geqslant 3$ and $\alpha(G) \leqslant \kappa(G)$ has a Hamilton cycle.*

Chapter 7

Extremal Graph Theory

How many edges can G of order n have and be triangle free?

Theorem 7.1 (Mantel 1907)**.** *The maximal number of edges a simple triangle free graph G can have is $\left\lfloor \frac{n^2}{4} \right\rfloor$, where $n = |G|$.*

Proof. Idea:

(a) Show a simple triangle free graph G has $\|G\| \leqslant \left\lfloor \frac{n^2}{4} \right\rfloor$, where $|G| = n$.

(b) Exhibit a triangle free graph G with $\|G\| = \left\lfloor \frac{n^2}{4} \right\rfloor$.

(a) Let G be a simple and triangle free.
Let $\Delta(G) = k$. Pick u with $\deg_G(u) = k$.
graph:
Since G is triangle free, $N(u)$ is an independent set.
So every edge is incident with at least one vertex in $V(G) - N(u)$
Hence

$$\|G\| \leqslant |G - N(u)| \cdot k = (n-k)k.$$

Therefore, $\|G\| \leqslant \max_k (n-k)k$, where the equality is attained for $n = 2k$ and $n(n-k) = \frac{n^2}{4}$.

(b) The graph we need is $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$.

□

Remark (Bipartite $K_{n,m}$)**.** Multipartite graphs k-partite graph.
We denote a complete k-partite graph by $K_{n_1,\dots,n_k}$, where n_i is cardinality of the i^{th} part.
All edges between distinct parts,

$$K_r^l = K_{r,\dots,r},$$

where the number of r's is l.

Definition 7.2. The Turan graph $T^r(n)$ is the unique n-vertex, complete r-partite simple graph whose partite sets dif and only ifer in cardinality by at most 1.

Example 7.3.

$$T^3(8) = K_{3,3,2}.$$

Proposition 7.4. Let $n, r \in \mathbb{N}$ and $n \geqslant r$ and choose l and $0 \leqslant j < r$ so that $n = rl + j$. Then the Turan graph $T^r(n)$ is defined as follows.

$$T^r(n) = K_{l,\ldots,l,l+1,\ldots,l+1},$$

where there are j $l+1$ and $r-j$ l.

Definition 7.5.

$$T^1(n) = \overline{K}_n,$$

which are n isolates.

Remark (Question)**.** Given n, r, can we find an r-partite graph having more edges than $T^r(n)$?

Lemma 7.6. Among all n-vertex simple r-partite graphs, $T^r(n)$ has the maximum number of edges.

Proof. Say G is r-partite with $|G| = n$ and $\|G\| \geqslant \|T^r(n)\|$.
Then there are parts L and S with $|L| - |S| \geqslant 2$.
Pick $v \in L$ and move it to S. Then the number of edges changes by $|L| - |S| - 1 \geqslant 1$. □

Remark. Denote

$$\|T^r(n)\| = \mathrm{t}_r(n).$$

Remark. Note that $T^2(n)$ is K^3 free.
In general, $T^{r-1}(n)$ is K^r free.
Each complete graph has at most 1 vertex in each part.

Remark. Is $\mathrm{t}_r(n)$ best possible? Is it the largest size of a graph of order n having no K^r subgraph? Is $T^{r-1}(n)$ the only such graph?
That is, what is the largest size for a graph G of order n with $G \not\supseteq K^r$.
More generally, let H be a graph with $|H| < n$. What is the largest size for a graph G on n vertices having $G \not\supseteq H$? Such a graph is called extremal for n and H. Its size is $\mathrm{ex}(n, H)$.

Remark (Question)**.** Is $\mathrm{ex}(n, K^r) = \mathrm{t}_{r-1}(n)$ and is $T^{r-1}(n)$ is the only graph that is extremal for n and K^r?

Theorem 7.7 (Turan 1941)**.** *For all integers r, n with $r > 1$, every $G \not\supseteq K^r$ with n vertices and $\mathrm{ex}(n, K^r)$ edges is $T^{r-1}(n)$.*

Proof. Let $G \not\supseteq K^r$ of order n.
We will construct an $r-1$ partite graph H with $V(H) = V(G)$ and show that $\|G\| \leqslant \|H\|$.
Then the result will follow from the lemma ($\|H\| \leqslant \mathrm{t}_{r-1}(n)$).
Induction on r.
$r = 2$. If $|G| = n$ and $G \not\supseteq K^2$, then $G = \overline{K}^n$.
So let $r \geqslant 3$.
Let $k = \Delta(G)$ and pick u with $d_G(u) = k$.
Let $G' = G[N_G(u)]$.

