

MATH 8510, Abstract Algebra I

Fall 2016

Exercises 6-2

Due date Thu 29 Sep 4:00PM

See the text for hints.

Exercise 1 (4.2.10). Prove that every non-abelian group of order 6 has a non-normal subgroup of order 2. Use this to classify all groups of order 6; specifically, provide a list G_1, G_2 of groups of order 6 such that (1) $G_1 \not\cong G_2$, and (2) for all groups G if $|G| = 6$, then $G \cong G_1$ or $G \cong G_2$. Justify your answers.

Proof. Let G be a group which is not abelian and $|G| = 6$.

Since $2|6$ and $3|6$ and $2, 3$ are primes,

by Cauchy Theorem, there exists $H, M \leq G$ and $|H| = 2$ and $|M| = 3$.

Assume all the subgroups of G of order 2 are normal subgroups of G .

Consider $H = \{e_G, h\} \leq G$, where $h \in G$ and $h \neq e_G$ and $h^2 = 1$.

Then $|H| = 2$ and $H \trianglelefteq G$ by assumption.

As a result, $N_G(H) = G$.

So for any $g \in G$, $gH = Hg$.

Since $h \in H$, we have $gh = hg$ for any $g \in G$.

So $h \in Z(G)$.

Since $|G| = 2 \times 3$ and $2, 3$ are primes, we have G is abelian or $Z(G) = \{e_G\}$ by the conclusion from Exercise 1 in homework 3.

Given G is not abelian, we have $Z(G) = \{e_G\}$, which is a contradiction since we already find $h \in Z(G)$ and $h \neq e_G$.

Thus, there exists a non-normal subgroup of G of order 2.

G is arbitrary, so every non-abelian group of order 6 has a non-normal subgroup of order 2.

Next we show G is isomorphic to S_3 .

Let $H \leq G$ of order 2 and be the non-normal subgroup of G .

Then G acts transitively on G/H .

Let $\pi_H : G \rightarrow S_{G/H}$ be the associated permutation representation.

Then

$$\text{Ker}(\pi_H) = \bigcap_{x \in G} xHx^{-1} \subset H.$$

So $|\text{Ker}(\pi_H)| = 1$ or 2 .

If $|\text{Ker}(\pi_H)| = 2$, then $\text{Ker}(\pi_H) = H$, which is a contradiction since $\text{Ker}(\pi_H) \trianglelefteq G$ and H is not a normal subgroup of G by assumption. So $|\text{Ker}(\pi_H)| = 1$.

Thus, π_H is 1-1.

Since $|G/H| = \frac{|G|}{|H|} = \frac{6}{2} = 3$,

$$S_{G/H} \cong S_3.$$

Then $|S_{G/H}| = |S_3| = 6$.

Since $|G| = 6$, we have π_H is onto since it is also 1-1.

Thus, π_H is bijective.

As a result, π_H is an isomorphism.

So

$$G \cong S_{GH}.$$

Therefore, for each non-abelian group G of order 6, we have

$$G \cong S_3.$$

Next we claim that every abelian group of order 6 is cyclic.

Let G be an abelian group of order 6.

By Cauchy Theorem, there exists $x, y \in G$ and $|x| = 2, |y| = 3$.

Then $x \neq y$.

Besides, $x^{-1} = x$ and $y^{-1} = y^2 \in G$.

Then $x \neq y^2$, otherwise, $x = y$.

So we find $\{e_G, x, y, y^2\} \subset G$.

We can verify that two distinct elements $xy, xy^2 \in G$ but $xy, xy^2 \notin \{e_G, x, y, y^2\}$.

Since $|G| = 6$, we have

$$G = \{e_G, x, y, y^2, xy, xy^2\}.$$

We claim $G = \langle xy \rangle$.

Since G is abelian,

$$\begin{aligned} (xy)^1 &= xy = e_G. \\ (xy)^2 &= x^2y^2 = y^2, \\ (xy)^3 &= x^3y^3 = x, \\ (xy)^4 &= x^4y^4 = y, \\ (xy)^5 &= x^5y^5 = xy^2, \\ (xy)^6 &= x^6y^6 = e_G. \end{aligned}$$

Thus, G is cyclic.

So every abelian group of order 6 is cyclic.

We know if a group G is cyclic of order 6, then

$$G \cong \mathbb{Z}/6\mathbb{Z}.$$

So for each abelian group G of order 6, we have

$$G \cong \mathbb{Z}/6\mathbb{Z}.$$

Since S_3 is not cyclic,

$$S_3 \not\cong \mathbb{Z}/6\mathbb{Z}.$$

As result, we have for any non-abelian group G of order 6,

$$G \cong S_3,$$

and for any abelian group G of order 6,

$$G \cong \mathbb{Z}/6\mathbb{Z},$$

where $S_3 \not\cong \mathbb{Z}/6\mathbb{Z}$.

□

Exercise 2 (4.3.6). Assume that G is a non-abelian group of order 15. Prove that $Z(G) = \{e\}$. Use the fact that $\langle g \rangle \leq C_G(g)$ to show that there is at most one possible class equation for G ; in other words, in the notation of Theorem 4.2.4 of the notes, find r and $|Z(G)|$ and $[G : C_G(g_1)], \dots, [G : C_G(g_r)]$. Justify your answers.