MATH 8510, Abstract Algebra I
Fall 2016
Exercises 2-1
Shuai Wei

Exercise 1. Let $n \in \mathbb{N}$, and consider the complex number

$$e^{2\pi i/n} = \cos(2\pi/n) + \sin(2\pi/n)i \neq 0$$

as an element of the multiplicative abelian group $\mathbb{C}^{\times}$. Compute the order $|e^{2\pi i/n}|$.

$$e = e^{2k\pi i/n} = 1_{\mathbb{R}} + 0_{\mathbb{R}}i \in \mathbb{C}^{\times}.$$

where $k \in \mathbb{Z}$.

$$\forall n \in \mathbb{N},\ e^{2\pi i/n}e = e^{2\pi i/n}(1_{\mathbb{R}} + 0_{\mathbb{R}}i) = e^{2\pi i/n}1_{\mathbb{R}} = e^{2\pi i/n}.$$

(1) If $n = 1, e^{2\pi i/n} = e^{2\pi i} = 1_{\mathbb{R}} + 0_{\mathbb{R}}i = e$, so $|e^{2\pi i/n}| = n = 1$.
So the order $|e^{2\pi i/n}|$ is 1.

(2) If $n > 1$,

$$(e^{2\pi i/n})^1 = \cos(2\pi/n) + \sin(2\pi/n)i \neq 1_{\mathbb{R}} + 0_{\mathbb{R}}i;$$

For $1 < m < n$,

$$(e^{2\pi i/n})^m = \cos(2m\pi/n) + \sin(2m\pi/n)i \neq 1_{\mathbb{R}} + 0_{\mathbb{R}}i;$$

since $m/n \notin \mathbb{Z}$.
But

$$(e^{2\pi i/n})^n = e^{2\pi i} = \cos(2\pi) + \sin(2\pi)i = 1_{\mathbb{R}} + 0_{\mathbb{R}}i = e.$$

So the order $|e^{2\pi i/n}|$ is n.

In summary, the order $|e^{2\pi i/n}|$ is n.

Exercise 2. Let A and B be groups. Prove that A and B are both abelian if and only if the cartesian product $A \times B$ is abelian.

Proof. $\forall a1, a2 \in A, b1, b2 \in B$, we have $(a1, b1), (a2, b2) \in A \times B$.
Then $(a1, b1), (a2, b2)$ are abitrary two elements from $A \times B$.
By definition, $(a1, b1)(a2, b2) = (a1a2, b1b2), (a2, b2)(a1, b1) = (a2a1, b2b1)$.

$$A \times B \text{ is abelian group.}$$

$$\Leftrightarrow (a1, b1)(a2, b2) = (a2, b2)(a1, b1).$$

$$\Leftrightarrow (a1a2, b1b2) = (a2a1, b2b1).$$

$$\Leftrightarrow a1a2 = a2a1 \text{ and } b1b2 = b2b1.$$

$$\Leftrightarrow \text{A and B are both abelian.}$$

□

Exercise 3. Let G be a group, and let $x \in G$ be an element with finite order n. Prove that the elements $1, x, x^2, \ldots, x^{n-1}$ are distinct in G. Deduce that $|x| \leq |G|$.

Proof. $x \neq 0$ since $0^n = 0 \neq 1$ for positive integer n, namely, 0 has not finite order.
Then we can write 1 as x^0.
Assume there exists $0 \leq i < j < n$ such that $x^i = x^j$.
Let $(x^i)^{-1} = x^{-i}$ be the inverse of x^i.
Then multiply x^{-i} in two sides of $x^i = x^j$, we have $1 = x^{i-i} = x^{j-i}$.
Thus $j - i >= n$ since n is the order of x.
It is a contradiction since $j - i \leq n - 1$ by the assumption $0 \leq i < j < n$.
Therefore, the elements $1, x, x^2, \ldots, x^{n-1}$ are distinct in G.
As a result, we get G has at least n distinct elements, $|G| \geq n = |x|$, which completes the proof. $\square$