MATH 8510, Abstract Algebra I
Fall 2016
Exercises 4-1
Due date Thu 15 Sep 4:00PM

Exercise 1. Let G be a group. The *commutator subgroup* of G is the subgroup $[G, G]$ of G generated by the set of all elements of the form $xyx^{-1}y^{-1}$:

$$[G, G] := \langle xyx^{-1}y^{-1} \mid x, y \in G\rangle.$$

Let $f: G \to H$ be a group homomorphism.

(a) Prove that $A \subseteq G \implies f(\langle A\rangle) = \langle f(A)\rangle$.

Proof.
First we show $f(\langle A\rangle)$ is a subgroup of H.
$e_G \in \langle A\rangle$ since $\langle A\rangle$ is a subgroup of G.
Then $f(e_G) = e_H \in f(\langle A\rangle)$ since f is homomorphism.
So $f(\langle A\rangle) \neq \emptyset$.
Let $y_1, y_2 \in f(\langle A\rangle)$.
Then $\exists x_1, x_2 \in \langle A\rangle$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.
$\langle A\rangle$ is a subgroup of G, so $x_2^{-1} \in \langle A\rangle$.
So $x_1x_2^{-1} \in \langle A\rangle$ and then $f(x_1x_2^{-1}) \in f(\langle A\rangle)$.
Since f is a homomorphism,
$f(x_1x_2^{-1}) = f(x_1)f(x_2^{-1}) = y_1f(x_2)^{-1} = y_1y_2^{-1} \in f(\langle A\rangle)$.
Besides, $f(\langle A\rangle) \subset H$.
Thus, $f(\langle A\rangle)$ is a subgroup of H.
$f(A) \subseteq f(\langle A\rangle)$, so

$$\langle f(A)\rangle \subseteq f(\langle A\rangle).$$

Let $x \in \langle A\rangle$, then $x = a_1^{\epsilon_1}a_2^{\epsilon_2}...a_n^{\epsilon_n}$ for some $n \in \mathbb{N}$ and $a_1, a_2, ...a_n \in A$ and $\epsilon_1, \epsilon_2, ...\epsilon_n \in \mathbb{Z}$.

$$\begin{aligned} f(x) &= f(a_1^{\epsilon_1}a_2^{\epsilon_2}...a_n^{\epsilon_n}) \\ &= f(a_1)^{\epsilon_1}f(a_2)^{\epsilon_2}...f(a_n)^{\epsilon_n} \\ &\in f(A) \subseteq \langle f(A)\rangle \end{aligned}$$

since f is a homomorphism and $f(a_1)^{\epsilon_1}, f(a_2)^{\epsilon_2}, ..., f(a_n)^{\epsilon_n} \in f(A)$.
So we have

$$f(\langle A\rangle) \subseteq \langle f(A)\rangle.$$

Since both of $f(\langle A\rangle)$ and $\langle f(A)\rangle$ are subgroups of H,
$f(\langle A\rangle) = \langle f(A)\rangle$ □

(b) Prove that $B \subseteq C \subseteq H \implies \langle B\rangle \subseteq \langle C\rangle$.

Proof.
$B \subseteq C \subseteq H$, so $B \subseteq C \subseteq \langle C\rangle \subseteq H$.
So $B \subseteq \langle C\rangle$.
Thus,$\langle B\rangle \subseteq \langle C\rangle$ since $\langle C\rangle$ is a subgroup of H. □

(c) Prove that G is abelian if and only if $[G, G] = \{e\}$.

Proof.

(a) Assume G is abelian.
Then

$$\begin{aligned}[G,G] &= \langle xyx^{-1}y^{-1} | x,y \in G \langle \\ &= \langle (xx^{-1})(yy^{-1}) | x,y \in G \langle \\ &= \langle e | x,y \in G \langle \\ &= \{e\}\end{aligned}$$

(b)

$\square$

(d) Prove that $f([G,G]) \subseteq [H,H]$.
(e) Prove that if H is abelian, then $[G,G] \subseteq \mathrm{Ker}(f)$.

Exercise 2 (2.4.14)**.** See the text for a hint for this exercise.

(a) Prove that every finitely generated subgroup of $(\mathbb{Q},+)$ is cyclic.
(b) Prove that $(\mathbb{Q},+)$ is not cyclic. Conclude that $(\mathbb{Q},+)$ is not finitely generated.