STAT 8101 Stochastic Processes Final

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Exercise 1. *Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?*

Solution: Denote the three urns by U_R, U_B, U_W which contain the following balls:

$$U_R = \{1 \text{ red, } 4 \text{ blue}\},$$

$$U_B = \{2 \text{ red, } 2 \text{ blue, } 3 \text{ white}\},$$

$$U_W = \{3 \text{ red, } 2 \text{ blue, } 4 \text{ white}\}.$$

From above we have total balls in $U_R = 5$, $U_B = 9$, and $U_W = 7$. We define a Markov Chain whose state is the current urn. Since, from the problem statement we start in U_R, we let State 0 = red. Then we randomly define State 1 = blue and State 2 = white. From the fraction of each color ball per urn, we can define the state transition matrix as:

$$P = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 \\ \frac{3}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \end{bmatrix}.$$

To find the proportion of each ball selected in the long run, we calculate the limiting probabilities π_0, π_1, and π_2 using the following equations:

$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = \frac{1}{5}\pi_0 + \frac{3}{9}\pi_1 + \frac{2}{7}\pi_2,$$

$$\pi_1 = \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21} = \frac{4}{5}\pi_0 + \frac{2}{9}\pi_1 + \frac{2}{7}\pi_2,$$

$$\pi_2 = \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} = \frac{4}{9}\pi_1 + \frac{3}{7}\pi_2,$$

subject to the constraint $\pi_0 + \pi_1 + \pi_2 = 1$. From the above equations, we get the following system:

$$\begin{bmatrix} \frac{1}{5} - 1 & \frac{3}{9} & \frac{2}{7} \\ \frac{4}{5} & \frac{2}{9} - 1 & \frac{2}{7} \\ \frac{2}{7} & \frac{4}{9} & \frac{3}{7} - 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

From the equation for π_2 we have

$$\pi_2 = \frac{4}{9}\pi_1 + \frac{3}{7}\pi_2 \Rightarrow \pi_2 = \frac{7}{9}\pi_1.$$

Substituting this into the equation for π_1 gives us

$$\pi_1 = \frac{4}{5}\pi_0 + \frac{2}{9}\pi_1 + \frac{2}{9}\pi_1 \Rightarrow \pi_0 = \frac{25}{36}\pi_1.$$

Next, we use the constraint that all π_i must sub to 1:

$$\pi_0 + \pi_1 + pi_2 = \frac{25}{36}\pi_1 + \pi_1 + \frac{7}{9}\pi_1 = 1 \Rightarrow \pi_1 = \frac{36}{89}.$$

Finally we get

$$\text{Proportion of selected balls are red} = \pi_0 = \frac{25}{89}$$

$$\text{Proportion of selected balls are blue} = \pi_1 = \frac{36}{89}$$

$$\text{Proportion of selected balls are white} = \pi_2 = \frac{28}{89}.$$

Exercise 2. *An individual possesses r umbrellas that he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability p.*

(a) Define a Markov chain with $r+1$ states, which will help us to determine the proportion of time that our man gets wet. (Note: He gets wet if it is raining, and all umbrellas are at his other location.)

(b) Find the limiting probabilities π_i.

(c) What fraction of time does our man get wet?

(d) When $r = 3$, what value of p maximizes the fraction of time he gets wet?

Solution for (a): We have r total umbrellas and at either location we can have a total of $0, \cdots, r$ umbrellas. We define a Markov Chain in the following way: let the state of the chain correspond to the number of umbrellas at the current location. This gives us a Markov Chain with $r+1$ states. We see that the Markovian property is satisfied since the probability that it rains on a given day is independent of past days, and thus the umbrella distribution on day $n+1$ only depends on the distribution on day n.

