MATH 8170,

Fall 2016

HW 2

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**1.**

Let $X_A(t)$ denote the number of organisms in state A at time $t$.

Let $X_B(t)$ denote the number of organisms in state B.

Then $X = \{X_A, X_B\}$ is a CTMC with sojourn time rate $\upsilon_{ij} = \alpha i + \beta j$ for each state $(i, j) \in E$, where the state space is

$$E = \{(i, j)\}_{i,j \geq 0, i+j \geq 1},$$

where $i, j \geq 0$ and $i + j \geq 1$.

The sojourn time $T_{ij}$ for the state $(i, j)$ is exponentially distributed with rate $\alpha i + \beta j$.

The transition probability is :

$$P_{(i,j)(i+2,j-1)} = \frac{\beta j}{\alpha i + \beta j};$$

$$P_{(i,j)(i-1,j+1)} = \frac{\alpha i}{\alpha i + \beta j};$$

**2.**

Let $X(t)$ denote the number of customers in the system at time $t$.

The time until the next entering/birth $X \sim \exp(\lambda\alpha_i)$, where $i$ is the number of customers already in the system.

The time until the next service ending/death $Y \sim \exp(\mu)$.

The birth rate $\lambda\alpha_i$, the death rate is $\mu$.

Then $X(t)$ is a CTMC with sojourn time rate $\upsilon_i = \mu + \lambda\alpha_i$ for state $i$.

Since it can only change its state by increasing by one or decreasing by one, it is a birth and death procee.

The state space is

$$E = \{0, 1, 2, 3, ...\}.$$

The transition probability is :

$$P_{ij} = \begin{cases} P(X < Y), & j = i+1, i \geq 1 \\ P(Y < X), & j = i-1, i \geq 1 \end{cases} = \begin{cases} \frac{\lambda\alpha_i}{\mu+\lambda\alpha_i}, & j = i+1, i \geq 1 \\ \frac{\mu}{\mu+\lambda\alpha_i}, & j = i-1, i \geq 1 \end{cases}$$

When $i = 0$, $P_{01} = 1$.

**3.**

Yes, it is a CTMC.

Assume the number of people who have a infection is $k$ given $1 \leq k \leq N$, then the state space is

$$E = \{k, n+1, ..., N\}.$$

For $1 \leq n \leq N$, let $\tau_n$ be the time when a new infection occurs.

When a contact occurs at time $t$ with $X(t) = i$ , the probability of one being infected is

$$\frac{N(N-i)}{\binom{N}{2}} = \frac{2i(N-i)}{N(N-1)}.$$

Since contacts between two members of this population occur in accordance with a Poisson process having rate $\lambda$,

the sojourn time $T_n = \tau_{n+1} - \tau_n$ for $X(\tau_n) = i$ with $k \leq i < N$and $1 \leq n \leq N$ satisfies

$$T_n \sim \exp\left(\frac{2i(N-i)}{N(N-1)}\lambda\right).$$

For $k \leq i < N$,

$$\begin{aligned} P(X_{n+1} = i+1, T_{n+1} > t \mid X_n = i, (X_j, T_j), j < n) &= P(X_{n+1} = i+1, T_{n+1} > t \mid X_n = i) \\ &= P(X_1 = i+1, T_1 > t \mid X_0 = i) \\ &= \exp\left(-\frac{2i(N-i)}{N(N-1)}\lambda t\right). \end{aligned}$$

If the system enters the state $N$, it will stay there forever.

**4.**

Define a CTMC with $E = \{0, 1, 2, 3\}$.

The state 0 means both machine 1 and 2 operate.

The state 1 means machine 2 opertes but machine 1 is being repaired.

The state 2 means machine 1 opertes but machine 2 is being repaired.

The state 3 means both machine 1 and 2 are being repaired.

The operate time

$$O_i \sim \exp(\lambda_i)$$

for $i = 1, 2$.

The repair time

$$R_i \sim \exp(\mu i)$$

for $i = 1, 2$.

Then by the memoryless property of exponential distribution,

$$\begin{aligned}
P_{01} &= P(O_1 < O_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \quad P_{02} = P(O_2 < O_1) = \frac{\lambda_2}{\lambda_1 + \lambda_2}, \\
P_{10} &= P(R_1 < O_2) = \frac{\mu_1}{\mu_1 + \lambda_2}, \quad P_{13} = P(O_2 < R_1) = \frac{\lambda_2}{\mu_1 + \lambda_2}, \\
P_{20} &= P(R_2 < O_1) = \frac{\mu_2}{\lambda_1 + \mu_2}, \quad P_{23} = P(O_1 < R_2) = \frac{\lambda_1}{\lambda_1 + \mu_2}, \\
P_{31} &= P(R_2 < R_1) = \frac{\mu_2}{\mu_1 + \mu_2}, \quad P_{32} = P(R_1 < R_2) = \frac{\mu_1}{\mu_1 + \mu_2}.
\end{aligned}$$

