MATH 8510, Abstract Algebra I
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Exercises 6-1
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Exercise 1 (3.5.10)**.** Consider the alternating group A_4. Prove that there is a chain of normal subgroups $\{(1)\}N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = A_4$ such that each quotient N_i/N_{i-1} is abelian. (This says that A_4 is *solvable*.)

Hint: Set $N = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subseteq A_4$, and prove the following:

(a) Prove that $N \leq A_4$.

Proof. (i) It is obvious $N \subseteq A_4$

(ii) N is not empty since $e_{A_4} = (1) \in N$.

(iii) Since $N = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$,
$((1\ 2)(3\ 4))\,((1\ 3)(2\ 4)) = (1\ 4)(2\ 3) \in N$, and
$((1\ 2)(3\ 4))\,((1\ 4)(2\ 3)) = (1\ 3)(2\ 4) \in N$, and
$((1\ 3)(2\ 4))\,((1\ 4)(2\ 3)) = (1\ 2)(3\ 4) \in N$.
Similarly,
$((1\ 3)(2\ 4))\,((1\ 2)(3\ 4)) = (1\ 4)(2\ 3) \in N$, and
$((1\ 4)(2\ 3))\,((1\ 2)(3\ 4)) = (1\ 3)(2\ 4) \in N$, and
$((1\ 4)(2\ 3))\,((1\ 3)(2\ 4)) = (1\ 2)(3\ 4) \in N$.
So N is abelian.

(iv) $((1\ 2)(3\ 4))\,((1\ 2)(3\ 4)) = (1) \in N$.
$((1\ 3)(2\ 4))\,((1\ 3)(2\ 4)) = (1) \in N$.
$((1\ 4)(2\ 3))\,((1\ 4)(2\ 3)) = (1) \in N$.
So the inverses of $(1\ 2)(3\ 4), (1\ 3)(2\ 4)$ and $(1\ 4)(2\ 3)$ are themselves, respectively, which are obviously in N.

Thus, N is a subgroup of A_4 □

(b) Prove that $N \smallsetminus \{(1)\} = \{\tau \in A_4 \mid |\tau| = 2\}$.

Proof. Let $D = \{(a\ b\ c), a, b, c \in \{1, 2, 3, 4\}, a \neq b \neq c\}$.
It is obvious that $(1) \notin D$.
Then $D \subsetneq A_4$ since $(a\ b\ c) = (a\ c)(a\ b)$.
We have $(a\ b\ c)(a\ b\ c) = (a\ c\ b) \neq (1)$ and $(a\ b\ c)(a\ b\ c)(a\ b\ c) = (1)$,
so $|(a\ b\ c)| = 3$.
Beside, we have $|D| = \frac{4!}{3} = 8$.
Since $|N| + |D| = 12 = |A_4|$ and $N \cap D = \emptyset$,
$A_4 = N \cup D$.
So we know all the elements with order 2 are in A_4.
Moreover, all elements in A_4 has order 2 except element (1). Thus,

$$N \smallsetminus \{(1)\} = \{\tau \in A_4 \ \mid\ |\tau| = 2\}.$$

□

(c) Prove that for all $\sigma \in A_4$, for all $\tau \in N \smallsetminus \{(1)\}$, the element $\sigma\tau\sigma^{-1}$ has order 2, so it is in N.

Proof. Suppose $|\sigma\tau\sigma^{-1}| = 1$, then $\sigma\tau\sigma^{-1} = (1)$ for all $\sigma \in A_4$ and all $\tau \in N \setminus \{(1)\}$.
So $\sigma\tau = \sigma$ for all $\sigma \in A_4$ and all $\tau \in N \setminus \{(1)\}$.
Then $\tau = (1)$, which is a contradiction since $\tau \in N \setminus \{(1)\}$.
So $|\sigma\tau\sigma^{-1}| > 1$.
For all $\tau \in N \setminus \{(1)\}$, we have $\tau\tau = (1)$ since the order of any element of $N \setminus \{(1)\}$ is 2.
For all $\sigma \in A_4$ and all $\tau \in N \setminus \{(1)\}$,

$$\begin{aligned}(\sigma\tau\sigma^{-1})(\sigma\tau\sigma^{-1}) &= \sigma\tau\tau\sigma^{-1}\\ &= \sigma(1)\sigma^{-1}\\ &= (1).\end{aligned}$$

So the element $\sigma\tau\sigma^{-1}$ has order 2, and then $\sigma\tau\sigma^{-1} \in N$.
Thus, $N \trianglelefteq A_4$.
Since $|A_4/N| = \frac{|A_4|}{|N|} = 3$ by Lagrange Theorem, A_4/N is simple and cyclic.
Then A_4/N is abelian.
As a result, we have a trivial chain $N \trianglelefteq A_4$.
Since $|N| = 4$, by Jordan-Hölder theorem, there is a chain of subgroups

$$\{(1)\} = N_0 \trianglelefteq N_1 ... \trianglelefteq N_k = N.$$

N is abelian by part (i), so $N_0, N_1, ..., N_{k-1}$ are also abelian since they are subgroups of N.
Thus, $N_1/N_0, N_2/N_1 .., N/N_{n-1}$ are abelian.
Combine the chain $\{(1)\} = N_0 \trianglelefteq N_1 ... \trianglelefteq N_k = N$ with the chain $N \trianglelefteq A_4$, we get a new chain

$$N_0 \trianglelefteq N_1 ... \trianglelefteq N \trianglelefteq A_4,$$

where $N_1/N_0, N_2/N_1 .., N/N_{k-1}, A_4/N$ are abelian. □

Exercise 2 (4.1.9)**.** Assume that G acts transitively on a finite set A, and let $H \trianglelefteq G$. Note that H also acts on A. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the distinct orbits of H on A

(a) Prove that G permutes the sets $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ in the sense that for each $g \in G$ and each $i \in [r] = \{1, \ldots, r\}$, there is a j such that $g\mathcal{O}_i = \mathcal{O}_j$ where $g\mathcal{O} = \{ga \in A \mid a \in \mathcal{O}\}$. Prove that G acts transitively on $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}$. Deduce that all orbits of H on A have the same cardinality.

