1. Let $\{N(t); t \geq 0\}$ be a renewal process with common interrenewal probability mass function

$$P(X_n = 1) = 0.8, \; P(X_n = 2) = 0.2, \; n \geq 1.$$

Compute $P(N(t) = k)$ for $k \geq 1$.

2. Compute the renewal equation for $E(A(t)B(t))$, where $A(t) = t - T_{N(t)}$ is the age process at time t, and $B(t) = T_{N(t)+1} - t$ is the excess life process at time t for a renewal process $\{N(t); t \geq 0\}$. Show that the key renewal theorem is applicable and compute the limiting value of $E(A(t)B(t))$ as $t \to \infty$, assuming that the interrenewal CDF F has finite second moment, and it is nonarithmetic.

3. Compute the renewal equation for $P(N(t)$ is odd$)$ for a renewal process $\{N(t); t \geq 0\}$. Is this a renewal type equation? Solve it explicitely when the renewal process is a Poisson process with rate λ.

4. Let $\{N_D(t); t \geq 0\}$ be a delayed renewal process with interrenewals $\{X_n\}_{n \geq 1}$. Assume that X_1 has cdf G, and $X_n, n \geq 2$, has cdf F, and both G and F have positive finite mean. Show that

$$\lim_{t \to \infty} \frac{N_D(t)}{t} = \frac{1}{E(X_2)},$$

and

$$\lim_{t \to \infty} \frac{E(N_D(t))}{t} = \frac{1}{E(X_2)}.$$

5. Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time it takes the miner to become free.

   (a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$, and a stopping time N such that

   $$T = \sum_{i=1}^{N} X_i.$$

   Note: To define $X_1, X_2, \ldots$, you may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

   (b) Use Wald's equation to find $E(T)$.

   (c) Compute $E\left[\sum_{i=1}^{N} X_i | N = n\right]$, and note that it is not equal to $E\left[\sum_{i=1}^{N} X_i\right]$.

   (d) Use part (c) for a second derivation of $E[T]$.

6. Consider a single-server bank for which customers arrive in accordance with a Poisson process with rate λ. If a customer will enter the bank only if the server is free when he/she arrives, and if the service time of a customer has the distribution G, then what proportion of time (on the average) is the server busy?

7. Suppose that passengers arrive at a bus stop according to a Poisson process with rate λ. Suppose also that buses arrive according to a renewal process with interrenewal distribution function F, and that buses pick up all waiting passengers. Assume that the Poisson process of people arriving and the renewal process of buses arriving are independent.

   (a) Find the average number of people who are waiting for a bus, averaged over all time. Hint: You may construct a renewal reward process w.r.t. the arrival times of the bus, and between two consecutive arrival times, the reward would be the number of passengers.

   (b) Find the average amount of time that a passenger waits, averaged over all passengers. Hint: You may construct a discrete time renewal reward process $\{W_n\}_{n\geq 1}$ w.r.t $\{nN\}_{n\geq 1}$, where W_n is the waiting time of the nth passenger, and N is the number of passengers who arrive during the cycles in part (a). The reward between nN and $(n+1)N$ would be the total amount of waiting time $\sum_{k=nN+1}^{(n+1)N} W_k$.