MATH 8510, Abstract Algebra I
Fall 2016
Exercises 4-2
Due date Thu 15 Sep 4:00PM

Exercise 1. Let A and B be groups, and set $A' := \{(a, e_B) \in A \times B \mid a \in A\} \subseteq A \times B$.

(a) Prove that A' is a normal subgroup of $A \times B$.

Proof.
We show it by the definition of a subgroup.
First we show $A' \leq A \times B$.
Since A is a group, $e_A \in A$.
Then $(e_A, e_B) \in A'$.
So $A' \neq \emptyset$.
$\forall a_1, a_2 \in A, x = (a_1, e_B) \in A', y = (a_2, e_B) \in A'$,

$$\begin{aligned} xy^{-1} &= (a_1, e_B)(a_2, e_B)^{-1} \\ &= (a_1, e_B)({a_2}^{-1}, {e_B}^{-1}) \\ &= (a_1{a_2}^{-1}, e_B{e_B}^{-1}) \\ &= (a_1{a_2}^{-1}, e_B). \end{aligned}$$

Since A is a group and $a_1, a_2 \in A$, $a_1{a_2}^{-1} \in A$.
So $xy^{-1} = (a_1{a_2}^{-1}, e_B) \in A'$.
Besides, $A' \subseteq A \times B$ by the definition of A'.
Thus, $A' \leq A \times B$.

□

(b) Prove that $A' \cong A$ and $(A \times B)/A' \cong B$.

Proof.

(i) Define

$$\begin{aligned} f : A &\to A' \\ a &\mapsto (a, e_B). \end{aligned}$$

$\forall x, y \in A$,

$$\begin{aligned} f(xy) &= (xy, e_B) \\ &= (xy, e_Be_B) \\ &= (x, e_B)(y, e_B) \\ &= f(x)f(y) \end{aligned}$$

So f is a homomorphism.
Since

$$\begin{aligned} \forall a \in \mathrm{Ker}(f) &\iff f(a) = (e_A, e_B) \\ &\iff (a, e_B) = (e_A, e_B) \\ &\iff a = e_A, \end{aligned}$$

Ker$(f) = e_A$.
So f is 1-1.
Im$(f) = \{(a, e_B) | a \in A\} = \{(a, e_B) \in A \times B \mid a \in A\} = A'$
So f is onto.
Thus, f is an isomorphism.
Hence, $A' \cong A$.

(ii) Define

$$\begin{aligned} F : A \times B &\to B \\ (a, b) &\mapsto b \end{aligned}$$

$\forall a, b \in A, c, d \in B, ac, bd \in A \times B$, then

$$\begin{aligned} F\big((a, b)(c, d)\big) &= F\big((ac, bd)\big) \\ &= bd \\ &= F\big((a, b)\big)F\big((c, d)\big). \end{aligned}$$

So F is a homomorphism.
$\forall a \in A, (a, e_B) \in A', F\big((a, e_B)\big) = e_B$.
So $A' \subset \mathrm{Ker}(F)$
Then define

$$\begin{aligned} f : A \times B/A' &\to B. \\ \overline{(a, b)} &\mapsto F\big((a, b)\big) \end{aligned}$$

Then f is a well-defined homomorphism.
Besides, Im$(f) = $ Im$(F) = \{b | b \in B\} = B$.
So f is onto.
Since

$$\begin{aligned} \forall \overline{(a, b)} \in \mathrm{Ker}(f) &\iff f(\overline{(a, b)}) = e_B \\ &\iff F\big((a, b)\big) = e_B \\ &\iff b = e_B, a \in A \\ &\iff (a, b) \in A', \end{aligned}$$

Ker$(f) = A'$.
So f is 1-1.
Thus, f is an isomorphism.
Hence, $(A \times B)/A' \cong B$.

□

Exercise 2. Let G be a group, and consider the commutator subgroup $[G, G]$ from Exercise 4-1.1.

(a) Prove that $[G, G]$ is a normal subgroup of G.

Proof. $\forall x, y, g \in G, xyx^{-1}y^{-1} \in [G, G]$.
Then

$$\begin{aligned} g(xyx^{-1}y^{-1})g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gx^{-1}g^{-1})(gy^{-1}g^{-1}) \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1}. \end{aligned}$$

Furthermore, $gxg^{-1}, gyg^{-1} \in G$ since $x, y, g \in G$.
So $g(xyx^{-1}y^{-1})g^{-1} \in [G, G]$.
Thus, $[G, G]$ is a normal subgroup of G. $\square$

(Hint: simplify the following product $(gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1}$.)

(b) Prove that $G/[G, G]$ is abelian.

Proof. $\forall x, y \in G, xyx^{-1}y^{-1} \in [G, G]$.
Then

$$\begin{aligned} xyx^{-1}y^{-1} \in [G, G] &\iff xy(yx)^{-1} \in [G, G] \\ &\iff xy[G, G] = yx[G, G] \\ &\iff x[G, G]y[G, G] = y[G, G]x[G, G] \end{aligned}$$

since $[G, G]$ is a normal subgroup of G.
Thus, $G/[G, G]$ is abelian. $\square$

(c) Let N be a normal subgroup of G, and prove that G/N is abelian if and only if $[G, G] \subseteq N$.

Proof.
Since N is a normal subgroup of G and $[G, G] = \{xyx^{-1}y^{-1} | x, y \in G\}$,
$\forall x, y \in G$,

$$\begin{aligned} G/N \text{ is abelian } &\iff xNyN = yNxN \\ &\iff xyN = yxN \\ &\iff xy(yx)^{-1} \in N \\ &\iff xyx^{-1}y^{-1} \in N \\ &\iff [G, G] \subseteq N. \end{aligned}$$

$\square$