

MATH 8510, Abstract Algebra I

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Exercises 1

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**Exercise 1.** Prove that  $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$  is a field under the usual addition and multiplication of  $\mathbb{R}$ .

*Proof.*  $\forall a + b\sqrt{2}, c + d\sqrt{2}, e + f\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ , we have  $a, b, c, d, e, f \in \mathbb{Q}$ .

(a)

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

since  $a + c, b + d \in \mathbb{Q}$ .

$$(a + b\sqrt{2})(c + d\sqrt{2}) = a(c + d\sqrt{2}) + b\sqrt{2}(c + d\sqrt{2}),$$

where we use the distributive law of the field  $\mathbb{R}$  since  $a, b\sqrt{2}, c, d\sqrt{2} \in \mathbb{R}$ . Besides,  $b\sqrt{2}c = bc\sqrt{2}$  and  $(b\sqrt{2})(d\sqrt{2}) = (bd)(\sqrt{2}\sqrt{2}) = bd(2) = 2bd$ .

Then

$$\begin{aligned}(a + b\sqrt{2})(c + d\sqrt{2}) &= (ac + ad\sqrt{2}) + (bc\sqrt{2} + 2bd) \\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q}(\sqrt{2})\end{aligned}$$

where we use the multiplication associative, commutative and distribution law of the field  $\mathbb{R}$ .

(b)

$$\begin{aligned}(a + b\sqrt{2}) + (c + d\sqrt{2}) &= (a + c) + (b + d)\sqrt{2} \\ &= (c + a) + (d + b)\sqrt{2} \\ &= (c + d\sqrt{2}) + (a + b\sqrt{2}).\end{aligned}$$

$$\begin{aligned}\left((a + b\sqrt{2}) + (c + d\sqrt{2})\right) + (e + f\sqrt{2}) &= \left((a + c) + (b + d)\sqrt{2}\right) + (e + f\sqrt{2}) \\ &= (a + c + e) + (b + d + f)\sqrt{2} \\ &= \left(a + (c + e)\right) + \left(b + (d + f)\right)\sqrt{2} \\ &= (a + b\sqrt{2}) + \left((c + e) + (d + f)\sqrt{2}\right) \\ &= (a + b\sqrt{2}) + \left((c + d\sqrt{2}) + (e + f\sqrt{2})\right).\end{aligned}$$

$$0_{\mathbb{Q}(\sqrt{2})} = 0_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} = 0 + 0\sqrt{2}.$$

$$(0 + 0\sqrt{2}) + (a + b\sqrt{2}) = (0 + a) + (0 + b)\sqrt{2} = a + b\sqrt{2}.$$

$$-(a + b\sqrt{2}) = (-a) + (-b)\sqrt{2} = -a - b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$$

since  $-a, -b \in \mathbb{Q}$ . Check

$$(a + b\sqrt{2}) + (-a - b\sqrt{2}) = (a - a) + (b - b)\sqrt{2} = 0_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} = 0_{\mathbb{Q}(\sqrt{2})} = 0.$$

(c)

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$$

by previous steps .

Similarly, we can get

$$\begin{aligned} (c + d\sqrt{2})(a + b\sqrt{2}) &= (ca + 2db) + (da + cb)\sqrt{2} \\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \\ &= (a + b\sqrt{2})(c + d\sqrt{2}). \end{aligned}$$

$$\begin{aligned} \left( (a + b\sqrt{2}) + (c + d\sqrt{2}) \right) + (e + f\sqrt{2}) &= \left( (a + c) + (b + d)\sqrt{2} \right) + (e + f\sqrt{2}) \\ &= (a + c + e) + (b + d + f)\sqrt{2}. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} (a + b\sqrt{2}) + \left( (c + d\sqrt{2}) + (e + f\sqrt{2}) \right) &= (a + c + e) + (b + d + f)\sqrt{2} \\ &= (a + c + e) + (b + d + f)\sqrt{2} \\ &= \left( (a + b\sqrt{2}) + (c + d\sqrt{2}) \right) + (e + f\sqrt{2}). \end{aligned}$$

$$1_{\mathbb{Q}(\sqrt{2})} = 1_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} = 1 + 0\sqrt{2}.$$

$$(1 + 0\sqrt{2})(a + b\sqrt{2}) = (a + b\sqrt{2}) + (0a\sqrt{2} + 0(2b)) = a + b\sqrt{2}$$

where we use the multiplication associative, commutative and distributive law of field  $\mathbb{R}$ . Consider  $a + b\sqrt{2} \neq 0$ , then  $a \neq -b\sqrt{2}$ .

