MATH 8170,
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HW 1
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1.

$$\begin{aligned} P\big(N(t) = k\big) &= P(T_k \le t < T_{k+1}) \\ &= P(T_k \le t) - P(T_{k+1} \le t) \end{aligned}$$

$\{T_1 = X_1\}$ may be interpreted as the rewarded recieved after conducted one bernoulli trail. $X_1 = 1$ if trial fails and $X_1 = 2$ if it succeeds.
$\{T_n = X_n\}$ may be interpreted as the rewards n recieved after conducting k bernoulli trails, and then we have $n-k$ succeses and $k-(n-k) = 2k-n$ fails. Moreover, the rewards n range from k to $2k$.

$$P(T_k = n) = \binom{k}{n-k} 0.2^{n-k} 0.8^{2k-n}, n = k, k+1, ..., 2k$$

So

$$\begin{aligned} P\big(N(t) = k\big) &= \sum_{n=k}^{\min(\lfloor t \rfloor, 2k)} P(T_k = n) - \sum_{n=k+1}^{\lfloor t \rfloor} P(T_{k+1} = n) \\ &= \sum_{n=k}^{\min(\lfloor t \rfloor, 2k)} \binom{k}{n-k} 0.2^{n-k} 0.8^{2k-n} - \sum_{n=k+1}^{\min(\lfloor t \rfloor, 2k+2)} \binom{k+1}{n-k-1} 0.2^{n-k-1} 0.8^{2k+2-n} \end{aligned}$$

2.
Assume $T_0 = 0$.

$$\begin{aligned} E(A(t)B(t)) &= \int_0^\infty E(A(t)B(t)|T_1 = s)dF(s) \\ &= \int_0^t E(A(t)B(t)|T_1 = s)dF(s) + \int_t^\infty E(A(t)B(t)|T_1 = s)dF(s) \\ &= \int_0^t E(A(t-s)B(t-s)dF(s) + E\Big((t - T_{N(t)})(T_{N(t)+1} - t)|T_1 \ge t\Big) \\ &= \int_0^t E(A(t-s)B(t-s)dF(s) + E\big(t(T_1 - t)|T_1 \ge t\big) \end{aligned}$$

Then $h(t) = E\big(t(T_1 - t)|T_1 \ge t\big)$, so h is continuous a.e on $[0, \infty)$.
so for $t \ge 1$,

$$\begin{aligned} |h(t)| &= E\big(t(T_1 - t)|T_1 \ge t) \le E\big(T_1/2(T_1 - T_1/2)|T_1 \ge t\big) \\ &= E\big(T_1^2/4\big|T_1 \ge t) \\ &= \frac{1}{4} E(T_1^2|T_1 \ge t) = b(t). \end{aligned}$$

It is obvious that $b(t)$ is directly Riemann integrable since F has finite second moment.
Besides, F is nonarithmetic.
So the key renewal theorem is applicable, and

$$
\begin{aligned}
\lim_{t\to\infty} E(A(t)B(t)) &= \frac{1}{E(X_1)}\int_0^\infty E\big(t(T_1-t)|T_1\geq t\big)dt \\
&= \frac{1}{E(X_1)}\int_0^\infty E(tT_1-t^2|T_1\geq t)dt \\
&= \frac{1}{E(X_1)}\int_0^\infty\int_t^\infty E(tT_1-t^2|T_1=s)dF(s)dt \\
&= \frac{1}{E(X_1)}\int_0^\infty\int_t^\infty E(ts-t^2)dF(s)dt \\
&= \frac{1}{E(X_1)}\int_0^\infty\int_t^\infty ts-t^2dF(s)dt \\
&= \frac{1}{E(X_1)}\int_0^\infty\left(t\int_t^\infty sdF(s)-t^2\int_t^\infty dF(s)\right)dt \\
&= \frac{1}{\int_0^\infty sdF(s)}\int_0^\infty\left(t\int_t^\infty sdF(s)-t^2\big(1-F(t)\big)\right)dt
\end{aligned}
$$

3.

$$
\begin{aligned}
P(N(t)\text{ is odd }) &= P\big(N(t)\text{ is odd },T_1>t\big)+P(N(t)\ N(t)\text{ is odd },T_1\leq t) \\
&= P\big(N(t)\text{ is odd },T_1\leq t\big) \\
&= \int_0^t P\big(N(t)\text{ is odd }|T_1=s\big)dF(s) \\
&= \int_0^t P\big(\ N(t-s)\text{ is even }\big)dF(s) \\
&= \int_0^t \Big(1-P\big(\ N(t-s)\text{ is odd }\big)\Big)dF(s) \\
&= \int_0^t dF(s)-\int_0^t P\big(\ N(t-s)\text{ is odd }\big)dF(s) \\
&= F(t)-\int_0^t P\big(\ N(t-s)\text{ is odd }\big)dF(s)
\end{aligned}
$$