The transition probability $P_{i,j}$ is the probability that given i umbrellas at the current location, what is probability we have j umbrellas at the next location. The transitions follow as:

$$P_{0,j} = 0 \ \forall j < r, \ \ P_{0,r} = 1,$$

$$P_{i,r-i} = 1-p, \ \ P_{i,r-i+1} = p, \ \ , \ \ \forall 1 \leq i \leq r.$$

The first line follows since, given 0 umbrellas at the current location, the only possibility is r umbrellas at the other. The second line follows since, given i umbrellas at the current location, there will be at least $r-i$ umbrellas at the next location. If it rains (probability p) we will bring an umbrella, giving us $r-i+1$ total at the next location. If it does not rain (probability $1-p$), we will not bring an umbrella and therefor the next location will have $r-i$ umbrellas. Finally, we get the state transition matrix:

$$P = \begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1-p & p \\ 0 & \cdots & 1-p & p & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1-p & p & 0 & \cdots & 0 \end{bmatrix}.$$

Solution for (b): We use the following formulas

$$\pi_j = \sum_{i=0}^{r} \pi_i P_{i,j},$$

$$\sum_{i=0}^{r} \pi_i = 1,$$

and compute the limiting probabilities for three cases: π_0, π_i, and π_r. From the above transition matrix, we note that each column contains 1 or 2 elements. Thus, we expect the limiting probabilities to contain either 1 or 2 terms.

$$\pi_0 = \sum_{j=0}^{r} \pi_j P_{j,0} = \pi_r(1-p),$$

$$\pi_i = \sum_{j=0}^{r} \pi_j P_{j,i} = \pi_{r-i+1}p + \pi_{r-i}(1-p),$$

$$\pi_r = \sum_{j=0}^{r} \pi_j P_{j,r} = \pi_0 + \pi_1 p.$$

From this we get

$$\pi_0 = (1-p)\pi_1,$$

and therefore $\pi_1 = \pi_r$. Then

$$\pi_1 = (1-p)\pi_{r-1} + p\pi_r = (1-p)\pi_{r-1} + p\pi_1 \Rightarrow \pi_1 = \pi_{r-1}.$$

Similarly,

$$\pi_{r-1} = (1-p)\pi_1 + p\pi_2 = (1-p)\pi_{r-1} + p\pi_2 \Rightarrow \pi_1 = \pi_2.$$

We repeat this process, considering π_2, then $\pi_{r-2}, \pi_3, \pi_{r-3}$, etc. Doing this results in

$$\pi_1 = \pi_2 = \cdots = \pi_r.$$

Using the constraint that all limiting probabilities must sum to 1, we get

$$1 = \sum_{i=0}^{r} \pi_i = \pi_0 + r\pi_1 = \pi_1(1-p) + r\pi_1 \Rightarrow \pi_1 = \frac{1}{r+1-p}.$$

Plugging this into our equation for π_0 we get

$$\pi_0 = \frac{1-p}{r+1-p}$$

Solution for (c): We know that the person will only get wet if it rains and there are no umbrellas at their current location. The only situation where there are zero umbrellas at the current location is π_0. This, combined with the probability p that it will rain, gives us that the person will get wet with probability

$$\pi_0 p = \frac{p(1-p)}{r+1-p}.$$

Solution for (d): Let $r = 3$. We take the results from (c), and find the value that maximizes this by differentiating (using the quotient rule) and setting equal to 0:

$$\frac{d}{dp}\frac{p(1-p)}{r+1-p}) = \frac{(4-p)(1-2p) + p(1-p)}{(4-p)^2} = \frac{p^2 - 8p + 4}{(4-p)^2} = 0$$

Then we take $p^2 - 8p + 4 = 0$ which gives us $p = 4 - 2\sqrt{3} \approx 0.54$.

To see if this is actually a minimum, we take the second derivative and plug in the value of 0.54:

$$\frac{d}{dp}(\frac{p^2 - 8p + 4}{(4-p)^2}) = \frac{(4-p)^2(2p-8) + 2(4-p)(p^2 - 8p + 4}{(4-p)^4}\Big|_{p=0.54} \approx -0.58 < 0.$$

Since the second derivative is less than 0, we see that we do indeed have a minimum.

