Homework 4, MATH 8010

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2.

(b) *Proof.* Let T_n be the n^{th} replacement for $n \geq 1$ and set $T_0 = 0$.

Then $\{T_n - T_{n-1}\}_{n=1}^{\infty}$ is an *iid* sequence of nonnegative random variables.

Let S_1 be the first in-use failure.

Let $N = \inf\{n \geq 1 : T_n - T_{n-1} < T\}$.

Since $\mathbb{1}(N \leq n) = 1 - \mathbb{1}(N > n) = 1 - \mathbb{1}(T_1 - T_0 \geq T, T_2 - T_1 > T, ..., T_n - T_{n-1} \geq T)$, N is a stopping time w.r.t. $\{T_n - T_{n-1}\}_{n=0}^{\infty}$.

Then

$$S_1 = \sum_{n=1}^{N} (T_n - T_{n-1}).$$

By Wald's identity, we have

$$E(S_1) = E(N)E(T_1).$$

Since for $n \in \mathbb{N}$,

$$\begin{aligned} P(N = n) &= P(T_n - T_{n-1} < T) \prod_{k=1}^{n-1} P(T_k - T_{k-1} \geq T) \\ &= P(T_1 < T) \prod_{k=1}^{n-1} P(T_1 \geq T) \\ &= F(T) \left(1 - F(T)\right)^{n-1}, \end{aligned}$$

$$E(N) = \sum_{n=1}^{\infty} nP(N = n) = F(T) \sum_{n=1}^{\infty} n \left(1 - F(T)\right)^{n-1}.$$

We know the Maclaurin series of $\frac{1}{(1-x)^2}$ is

$$\sum_{n=1}^{\infty} nx^{n-1}$$

for $-1 < x < 1$.

The density f is continuous on $(0, \infty)$, so $0 < F(T) < 1$ for $T \in \mathbb{R}^+$.

Then we have $0 < 1 - F(T) < 1$.

So, replace x with $1 - F(T)$, we have

$$\sum_{n=1}^{\infty} n\left(1 - F(T)\right)^{n-1} = \frac{1}{1 - \left(1 - F(T)\right)^2} = \frac{1}{F^2(T)}.$$

Then

$$E(N) = F(T)\frac{1}{F^2(T)} = \frac{1}{F(T)}.$$

Thus,the averge time between two successive in-use failures is

$$E(S_1) = E(N)E(T_1) = \frac{\int_0^T xf(x)dx + T\left(1 - F(T)\right)}{F(T)}.$$

$\square$

3.

(a) Assume $p \neq q$.

Let $\{X_n\}$ be the winning of the n^{th} bet for $n \geq 1$.

Then for $n \geq 1$,

$$X_n = \begin{cases} 1, & \text{with probability } p; \\ -1, & \text{with probability } 1-p. \end{cases}$$

So $\{X_n\}_{n=1}^{\infty}$ is an *iid* sequence of nonnegative random variables.

Then $E(X_n) = p - q$.

Let $T = \inf\{n \geq 1 : \sum_{k=1}^{n} X_k = N \text{ or } \sum_{k=1}^{n} X_k = 0\}$.

Since

$$\begin{aligned} \mathbb{1}(T \leq n) &= 1 - \mathbb{1}(T > n) \\ &= 1 - \mathbb{1}(0 < \sum_{k=1}^{1} X_k < N, 0 < \sum_{k=1}^{2} X_k < N, ..., 0 < \sum_{k=1}^{n} X_k < N), \end{aligned}$$

T is a stopping time w.r.t $\{\sum_{k=1}^{n} X_k\}_{n=1}^{\infty}$.

Let Y be the final winning of the gambler when he stops.

So given the initial fortune i \$,

$$Y = \begin{cases} N-i, & \text{with probability } P_i = \frac{1-(q/p)^i}{1-(q/p)^N}; \\ -i, & \text{with probability } 1-P_i = \frac{(q/p)^i-(q/p)^N}{1-(q/p)^N}. \end{cases}$$

So

$$E\left(\sum_{n=1}^{T} X_n\right) = E(Y) = (N-i)P_i - i(1-P_i).$$

By Wald's identity, we also have

$$E\left(\sum_{n=1}^{T} X_n\right) = E(T)E(X_1) = (p-q)E(T).$$

Thus,

$$E(T) = \frac{(N-i)P_i - i(1-P_i)}{p-q}.$$

(b) Assume $p = q$.

Let $E_i(T)$ be $E(T)$ when given initial fortune is i \$.

Then

$$\begin{aligned}
E_i(T) &= E\left(E_i(T|X_1)\right) \\
&= E(T|X_1 = 1)\frac{1}{2} + E(T|X_1 = -1)\frac{1}{2} \\
&= \frac{1}{2}\left(1 + E_{i+1}(T)\right) + \frac{1}{2}\left(1 + E_{i-1}(T)\right) \\
&= 1 + \frac{1}{2}E_{i+1}(T) + \frac{1}{2}E_{i-1}(T)
\end{aligned}$$

Let $d_i = E_i(T) - E_{i-1}(T)$.

Then $d_{i+1} = d_i - 2$.