(1) If  $b = 0$ , then  $a \neq 0$ ,

$$(a + b\sqrt{2})^{-1} = \frac{1}{a} = \frac{1}{a} + 0\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

(2) If  $a = b\sqrt{2} \neq 0$ , then  $b \neq 0$ ,

$$\begin{aligned} (a + b\sqrt{2})^{-1} &= (2b\sqrt{2})^{-1} \\ &= \frac{1}{4b}\sqrt{2} = 0 + \frac{1}{4b}\sqrt{2} \in \mathbb{Q}(\sqrt{2}). \end{aligned}$$

(3) If  $b \neq 0$  and  $a \neq b\sqrt{2}$ , then  $a^2 \neq 2b^2$  and

$$(a + b\sqrt{2})^{-1} = \frac{1}{a^2 - 2b^2}(a - b\sqrt{2}) = \frac{a}{a^2 - 2b^2} + \frac{-b}{a^2 - 2b^2}\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

since  $\frac{a}{a^2-2b^2}, \frac{-b}{a^2-2b^2} \in \mathbb{Q}$ . Besides,

$$\begin{aligned}
 (a + b\sqrt{2})\left(\frac{1}{a^2-2b^2}(a - b\sqrt{2})\right) &= (a + b\sqrt{2})\left(\frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}\right) \\
 &= (a + b\sqrt{2})\left(\frac{a}{a^2-2b^2}\right) + (a + b\sqrt{2})\left(\frac{-b}{a^2-2b^2}\sqrt{2}\right) \\
 &= \frac{a^2}{a^2-2b^2} + \frac{ab\sqrt{2}}{a^2-2b^2} + \frac{-ab\sqrt{2}}{a^2-2b^2} + \frac{-2b^2}{a^2-2b^2} \\
 &= \frac{a^2-2b^2}{a^2-2b^2} + \frac{ab-ab}{a^2-2b^2}\sqrt{2} = 1_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} \\
 &= 1
 \end{aligned}$$

where we use the multiplication associative, commutative and distributive law of the field  $\mathbb{R}$  since  $a, b\sqrt{2}, \frac{a}{a^2-2b^2}, \frac{-b}{a^2-2b^2}\sqrt{2} \in \mathbb{R}$ .

(d)

$$\begin{aligned}
 (a + b\sqrt{2})\left((c + d\sqrt{2}) + (e + f\sqrt{2})\right) &= (a + b\sqrt{2})\left((c + e) + (d + f)\sqrt{2}\right) \\
 &= a(c + e) + a(d + f)\sqrt{2} + b(c + e)\sqrt{2} + 2b(d + f) \\
 &= (ac + ae + 2bd + 2bf) + (ad + af + bc + be)\sqrt{2}
 \end{aligned}$$

where we use the multiplication commutative, associative and distributive law of the field  $\mathbb{R}$ .

$$\begin{aligned}
 &(a + b\sqrt{2})(c + d\sqrt{2}) + (a + b\sqrt{2})(e + f\sqrt{2}) \\
 &= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd + ae + af\sqrt{2} + be\sqrt{2} + 2bf \\
 &= (ac + ae + 2bd + 2bf) + (ad + af + bc + be)\sqrt{2} \\
 &= (a + b\sqrt{2})\left((c + d\sqrt{2}) + (e + f\sqrt{2})\right)
 \end{aligned}$$

where we use the multiplication commutative, associative and distributive law of the field  $\mathbb{R}$ .

Similarly,

$$\begin{aligned}
 \left((a + b\sqrt{2}) + (c + d\sqrt{2})\right)(e + f\sqrt{2}) &= (e + f\sqrt{2})\left((a + b\sqrt{2}) + (c + d\sqrt{2})\right) \\
 &= (e + f\sqrt{2})(a + b\sqrt{2}) + (e + f\sqrt{2})(c + d\sqrt{2}) \\
 &= (a + b\sqrt{2})(e + f\sqrt{2}) + (c + d\sqrt{2})(e + f\sqrt{2})
 \end{aligned}$$

according to previous proof steps.

