MATH 8510, Abstract Algebra I
Fall 2016
Exercises 3-2
Due date Thu 08 Sep 4:00PM

Exercise 1 (2.3.18–19 +5ϵ)**.** Let G be a group, and let $g \in G$.

(a) Prove that there exists a unique group homomorphism $f_g \colon \mathbb{Z} \to G$ such that $f_g(1) = g$.

Proof. Define

$$\begin{aligned} f_g : \mathbb{Z} &\to G \\ n &\mapsto g^n \end{aligned}$$

Then f_g is a homomorphism since $\forall m, n \in \mathbb{Z}, f_g(m+n) = g^{m+n} = g^m g^n = f_g(m)f_g(n)$ and satisfies $f_g(1) = g$.
So there exists a group homomorphism $f_g \colon \mathbb{Z} \to G$ such that $f_g(1) = g$.
Next we show it is unique.
Assume there exists another homomGorphism h_g differing from f_g such that $h_g(1) = g$.

(a) If $n > 0$, $h_g(n) = h_g(\sum_{i=1}^{n} 1) = \prod_{i=1}^{n} h_g(1) = \prod_{i=1}^{n} g = g^n$ since f_g is homomorphism.
(b) If $n < 0$, then $-n > 0$ and $h_g(n) = h_g(-(-n)) = (h_g(-n))^{-1} = (g^{-n})^{-1} = g^n$.
(c) If $n = 0$, then $h_g(0) = h_g(n-n) = h_g(n)h_g(-n) = g^n g^{-n} = g^{n-n} = g^0$.
So $h_g(n) = g^n = f_g(n), \forall n \in \mathbb{Z}$.

Thus, $f_g = h_g$, which is contradicted by the assumption.
Hence, such group homomorphism is unique. □

(b) Prove that $\mathrm{Im}(f_g) = \langle g \rangle$.

Proof.
$Im(f_g) = \{g^n | n \in \mathbb{Z}\} \subset G$.
So $g^n \in G, \forall n \in \mathbb{Z}$.
Thus, $\langle g \rangle = \{g^n \in G | n \in \mathbb{Z}\} = \{g^n | n \in \mathbb{Z}\} = Im(f_g)$. □

(c) Prove that f_g is a monomorphism if and only if $|g| = \infty$.

Proof.

(a) Assume f_g is a monomorphism, then f_g is 1-1 since f_g is homomorphism.
Then $\infty = |\mathbb{Z}| = |Im(f_g)| = |\langle g \rangle| = |g|$.
So $|g| = \infty$.
(b) Assume $|g| = \infty$.
Suppose f_g is not a homomorphism, then f_g is not 1-1.
So $\exists$ different $m, n (m > n) \in \mathbb{Z}$ such that $f_g(m) = g^m = g^n = f_g(n)$.
Then $g^{m-n} = e_G$.
So $|g| = |\langle g \rangle| \leq m - n < \infty$ since $0 < m - n < \infty$.
It is a contradion since $|g| = \infty$ by assumption.
So f_g is a homomorphism.

$\square$

(d) Assume that $|g| = n < \infty$.

(1) Prove that $\mathrm{Ker}(f_g) = n\mathbb{Z} := \{nm \in \mathbb{Z} \mid m \in \mathbb{Z}\}$.

Proof. $Ker(f_g) = m \in Z | f_g(m) = g^m = e_G$.
Since $|g| = n$, $g^n = e_G$ and $g^{mn} = (g^n)^m = (e_G)^m = e_G, \forall m \in \mathbb{Z}$.
Then $mn \in Ker(f_g), \forall m \in \mathbb{Z}$.
Moreover, for $k = 1, 2, ...n-1$, $g^k \neq e_G$. Otherwise, it is contradicted by $|g| = n$. So $g^{mk} \neq e_G, \forall m \in \mathbb{Z}$ and $m \neq 0$.
Thus, $Ker(f_g) = n\mathbb{Z}$. $\square$

(2) Prove that there is a unique group monomorphism $\phi_g \colon \mathbb{Z}/n\mathbb{Z} \to G$ such that $\phi_g(\bar{1}) = g$.

Proof.
Define

$$\begin{aligned} \phi_g : \mathbb{Z}/n\mathbb{Z} &\to G \\ \bar{m} &\mapsto g^m \end{aligned}$$

Then $\phi_g(\bar{1}) = g^1 = g$.
Let $m, k \in \mathbb{Z}$ and $0 \leq m \leq k \leq n-1$, and $\phi(\bar{m}) = g^m = g^k = f(\bar{k})$.
Then $g^{k-m} = e_G$.
So $k - m = 0$ since $0 \leq k - m \leq n-1$.
Thus, $k = m$.
So $\bar{m} = \bar{k}$.
Hence, ϕ_g is 1-1 and so it is a group monomorphism.
Suppose there exists another group monomorphism $h_g \colon \mathbb{Z}/n\mathbb{Z} \to G$ such that $h_g(\bar{1}) = g$.
Then when $1 < k \leq n-1, h_g(\bar{k}) = h_g(\sum_{i=1}^{k} \bar{1}) = \prod_{i=1}^{k} h_g(1) = g^k$ since h_g is a monomorphism.
Besides, $h_g(\bar{0}) = h_g(\bar{n}) = (h_g(1))^n = g^n = e_G = \phi_g$.
So $\phi_g(\bar{k}) = h_g(\bar{k})$ for all $0 \leq k \leq n-1$.
Therefore, $\phi_g = h_g$.
We conclude that such a group monomorphism is unique. $\square$

(3) Prove that $\mathrm{Im}(\phi_g) = \langle g \rangle$.

(4) We say that a diagram of group homomorphisms

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & B \arrow[d, "\beta"] \\ & C \end{tikzcd}$$

"commutes" when $\beta \circ \alpha = \alpha'$. Let $\pi \colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical epimorphism, and prove that the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\pi"] \arrow[dr, "f_g"'] & \mathbb{Z}/n\mathbb{Z} \arrow[d, "\phi_g"] \\ & G \end{tikzcd}$$

Exercise 2. In your free time, read the statements of the exercises from Section 2.3.