

MATH 8170,  
Fall 2016  
HW 1  
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1.

$$\begin{aligned} P(N(t) = k) &= P(T_k \leq t < T_{k+1}) \\ &= P(T_k \leq t) - P(T_{k+1} \leq t) \end{aligned}$$

$\{T_1 = X_1\}$  may be interpreted as the reward received after conducting one bernoulli trial.  $X_1 = 1$  if trial fails and  $X_1 = 2$  if it succeeds.

$\{T_n = X_n\}$  may be interpreted as the rewards  $n$  received after conducting  $k$  bernoulli trials, and then we have  $n - k$  successes and  $k - (n - k) = 2k - n$  fails. Moreover, the rewards  $n$  range from  $k$  to  $2k$ .

$$P(T_k = n) = \binom{k}{n-k} 0.2^{n-k} 0.8^{2k-n}, n = k, k+1, \dots, 2k$$

So

$$\begin{aligned} P(N(t) = k) &= \sum_{n=k}^{\min(\lfloor t \rfloor, 2k)} P(T_k = n) - \sum_{n=k+1}^{\lfloor t \rfloor} P(T_{k+1} = n) \\ &= \sum_{n=k}^{\min(\lfloor t \rfloor, 2k)} \binom{k}{n-k} 0.2^{n-k} 0.8^{2k-n} - \sum_{n=k+1}^{\min(\lfloor t \rfloor, 2k+2)} \binom{k+1}{n-k-1} 0.2^{n-k-1} 0.8^{2k+2-n} \end{aligned}$$

2.

Assume  $T_0 = 0$ .

$$\begin{aligned} E(A(t)B(t)) &= \int_0^\infty E(A(t)B(t)|T_1 = s) dF(s) \\ &= \int_0^t E(A(t)B(t)|T_1 = s) dF(s) + \int_t^\infty E(A(t)B(t)|T_1 = s) dF(s) \\ &= \int_0^t E(A(t-s)B(t-s) dF(s) + E((t - T_{N(t)})(T_{N(t)+1} - t) | T_1 \geq t) \\ &= \int_0^t E(A(t-s)B(t-s) dF(s) + E(t(T_1 - t) | T_1 \geq t) \end{aligned}$$

Then  $h(t) = E(t(T_1 - t) | T_1 \geq t)$ , so  $h$  is continuous a.e on  $[0, \infty)$ .  
so for  $t \geq 1$ ,

$$\begin{aligned} |h(t)| &= E(t(T_1 - t) | T_1 \geq t) \leq E(T_1/2(T_1 - T_1/2) | T_1 \geq t) \\ &= E(T_1^2/4 | T_1 \geq t) \\ &= \frac{1}{4} E(T_1^2 | T_1 \geq t) = b(t). \end{aligned}$$

It is obvious that  $b(t)$  is directly Riemann integrable since  $F$  has finite second moment.

Besides,  $F$  is nonarithmetic.

So the key renewal theorem is applicable, and

$$\begin{aligned}
\lim_{t \rightarrow \infty} E(A(t)B(t)) &= \frac{1}{E(X_1)} \int_0^\infty E(t(T_1 - t) | T_1 \geq t) dt \\
&= \frac{1}{E(X_1)} \int_0^\infty E(tT_1 - t^2 | T_1 \geq t) dt \\
&= \frac{1}{E(X_1)} \int_0^\infty \int_t^\infty E(tT_1 - t^2 | T_1 = s) dF(s) dt \\
&= \frac{1}{E(X_1)} \int_0^\infty \int_t^\infty E(ts - t^2) dF(s) dt \\
&= \frac{1}{E(X_1)} \int_0^\infty \int_t^\infty ts - t^2 dF(s) dt \\
&= \frac{1}{E(X_1)} \int_0^\infty \left( t \int_t^\infty s dF(s) - t^2 \int_t^\infty dF(s) \right) dt \\
&= \frac{1}{\int_0^\infty s dF(s)} \int_0^\infty \left( t \int_t^\infty s dF(s) - t^2 (1 - F(t)) \right) dt
\end{aligned}$$

**3.**

$$\begin{aligned}
P(N(t) \text{ is odd}) &= P(N(t) \text{ is odd}, T_1 > t) + P(N(t) \text{ is odd}, T_1 \leq t) \\
&= P(N(t) \text{ is odd}, T_1 \leq t) \\
&= \int_0^t P(N(t) \text{ is odd} | T_1 = s) dF(s) \\
&= \int_0^t P(N(t-s) \text{ is even}) dF(s) \\
&= \int_0^t (1 - P(N(t-s) \text{ is odd})) dF(s) \\
&= \int_0^t dF(s) - \int_0^t P(N(t-s) \text{ is odd}) dF(s) \\
&= F(t) - \int_0^t P(N(t-s) \text{ is odd}) dF(s)
\end{aligned}$$

