MATH 8510, Abstract Algebra I
Fall 2016
Exercises 2-2
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Exercise 1 (1.6.20)**.** Let G be a group, and let $\mathrm{Aut}(G)$ be the set of all isomorphisms $G \xrightarrow{\cong} G$. Prove that $\mathrm{Aut}(G)$ is a group under function composition. (The elements of $\mathrm{Aut}(G)$ are *automorphism* of G, and $\mathrm{Aut}(G)$ is the *automorphism group of* G.)

Proof. $\forall g, h, k \in \mathrm{Aut}(G)$,

(1) $g : G \xrightarrow{\cong} G, h : G \xrightarrow{\cong} G$, then $gh : G \xrightarrow{\cong} G$ since $\cong$ is an equivalent relation.
So $g \circ h \in \mathrm{Aut}(G)$.

(2) Define $e : G \xrightarrow{\cong} G$ by $e(x) := x$.
Then $e \in \mathrm{Aut}(G)$.
Define $h : G \xrightarrow{\cong} G$ by $h(x) = y_x$.
$\forall x \in G$, we have $h \circ e(x) = hx$, so $h \circ e = e$.
and $e \circ h(x) = e(y_x) = y_x = h(x)$, so $e \circ h = h$.
Thus $\exists\, e \in \mathrm{Aut}(G)$ such that $e \circ h = e = h \circ e$.

(3) $\forall x \in G$,
$g \circ (h \circ k)(x) = g \circ h \circ k(x) = (g \circ h) \circ k(x)$ by the definition of function composition.
So $g \circ (h \circ k) = (g \circ h) \circ k$.

(4) $\forall x \in G, h(x) = y_x$ using the definition of h from (2).
Define $h^{-1} : G \xrightarrow{\cong} G$ by $h^{-1}(y_x) := x$.
Then $h^{-1} \in \mathrm{Aut}(G)$.
So $h^{-1} \circ h(x) = h^{-1}(y_x) = x = e(x)$, so $h^{-1} \circ h = e$
and $h \circ h^{-1}(y_x) = h(x) = y_x = e(y_x)$, so $h \circ h^{-1} = e$.
Thus, $\exists\, h^{-1} \in \mathrm{Aut}(G)$ such that $h \circ h^{-1} = e = h^{-1} \circ h$.

□

Exercise 2 (1.7.16)**.** Let G be a group.

(a) Prove that the formula $g \cdot x := gxg^{-1}$ defines a group action of G on itself. (This action is called the *conjugation*. You may wish to compare it to similarity from linear algebra.)

Proof. $\forall h, g, a, \in G, x \in G$,
let $e_G \in G$ be the identity and then $e_G g = e_G = g e_G$.
Then
(a) $e_G \cdot a = e_G^{-1} a e_G = e_G a e_G = a$.
(b) $h \cdot (g \cdot a) = h \cdot (gag^{-1}) = h(gag^{-1})h^{-1} = hgag^{-1}h^{-1}$;
$(hg) \cdot a = hga(hg)^{-1} = hgag^{-1}h^{-1}$.
So $h \cdot (g \cdot a) = (hg) \cdot a$.

□

(b) For each $g \in G$, define $\sigma_g \colon G \to G$ by the formula $\sigma_g(x) := gxg^{-1}$. Prove that the rule $g \mapsto \sigma_g$ defines a group homomorphism $G \to \mathrm{Aut}(G)$. (An automorphism of the form σ_g using the conjugation action is called an *inner* automorphism.)

Proof. $\forall g, h, a \in G$,
let f be defined as $f : G \to \mathrm{Aut}(G)$ with the rule $g \mapsto \sigma_g$.
Then $f(g) = \sigma_g$ and
$f(gh) = \sigma_{gh}$ since $gh \in G$
$f(gh)(a) = \sigma_{gh}(a) = (gh)a(gh)^{-1} = ghag^{-1}h^{-1}$.
$f(g) \circ f(h)(a) = \sigma_g \circ \sigma_h(a) = \sigma_g(hah^{-1}) = g(hah^{-1})g^{-1} = ghah^{-1}g^{-1}$.
So $\forall a \in G$, we have $f(gh)(a) = f(g) \circ f(h)(a)$.
Thus, $f(gh) = f(g) \circ f(h)$.
As a result, f defines a group homomorphism. □

Exercise 3. In your free time, read the statements of the following exercises.
1.1: 19, 20, 22–25, 31–36
1.6: 1–6, 8, 10–12, 17–23
1.7: 1–3, 8–10, 14–18