Since $G \not\supseteq K^r$, $G' \not\supseteq K^{r-1}$.
By induction, there exists an $(r-2)$-partite graph H' with $V(H') = N_G(u)$ and $\|G'\| \leqslant \|H'\|$.
Construct H as follows.
$\vdots$ $\square$

Remark. Uniquely so, $\left\|T^{r-1}(n)\right\| = \mathrm{t}_{r-1}(n)$. This generates Mantel's Theorem.

Definition 7.8. For a graph H with $|H| \leqslant n$, $\mathrm{ex}(n, H)$ is the largest number of edges of a graph G of order n, can have and still not contain a subgraph H. Such a graph G is called extremal in n and H.

Definition 7.9. Let $|G| = n$. Let density of a graph G be $\frac{\|G\|}{\binom{n}{2}}$, where $n = |G|$. If $\|G\|$ is of order n^2, then G is dense. Otherwise, G is sparse.

Remark. Turan graphs are dense. Specifically,

$$\mathrm{t}_{r-1}(n) \leqslant \frac{1}{2}n^2\frac{r-2}{r-1},$$

with equality when $r - 1 \mid n$.

Proof. Hint: choose k and i so that $n = (r-1)k + i$, where $0 \leqslant i < r - 1$.
When $i = 0$, the number of edges in $T^{r-1}(n)$ is

$$\binom{r-1}{2}k^2 = \frac{(r-1)(r-2)}{2}\frac{n^2}{(r-1)^2} = \frac{1}{2}n^2\frac{r-2}{r-1}.$$

For $i \neq 0$, show that

$$\mathrm{t}_{r-1}(n) = \frac{1}{2}\frac{r-2}{r-1}(n^2 - i^2) + \binom{i}{2} < \frac{1}{2}n^2\frac{r-2}{r-1}.$$

$\square$

Remark. What happens when we add edges to $T^{r-1}(n)$?
Surprising answer: Just a few more edges not only forces a K^r but forces many copies of K^r in the form of a subgraph $K^r_s = K_{s,\ldots,s}$ for some s.
Any set of vertices with exactly one vertex in each part induces a K^r.
Specifically: fix $\epsilon \in \mathbb{R}^r$, fix $s \in \mathbb{Z}^+$, then there exists r_0 so that for any $n \geqslant n_0$, adding ϵn^2 edges to $T^{r-1}(n)$ forces a K^r_s.

Theorem 7.10 (Erdős Stone)**.** *For all $r \geqslant 2$ and $s \geqslant 1$ and every $\epsilon \in \mathbb{R}^+$, there exists an integer n_0 so that every graph with $n \geqslant n_0$ vertices and at least $\mathrm{t}_{r-1})(n) = \epsilon n^2$ edges contains K^r_s as a subgraph.*

Definition 7.11. Given a graph H with $|H| \leqslant n$, $h_n = \frac{\mathrm{ex}(n,H)}{\binom{n}{2}}$, a critical number. This is maximum edge density that an n-vertex graph can have without containing H as a subgraph.

Remark. What happens to this critical number as $n \to \infty$. It converges to a number that depends only on $\chi(H)$.

Lemma 7.12.

$$\lim_{n\to\infty} \frac{\mathrm{t}_{r-1}(n)}{\binom{n}{2}} = \frac{r-2}{r-1}.$$

Corollary 7.13. For every graph H with at least one edge,

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Proof. Let H be a graph with at least one edge.
Let $r := \chi(H)$.

- Note $H \nsubseteq T^{r-1}(n)$ for any $n \in \mathbb{N}$. Otherwise, H would be $(r-1)$-colorable.
  Since $T^{r-1}(n)$ has no H-subgraph, $\mathrm{t}_{r-1}(n) \leqslant \mathrm{ex}(n,H)$
- Note $H \subseteq K_s^r$ for sufficiently large s. So $\mathrm{ex}(n,H) \leqslant \mathrm{ex}(n,K_s^r)$ for sufficiently large s.
- Fix such an s. By Erdós Stone, $\mathrm{ex}(n,K_s^r) < \mathrm{t}_{r-1}(n) + \epsilon n^2$ for n big enough.
  So

$$\mathrm{t}_{r-1}(n)/\binom{n}{2} \leqslant \mathrm{ex}(n,H)/\binom{n}{2} \leqslant \mathrm{ex}(n,K_s^r)/\binom{n}{2} < \frac{\mathrm{t}_{r-1}(n)}{\binom{n}{2}} + \frac{\epsilon n^2}{\binom{n}{2}}$$
$$= \frac{\mathrm{t}_{r-1}(n)}{\binom{n}{2}} + \frac{2\epsilon n^2}{\binom{n}{2}} = \frac{\mathrm{t}_{r-1}(n)}{\binom{n}{2}} + .$$

□

Remark (Conjecture Hadwiger 1993)**.** For every $r \in \mathbb{N}$ and every graph G, if $\chi(G) = r$, then $G \geqslant K^r$.
$r = 1,2,3,4$ has been proved.