Let  $H(t) = P(N(t) \text{ is odd})$ , then  $H(0) = 0$  and  $F(t) = 1 - e^{-\lambda t}$  and  $dF(s) = \lambda e^{-\lambda s} ds$ .

$$\begin{aligned} H(t) &= F(t) - \int_0^t H(t-s) dF(s) \\ &= 1 - e^{-\lambda t} - \int_0^t H(t-s) \lambda e^{-\lambda s} ds \\ &= 1 - e^{-\lambda t} - \lambda e^{-\lambda t} \int_0^t H(x) e^{\lambda x} dx \end{aligned}$$

Take derivatives on both sides, we have

$$\begin{aligned} H'(t) &= \lambda e^{-\lambda t} + \lambda^2 e^{-\lambda t} \int_0^t H(x) e^{\lambda x} dx - \lambda e^{-\lambda t} H(t) e^{\lambda t} \\ &= \lambda e^{-\lambda t} + \lambda^2 e^{-\lambda t} \int_0^t H(x) e^{\lambda x} dx - \lambda H(t) \\ &= \lambda e^{-\lambda t} + \lambda(1 - e^{-\lambda t} - H(t)) - \lambda H(t) \\ &= \lambda - 2\lambda H(t) \end{aligned}$$

So

$$H'(t) + 2\lambda H(t) = \lambda$$

It can be found that  $\frac{\lambda}{2}$  is the special solution and  $Ce^{-2\lambda t}, C \neq 0$  are the general solutions.

So  $H(t) = Ce^{-2\lambda t} + \frac{1}{2}$  is the general form of the solution.

Since  $H(0) = C + \frac{1}{2} = 0$ , we have  $C = -\frac{1}{2}$ .

Thus,

$$H(t) = -\frac{1}{2}e^{-2\lambda t} + \frac{1}{2}.$$

Namely,

$$P(N(t) \text{ is odd}) = -\frac{1}{2}e^{-2\lambda t} + \frac{1}{2}.$$

*Proof.* (a)

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{T_n^D}{n} &= \lim_{n \rightarrow \infty} \frac{X_1 + \sum_{i=2}^n X_i}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\sum_{i=2}^n X_i}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\sum_{i=2}^n X_i}{n-1} \frac{n-1}{n} \\
&= \lim_{n \rightarrow \infty} \frac{\sum_{i=2}^n X_i}{n-1} \lim_{n \rightarrow \infty} \frac{n-1}{n} \\
&= E[X_2] \quad w.p.1
\end{aligned}$$

by central limit theorem since  $X_n, n \geq 2$  are *iid*.

Then there exists an event  $A \subseteq \Omega$  with  $P(A) = 1$  and  $\lim_{t \rightarrow \infty} \frac{T_n^D(\omega)}{n} = \mu$  for each  $\omega \in A$ .

We observe that  $T_n^D(\omega) < \infty$  for each  $n$  and  $T_n^D(\omega) \rightarrow \infty$  as  $n \rightarrow \infty$ , which implies  $\lim_{t \rightarrow \infty} N^D(t, \omega) = \infty$ .

Fix an  $\omega \in A$ . For each  $t > 0$ ,

$$T_{N^D(t)}^D(\omega) \leq t < T_{N^D(t)+1}^D(\omega).$$

Let  $t$  large enough such that  $N^D(t, \omega) \geq 1$ , and divide the above equalities by  $N^D(t, \omega)$ , so we have

$$\frac{T_{N^D(t)}^D(\omega)}{N^D(t, \omega)} \leq \frac{t}{N^D(t, \omega)} < \frac{T_{N^D(t)}^D(\omega) + 1}{N^D(t, \omega)}$$

Namely,

$$\frac{T_{N^D(t)}^D(\omega)}{N^D(t, \omega)} \leq \frac{t}{N^D(t, \omega)} < \frac{T_{N^D(t)}^D(\omega) + 1}{N^D(t, \omega) + 1} \frac{N^D(t, \omega) + 1}{N^D(t, \omega)}$$

Let  $t \rightarrow \infty$ , then

$$\frac{T_{N^D(t)}^D(\omega)}{N^D(t, \omega)} \rightarrow E[X_2] \text{ and } \frac{T_{N^D(t)}^D(\omega) + 1}{N^D(t, \omega) + 1} \frac{N^D(t, \omega) + 1}{N^D(t, \omega)} \rightarrow E[X_2].$$

So

$$\lim_{t \rightarrow \infty} \frac{N_D(t)}{t} = \frac{1}{E[X_2]}$$

(b) Step1:

Note that the *r.v*  $N^D(t) + 1$  is a stopping time *w.r.t*  $\{X_n\}_{n=1}^\infty$ .