Let $H(t) = P(N(t) \text{ is odd })$, then $H(0) = 0$ and $F(t) = 1 - e^{-\lambda t}$ and $dF(s) = \lambda e^{-\lambda s} ds$.

$$
\begin{aligned}
H(t) &= F(t) - \int_0^t H(t-s) dF(s) \\
&= 1 - e^{-\lambda t} - \int_0^t H(t-s) \lambda e^{-\lambda s} ds \\
&= 1 - e^{-\lambda t} - \lambda e^{-\lambda t} \int_0^t H(x) e^{\lambda x} dx
\end{aligned}
$$

Take derivatives on both sides, we have

$$
\begin{aligned}
H'(t) &= \lambda e^{-\lambda t} + \lambda^2 e^{-\lambda t} \int_0^t H(x) e^{\lambda x} dx - \lambda e^{-\lambda t} H(t) e^{\lambda t} \\
&= \lambda e^{-\lambda t} + \lambda^2 e^{-\lambda t} \int_0^t H(x) e^{\lambda x} dx - \lambda H(t) \\
&= \lambda e^{-\lambda t} + \lambda (1 - e^{-\lambda t} - H(t)) - \lambda H(t) \\
&= \lambda - 2\lambda H(t)
\end{aligned}
$$

So

$$H'(t) + 2\lambda H(t) = \lambda$$

It can be found that $\frac{\lambda}{2}$ is the special solution and $Ce^{-2\lambda t}, C \neq 0$ are the general solutions.
So $H(t) = Ce^{-2\lambda t} + \frac{1}{2}$ is the general form of the solution.
Since $H(0) = C + \frac{1}{2} = 0$, we have $C = -\frac{1}{2}$.
Thus,

$$H(t) = -\frac{1}{2} e^{-2\lambda t} + \frac{1}{2}.$$

4.

Proof. (a)

$$
\begin{aligned}
\lim_{n\to\infty} \frac{T_n^D}{n} &= \lim_{n\to\infty} \frac{X_1 + \sum_{i=2}^n X_i}{n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=2}^n X_i}{n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=2}^n X_i}{n-1} \frac{n-1}{n} \\
&= \lim_{n\to\infty} \frac{\sum_{i=2}^n X_i}{n-1} \lim_{n\to\infty} \frac{n-1}{n} \\
&= E[X_2] \quad w.p.1
\end{aligned}
$$

by central limit theorem since $X_n, n \geq 2$ are *iid*.
Then there exists an event $A \subseteq \Omega$ with $P(A) = 1$ and $\lim_{t\to\infty} \frac{T_n^D(\omega)}{n} = \mu$ for

each $\omega \in A$.
We observe that $T_n^D(\omega) < \infty$ for each n and $T_n^D(\omega) \to \infty$ as $n \to \infty$, which implies $\lim_{t\to\infty} N^D(t,\omega) = \infty$.
Fix an $\omega \in A$. For each $t > 0$,

$$T^D_{N^D(t)}(\omega) \le t < T^D_{N^D(t)+1}(\omega).$$

Let t large enough such that $N^D(t,\omega) \ge 1$, and divide the above equalities by $N^D(t,\omega)$, so we have

$$\frac{T^D_{N^D(t)}(\omega)}{N^D(t,\omega)} \le \frac{t}{N^D(t,\omega)} < \frac{T^D_{N^D(t)}(\omega)+1}{N^D(t,\omega)}$$

Namely,

$$\frac{T^D_{N^D(t)}(\omega)}{N^D(t,\omega)} \le \frac{t}{N^D(t,\omega)} < \frac{T^D_{N^D(t)}(\omega)+1}{N^D(t,\omega)+1}\frac{N^D(t,\omega)+1}{N^D(t,\omega)}$$

Let $t \to \infty$, then

$$\frac{T^D_{N^D(t)}(\omega)}{N^D(t,\omega)} \to E[X_2] \text{ and } \frac{T^D_{N^D(t)}(\omega)+1}{N^D(t,\omega)+1}\frac{N^D(t,\omega)+1}{N^D(t,\omega)} \to E[X_2].$$

So

$$\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{E[X_2]}$$

(b) Step1:
Note that the $r.v$ $N^D(t)+1$ is a stopping time $w.r.t$ $\{X_n\}_{n=1}^{\infty}$.
Since

$$1\big(N^D(t)+1 \ge k\big) = 1 - 1\big(N^D(t)+1 < k\big) = 1 - 1\big(N^D(t)+1 \le k-1\big),$$

$1\big(N^D(t)+1 \ge k\big)$ is a function of $X_0, X_1, ...X_{k-1}$, and $1\big(N^D(t) \ge k\big)$ is independent of X_k.