- $r = 1$: G contains a vertex.
- $r = 2$: G contains a edge.
- $r = 3$: G contains a cycle, which implies K^s minor.
- $r = 4$: need a few work.

Proposition 7.14. A graph G with $|G| \geqslant 3$ is edge-maximal with no K^4 minor if and only if it can be considered by recursively pasting triangles.
(Note any subgraph has $2|G|-3$ cycles.)

Proof. "$\Leftarrow$". Exercise (For $|G| > 3$).
"$\Rightarrow$". WTS if G is maximal with no K^4, then G is triangle-pasted.
Induction on $|G|$.
If $|G| = 3$, done.
Let $|G| \geqslant 4$ and G is maximal with no K^4 minor but not triangle-pasted.
If G is not complete, done.
Let S be a separator with $|S| = \kappa(G)$.

Case 1: $\kappa(G) \geqslant 3$.
Graph
There exists P_1, P_2, P_3, $G - \{v_1, v_2, v_3\}$ is connected.
There exists a shorstest path P connected two of P_1, P_2, P_3.
Graph. K^4 minor.
So $\kappa(G) \leqslant 2$.
Use fact K^4 minor $\cong TK^4$. (Lemma 4.4.4) $\square$

Corollary 7.15. Hadwiger holds for $r = 4$.
Graph.
$\chi(G) = \max(\chi(G_1), \chi(G_2))$.
Use induction on $|G|$ and Thm 7.3.1 to show all edge maximal graphs.

Chapter 8

Ramsey Theory for Graphs

Remark. We've see that $\mathrm{tr}(n)$ edges forces a K^r in G for $|G| = n$. What if we want to know how to force a K^r or a $\overline{K}^r$.

Theorem 8.1 (Ramsey 1930)**.** *For every $r \in N$, there exists $n \in \mathbb{N}$ so that if $|G| \geqslant n$, then G contains either K^r or $\overline{K}^r$ as a subgraph.*

Remark. Trivial for $r \leqslant 1$.
Let $n = 2^{2r-3}$ and $|G| = n$.
Define a sequence of subsets of $V(G)$ $V_1, \ldots, V_{2r-2}$ with $V_1 \supseteq V_2 \supseteq \cdots \supseteq V_{2r-2}$ and with $v_i \in V_i - V_{i-1}$ as follows: pick $V_1 \subseteq V(G)$ with $|V_1| = 2^{2r-3}$ and let $v_1 \in V_1$.
Let $A = N(v_1) \cap V_1$ and $B = (V_1 - \{v_1\}) - A$.
Then A or B contains at least 2^{2r-4} vertices.
Let V_2 be 2^{2r-4} of the vertices in that set.
So either $v_1 \sim w$ for any $w \in V_2$ or $w \not\sim w$ for any $w \in V_2$.
Pick v_2 arbitrary. Continue the process, $|V_3| = 2^{2r-5}$. Pick v_3.
So $V_i = 2^{2r-2-i}$ and v_{i-1} is either adjacent to all vertices in V_i or $v_i \not\sim w$ for any $w \in V_i$.
Among the vertices $v_1, \ldots, v_{2r-2}$, at least $r-1$ showed the same behavior when viewed as v_{i-1} when choosing V_i.
So this set of $r-1$ vertices together with the last one either induces a K^r or a $\overline{K}^r$.

Definition 8.2. Define $R(r)$ to be the least number n so that $|G| \geqslant n$ so that $G \supseteq K^r$ or $G \supseteq \overline{K}^r$. We showed that $R(r) \leqslant 2^{2r-3}$, can't say much more. We'll show that $R(r) \leqslant 2^{r/2}$ using probabilistic method.

Definition 8.3. Define $R(H_1, H_2)$ to be the least number n so that $|G| \geqslant n$ so that $G \supseteq H_1$ or $G \supseteq \overline{H}_2$.

Remark.

$$R(r) = R(K^r, K^r).$$

Remark. Trees-an exception-not so hard.

Theorem 8.4. *Let s, t be positive integers and let T be a tree of order s. Thenn $R(T, K^s) = (s-1)(t-1) + 1$.*

Proof. Prove part of this.
Consider the graph G build as the disjoint union of $s-1$ copies of K^{t-1}.
Then $G \not\supseteq T$.
graph.
$s-1$ of these because the largest component of G has order $t-1$.
$G \not\supseteq \overline{K}^s$ (if and only if $\overline{G} \not\subseteq K^s$) because the largest independent set of G has cardinality $s-1$.
So $R(T, K^s) > (s-1)(t-1)$.
To show $R(T, K^s) = (s-1)(t-1)+1$, consider a graph G containing no $\overline{K}^s$, then show that $G \not\supseteq T$.
Hint: consider a proper coloring with $\chi(G)$ colors. $\square$

Chapter 9

Random Graph

Remark. Intuitively, we build a random graph G on n vertices by performing an experiment for each possible edge e in G. Fix $0 < p < 1$, let $P(e \in E(G)) = p$ and $P(e \notin E(G)) = 1 - p$.