(e) Since

$$\begin{aligned}
 0_{\mathbb{Q}(\sqrt{2})} &= 0_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2} \\
 1_{\mathbb{Q}(\sqrt{2})} &= 1_{\mathbb{Q}} + 0_{\mathbb{Q}}\sqrt{2}, \\
 1_{\mathbb{Q}(\sqrt{2})} &\neq 0_{\mathbb{Q}(\sqrt{2})}
 \end{aligned}$$

given  $1_{\mathbb{Q}} \neq 0_{\mathbb{Q}}$ .

□

**Exercise 2.** Is the set  $\mathbb{R}^{2 \times 2} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$  is a field under the usual addition and multiplication of matrices?

If  $\mathbb{R}^{2 \times 2}$  is a field, prove it. Otherwise, prove the field axioms that do hold, and give specific counterexamples for the axioms that fail.

No, it is not a field.

*Proof.*  $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \begin{pmatrix} i & j \\ k & l \end{pmatrix} \in \mathbb{R}^{2 \times 2},$

(a)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

since  $a+e, b+f, c+g, d+h \in \mathbb{R}$ .

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

since  $ae+bg, af+bh, ce+dg, cf+dh \in \mathbb{R}$ .

(b)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\begin{aligned} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) + \begin{pmatrix} i & j \\ k & l \end{pmatrix} &= \left( \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \right) + \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\ &= \begin{pmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{pmatrix} \\ &= \begin{pmatrix} a+(e+i) & b+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left( \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix} \right). \end{aligned}$$

$$0_{\mathbb{R}^{2 \times 2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + 0_{\mathbb{R}^{2 \times 2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

Check

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a+a & -b+b \\ -c+c & -d+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{\mathbb{R}^{2 \times 2}}.$$

(c) It does not satisfy multiplication commutative law. For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix},$$

but

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

$$\begin{aligned}
\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \begin{pmatrix} i & j \\ k & l \end{pmatrix} &= \left( \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \right) \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\
&= \begin{pmatrix} (ae+bg)i+(af+bh)k & (ae+bg)j+(af+bh)l \\ (ce+dg)i+(cf+dh)k & (ce+dg)j+(cf+dh)l \end{pmatrix} \\
&= \begin{pmatrix} aei+afk+bgi+bhk & aej+afj+bgj+bhl \\ cei+cfk+dgi+dhk &cej+cfl+dgj+dhl \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} \right) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix} \\
&= \begin{pmatrix} a(ei+fk)+b(gi+hk) & a(ej+fl)+b(gj+hl) \\ c(ei+fk)+d(gi+hk) & c(ej+fl)+d(gj+hl) \end{pmatrix} \\
&= \begin{pmatrix} aei+afk+bgi+bhk & aej+afj+bgj+bhl \\ cei+cfk+dgi+dhk &cej+cfl+dgj+dhl \end{pmatrix} \\
&= \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \begin{pmatrix} i & j \\ k & l \end{pmatrix}.
\end{aligned}$$

$$1_{\mathbb{R}^{2 \times 2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} 1_{\mathbb{R}^{2 \times 2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b0 & a0+b1 \\ c1+d0 & c0+d1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Not every element of  $\mathbb{R}^2$  has a multiplicative inverse, for instance, for  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ , assume we can find an element  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2 \times 2}$  such that  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , then we have  $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , which is impossible since  $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ .

(d)

$$\begin{aligned}
\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \begin{pmatrix} i & j \\ k & l \end{pmatrix} &= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\
&= \begin{pmatrix} (a+e)i+(b+f)k & (a+e)j+(b+f)l \\ (c+g)i+(d+h)k & (c+g)j+(d+h)l \end{pmatrix} \\
&= \begin{pmatrix} ai+ei+bk+fk & aj+ej+bl+fl \\ ci+gi+dk+hk & cj+gj+dl+hl \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} &= \begin{pmatrix} ai+bk & aj+bl \\ ci+dk & cj+dl \end{pmatrix} + \begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix} \\
&= \begin{pmatrix} ai+ei+bk+fk & aj+ej+bl+fl \\ ci+gi+dk+hk & cj+gj+dl+hl \end{pmatrix} \\
&= \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \begin{pmatrix} i & j \\ k & l \end{pmatrix}.
\end{aligned}$$

(e) Since

$$0_{\mathbb{R}^{2 \times 2}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$1_{\mathbb{R}^{2 \times 2}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$0_{\mathbb{R}^{2 \times 2}} \neq 1_{\mathbb{R}^{2 \times 2}}$$

give  $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ .

□