Besides,

$$P_{03} = P_{30} = P_{12} = P_{21} = 0.$$

We know

$$\upsilon_0 = \lambda_1 + \lambda_2, \ \upsilon_1 = \mu_1 + \lambda_2, \ \upsilon_2 = \lambda_1 + \mu_2, \ \upsilon_3 = \mu_1 + \mu_2.$$

So

$$q_{01} = \upsilon_0 P_{01} = \lambda_1, \ q_{02} = \upsilon_0 P_{02} = \lambda_2,$$

$$q_{10} = \upsilon_1 P_{10} = \mu_1, \ q_{13} = \upsilon_1 P_{13} = \lambda_2,$$

$$q_{20} = \upsilon_2 P_{20} = \mu_2,\ q_{23} = \upsilon_2 P_{23} = \lambda_1,$$

$$q_{31} = \upsilon_3 P_{31} = \mu_2,\ q_{32} = \upsilon_3 P_{32} = \mu_1,$$

and

$$q_{03} = q_{30} = q_{12} = q_{21} = 0.$$

Moreover, $q_{ii} = \upsilon_i$ for $i = 0, 1, 2, 3$. Thus, the transition rate matrix is

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -(\mu_1 + \lambda_2) & 0 & \lambda_2 \\ \mu_2 & 0 & -(\lambda_1 + \mu_2) & \lambda_1 \\ 0 & \mu_2 & \mu_1 & -(\mu_1 + \mu_2) \end{pmatrix} \end{array}$$

Next we compute the transition matrix.

Consider first the case that there is just the machine 1.

Definte a CTMC with state space $E = \{o, r\}$.

State $o$ means it operates and $r$ means it is being repaired.

Then $\upsilon_o = \lambda_1$ and $\upsilon_r = \mu_1$.

So the transition rate matrix is

$$Q_1 = \begin{array}{c} \\ o \\ r \end{array} \begin{array}{c} \begin{array}{cc} o & r \end{array} \\ \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \mu_1 & -\mu_1 \end{pmatrix} \end{array}$$

According to the computational results from a similar example in class, we have

$$P^1_{oo}(t) = \frac{\mu_1}{\lambda_1 + \mu_1} + \frac{\lambda_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t},\ P^1_{or}(t) = \frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\lambda_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t},$$
$$P^1_{rr}(t) = \frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\mu_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t},\ P^1_{ro}(t) = \frac{\mu_1}{\lambda_1 + \mu_1} - \frac{\mu_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t}.$$

In addtion, we have the similar result for the machine 2.

Since the machines act independently of each other, we have

$$P_{00}(t) = P^1_{oo}(t)P^2_{oo}(t) = \left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right),$$

$$P_{01}(t) = P^1_{or}(t)P^2_{oo}(t) = \left(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right),$$

$$P_{02}(t) = P^1_{oo}(t)P^2_{or}(t) = \left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\lambda_2}{\lambda_2+\mu_2} - \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right),$$

$$P_{03}(t) = P^1_{or}(t)P^2_{or}(t) = \left(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\lambda_2}{\lambda_2+\mu_2} - \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right),$$

$$P_{10}(t) = P^1_{ro}(t)P^2_{oo}(t) = \left(\frac{\mu_1}{\lambda_1+\mu_1} - \frac{\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right),$$

$$P_{20}(t) = P^1_{oo}(t)P^2_{ro}(t) = \left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} - \frac{\mu_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right),$$

$$P_{30}(t) = P^1_{ro}(t)P^2_{ro}(t) = \left(\frac{\mu_1}{\lambda_1+\mu_1} - \frac{\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} - \frac{\mu_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right).$$

Then the transition matrix

$$P(t) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 0 \quad & \quad 1 \quad & \quad 2 \quad & \quad 3 \quad \end{array} \\ \begin{pmatrix} P^1_{or}P^2_{oo} & P^1_{or}P^2_{oo} & P^1_{oo}P^2_{or} & P^1_{or}P^2_{or} \\ P^1_{ro}P^2_{oo} & P^1_{rr}P^2_{oo} & P^1_{ro}P^2_{or} & P^1_{rr}P^2_{or} \\ P^1_{oo}P^2_{ro} & P^1_{or}P^2_{ro} & P^1_{oo}P^2_{rr} & P^1_{or}P^2_{rr} \\ P^1_{ro}P^2_{ro} & P^1_{rr}P^2_{ro} & P^1_{ro}P^2_{rr} & P^1_{rr}P^2_{rr} \end{pmatrix} \end{array}$$