Since

$$1(N^D(t) + 1 \geq k) = 1 - 1(N^D(t) + 1 < k) = 1 - 1(N^D(t) + 1 \leq k - 1),$$

$1(N^D(t) + 1 \geq k)$  is a function of  $X_0, X_1, \dots, X_{k-1}$ , and  $1(N^D(t) \geq k)$  is independent of  $X_k$ .

Therefore,

$$\begin{aligned}
E(T_{N^D(t)+1}^D) &= E\left(\sum_{k=1}^{N^D(t)+1} X_k\right) \\
&= E\left(\sum_{k=1}^{\infty} X_k 1(N^D(t) + 1 \geq k)\right) \\
&= \sum_{k=1}^{\infty} E\left(X_k 1(N^D(t) + 1 \geq k)\right) \\
&= \sum_{k=1}^{\infty} E(X_k) P(N^D(t) + 1 \geq k) \\
&= E(X_1) P(N^D(t) + 1 \geq 1) + \sum_{k=2}^{\infty} E(X_k) P(N^D(t) + 1 \geq k) \\
&= E(X_1) + \sum_{k=1}^{\infty} E(X_{k+1}) P(N^D(t) + 1 \geq k+1) \\
&= E(X_1) + \sum_{k=1}^{\infty} E(X_{k+1}) P(N^D(t) \geq k) \\
&= E(X_1) + E(X_2) \sum_{k=1}^{\infty} P(N^D(t) \geq k) \\
&= E(X_1) + E(X_2) E(N^D(t)).
\end{aligned}$$

So

$$t < E(T_{N^D(t)+1}^D) = E(X_1) + E(X_2) E(N^D(t)).$$

Then

$$\frac{E(N^D(t))}{t} > \frac{1}{E(X_2)} - \frac{E(X_1)}{E(X_2)} \frac{1}{t}.$$

Thus,

$$\lim_{t \rightarrow \infty} \frac{E(N^D(t))}{t} \geq \frac{1}{E(X_2)}.$$

Step2:

Fix  $a > 0$ , and define  $X_n^{(a)} = \min\{X_n, a\}$ .

Let  $T_n^{D(a)} = \sum_{k=1}^n X_k^{(a)}$ , and define a new renewal process  $\{N^{D(a)}(t)\}$  whose points are given by  $\{T_n^{D(a)}\}$ .

Then

$$E(T_{N^{D(a)}(t)+1}^{D(a)}) = E(X_1^a) + E(X_2^{(a)}) E(N^{D(a)}(t)).$$

by following the similat process with Step 1.

Next we note that

$$T_{N^{D(a)}(t)+1}^{D(a)} = T_{N^{D(a)}(t)}^{D(a)} + X_{N^{D(a)}(t)+1}^{(a)} \leq t + a.$$

So

$$\begin{aligned}
E\left(T_{N^{D(a)}(t)+1}^{D(a)}\right) &\leq t + a \implies E(X_1^a) + E(X_2^{(a)})E(N^{D(a)}(t)) \leq t + a \\
&\implies \frac{E(N^{D(a)}(t))}{t} \leq \frac{1}{E(X_2^{(a)})} + \frac{a - E(X_1^a)}{E(X_2^{(a)})} \frac{1}{t} \\
&\implies \lim_{t \rightarrow \infty} \sup \frac{E(N^{D(a)}(t))}{t} \leq \frac{1}{E(X_2^{(a)})}
\end{aligned}$$

Since  $N^D(t) < N^{D(a)}(t)$ , we have

$$\lim_{t \rightarrow \infty} \sup \frac{E(N^D(t))}{t} \leq \frac{1}{E(X_2^{(a)})}.$$

Let  $a \rightarrow \infty$ , and we have

$$\lim_{t \rightarrow \infty} \sup \frac{E(N^D(t))}{t} \leq \frac{1}{E(X_2)}$$

According to the above two steps, we have

$$\lim_{t \rightarrow \infty} \frac{E(N^D(t))}{t} = \frac{1}{E(X_2)}$$

□

## 5.

- (a) Define  $X_i$  as how long it takes the miner to make the  $i$ -th journey.  
Define the stopping time  $N$  as the first journey which takes the miner 2 days.  
Verify:

$$1(N = n) = 1(X_1 = 4 \text{ or } 6 \text{ for } i = 1, 2, \dots, n-1, X_{n-1} = 2)$$