Exercise 3. *There are two servers available to process n jobs. Initially, each server begins work on a job. Whenever a server completes work on a job, that job leaves the system and the server begins processing a new job (provided there are still jobs waiting to be processed). Let T denote the time until all jobs have been processed. If the time that it takes server i to process a job is exponentially distributed with rate μ_i, $i = 1, 2$, find $E(T)$ and $Var(T)$.*

Solution: Let T denote the time until all n jobs have been processed and let T_i denote the time in between the $(i-1)$th and ith job. Then we have that $T = \sum_{i=1}^{n} T_i$. Also, the time it takes a server to process a job is exponentially distributed with parameters μ_1, μ_2. Therefore, T_i, for $0 \leq i \leq n-1$, is the minimum time of server 1 or 2, and, thus, is exponentially distributed with parameter $\mu_1 + \mu_2$. Note, that these T_i are independent.

We cannot have both servers working on the nth job simultaneously, either server 1 works on the nth job or server 2. Thus, T_n cannot be the minimum time for server 1 and 2. If server 1 is processing job n, we have that T_n is Exp(μ_1). This will happen with probability $\frac{\mu_2}{\mu_1+\mu_2}$. Similarly, if server 2 is processing job n, we have that T_n is Exp(μ_2) and this will happen with probability $\frac{\mu_1}{\mu_1+\mu_2}$.

From this we have

$$E\{T_i\} = \frac{1}{\mu_1 + \mu_2}, \ \ 0 \leq i \leq n-1,$$

$$E\{T_n\} = E\{T_n | T_n \sim Exp(\mu_1)\} P(T_n \sim Exp(\mu_1)) + E\{T_n | T_n \sim Exp(\mu_2)\} P(T_n \sim Exp(\mu_2))$$

$$= \frac{\mu_2}{\mu_1 + \mu_2}\frac{1}{\mu_1} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{1}{\mu_2}.$$

Putting this together gives us

$$E\{T\} = E\{\sum_{i=1}^{n} T_i\} = \frac{n-1}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2}\frac{1}{\mu_1} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{1}{\mu_2}.$$

Next, we compute Var$\{T\}$, noting that

$$\text{Var}\{T_i\} = \frac{1}{(\mu_1 + \mu_2)^2}, \ \ 0 \leq i \leq n-1,$$

$$\text{Var}\{T_n\} = E\{T_n^2\} - E\{T_n\}^2$$

$$= E\{T_n^2 | T_n \sim Exp(\mu_1)\} P(T_n \sim Exp(\mu_1)) + E\{T_n^2 | T_n \sim Exp(\mu_2)\} P(T_n \sim Exp(\mu_2)) - E\{T_n\}^2$$

$$= \frac{\mu_2}{\mu_1 + \mu_2}\frac{2}{\mu_1^2} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{2}{\mu_2^2} - (\frac{\mu_2}{\mu_1 + \mu_2}\frac{1}{\mu_1} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{1}{\mu_2})^2$$

Then we can write

$$\text{Var}\{T\} = \text{Var}\{\sum_{i=1}^{n} T_i\} =$$

$$= \frac{n-1}{(\mu_1 + \mu_2)^2} + \frac{\mu_2}{\mu_1 + \mu_2}\frac{2}{\mu_1^2} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{2}{\mu_2^2} - (\frac{\mu_2}{\mu_1 + \mu_2}\frac{1}{\mu_1} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{1}{\mu_2})^2.$$

Exercise 4. *Consider a two-state continuous time Markov chain with states 0 and 1. Suppose the birth rate is λ and the death rate is μ. Starting in state 0, find Cov$[X(s), X(t)]$.*