Proof. For each $g \in G$, and each $i \in [r] = \{1, \ldots, r\}$, there exists $a_i \in A$ such that $\mathcal{O}_i = H \cdot a_i = \{h \cdot a_i \in A \mid h \in H\}$.

$$\begin{aligned}g\mathcal{O}_i &= g\{h \cdot a_i \in A \mid h \in H\}\\ &= \{gh \cdot a_i \in A \mid h \in H\}\\ &= \{h(g \cdot a_i) \in A \mid h \in H\}\end{aligned}$$

since $H \trianglelefteq G$.
Besides, $g \cdot a_i \in A$,so there exists $a \in A$ such that $g \cdot a_i = a$.
There exists some $j \in [r]$ such that $a \in \mathcal{O}_j$ since $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ are disjoint orbits of H on A.

Set $a_j = a$ for such $j \in [r]$ such that $a \in \mathcal{O}_j$.
Let $H \cdot a_j = \mathcal{O}_j = \{h \cdot a_j \in A \mid h \in H\}$ and then

$$\begin{aligned} g\mathcal{O}_i &= \{h(g \cdot a_i) \in A \mid h \in H\} \\ &= \{ha_j \in A \mid h \in H\} \\ &= \mathcal{O}_j \end{aligned}$$

Next we show G acts transitively on $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r\}$.
Since G acts transitively on A, for any $j \in [r]$, there exists $g_j \in G$ such that $a_j = g_j a_1$, where $a_j, j \in [r]$ and a_1 is already defined by us.
So for any $\mathcal{O}_j \in \{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r\}$,

$$\begin{aligned} \mathcal{O}_j &= \{h \cdot a_j \in A \mid h \in H\} \\ &= \{h \cdot g_j a_1 \in A \mid h \in H\} \\ &= \{g_j h \cdot a_1 \in A \ \mid h \in H\} \\ &= g_j\{h \cdot a_1 \in A \mid h \in H\} \\ &= g_j \mathcal{O}_1 \end{aligned}$$

Therefore, G acts transitively on $\{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r\}$.
Since $\mathcal{O}_j = g_j \mathcal{O}_1$ for any $j \in [r]$,
we have for any $j \in [r]$,

$$|\mathcal{O}_j| = |g_j \mathcal{O}_1| = |\mathcal{O}_1|.$$

Thus, we conclude that all orbits of H on A have the same cardinality. □

(b) Prove that if $a \in \mathcal{O}_1$, then $|\mathcal{O}_1| = [H\colon H \cap G_a]$, and prove that $r = [G\colon HG_a]$.

Proof. We claim $H \cap G_a$ is the stablier of a in H, namely, $H_a = H \cap G_a$.

(a) Let $s \in H \cap G_a$, then $s \in H$ and $sa = a$ since $s \in G_a$.
So $s \in H_a$.
As are result, $H \cap G_a \subset H_a$.
(b) Let $h \in H_a$, then $h \in H$ and $ha = a$.
So $h \in G_a$ since $h \in H_a \subset H \trianglelefteq G$.
Thus, $h \in H \cap G_a$.
Then $H_a \subset H \cap G_a$.

Therefore, $H_a = H \cap G_a$.
Since $\mathcal{O}_1$ is one of the orbits of H on A,
given $a \in \mathcal{O}_1$, we have

$$\begin{aligned} |\mathcal{O}_1| &= |\mathcal{O}_a| \\ &= [H : H_a] \\ &= [H : H \cap G_a] \end{aligned}$$

Then we will show $r = [G : HG_a]$.
Since $G \leq H$ and $G_a \leq G$, we have

$$\begin{aligned} |HG_a| &= \frac{|H||G_a|}{|H \cap G_a|} \\ &= \frac{|H||G_a|}{|H_a|} \end{aligned}$$

Then

$$[G : HG_a] = \frac{|G|}{|HG_a|}$$
$$= \frac{|G||H_a|}{|H||G_a|}$$

We have shown $|\mathcal{O}_1| = [H : H_a]$, so $|H| = |H_a||\mathcal{O}_1|$.
Then

$$[G : HG_a] = \frac{|G|}{|G_a||\mathcal{O}_1|}$$

We kown all orbits of H on A have the same cardinality.
So $r|\mathcal{O}_1| = |A|$.
Then

$$[G : HG_a] = \frac{|G|}{|G_a||A|}r$$

Since G acts transitively on A, for $a \in A$.

$$A = G \cdot a.$$

Then

$$|A| = |G \cdot a|$$
$$= [G : G_a]$$
$$= \frac{|G|}{|G_a|}$$

according to what we have shown in class.
Then

$$[G : HG_a] = \frac{|G|||G_a}{|G_a||G|}r = r.$$

Namely,

$$r = [G : HG_a]$$

$\square$