MATH 8510, Abstract Algebra I
Fall 2016
Exercises 6-1
Due date Thu 29 Sep 4:00PM

Exercise 1 (3.5.10)**.** Consider the alternating group A_4. Prove that there is a chain of normal subgroups $\{(1)\}N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_k = A_4$ such that each quotient N_i/N_{i-1} is abelian. (This says that A_4 is *solvable*.)

Hint: Set $N = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subseteq A_4$, and prove the following:

(a) Prove that $N \leq A_4$.

Proof. (i) It is obvious $N \subseteq A_4$

(ii) N is not empty since $e_{A_4} = (1) \in N$.

(iii) Since $N = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$,
$((1\ 2)(3\ 4))\,((1\ 3)(2\ 4)) = (1\ 4)(2\ 3) \in N$, and
$((1\ 2)(3\ 4))\,((1\ 4)(2\ 3)) = (1\ 3)(2\ 4) \in N$, and
$((1\ 3)(2\ 4))\,((1\ 4)(2\ 3)) = (1\ 2)(3\ 4) \in N$.
Similarly,
$((1\ 3)(2\ 4))\,((1\ 2)(3\ 4)) = (1\ 4)(2\ 3) \in N$, and
$((1\ 4)(2\ 3))\,((1\ 2)(3\ 4)) = (1\ 3)(2\ 4) \in N$, and
$((1\ 4)(2\ 3))\,((1\ 3)(2\ 4)) = (1\ 2)(3\ 4) \in N$.
So N is abelian.

(iv) $((1\ 2)(3\ 4))\,((1\ 2)(3\ 4)) = (1) \in N$.
$((1\ 3)(2\ 4))\,((1\ 3)(2\ 4)) = (1) \in N$.
$((1\ 4)(2\ 3))\,((1\ 4)(2\ 3)) = (1) \in N$.
So the inverses of $(1\ 2)(3\ 4), (1\ 3)(2\ 4)$ and $(1\ 4)(2\ 3)$ are themselves, respectively, which are in N.

Thus, N is a subgroup of A_4 □

(b) Prove that $N \setminus \{(1)\} = \{\tau \in A_4 \mid |\tau| = 2\}$.

Proof. Let $D = \{(a\ b\ c), a, b, c \in \{1, 2, 3, 4\}, a \neq b \neq c\}$.
Then $D \subsetneq A_4$ since $(a\ b\ c) = (a\ c)(a\ b)$.
We have $(a\ b\ c)(a\ b\ c) = (a\ c\ b)$ and $(a\ b\ c)(a\ b\ c)(a\ b\ c) = (1)$, so $|(a\ b\ c)| = 3$.
Beside, we have $|D| = \frac{4!}{3} = 8$.
Since $|N| + D = 12 = |A_4|$ and $N \cap D = \emptyset$,
$A_4 = N \cup D$.
So we know all the elements with order 2 are in A_4.
Moreover, all elements in A_4 has order 2 except element (1). Thus,

$$N \setminus \{(1)\} = \{\tau \in A_4 \ \mid\ |\tau| = 2\}.$$

□

(c) Prove that for all $\sigma \in A_4$, for all $\tau \in N \setminus \{(1)\}$, the element $\sigma\tau\sigma^{-1}$ has order 2, so it is in N.

Proof. For all $\tau \in N \setminus \{(1)\}$, we have $\tau\tau = (1)$ since the order of the elment of $N \setminus \{(1)\}$ is 2.

For all $\sigma \in A_4$, $\tau \in N \setminus \{(1)\}$,

$$\begin{aligned}(\sigma\tau\sigma^{-1})(\sigma\tau\sigma^{-1}) &= \sigma\tau\tau\sigma^{-1}\\ &= \sigma(1)\sigma^{-1}\\ &= (1).\end{aligned}$$

So the element $\sigma\tau\sigma^{-1}$ has order 2, and then $\sigma\tau\sigma^{-1} \in N$.
Thus, $N \trianglelefteq A_4$.
Since $|A_4/N| = \frac{|A_4|}{N} = 3$ by Lagrange Theorem, A_4/N is simple and cyclic.
Then A_4/N is abelian.
As a result, we have a trivial chain $N \trianglelefteq A_4$.
Since $|N| = 4$, by Jordan-Hölder theorem, there is a chain of subgroups

$$\{(1)\} = N_0 \trianglelefteq N_1 ... \trianglelefteq N_k = N.$$

N is abelian by part (i), so $N_0, N_1, ..., N_{k-1}$ are also abelian since they are subgroups of N.
Thus, $N_1/N_0, N_2/N_1 .., N/N_{n-1}$ are abelian.
Combine the chain $\{(1)\} = N_0 \trianglelefteq N_1 ... \trianglelefteq N_k = N$ with the chain $A_4 \trianglelefteq N$,
we get a new chain

$$N_0 \trianglelefteq N_1 ... \trianglelefteq N \trianglelefteq A_4,$$

where $N_1/N_0, N_2/N_1 .., N/N_{k-1}, A_4/N$ are abelian. □

Exercise 2 (4.1.9)**.** Assume that G acts transitively on a finite set A, and let $H \trianglelefteq G$. Note that H also acts on A. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the distinct orbits of H on A

(a) Prove that G permutes the sets $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ in the sense that for each $g \in G$ and each $i \in [r] = \{1, \ldots, r\}$, there is a j such that $g\mathcal{O}_i = \mathcal{O}_j$ where $g\mathcal{O} = \{ga \in G \mid a \in \mathcal{O}\}$. Prove that G acts transitively on $\{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r\}$. Deduce that all orbits of H on A have the same cardinality.
(b) Prove that if $a \in \mathcal{O}_1$, then $|\mathcal{O}_1| = [H\colon H \cap G_a]$, and prove that $r = [G\colon HG_a]$.