Remark. A latter model by Erdós-Renyi, $G(n, m)$. Think of this as a process. Start with $G_{n,0}$ with no edges. At step we add 1 more edge so that all possible new edges are equally likely.

$$G_{n,0} \subseteq G_{n,1} \subseteq \cdots \subseteq G_{r,\binom{r}{2}}.$$

What kind of questions can we answer?

(a) Deterministic question.

- What is a better bound on $R(r)$? $(2^{r/2})$.
- What is a bound on the number of crossings in a graph with $\|G\| \geqslant 4|G|$?

(b) Erdó-Renyi.
How big should m be to ensure $G_{n,m}$ is Hamiltonian? Same question because $\Delta(G_{n,m}) = 2$?

Theorem 9.1 (Erdós 1947)**.** *For every integer $k \geqslant 3$, $R(k) > 2^{k/2}$.*

Proof. For $k = 3$, the statement is $R(3) > 2^{3/2}$. $R(3) = 6 > 2^{3/2}$.
So let $k \geqslant 4$, let $k \leqslant 2^{k/2}$.
We will show there exists a graph of order n with no K^k or $\overline{K}^k$ subgraph.
Take a random graph on n vertices $G(n, p)$.
Let $p = 1/2$. $P(\alpha(G) \geqslant k)$ and $P(\omega(G) \geqslant k)$ are each since $1/k! < 1/2^k$,

$$\leqslant \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}} < \frac{n^k}{2^k} 2^{-\frac{1}{2}k(k-1)} \left(\frac{n(n-1)\cdots(n-k+1)}{k!} < \frac{n^k}{2^k}\right) \leqslant \frac{(2^{k/2})^k}{2} 2^{k(k-1)}$$
$$= 2^{k^2/2-k-k^2/+k/2} = 2^{-k/2} < 1/2.$$

So $P(\alpha(G) \geqslant k)$ or $P(\omega(G) \geqslant k) < 1/2 + 1/2 = 1$.
Then the probability that a graph $G(n, p)$ has either a K^k or $\overline{K}^k$ subgraph is less than 1.
So there exists a graph of order n having no K^k or $\overline{K}^k$ subgraph.
Thus, $R(k) > 2^{k/2}$. □

Remark (Backgraph)**.** We have

- Euler's formula: For planar graph, $n - m + l = 2$.
- For a planar graph, $m \leqslant 3n - 6$.
- We can embed. any graph in the plane so that each crossing point is incident with at least 2 edges.
- Linearity of expectation $E(X+Y) = E(X) + E(Y)$.
- From any graph G, we can construct a new graph H: Assume G is emdeded in plane. $V(H) = V(G)+$crossing points. $E(H) =$ all pieces of the original edges. $N(H) = n+\operatorname{cr}(G)$. $E(H) = m+2\operatorname{cr}(G)$. So $m+\operatorname{cr}(G) \leqslant 3(n+\operatorname{cr}(G)-6)$. Hence $\operatorname{cr}(G) \geqslant m-3n-6$. Thus, $\operatorname{cr}(G)-m-3n \geqslant 6 > 0$.

Theorem 9.2. *If G is simple with n vertices and m edges, where $m \geqslant 4n$, then* $\operatorname{cr}(G) \geqslant \frac{1}{64}\frac{m^3}{n^2}$.

Proof. Let $0 < p < 1$. Start with a graph G drawn in the plane with $\operatorname{cr}(G)$ crossings.
Generate G_p: Pick vertices independently with probability p and consider the resulting induced subgraph.
Let n_p be the number of G_p and m_p be the number of edges of G_p and X_p be the number of crossing points of G_p.
By previous result, $E(X_p - m_p + 3n_p) \geqslant 0$, $E(n_p) = pn$, $E(m_p) = p^2 m$ and $E(X_p) = p^4 \operatorname{cr}(G)$.
We get

$$0 \leqslant E(X_p) - E(m_p) + 3E(n_p),$$

i.e.,

$$0 \leqslant p^4 \operatorname{cr}(G) - p^2 n + 3pn,$$

i.e.,

$$\operatorname{cr}(G) \geqslant \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}.$$

Hence where we pick $p = \frac{4n}{m}$, plugging in it, we get

$$\operatorname{cr}(G) \geqslant \frac{1}{64}\frac{m^3}{n^2}.$$

□

9.1 Properties of almost all graphs