MATH 8510, Abstract Algebra I
Fall 2016
Exercises 3-2
Due date Thu 08 Sep 4:00PM

Exercise 1 (2.3.18–19 $+5\epsilon$)**.** Let G be a group, and let $g \in G$.

(a) Prove that there exists a unique group homomorphism $f_g\colon \mathbb{Z} \to G$ such that $f_g(1) = g$.

Proof. Define

$$\mathbb{Z} \to G$$
$$f_g(n) \mapsto g^n$$

Then f_g is a homomorphism since $\forall m, n \in \mathbb{Z}, f_g(m+n) = g^{m+n} = g^m g^n = f_g(m)f_g(n)$ and satisfies $f_g(1) = g$.
So there exists a group homomorphism $f_g\colon \mathbb{Z} \to G$ such that $f_g(1) = g$.
Next we show it is unique.
Assume there exists another homomorphism h_g differing from f_g such that $h_g(1) = g$.

(a) If $n > 0$, $h_g(n) = h_g(\sum_{i=1}^n 1) = \prod_{i=1}^n h_g(1) = \prod_{i=1}^n g = g^n$ since f_g is homomorphism.
(b) If $n < 0$, then $-n > 0$ and $h_g(n) = h_g(-(-n)) = (h_g(-n))^{-1} = (g^{-n})^{-1} = g^n$.
(c) If $n = 0$, then $h_g(0) = h_g(n-n) = h_g(n)h_g(-n) = g^n g^{-n} = g^{n-n} = g^0$.
So $h_g(n) = g^n = f_g(n), \forall n \in \mathbb{Z}$.

Thus, $f_g = h_g$, which is contradicted by the assumption.
Hence, such group homomorphism is unique. □

(b) Prove that $\text{Im}(f_g) = \langle g \rangle$.

Proof.
$Im(f_g) = \{g^n | n \in \mathbb{Z}\} \subset G$.
So $g^n \in G, \forall n \in \mathbb{Z}$.
Thus, $< g >= \{g^n \in G | n \in \mathbb{Z}\} = \{g^n | n \in \mathbb{Z}\} = Im(f_g)$. □

(c) Prove that f_g is a monomorphism if and only if $|g| = \infty$.

Proof.

(a) Assume f_g is a monomorphism, then f_g is 1-1 since f_g is homomorphism.
Then $\infty = |\mathbb{Z}| = |Im(f_g)| = | < \langle g \rangle| = |g|$.
So $|g| = \infty$.
(b) Assume $|g| = \infty$.
Suppose f_g is not homomorphism, then f_g is not 1-1.
So $\exists$ different $m, n \in \mathbb{Z}$ such that $f_g(m) = g^m = g^n = f_g(n)$.
Then $g^(m-n) = e_G$.
So $|g| = | < g > | \leq m - n < \infty$ since $0 < m - n < \infty$.

□

(d) Assume that $|g| = n < \infty$.
  (1) Prove that $\mathrm{Ker}(f_g) = n\mathbb{Z} := \{nm \in \mathbb{Z} \mid m \in \mathbb{Z}\}$.
  (2) Prove that there is a unique group monomorphism $\phi_g \colon \mathbb{Z}/n\mathbb{Z} \to G$ such that $\phi_g(\overline{1}) = g$.
  (3) Prove that $\mathrm{Im}(\phi_g) = \langle g \rangle$.
  (4) We say that a diagram of group homomorphisms

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & B \arrow[d, "\beta"] \\ & C \end{tikzcd}$$

"commutes" when $\beta \circ \alpha = \alpha'$. Let $\pi \colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical epimorphism, and prove that the following diagram commutes.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\pi"] \arrow[dr, "f_g"'] & \mathbb{Z}/n\mathbb{Z} \arrow[d, "\phi_g"] \\ & G \end{tikzcd}$$

Exercise 2. In your free time, read the statements of the exercises from Section 2.3.