Solution: Using the covariance formula we have

$$\text{Cov}[X(s), X(t)] = E[X(s), X(t)] - E[X(s)]E[X(t)].$$

Therefore, we consider each term separately: $E[X(s), X(t)]$, $E[X(s)]$, and $E[X(t)]$.

$$\begin{aligned} E\{X(t)\} &= E\{X(t)\big|X(0) = 1\}P(X(0) = 1) + E\{X(t)\big|X(0) = 0\}P(X(0) = 0) \\ &= E\{X(t)\big|X(0) = 0\} = \sum_{x=0}^{1} P(X(t) = x\big|X(0) = 0) = P_{00}(t). \end{aligned}$$

For a continuous time Markov Chain, with two states, the probability $P_{00}(t)$ was derived in class and is equal to:

$$P_{00}(t) = \frac{1}{\mu + \lambda}\big(\mu + \lambda e^{-(\mu+\lambda)t}\big)$$

Thus, we have

$$E[X(t)]E[X(s)] = \frac{1}{(\mu + \lambda)^2}\big(\mu + \lambda e^{-(\mu+\lambda)t}\big)\big(\mu + \lambda e^{-(\mu+\lambda)s}\big).$$

Next, we compute $E[X(s), X(t)]$. Since we are dealing with two states, 0 and 1, we see that $X(s)X(t) = 1$ if and only if $X(s) = X(t) = 1$. Following similar steps as above we get

$$E[X(s), X(t)] = P(X(t) = 1, X(s) = 1\big|X(0) = 0)$$

which can be rewritten as, since we are dealing with Markov Chains,

$$P(X(t) = 1, X(s) = 1\big|X(0) = 0) = P_{00}(t - s)P_{00}(s).$$

Using the same formula as above, we get

$$E[X(s), X(t)] = \frac{1}{(\mu + \lambda)^2}\big(\mu + \lambda e^{-(\mu+\lambda)(t-s)}\big)\big(\mu + \lambda e^{-(\mu+\lambda)s}\big).$$

Putting this all together gives us

$$\text{Cov}[X(s), X(t)] = \frac{1}{(\mu + \lambda)^2}\big(\mu + \lambda e^{-(\mu+\lambda)s}\big)\Big(\big(\mu + \lambda e^{-(\mu+\lambda)(t-s)}\big) + \big(\mu + \lambda e^{-(\mu+\lambda)(t)}\big)\Big).$$

Exercise 5. *Consider two machines. Machine i operates for an exponential time with rate λ_i and then fails; its repair time is exponential with rate μ_i, $i = 1, 2$. The machines act independently of each other. Define a four-state continuous time Markov chain that jointly describes the condition of the two machines. Use the assumed independence to compute the transition probabilities for this chain and then verify these transition probabilities satisfy the forward and backward equations.*

Solution: First, we consider each machine separately. Since both machines act in the same way, we can consider machine i. Following Example 6.11 (which was also worked out in class), we have for a continuous time Markov Chain with two states, the following transition probabilities:

$$P_{00}(t) = \frac{1}{\mu_i + \lambda_i}\left[\mu_i + \lambda_i e^{-(\mu_i+\lambda_i)t}\right],$$

$$P_{10}(t) = \frac{1}{\mu_i + \lambda_i}\left[\mu_i + \mu_i e^{-(\mu_i+\lambda_i)t}\right].$$

In a similar fashion (using the backwards equations of the birth / death process given by equation 6.9 in Ross and then following step by step the same derivation as in Example 6.11), one can derive the following equations for P_{01} and P_{11}:

$$P_{01}(t) = \frac{1}{\mu_i + \lambda_i}\left[\lambda_i + \mu_i e^{-(\mu_i+\lambda_i)t}\right],$$

$$P_{11}(t) = \frac{1}{\mu_i + \lambda_i}\left[\lambda_i + \lambda_i e^{-(\mu_i+\lambda_i)t}\right].$$

