MATH 8510, Abstract Algebra I
Fall 2016
Exercises 5-2
Due date Thu 22 Sep 4:00PM

Exercise 1 (3.4.4)**.** Let G be a finite abelian group, and let n be a divisor of $|G|$. Use Cauchy's Theorem to show that G has a subgroup of order n.

Proof. (1) If $|G| = 1$, then $G = \{e_G\}$ and only 1 is a divisor of $|G|$.
So G with order 1 is a subgroup of G.

(2) Consider $1 < |G| < \infty$.
We will show it by induction.
Since $|G| \geq 1 and |G| \in \mathbb{N}$, $|G|$ can be written as $p_1p_2...p_{k_p}$, where $p_1, p_2, ...p_{k_p}$ are primes.

(a) If $|G| = p$ is a prime, then the divisor n can only be 1 or p.
So when $n = 1$, just let $H = \{e_G\} \leq G$; when $n = p$, we let $H = G \leq G$.
Namely, it holds for the basic case.

(b) Suppose every group G with $|G| = p_1, p_2, ...p_m$ has the property that if n is a divisor of $|G|$, then there exists a subgroup $H \leq G$ with $|H| = n$.

(c) Let G be a group such that $|G| = p_1, p_2, ...p_{k_p}p_{m+1}$.
If $n = 1$, just let $H = e_G \leq G$.
If $n > 1$, then withour loss of generality, we assume $p_1|n$.
By Cauchy's theorem, there exists $x \in G$ with $|x| = p_1$.
Since G is abelian, we have $\langle x\rangle$ is normal.
Then $\langle x\rangle \trianglelefteq G$ since $x \in G$.
Then $|G/\langle x\rangle| = \frac{|G|}{|\langle x\rangle|} = \frac{p_1,p_2,...p_m}{p_1} = p_2, ...p_{m+1}$.
Since n is a divisor of $|G| = p_1, p_2, ...p_{k_p}p_{m+1}$ and $p_1|d$, then $\frac{n}{p_1}$ is a divisor of $p_2, p_3, ...p_{m+1}$.
Thus, by assumption, there exists a subgroup $H/\langle x\rangle$ of $G/\langle x\rangle$ with order $|H/\langle x\rangle| = \frac{n}{p_1}$.
Besides, by the fourth isomophism theorem, we have $H \leq G$.
As last, by Lagrange's theorem, we have $|H| = |H/\langle x\rangle||\langle x\rangle| = \frac{n}{p_1}p_1 = n$.
Therefore, our assumption also holds for $|G| = p_1, p_2, ...p_{k_p}p_{m+1}$.

Hence, the claim holds.

□

Exercise 2. Let G be a finite abelian group, written additively. Prove that there is a chain of subgroups $0 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = G$ such that each quotient N_i/N_{i-1} is cyclic of prime order.

Proof. Since G is a finite group, then by Jordan-Hölder theorem, G has a composition series.
Namely, there is a chain of subgroups $0 = N_0 \trianglelefteq N_1 \trianglelefteq ... \trianglelefteq N_k = G$.
Then we show $N_n/N_{n-1}, n = 1, 2, ..k$ are abelian.
Let $gN_{n-1}, hN_{n-1} \in N_n/N_{n-1}$, where $g, h \in N_n$.
Since $N_{n-1} \trianglelefteq N_n \trianglelefteq G$ for $n = 1, 2, ...k$ and G is abelian,

$$\begin{aligned} gN_{n-1}hN_{n-1} &= ghN_{n-1} \\ &= hgN_{n-1} \\ &= hN_{n-1}gN_{n-1} \end{aligned}$$

So $N_n/N_{n-1}, n = 1, 2, ..k$ are abelian.
Assume there exsit $n \in \mathbb{Z}$ and $1 \leq n \leq k$ such that N_n/N_{n-1} is of prime order.
Then $|N_n/N_{n-1}| = p_1p_2..p_m$, where $p_1, p_2, ..., p_m$ are primes and $1 < m < \infty, m \in \mathbb{N}$.
By Exercise 1, we know there exists $H \leq N_n/N_{n-1}$ with order $|H| = p_1$.
Since $|\{e_{N_n/N_{n-1}}\}| = 1 < p_1 < m = |N_n/N_{n-1}|$.
Thus, $e_{e_{N_n/N_{n-1}}} \leq H \leq N_n/N_{n-1}$.
So we have N_n/N_{n-1} is not simple, which is contradicted by the chain $0 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = G$ is a composition series.
Therefore, each quotient N_i/N_{i-1} for $i = 1, 2, .., n$ is of prime order.
We have shown in Exercise 1(b) of homework 5-1, that G is cyclic if the order of p is a prime.
As a result, each quotient N_i/N_{i-1} for $i = 1, 2, .., n$ is of prime order $\square$