MATH 8510, Abstract Algebra I
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Exercises 3-1
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Exercise 1 (2.1.8)**.** Let H and K be subgroups of a group G. Prove that $H \cup K \leq G$ if and only if $H \subseteq K$ or $K \subseteq H$.

Proof. (a) Assume $H \subseteq K$ or $K \subseteq H$.
Then for convenience, we can suppose $H \subseteq K$.
$e_G \in H \subset H \cup K$. So $H \cup K$ is not empty.
$\forall x, y \in H \cup K$, we have $x, y \in H$ or K.
Since $H \subseteq K$, we have $x, y \in K \subset H \cup K$.
K is a group, so $xy^{-1} \in K \subset H \cup K$.
Furthermore, since $H, K \leq G$, $H, K \subset G$ and then $H \cup K \subset G$.
Thus, $H \cup K \leq G$.

(b) Assume $H \cup K \leq G$.
$\forall x \in H \subset H \cup K, y \in K \subset H \cup K$,
we have $xy^{-1} \in H \cup K \leq G$.
Then $xy^{-1} \in H$ or K.
(1) If $xy^{-1} \in H$, since $x^{-1} \in H$, we have $x^{-1}xy^{-1} = y^{-1} \in H$.
Then $(y^{-1})^{-1} = y \in H$.
$y \in K$ is arbitrary,
so $K \subseteq H$.
(2) If $xy^{-1} \in K$, since $y \in K$, we have $xy^{-1}y = x \in K$
$x \in H$ is arbitrary,
so $H \subseteq K$.
Hence, $H \subseteq K$ or $K \subseteq H$.

□

Exercise 2 (2.2.8)**.** Fix $i \in [n] = \{1, \ldots, n\}$ and set $G_i := \{\sigma \in S_n \mid \sigma(i) = i\}$. (In other words, G_i is the stabilizer of i in $G = S_n$.) Prove that $S_{n-1} \cong G_i \leq S_n$.

Proof.

(1) Since S_i is a stabilizer of i in S_n,
G_i consists of all of the permutations of $\{i, 1, 2, ...i-1, i+1, ..., n\}$ with i fixed.
Namely, G_i is all the permutations of $\{1, 2, ..., i-1, i+1, ...n\}$.
So it is obvious that G_i is isomorphic to S_{n-1}.
(2) For any element in S_{n-1}, it is a permutation in S_n with n fixed,
so $S_{n-1} \subset S_n$.
Besides, both of S_{n-1} and S_n are symmetric groups, which has same group operation, i.e, permutation composition.
So $S_{n-1} \leq S_n$.

(3) By the definition of G_i, we have $G_i \subset S_n$.
Let e_{S_n} be the permutation which fixes $1, 2, ...n$, then $e_{S_n} \in S_n$ and $e_{S_n} \in G_i$ since it also fixes i.

So $G_i \neq \emptyset$.
$\forall x, y \in G_i \subset S_n, xy^{-1} \in S_n$ since S_n is a group.
$y^{-1} \in G_i$ since y^{-1} just the reverse order of the permutation y, which also keeps i fixed.
So $xy^{-1}(i) = x(y^{-1}(i)) = x(i) = i$.
Thus, $xy^{-1} \in G_i$.
Hence $G_i \leq S_n$.

$\square$

Exercise 3. In your free time, read the statements of the following exercises.
2.1: 1, 4, 6–8, 10–13, 15–17
2.2: 1–3, 5(a), 6, 8–11