Let 0 denote a working machine and 1 denote a broken machine. Then we define a Markov Chain with states $(0,0), (0,1), (1,0)$, and $(1,1)$, where the left index corresponds to the state of machine 1 and the right index corresponds to the state of machine 2. Since each machine operates with exponential times and the machines are independent, we get the following state transition rates, which will be used in the Forward / Backward equations:

$$v_{(0,0)} = \lambda_1 + \lambda_2, \;\; v_{(0,1)} = \lambda_1 + \mu_2,$$

$$v_{(1,0)} = \mu_1 + \lambda_2, \;\; v_{(1,1)} = \mu_1 + \mu_2.$$

The above comes from the fact that we are considering minimum times of exponentials. From these transition rates, we can calculate the instantaneous transition rates, using $q_{ij} = v_i P_{ij}$:

$$q_{(0,0)(0,1)} = v_{(0,0)} P_{(0,0)(0,1)} = (\lambda_1 + \lambda_2)\frac{\lambda_2}{\lambda_1 + \lambda_2} = \lambda_2,$$

$$q_{(0,0)(1,0)} = v_{(0,0)} P_{(0,0)(1,0)} = (\lambda_1 + \lambda_2)\frac{\lambda_1}{\lambda_1 + \lambda_2} = \lambda_1,$$

$$q_{(0,1)(0,0)} = v_{(0,1)} P_{(0,1)(0,0)} = (\lambda_1 + \mu_2)\frac{\mu_2}{\lambda_1 + \mu_2} = \mu_2,$$

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$$q_{(1,1)(1,0)} = v_{(1,1)} P_{(1,1)(1,0)} = (\mu_1 + \mu_2) \frac{\lambda_2}{\mu_1 + \mu_2} = \lambda_2.$$

From the independence of the two machines, we can write the probability transitions of the 4 state Markov Chain as the product of each machine's probability transitions

$$P_{(a,b),(c,d)}(t) = P^{[1]}_{(a,c)}(t) P^{[2]}_{(b,d)}(t),$$

where $P^{[i]}_{(a,b)}$ refers to the ith machine. Given our chain has 4 states, we now have 16 transition probabilities. The derivation of each, is essentially the same, so we consider the forward equation only for $P_{(1,1),(1,1)}(t)$:

$$\begin{aligned} P'_{(1,1),(1,1)}(t) &= q_{(0,1)(1,1)} P_{(1,1),(0,1)}(t) + q_{(1,0)(1,1)} P_{(1,1),(1,0)}(t) - v_{(1,1)} P_{(1,1),(1,1)}(t) \\ &= \lambda_1 P^{[1]}_{(1,0)}(t) P^{[2]}_{(1,1)}(t) + \lambda_2 P^{[1]}_{(1,1)}(t) P^{[2]}_{(1,0)}(t) - (\mu_1 + \mu_2) P^{[1]}_{(1,1)}(t) P^{[2]}_{(1,1)}(t) \end{aligned}$$

$$\begin{aligned} &= \lambda_1 \Big(\frac{1}{\mu_1 + \lambda_1} \big[\mu_1 + \mu_1 e^{-(\mu_1 + \lambda_1)t}\big]\Big)\Big(\frac{1}{\mu_2 + \lambda_2}\big[\lambda_2 + \lambda_2 e^{-(\mu_2 + \lambda_2)t}\big]\Big) \\ &\qquad + \lambda_2 \Big(\frac{1}{\mu_1 + \lambda_1} \big[\lambda_1 + \lambda_1 e^{-(\mu_1 + \lambda_1)t}\big]\Big)\Big(\frac{1}{\mu_2 + \lambda_2}\big[\mu_2 + \mu_2 e^{-(\mu_2 + \lambda_2)t}\big]\Big) \\ &\qquad\qquad - (\mu_1 + \mu_2) \Big(\frac{1}{\mu_1 + \lambda_1} \big[\lambda_1 + \lambda_1 e^{-(\mu_1 + \lambda_1)t}\big]\Big)\Big(\frac{1}{\mu_2 + \lambda_2}\big[\lambda_2 + \lambda_2 e^{-(\mu_2 + \lambda_2)t}\big]\Big) \\ &\qquad\qquad = P^{[1]}_{(a,c)}(t) \frac{d}{dt} P^{[2]}_{(b,d)}(t) + \frac{d}{dt} P^{[1]}_{(a,c)}(t) P^{[2]}_{(b,d)}(t) \end{aligned}$$