$$P(N = i) = \left(\frac{2}{3}\right)^{i-1} \frac{1}{3}, \quad i = 1, 2, \dots$$

So  $N \sim \text{Geo}\left(\frac{1}{3}\right)$  and

$$E(N) = 3$$

Moreover,

$$P(X_i = j) = \frac{1}{3} \text{ for } j = 2, 4, 6.$$

So for  $i = 1, 2, \dots$ ,

$$E(X_i) = \frac{1}{3}(2 + 4 + 6) = 4.$$

- (b) By Wald's equality,

$$E(T) = E(N)E(X_i) = 3 \times 4 = 12$$

(c)

$$\begin{aligned}
E\left(\sum_{i=1}^N X_i | N = n\right) &= E\left(\sum_{i=1}^n X_i\right) \\
&= \sum_{i=1}^n E(X_i) \\
&= 4n
\end{aligned}$$

since  $n$  is a constant and  $X_i$  are *iid*.

(d) By part (c), we have

$$E(T|N) = E\left(\sum_{i=1}^N X_i | N\right) = 4N.$$

So

$$E(T) = E(E(T|N)) = E(4N) = 4E(N) = 4 \times 3 = 12.$$

## 6.

Let  $\mu_G$  denote the mean service time.

Then the mean time of a cycle is  $\mu_G + \frac{1}{\lambda}$  by the memoryless property of the Poisson process.

So the proportion of time (on the average) the server is busy is  $\frac{\mu_G}{\mu_G + \frac{1}{\lambda}}$ .

## 7.

(a) let  $\{N(t), t \geq 0\}$  be the Poisson process that passengers arrive at the bus stop. Fix  $t$ , then  $N(t) \sim \text{Poisson}(\lambda t)$  and  $E(N(t)) = \lambda t$ .

let  $M(t)$  be the renewal process that the buses arrive with points  $\{T_n\}$  and interrenewals  $X_n$ , assume at the  $T_n$ , the total amount of passengers waiting for buses is  $R_n$ .

Then  $\{(X_n, R_n)\}$  is an *iid* sequence and  $0 < E(X_1) < \infty$  and  $E(R_1) < \infty$ .

Let  $R(t)$  be the amount of rewards accumulated in  $[0, t]$ .

So we have the average number of people who are waiting for a bus

$$\lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{E(R_1)}{E(X_1)},$$

and

$$E(R_1 | T_1 = t, N(T_1) = n) = \frac{nt}{2}$$

since there are  $n$  arrivals by time  $t$  and the set of arrival times are distributed as  $n$  independent uniform  $(0, t)$  random variables, and so the average amount received per passenger is  $\frac{t}{2}$ .

Thus,

$$\begin{aligned}
E(R_1) &= E\left(E(R_1|T_1, N(T_1))\right) \\
&= \frac{1}{2}E(T_1 N(T_1)) \\
&= \frac{1}{2}E\left(E(T_1 N(T_1)|T_1)\right) \\
&= \frac{1}{2}E\left(T_1 E(N(T_1)|T_1)\right) \\
&= \frac{1}{2}E(T_1 \lambda T_1) \\
&= \frac{\lambda}{2}E(T_1^2)
\end{aligned}$$

since  $N(T_1) \sim \text{Poisson}(\lambda T_1)$ .

Therefore, the average number of people who are waiting for a bus is

$$\begin{aligned}
\frac{E(R_1)}{E(X_1)} &= \frac{\lambda E(T_1^2)}{2E(T_1)} \\
&= \frac{\lambda \int_0^\infty t^2 dF(t)}{2 \int_0^\infty t dF(t)}
\end{aligned}$$

- (b) We construct a discrete time renewal reward process  $\{W_n\}_{n \geq 1}$  w.r.t  $\{nN\}_{n \geq 1}$ , where  $W_n$  is the waiting time of the  $n$ th passenger, and  $N$  is the number of passengers who arrive during the cycles in part (a).

Let the total waiting time between  $nN$  and  $(n+1)N$  be

$$R_n = \sum_{k=nN+1}^{(n+1)N} W_k, n \geq 1$$

Then

$$\begin{aligned}
E(R_1) &= E\left(\sum_{k=n+1}^N W_k\right) \\
&= \frac{\lambda}{2}E(T_1^2)
\end{aligned}$$

by similar process with part (a).

$$\begin{aligned}
E(N) &= E(E(N|T_1)) \\
&= E(\lambda T_1) \\
&= \lambda E(T_1)
\end{aligned}$$

So the average amount of time that a passenger waits is

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{W_1 + \dots + W_n}{n} &= \frac{E(R_1)}{E(N)} \\
&= \frac{E(T_1^2)}{2E(T_1)} \\
&= \frac{\lambda \int_0^\infty t^2 dF(t)}{2 \int_0^\infty t dF(t)}
\end{aligned}$$