MATH 8510, Abstract Algebra I
Fall 2016
Exercises 4-1
Due date Thu 15 Sep 4:00PM

Exercise 1. Let G be a group. The *commutator subgroup* of G is the subgroup $[G,G]$ of G generated by the set of all elements of the form $xyx^{-1}y^{-1}$:

$$[G,G] := \langle xyx^{-1}y^{-1} \mid x, y \in G\rangle.$$

Let $f\colon G \to H$ be a group homomorphism.

(a) Prove that $A \subseteq G \implies f(\langle A\rangle) = \langle f(A)\rangle$.

Proof.
First we show $f(\langle A\rangle)$ is a subgroup of H.
$e_G \in \langle A\rangle$ since $\langle A\rangle$ is a subgroup of G.
Then $f(e_G) = e_H \in f(\langle A\rangle)$ since f is homomorphism.
So $f(\langle A\rangle) \neq \emptyset$.
Let $y_1, y_2 \in f(\langle A\rangle)$.
Then $\exists x_1, x_2 \in \langle A\rangle$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.
$\langle A\rangle$ is a subgroup of G, so $x_2^{-1} \in \langle A\rangle$.
So $x_1x_2^{-1} \in \langle A\rangle$ and then $f(x_1x_2^{-1}) \in f(\langle A\rangle)$.
Since f is a homomorphism,
$f(x_1x_2^{-1}) = f(x_1)f(x_2^{-1}) = y_1f(x_2)^{-1} = y_1y_2^{-1} \in f(\langle A\rangle)$.
Besides, $f(\langle A\rangle) \subset H$.
Thus, $f(\langle A\rangle)$ is a subgroup of H.
$f(A) \subseteq f(\langle A\rangle)$, so

$$\langle f(A)\rangle \subseteq f(\langle A\rangle).$$

Let $x \in \langle A\rangle$, then $x = a_1^{\epsilon_1}a_2^{\epsilon_2}...a_n^{\epsilon_n}$ for some $n \in \mathbb{N}$ and $a_1, a_2, ...a_n \in A$ and $\epsilon_1, \epsilon_2, ...\epsilon_n \in \mathbb{Z}$.

$$\begin{aligned} f(x) &= f(a_1^{\epsilon_1}a_2^{\epsilon_2}...a_n^{\epsilon_n}) \\ &= f(a_1)^{\epsilon_1}f(a_2)^{\epsilon_2}...f(a_n)^{\epsilon_n} \\ &\in f(A) \subseteq \langle f(A)\rangle \end{aligned}$$

since f is a homomorphism and $f(a_1)^{\epsilon_1}, f(a_2)^{\epsilon_2}, ..., f(a_n)^{\epsilon_n} \in f(A)$.
So we have

$$f(\langle A\rangle) \subseteq \langle f(A)\rangle.$$

Since both of $f(\langle A\rangle)$ and $\langle f(A)\rangle$ are subgroups of H,
$f(\langle A\rangle) = \langle f(A)\rangle$ □

(b) Prove that $B \subseteq C \subseteq H \implies \langle B\rangle \subseteq \langle C\rangle$.

Proof.
$B \subseteq C \subseteq H$, so $B \subseteq C \subseteq \langle C\rangle \subseteq H$.
So $B \subseteq \langle C\rangle$.
Thus,$\langle B\rangle \subseteq \langle C\rangle$ since $\langle C\rangle$ is a subgroup of H. □

(c) Prove that G is abelian if and only if $[G,G] = \{e\}$.

Proof.

(a) Assume G is abelian.
Then

$$\begin{aligned}[G,G] &= \langle xyx^{-1}y^{-1} | x,y \in G\rangle \\ &= \langle (xx^{-1})(yy^{-1}) | x,y \in G\rangle \\ &= \langle e | x,y \in G\rangle \\ &= \{e\}\end{aligned}$$

(b) Assume $[G,G] = \{e\}$.
By the definition of $[G,G]$, we have $\forall x,y \in G$,

$$\begin{aligned}xyx^{-1}y^{-1} = e &\iff xyx^{-1} = y. \\ &\iff xy = yx.\end{aligned}$$

So G is abelian.

$\square$

(d) Prove that $f([G,G]) \subseteq [H,H]$.

Proof.
$\forall x,y \in G, xyx^{-1}y^{-1} \in [G,G]$,
$f(xyx^{-1}y^{-1}) = f(x)f(y)f(x^{-1})f(y^{-1}) = f(x)f(y)f(x)^{-1}f(y)^{-1}$.
since f is a homomorphism.
Moreover, $f(x), f(y) \in H$, so $f(x)f(y)f(x)^{-1}f(y)^{-1} \in [H,H]$.
Thus, $f([G,G]) \subset [H,H]$.