after some considerable manipulation. Thus, we get the following, which satisfies the forward equation

$$P'_{(1,1),(1,1)}(t) = P^{[1]}_{(a,c)}(t) \frac{d}{dt} P^{[2]}_{(b,d)}(t) + \frac{d}{dt} P^{[1]}_{(a,c)}(t) P^{[2]}_{(b,d)}(t) = \frac{d}{dt}\big[P^{[1]}_{(1,1)}(t) P^{[2]}_{(1,1)}(t)\big] = \frac{d}{dt}\big[P_{(1,1),(1,1)}(t)\big].$$

Again, the derivations for all 16 equations are essentially the same, so we consider the backward equation only for $P_{(1,1),(1,1)}(t)$:

$$\begin{aligned} P'_{(1,1),(1,1)}(t) &= q_{(1,1)(1,0)} P_{(1,0),(1,1)}(t) + q_{(1,1)(0,1)} P_{(0,1),(1,1)}(t) - v_{(1,1)} P_{(1,1),(1,1)}(t) \\ &= \lambda_2 P_{(1,0),(1,1)}(t) + \mu_1 P_{(0,1),(1,1)}(t) - (\mu_1 + \mu_2) P_{(1,1),(1,1)}(t) \end{aligned}$$

$$\begin{aligned} &= \lambda_2 \Big(\frac{1}{\mu_1 + \lambda_1} \big[\lambda_1 + \lambda_1 e^{-(\mu_1 + \lambda_1)t}\big]\Big)\Big(\frac{1}{\mu_2 + \lambda_2}\big[\lambda_2 + \mu_2 e^{-(\mu_2 + \lambda_2)t}\big]\Big) \\ &\qquad + \mu_1 \Big(\frac{1}{\mu_1 + \lambda_1} \big[\lambda_1 + \mu_1 e^{-(\mu_1 + \lambda_1)t}\big]\Big)\Big(\frac{1}{\mu_2 + \lambda_2}\big[\lambda_2 + \lambda_2 e^{-(\mu_2 + \lambda_2)t}\big]\Big) \\ &\qquad\qquad - (\mu_1 + \mu_2) \Big(\frac{1}{\mu_1 + \lambda_1} \big[\lambda_1 + \lambda_1 e^{-(\mu_1 + \lambda_1)t}\big]\Big)\Big(\frac{1}{\mu_2 + \lambda_2}\big[\lambda_2 + \lambda_2 e^{-(\mu_2 + \lambda_2)t}\big]\Big) \end{aligned}$$

$$= P^{[1]}_{(a,c)}(t)\frac{d}{dt}P^{[2]}_{(b,d)}(t) + \frac{d}{dt}P^{[1]}_{(a,c)}(t)P^{[2]}_{(b,d)}(t)$$

after some considerable manipulation. Thus, we get the following, which satisfies the forward equation

$$P'_{(1,1),(1,1)}(t) = P^{[1]}_{(a,c)}(t)\frac{d}{dt}P^{[2]}_{(b,d)}(t) + \frac{d}{dt}P^{[1]}_{(a,c)}(t)P^{[2]}_{(b,d)}(t) = \frac{d}{dt}\left[P^{[1]}_{(1,1)}(t)P^{[2]}_{(1,1)}(t)\right] = \frac{d}{dt}\left[P_{(1,1),(1,1)}(t)\right].$$