We can find that

$$\begin{aligned}
\begin{pmatrix} d_{i+1} \\ -2 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d_i \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2\begin{pmatrix} d_{i-1} \\ -2 \end{pmatrix} \\
&= \dots \\
&= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i\begin{pmatrix} d_1 \\ -2 \end{pmatrix}
\end{aligned}$$

We claim

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}.$$

Basic case:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Assume

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{i+1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & i+1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

So the claim holds. Then we get

$$\begin{pmatrix} d_{i+1} \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ -2 \end{pmatrix}$$

So

$$d_{i+1} = d_1 - 2i.$$

So

$$d_N = d_1 - 2(N-1).$$

We know $d_1 = E_1(T) - E_0(T) = E_1(T)$, and

$d_N = E_N(T) - E_{N-1}(T) = -E_{N-1}(T)$.

Since $p = q$, by symmetry, we have

$$E_1(T) = E_{N-1}(T).$$

So

$$d_N = -d_1.$$

Combine it with $d_N = d_1 - 2(N-1)$, we have

$$d_1 = N - 1.$$

Then

$$d_i = d_1 - 2(i-1) = N - 2i + 1.$$

As a result, given intial fortune i \$, we have

$$\begin{aligned}
E(T) = E_i(T) &= d_i + E_{i-1}(T) \\
&= d_i + d_{i-1} + E_{i-2}(T) \\
&= \ldots \\
&= d_i + d_{i-1} + \ldots + d_1 + E_0(T) \\
&= (N - 2i + 1) + (N - 2(i-1) + 1) + \ldots + N - 1 + 0 \\
&= Ni - i^2.
\end{aligned}$$

4. Consider the regenerating process w.r.t. the arrival times of the train.

For $n \geq 1$, let T_n be the arrival time of the n^{th} customer.

Then the interrenewals $\{T_n - T_{n-1}\}$ is a *iid* sequence of nonnegative random variables.

Besides, we know $(T_n - T_{n-1}) \sim exp(\lambda)$ for $n \geq 1$, so $E(T_1) = \frac{1}{\lambda}$ given $T(0) = 0$.

Since

$$T_N = \sum_{n=1}^{N} (T_n - T_{n-1}),$$

$$E(T_N) = \sum_{n=1}^{N} E(T_n - T_{n-1}) = NE(T_1) = \frac{N}{\lambda}.$$

Let τ_1 be the first arrival time of the train.

Then $\tau_1 = T_N + K$.

So

$$E(\tau_1) = E(T_N + K) = \frac{N}{\lambda} + K.$$

Suppose each customer pays us money at a rate of c per unit time.

So at T_N, the n^{th} customer will pay us $c(T_N - T_n)$ for $1 \leq n \leq N$.

Let R_1 be the reward earned when between time 0 and T_N.

Then

$$R_1 = \sum_{n=1}^{N} c(T_N - T_n).$$

Since for $1 \leq n \leq N$,

$$E(T_N - T_n) = E(T_{N-n}) = \frac{N-n}{\lambda},$$

$$E(R_1) = c\sum_{n=1}^{N} E(T_N - T_n) = c\sum_{n=1}^{N} \frac{N-n}{\lambda} = \frac{c}{2\lambda}N(N-1).$$

Let R_2 be the reward earned between T_N and τ_1.

Then

$$R_2 = cNK + \sum_{l=1}^{N(K)} c(K - T_l).$$

Since given $N(K) = n$ for $n \in \mathbb{N}$, we have $T_l \sim \text{Uniform}(0, K)$ for $1 \leq l \leq n$,

$$E\left(T_l\right) = \int_0^K x \frac{1}{K} dx = \frac{K}{2}.$$

So

$$\begin{aligned}
E\left(\sum_{l=1}^{N(K)} c(K - T_l) \mid N(K) = n\right) &= E\left(\sum_{l=1}^{n} c(K - T_l)\right) \\
&= c\sum_{l=1}^{n} E(K - T_l) \\
&= c\sum_{l=1}^{n} (K - E(T_l)) \\
&= cKn - c\sum_{l=1}^{n} E(T_l) \\
&= cKn - c\sum_{l=1}^{n} \frac{K}{2} \\
&= \frac{cKn}{2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
E\left(\sum_{l=1}^{N(K)} c(K - T_l)\right) &= E\left(E\left(\sum_{l=1}^{N(K)} c(K - T_l) \mid N(K) = n\right)\right) \\
&= E\left(\frac{cKN(K)}{2}\right) \\
&= \frac{cK}{2} E\left(N(K)\right) \\
&= \frac{cK}{2} \lambda K \\
&= \frac{c\lambda K^2}{2}
\end{aligned}$$

since given K, $N(K) \sim$Poisson(λK).

So

$$E(R_2) = cNK + E\left(\sum_{l=1}^{N(K)} c(K - T_l)\right) = cNK + \frac{c\lambda K^2}{2}.$$

Since the total cost during the first cycle τ_1 is

$$C_1 = R_1 + R_2,$$

$$E(C_1) = E(R_1) + E(R_2) = \frac{c}{2\lambda} N(N-1) + cNK + \frac{c\lambda K^2}{2}.$$

As a result, the long-run average cost per unit time is

$$\begin{aligned}\frac{E(C_1)}{E(\tau_1)} &= \frac{\frac{c}{2\lambda}N(N-1) + cNK + \frac{c\lambda K^2}{2}}{\frac{N}{\lambda} + K} \\ &= \frac{cN(N-1) + 2c\lambda NK + c\lambda^2 K^2}{2N + 2\lambda K}.\end{aligned}$$