Homework 4, MATH 8010

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2.

(b) *Proof.* Let T_n be the k^{th} replacement for $k \geq 1$ and set $T_0 = 0$.

Then $\{T_n - T_{n-1}\}_{n=1}^{\infty}$ is an *iid* sequence of nonnegative random variables.

Let S_1 be the first in use failure.

Let $N = \inf\{n \geq 1 : T_n - T_{n-1} < T\}$.

Since $\mathbb{1}(N \leq n) = 1 - \mathbb{1}(N > n) = 1 - \mathbb{1}(T_1 - T_0 \geq T, T_2 - T_1 > T, ..., T_n - T_{n-1} \geq T)$, N is a stopping time w.r.t. $\{T_n - T_{n-1}\}_{n=0}^{\infty}$.

Then

$$S_1 = \sum_{n=1}^{N} (T_n - T_{n-1}).$$

By Wald's identity, we have

$$E(S_1) = E(N)E(T_1).$$

Since for $n \in \mathbb{N}$,

$$\begin{aligned} P(N = n) &= P(T_n - T_{n-1} < T) \prod_{k=1}^{n-1} P(T_k - T_{k-1} \geq T) \\ &= P(T_1 < T) \prod_{k=1}^{n-1} P(T_1 \geq T) \\ &= F(T) \left(1 - F(T)\right)^{n-1}, \end{aligned}$$

$$E(N) = \sum_{n=1}^{\infty} nP(N = n) = F(T) \sum_{n=1}^{\infty} n \left(1 - F(T)\right)^{n-1}.$$

We know the Maclaurin series of $\frac{1}{(1-x)^2}$ is

$$\sum_{n=1}^{\infty} nx^{n-1}$$

for $-1 < x < 1$.

The density f is continuous on $(0, \infty)$, so $0 < F(T) < 1$ for $T \in \mathbb{R}^+$.

Then we have $0 < 1 - F(T) < 1$.

So, replace x with $1 - F(T)$, we have

$$\sum_{n=1}^{\infty} n \left(1 - F(T)\right)^{n-1} = \frac{1}{1 - \left(1 - F(T)\right)^2} = \frac{1}{F^2(T)}.$$

Then

$$E(N) = F(T)\frac{1}{F^2(T)} = \frac{1}{F(T)}.$$

Thus,the averge time between two successive in-use failures is

$$E(S_1) = E(N)E(T_1) = \frac{\int_0^T xf(x)dx + T\left(1 - F(T)\right)}{F(T)}.$$

$\square$