Therefore,

$$
\begin{aligned}
E\big(T^D_{N^D(t)+1}\big) &= E\left(\sum_{k=1}^{N^D(t)+1} X_k\right) \\
&= E\Big(\sum_{k=1}^{\infty} X_k 1\big(N^D(t)+1 \geq k\big)\Big) \\
&= \sum_{k=1}^{\infty} E\bigg(X_k 1\big(N^D(t)+1 \geq k\big)\bigg) \\
&= \sum_{k=1}^{\infty} E(X_k) P\big(N^D(t)+1 \geq k\big) \\
&= E(X_1) P\big(N^D(t)+1 \geq 1\big) + \sum_{k=2}^{\infty} E(X_k) P\big(N^D(t)+1 \geq k\big) \\
&= E(X_1) + \sum_{k=1}^{\infty} E(X_{k+1}) P\big(N^D(t)+1 \geq k+1\big) \\
&= E(X_1) + \sum_{k=1}^{\infty} E(X_{k+1}) P\big(N^D(t) \geq k\big) \\
&= E(X_1) + E(X_2) \sum_{k=1}^{\infty} P\big(N^D(t) \geq k\big) \\
&= E(X_1) + E(X_2) E\big(N^D(t)\big).
\end{aligned}
$$

So

$$
t < E\big(T^D_{N^D(t)+1}\big) = E(X_1) + E(X_2) E\big(N^D(t)\big).
$$

Then

$$
\frac{E\big(N^D(t)\big)}{t} > \frac{1}{E(X_2)} - \frac{E(X_1)}{E(X_2)} \frac{1}{t}.
$$

Thus,

$$
\lim_{t \to \infty} \frac{E\big(N^D(t)\big)}{t} \geq \frac{1}{E(X_2)}.
$$

Step2:
Fix $a > 0$, and define $X_n^{(a)} = \min\{X_n, a\}$.
Let $T_n^{D(a)} = \sum_{k=1}^{n} X_n^{(a)}$, and define a new renewal process $\{N^{D(a)}(t)\}$ whose points are given by $\{T_n^{D(a)}\}$.
Then

$$
E\Big(T^{D(a)}_{N^{D(a)}(t)+1}\Big) = E\big(X_1^a\big) + E\big(X_2^{(a)}\big) E\big(N^{D(a)}(t)\big).
$$

by following the similat process with Step 1.
Next we note that

$$
T^{D(a)}_{N^{D(a)}(t)+1} = T^{D(a)}_{N^{D(a)}(t)} + X^{(a)}_{N^{D(a)}(t)+1} \leq t + a.
$$

So

$$E\Big(T^{D(a)}_{N^{D(a)}(t)+1}\Big) \le t + a \implies E(X_1^a) + E(X_2^{(a)})E\big(N^{D(a)}(t)\big) \le t + a$$

$$\implies \frac{E\big(N^{D(a)}(t)\big)}{t} \le \frac{1}{E\big(X_2^{(a)}\big)} + \frac{a - E\big(X_1^a\big)}{E\big(X_2^{(a)}\big)}\frac{1}{t}$$

$$\implies \lim_{t\to\infty} sup \frac{E\big(N^{D(a)}(t)\big)}{t} \le \frac{1}{E\big(X_2^{(a)}\big)}$$

Since $N^D(t) < N^{D(a)}(t)$, we have

$$\lim_{t\to\infty} sup \frac{E\big(N^{D}(t)\big)}{t} \le \frac{1}{E\big(X_2^{(a)}\big)}.$$

Let $a \to \infty$, and we have

$$\lim_{t\to\infty} sup \frac{E\big(N^{D}(t)\big)}{t} \le \frac{1}{E\big(X_2\big)}$$

According to the above two steps, we have

$$\lim_{t\to\infty} \frac{E\big(N^{D}(t)\big)}{t} = \frac{1}{E(X_2)}$$

□

5.

(a) Define X_i as how long it takes the miner to make the i-th journey.
Define the stopping time N as the first journey which takes the miner 2 days.
Verify:

$$1(N = n) = 1(X_1 = 4 \text{ or } 6 \text{ for } i = 1, 2, ...n - 1, X_{n-1} = 2)$$

$$P(N = i) = \Big(\frac{2}{3}\Big)^{i-1}\frac{1}{3}, \quad i = 1, 2, ...$$

So $N \sim Geo\Big(\frac{1}{3}\Big)$ and

$$E(N) = 3$$

Morevoer,

$$P(X_i = j) = \frac{1}{3} \text{ for } j = 2, 4, 6.$$

So for $i = 1, 2, ...,$

$$E(X_i) = \frac{1}{3}(2 + 4 + 6) = 4.$$

(b) By Wald's equalitiy,

$$E(T) = E(N)E(X_i) = 3 * 4 = 12$$

(c)

$$
\begin{aligned}
E\left(\sum_{i=1}^{N} X_i | N = n\right) &= E\left(\sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n} E(X_i) \\
&= 4n
\end{aligned}
$$

since n is a constant and X_i are *iid*.