$$
\begin{aligned}
(QP)_{00} &= \sum_{k=0}^{3} q_{0k} P_{k0}(t) \\
&= q_{00}P_{00}(t) + q_{01}P_{10}(t) + q_{02}P_{20}(t) + q_{03}P_{30}(t) \\
&= -(\lambda_1+\lambda_2)\left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right) \\
&\quad + \lambda_1\left(\frac{\mu_1}{\lambda_1+\mu_1} - \frac{\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right) \\
&\quad + \lambda_2\left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} - \frac{\mu_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right) \\
&= -\frac{\lambda_1\mu_2(\lambda_1+\mu_1)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}e^{-(\lambda_1+\mu_1)t} + \frac{\lambda_2\mu_1(\lambda_2+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}e^{-(\lambda_2+\mu_2)t} \\
&\quad - \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}e^{-(\lambda_1+\lambda_2+\mu_1+\mu_2)t} \\
&= -\frac{\lambda_1\mu_2}{(\lambda_2+\mu_2)}e^{-(\lambda_1+\mu_1)t} - \frac{\lambda_2\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_2+\mu_2)t} \\
&\quad - \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}e^{-(\lambda_1+\lambda_2+\mu_1+\mu_2)t}.
\end{aligned}
$$

Besides,

$$
\begin{aligned}
{P_{00}}'(t) &= d\frac{P_{00}(t)}{dt} \\
&= d\frac{\left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right)}{dt} \\
&= -\lambda_1 e^{-(\lambda_1+\mu_1)t}\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2}e^{-(\lambda_2+\mu_2)t}\right) \\
&\quad - \lambda_2 e^{-(\lambda_2+\mu_2)t}\left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}\right) \\
&= -\frac{\lambda_1\mu_2}{(\lambda_2+\mu_2)}e^{-(\lambda_1+\mu_1)t} - \frac{\lambda_2\mu_1}{\lambda_1+\mu_1}e^{-(\lambda_2+\mu_2)t} \\
&\quad - \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}e^{-(\lambda_1+\lambda_2+\mu_1+\mu_2)t}.
\end{aligned}
$$

Thus,

$$
\sum_{k=0}^{3} q_{0k} P_{k0}(t) = {P_{00}}'(t).
$$

Similarly, for other $i, j \in E$, we have

$$\sum_{k=0}^{3} q_{ik} P_{kj}(t) = P_{ij}{}'(t).$$

As a result, the transition probability satisfies the forward equations.

$$\begin{aligned}
(PQ)_{00} &= \sum_{k=0}^{3} P_{0k}(t) q_{k0} \\
&= P_{00}(t)q_{00} + P_{01}(t)q_{10} + P_{02}(t)q_{20} + P_{03}(t)q_{30} \\
&= \left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2} e^{-(\lambda_2+\mu_2)t}\right)(-(\lambda_1+\lambda_2)) \\
&\quad + \left(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\mu_2}{\lambda_2+\mu_2} + \frac{\lambda_2}{\lambda_2+\mu_2} e^{-(\lambda_2+\mu_2)t}\right)\mu_1 \\
&\quad + \left(\frac{\mu_1}{\lambda_1+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}\right)\left(\frac{\lambda_2}{\lambda_2+\mu_2} - \frac{\lambda_2}{\lambda_2+\mu_2} e^{-(\lambda_2+\mu_2)t}\right)\mu_2 + 0 \\
&= -\frac{\lambda_1\mu_2(\lambda_1+\mu_1)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} e^{-(\lambda_1+\mu_1)t} - \frac{\lambda_2\mu_1(\lambda_2+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} e^{-(\lambda_2+\mu_2)t} \\
&\quad + \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} e^{-(\lambda_1+\lambda_2+\mu_1+\mu_2)t} \\
&= -\frac{\lambda_1\mu_2}{(\lambda_2+\mu_2)} e^{-(\lambda_1+\mu_1)t} - \frac{\lambda_2\mu_1}{\lambda_1+\mu_1} e^{-(\lambda_2+\mu_2)t} \\
&\quad - \frac{\lambda_1\lambda_2(\lambda_1+\lambda_2+\mu_1+\mu_2)}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} e^{-(\lambda_1+\lambda_2+\mu_1+\mu_2)t} \\
&= P_{00}{}'(t).
\end{aligned}$$

by previous steps.

Namely,

$$\sum_{k=0}^{3} P_{0k}(t) q_{k0} = P_{00}{}'(t).$$

Similarly, for other $i, j \in E$, we have

$$\sum_{k=0}^{3} P_{ik}(t) q_{kj} = P_{ij}{}'(t).$$

As a result, the transition probability satisfies the backward equations.

Let $X_t$ be the number of individuals at time $t$.

Assmue $X(0) = N \in bbn$.

Then the state space is $E = \{0, 1, 2, ..., N\}$.

Then when $1 \leq i \leq$ and $i \in N$,

$$P_{ii-1} = 1.$$

When the system enters the state 0, will stay there forever.

Then the transition rate

$$q_{ii} = -\mu$$

for $1 \leq i \leq$ and $i \in N$.

Besides,

$$q_{ii-1} = \mu P_{ii-1} = mu$$

for $1 \leq i \leq$ and $i \in N$.