Homework 2, MATH 9010
Due on Thursday, September 8

Problem 1 Suppose $\mathcal{C}$ is a collection of subsets of Ω that satisfies the following properties: (i) $\Omega \in \mathcal{C}$; (ii) if $A \in \mathcal{C}$, then $A^c \in \mathcal{C}$; (iii) If $\{A_i\}_{i=1}^{\infty}$ is a collection of disjoint subsets in $\mathcal{C}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{C}$.

Show that $\mathcal{C}$ is a λ-system.

Problem 2 Consider the semialgebra $\mathcal{S} := \{(a,b]; 0 \le a \le b \le 1\}$ of $\Omega = (0,1]$. Show that if $\{a_k\}_{k\ge1}$ and $\{b_k\}_{k\ge1}$ satisfy

$$0 \le a_n < b_n = a_{n+1} < b_{n+1} < 1$$

for each $n \ge 1$, then

$$\bigcup_{n=1}^{\infty} (a_n, b_n]$$

cannot be a member of $\mathcal{S}$. **Note:** This will be much more instructive after you read the construction of Lebesgue measure on $(0,1]$ in Section 2.5 of Resnick's text.

Problem 3 (Jacod and Protter, pg. 46 Problem 7.17) Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}\mathbf{1}_{[0,\infty)}(x) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(x) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(x).$$

Let $\mathbb{P}$ be given by

$$\mathbb{P}((-\infty, x]) = F(x), \qquad x \in \mathbb{R}.$$

Find the probabilities of the following events: $A = (-1/2, 1/2)$; $B = (-1/2, 3/2)$; $C = (2/3, 5/2)$; $D = [0,2)$; $E = (3, \infty)$.

Problem 4 (Jacod and Protter, page 45 Problem 7.14) Let $\{A_k\}_{k\ge1}$ be a sequence of null events, i.e. events where $\mathbb{P}(A_k) = 0$ for each $k \ge 1$. Show that $\cup_{k=1}^{\infty} A_k$ is also a null event.

Problem 5 (Resnick, pg. 63 Problem 1) Let Ω be a nonempty set, and let $\mathcal{F}_0$ be the collection of all subsets such that either A or A^c is finite.

Define, for each $A \in \mathcal{F}_0$, the set function $\mathbb{P}$, where

$$\mathbb{P}(A) = \begin{cases} 0, & \text{if } A \text{ is finite;} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

(a) If Ω is countably infinite, show $\mathbb{P}$ is additive on $\mathcal{F}_0$, but not countably additive.

(b) If Ω is uncountable, show $\mathbb{P}$ is countably additive on $\mathcal{F}_0$.

Problem 6 (Resnick, page 25 Problem 34) Suppose $\mathcal{B}$ is a σ-algebra of subsets of Ω, and suppose $A \notin B$. Show that $\sigma(\mathcal{B} \cup \{A\})$, the smallest σ-algebra containing both $\mathcal{B}$ and A,, consists of sets of the form

$$(A \cap B) \cup (A^c \cap B^{'}), \qquad B, B^{'} \in \mathcal{B}.$$

Problem 7 (Resnick, page 63 Problem 4) Suppose $\mathbb{P}$ is a probability measure on a σ-algebra $\mathcal{B}$ and suppose $A \notin \mathcal{B}$. Let $\mathcal{B}_1 := \sigma(\mathcal{B} \cup \{A\})$ and show that $\mathbb{P}$ has an extension to a probability measure $\mathbb{P}_1$ on $\mathcal{B}_1$ (Do this without applying an extension theorem).