

MATH 8510, Abstract Algebra I
 Fall 2016
 Exercises 5-2
 Due date Thu 22 Sep 4:00PM

Exercise 1 (3.4.4). Let G be a finite abelian group, and let n be a divisor of $|G|$. Use Cauchy's Theorem to show that G has a subgroup of order n .

Proof. (1) If $|G| = 1$, then $G = \{e_G\}$ and only 1 is a divisor of $|G|$.

So G with order 1 is a subgroup of G .

(2) Consider $1 < |G| < \infty$.

We will show it by induction.

Since $|G| \geq 1, |G| \in \mathbb{Z}$, k can be written as $p_1 p_2 \dots p_{k_p}$, where p_1, p_2, \dots, p_{k_p} are primes.

(a) If $|G| = p$ is a prime, then the divisor n can only be 1 or p .

So when $n = 1$, just let $H = \{e_G\} \leq G$; when $n = p$, we let $H = G \leq G$.

Namely, it holds for basic case.

(b) Suppose every group G with $|G| = p_1 p_2 \dots p_m$ has the property that if n is a divisor of $|G|$, then there exists a subgroup $H \leq G$ with $|H| = n$.

(c) Let G be a group such that $|G| = p_1 p_2 \dots p_{k_p} p_{m+1}$.

If $n = 1$, just let $H = e_G \leq G$.

If $n > 1$, then without loss of generality, we assume $p_1 | n$.

By Cauchy's theorem, there exists $x \in G$ with $|x| = p_1$.

Since G is abelian, we have $\langle x \rangle$ is normal.

Then $\langle x \rangle \trianglelefteq G$ since $x \in G$.

Then $|G/\langle x \rangle| = \frac{|G|}{|\langle x \rangle|} = \frac{p_1 p_2 \dots p_m}{p_1} = p_2 \dots p_{m+1}$.

Since n is a divisor of $|G| = p_1 p_2 \dots p_{k_p} p_{m+1}$ and $p_1 | d$, then $\frac{n}{p_1}$ is a divisor of $p_2 p_3 \dots p_{m+1}$.

Thus, by assumption, there exists a subgroup $H/\langle x \rangle$ of $G/\langle x \rangle$ with order $|H/\langle x \rangle| = \frac{n}{p_1}$.

Besides, by the fourth isomorphism theorem, we have $H \leq G$.

As last, by Lagrange's theorem, we have $|H| = |H/\langle x \rangle| |\langle x \rangle| = \frac{n}{p_1} p_1 = n$.

Therefore, our assumption also holds for $|G| = p_1 p_2 \dots p_{k_p} p_{m+1}$.

Hence, the claim holds.

□

Exercise 2. Let G be a finite abelian group, written additively. Prove that there is a chain of subgroups $0 = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_k = G$ such that each quotient N_i/N_{i-1} is cyclic of prime order.