(d) By part (c), we have

$$E(T|N) = E\left(\sum_{i=1}^{N} X_i | N\right) = 4N.$$

So

$$E(T) = E\big(E(T|N)\big) = E(4N) = 4E(N) = 4 * 3 = 12.$$

6.
Let μ_G denote the mean service time.
Then the mean time of a cycle is $\mu_G + \frac{1}{\lambda}$ by the memoryless property of the Poisson process.
So the proportion of time (on the average) the server is busy is $\frac{\mu_G}{\mu_G + \frac{1}{\lambda}}$.

7.

(a) let $\{N(t), t \geq 0\}$ be the Poisson process that passengers arrive at the bus stop. Fix t, then $N(t) \sim Possion(\lambda t)$ and $E\big(N(t)\big) = \lambda t$.
let $M(t)$ be the renewal process that the buses arrive with points $\{T_n\}$ and interrenewals X_n, assume at the T_n, the total amount of passengers waiting for buses is R_n.
Then $\{(X_n, R_n)\}$ is an *iid* sequence and $0 < E(X_1) < \infty$ and $E(R_1) < \infty$.
Let $R(t)$ be the amount of rewards accumulated in $[0, t]$.
So we have the average number of people who are waiting for a bus

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(R_1)}{E(X_1)},$$

and

$$E(R_1) = E\big(N(T_1)\big).$$

Since

$$E\big(N(T_1)|T_1 = t\big) = E\big(N(t)\big) = \lambda t,$$

$$E(R_1) = E\Big(E\big(N(T_1)|T_1\big)\Big) = E(\lambda T_1) = \lambda E(T_1)$$

So the average number of people who are waiting for a bus

$$\begin{aligned}\lim_{t\to\infty}\frac{R(t)}{t} &= \frac{E(R_1)}{E(X_1)}\\ &= \frac{\lambda E(T_1)}{E(X_1)}\\ &= \lambda\end{aligned}$$

(b) We construct a discrete time renewal reward process $\{W_n\}_{n\geq 1}$ w.r.t $\{nN\}_{n\geq 1}$, where N is the waiting time of the nth passenger, and N is the number of passengers who arrive during the cycles in part (a).
Then

$$E(N) = \lambda.$$

Let the total waiting time between nN and $(n+1)N$ be

$$R_n = \sum_{k=nN+1}^{(n+1)N} W_k, n \geq 1$$

$$\begin{aligned}E(R_n) &= E\left(\sum_{k=nN+1}^{(n+1)N} W_k\right)\\ &= E\left(E\left(\sum_{k=nN+1}^{(n+1)N} W_k|N\right)\right)\end{aligned}$$

Since

$$\begin{aligned}E\left(\sum_{k=nN+1}^{(n+1)N} W_k|N=m\right) &= E\left(\sum_{k=nm+1}^{(n+1)m} W_k\right)\\ &= E\left(\sum_{k=1}^{m} W_k\right)\\ &= \sum_{k=1}^{m} E(W_k)\\ &= \sum_{k=1}^{m} E\big(E(W_k|T_1)\big)\end{aligned}$$

Since

$$E(W_k|T_1=t) = t - \frac{k}{\lambda},$$

$$E(W_k|T_1) = T_1 - \frac{k}{\lambda}.$$

So

$$\begin{aligned} E\left(\sum_{k=nN+1}^{(n+1)N} W_k | N=m\right) &= \sum_{k=1}^{m} E\Big(T_1 - \frac{k}{\lambda}\Big) \\ &= \sum_{k=1}^{m} \Big(E(T_1) - \frac{k}{\lambda}\Big) \\ &= mE(T_1) - \frac{m(m+1)}{2\lambda}. \end{aligned}$$

Then

$$E\left(\sum_{k=nN+1}^{(n+1)N} W_k | N\right) = NE(T_1) - \frac{N(N+1)}{2\lambda}$$

Thus,

$$\begin{aligned} E(R_n) &= E\Big(\frac{N-1}{2}\Big) \\ &= \frac{E(N)-1}{2} \\ &= \frac{\lambda-1}{2} \end{aligned}$$

Thus, the average amount of time that a passenger waits is

$$\begin{aligned} \frac{E(R_n)}{E(N)} &= \frac{\lambda-1}{2\lambda} \\ &= \frac{1}{2} - \frac{1}{2\lambda} \end{aligned}$$