MATH 8510, Abstract Algebra I
Fall 2016
Exercises 3-2
Due date Thu 08 Sep 4:00PM

Exercise 1 (2.3.18–19 $+5\epsilon$)**.** Let G be a group, and let $g \in G$.

(a) Prove that there exists a unique group homomorphism $f_g \colon \mathbb{Z} \to G$ such that $f_g(1) = g$.

Proof. Define

$$\mathbb{Z} \to G$$
$$f_g(n) \mapsto g^n$$

Then f_g is a homomorphism since $\forall m, n \in \mathbb{Z}, f_g(m+n) = g^{m+n} = g^m g^n = f_g(m) f_g(n)$ and satisfies $f_g(1) = g$.

□

(b) Prove that $\mathrm{Im}(f_g) = \langle g \rangle$.

(c) Prove that f_g is a monomorphism if and only if $|g| = \infty$.

(d) Assume that $|g| = n < \infty$.

(1) Prove that $\mathrm{Ker}(f_g) = n\mathbb{Z} := \{nm \in \mathbb{Z} \mid m \in \mathbb{Z}\}$.

(2) Prove that there is a unique group monomorphism $\phi_g \colon \mathbb{Z}/n\mathbb{Z} \to G$ such that $\phi_g(\overline{1}) = g$.

(3) Prove that $\mathrm{Im}(\phi_g) = \langle g \rangle$.

(4) We say that a diagram of group homomorphisms

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & B \arrow[d, "\beta"] \\ & C \end{tikzcd}$$

"commutes" when $\beta \circ \alpha = \alpha'$. Let $\pi \colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical epimorphism, and prove that the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\pi"] \arrow[dr, "f_g"'] & \mathbb{Z}/n\mathbb{Z} \arrow[d, "\phi_g"] \\ & G \end{tikzcd}$$

Exercise 2. In your free time, read the statements of the exercises from Section 2.3.