$\square$

(e) Prove that if H is abelian, then $[G,G] \subseteq \mathrm{Ker}(f)$.

Proof.
$\forall x,y \in G, xyx^{-1}y^{-1} \in [G,G]$,
$f(xyx^{-1}y^{-1}) = f(x)f(y)f(x^{-1})f(y^{-1}) = f(x)f(y)f(x)^{-1}f(y)^{-1}$.
since f is a homomorphism.
Moreover, $f(x), f(y) \in H$ and H is abelian,
so $f(xyx^{-1}y^{-1}) = (f(x)f(x)^{-1})(f(y)f(y)^{-1}) = e_H$.
Thus, $[G,G] \subseteq \mathrm{Ker}(f)$. $\square$

Exercise 2 (2.4.14)**.** See the text for a hint for this exercise.

(a) Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.

Proof.
Assume H is finitely generated subgroup of $\mathbb{Q}$
Since $\forall q \in \mathbb{Q}, q = \frac{m}{n}$ for some $m,n \in \mathbb{N}$ and $n \neq 0$, H can be written as

$$H = \Big\langle \frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_n}{b_n} \Big\rangle,$$

where $n \in \mathbb{N}$ and $a_i, b_i \in \mathbb{N}$ for $i = 1, 2, ..., n$.
Since $(\mathbb{Q}, +)$ is abelian and $\frac{a_i}{b_i} \in \mathbb{Q}$ for $i = 1, 2, ..., n$,

$$\begin{aligned}H &= \Big\langle \frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_n}{b_n} \Big\rangle \\ &= \Big\{ \epsilon_1\big(\frac{a_1}{b_1}\big) + \epsilon_2\big(\frac{a_2}{b_2}\big) + ... + \epsilon_n\big(\frac{a_n}{b_n}\big) | \epsilon_1, \epsilon_2, ..., \epsilon_n \in \mathbb{Z} \Big\}.\end{aligned}$$

Let $k = b_1 b_2 ... b_n \neq 0$ since $b_i \neq 0$ for $i = 1, 2, ..., n$.
We claim $H \leq \langle \frac{1}{k} \rangle$.
Let $h \in H$, then

$$h = z_1(\frac{a_1}{b_1}) + z_2(\frac{a_2}{b_2}) + ... + z_n(\frac{a_n}{b_n}) \text{ for some } z_1, z_2, ..., z_n \in \mathbb{Z}.$$

So

$$h = \frac{z_1 a_1 b_2 ... b_n + z_2 a_2 b_1 b_3 ... b_n + z_n a_n b_1 b_2 ... b_{n-1}}{b_1 b_2 ... b_n} = \frac{z}{k},$$

where $z = z_1 a_1 b_2 ... b_n + z_2 a_2 b_1 b_3 ... b_n + z_n a_n b_1 b_2 ... b_{n-1} \in \mathbb{Z}$ since $z_i, a_i, b_i \in \mathbb{Z}$ for $i = 1, 2, ..., n$. Thus $h \in \langle \frac{1}{k} \rangle$.
Therefore $H \subseteq \langle \frac{1}{k} \rangle$.
As a result, our claim $H \leq \langle \frac{1}{k} \rangle$ holds since $H, \langle \frac{1}{k} \rangle$ are both groups.
It is obvious $\langle \frac{1}{k} \rangle$ is cyclic.
So H is also cyclic. □

(b) Prove that $(\mathbb{Q}, +)$ is not cyclic. Conclude that $(\mathbb{Q}, +)$ is not finitely generated.

Proof. Assume $(\mathbb{Q}, +)$ is cyclic
Then $\exists a, b \in \mathbb{Z}$ and $b \neq 0$ such that $\mathbb{Q} = \langle \frac{a}{b} \rangle$.
So for $\frac{a}{2b} \in (\mathbb{Q}, +)$, suppose $\frac{a}{2b} = n \frac{a}{b}$ for some $n \in bbz$.
Then we have $n = \frac{1}{2}$, which is contradicted by $n \in \mathbb{Z}$.
Thus, $(\mathbb{Q}, +)$ is not cyclic.
Hence, we conclude that $(\mathbb{Q}, +)$ is not finitely generated. Otherwise, by part (a), we have $(\mathbb{Q}, +)$ is cyclic, which is a contradiction since we have shown $(\mathbb{Q}, +)$ is not cyclic.

□