Homework 3 Solutions, MATH 9010

Problem 1 Let $(\Omega, \mathcal{B}, \mathbb{P}) = ((0,1], \mathcal{B}((0,1]), \lambda)$, where λ is Lebesgue measure. Define, for each $\omega \in \Omega$,

$$X_1(\omega) := 0, \qquad X_2(\omega) := \mathbf{1}_{\{1/2\}}(\omega), \qquad X_3(\omega) := \mathbf{1}_{\mathbb{Q}}(\omega).$$

where $\mathbb{Q} \subset (0,1]$ are the rational numbers in $(0,1]$. Note

$$\mathbb{P}(X_1 = X_2 = X_3 = 0) = 1$$

and give

$$\sigma(X_i), \quad i = 1, 2, 3.$$

Solution Observe that $\mathbb{P}(X_1 = 0) = \mathbb{P}(\Omega) = \lambda((0,1]) = 1$. Moreover, $\mathbb{P}(X_2 = 0) = \lambda(\{1/2\}^c) = 1$, and

$$\mathbb{P}(X_3 = 1) = \lambda(\mathbb{Q}) = \sum_{k=1}^{\infty} \lambda(\{r_k\}) = 0$$

since $\mathbb{Q}$ is countable, and so $\mathbb{Q}$ can be expressed as $\{r_1, r_2, r_3, \ldots\}$. Hence, $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_2 = 0) = \mathbb{P}(X_3 = 0) = 1$.

We can use Boole's inequality to show that $\mathbb{P}(X_1 = 0, X_2 = 0, X_3 = 0) = 1$. Here

$$\mathbb{P}\left(\left(\bigcap_{k=1}^{3} \{X_k = 1\}\right)^c\right) = \mathbb{P}\left(\bigcup_{k=1}^{3} \{X_k = 0\}\right) \leq \sum_{k=1}^{3} \mathbb{P}(X_k = 0) = 0.$$

Next, notice that since $\sigma(X_i)$ is the smallest σ-algebra containing all sets of the form $X_i^{-1}(\{k\})$ for $k = 0, 1$, we have

$$\sigma(X_1) = \{\emptyset, (0,1]\}$$

$$\sigma(X_2) = \{\emptyset, \{1/2\}, \{1/2\}^c, (0,1]\}$$

and

$$\sigma(X_3) = \{\emptyset, \mathbb{Q}, \mathbb{Q}^c, (0,1]\}.$$

Problem 2 Suppose $f : \mathbb{R}^k \to \mathbb{R}$, where f is $\mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$ measurable. Let $X_1, X_2, \ldots, X_k$ be random variables on $(\Omega, \mathcal{B})$. Then $f(X_1, X_2, \ldots, X_k)$ is $\sigma(X_1, X_2, \ldots, X_k)/\mathcal{B}(\mathbb{R})$-measurable.

Solution Observe that each random variable X_j is $\sigma(X_1, X_2, \ldots, X_k)$-measurable, by definition of $\sigma(X_1, X_2, \ldots, X_k)$, so it follows from Proposition 3.2.4 of Resnick that $(X_1, X_2, \ldots, X_k)$ is $\sigma(X_1, X_2, \ldots, X_k)/\mathcal{B}(\mathbb{R}^k)$ measurable.

Furthermore, since f is $\mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$ measurable, we can use Proposition 3.2.2 of Resnick to say that $f(X_1, X_2, \ldots, X_k)$ is $\sigma(X_1, X_2, \ldots, X_k)/\mathcal{B}(\mathbb{R})$ measurable, which proves the claim.

Problem 3 If X is a random variable, so is $|X|$. The converse may be false.

Proof Observe that the function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = |x|$ is continuous on $\mathbb{R}$, and $|X| = g \circ X$. Hence, $|X|$ is the composition of two measurable functions, so it is also a random variable.

Assuming our measurable space is denoted as $(\Omega, \mathcal{B})$, pick a subset B of Ω that is not a member of $\mathcal{B}$ (we have seen examples of σ-algebras that do not contain all subsets), and define $X := \mathbf{1}_B - \mathbf{1}_{B^c}$. Clearly X is not $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable, but

$$|X| = \mathbf{1}_B + \mathbf{1}_{B^c} = \mathbf{1}_\Omega = 1$$

and $|X|$ is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable. Hence,we have found an example where $|X|$ is a random variable, but X is not.

Problem 4 Suppose that $\{B_n\}_{n\geq 1}$ is a countable partition of Ω and define $\mathcal{B} = \sigma(B_n, n \geq 1)$. Show a function $X : \Omega \to (-\infty, \infty]$ is $\mathcal{B}$-measurable if and only if X is of the form

$$X = \sum_{i=1}^{\infty} c_i \mathbf{1}_{B_i}$$

for constants $\{c_i\} \subset \mathbb{R}$. (What is $\mathcal{B}$?)

Solution If X is of the form

$$X = \sum_{i=1}^{\infty} c_i \mathbf{1}_{B_i}$$

it is clearly measurable with respect to $\mathcal{B}$, since $c_i \mathbf{1}_{B_i}$ is measurable with respect to $\mathcal{B}$ for each $i \geq 1$.

Conversely, suppose that X is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable. For each integer $n \geq 1$, choose an element $\omega_n \in B_n$, and define $c_n := X(\omega_n)$. $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurability of X implies

$$X^{-1}(\{c_n\}) = \cup_{i\in I_n} B_i$$

where I_n consists of all indices k satisfying $c_k = c_n$. This also means that $X(\omega) = c_n$ for each $\omega \in \cup_{i\in I_n} B_i$.

Next, observe that

$$X^{-1}(\{c_1, c_2, c_3, \ldots\}) = \cup_{n=1}^{\infty} B_n = \Omega$$

and so if a real number a is not a member of $\{c_1, c_2, \ldots,\}$, then $X^{-1}(a) = \emptyset$: this is true because

$$\{a\} \cap \{c_1, c_2, \ldots\} = \emptyset$$

implies

$$X^{-1}(a) \cap X^{-1}(\{c_1, c_2, \ldots\}) = \emptyset$$

and since $X^{-1}(\{c_1, c_2, \ldots\}) = \Omega$, it must be the case that $X^{-1}(a) = \emptyset$. These observations imply that the range of X is $\{c_n\}_{n\geq 1}$. Hence,

$$X = \sum_{n=1}^{\infty} c_n \mathbf{1}_{B_n}.$$

Problem 5 Suppose $-\infty < a \leq b < \infty$. Show that the indicator function $\mathbf{1}_{(a,b]}(x)$ can be approximated by bounded and continuous functions; that is, show that there exist a sequence of continuous functions $\{f_n\}$, $0 \leq f_n \leq 1$ such that $f_n \to \mathbf{1}_{(a,b]}$ pointwise.

Hint: Approximate the rectangle of height one and base $(a, b]$ by a trapezoid of height 1 with base $(a, b + n^{-1}]$ whose top line extends from $a + n^{-1}$ to b.

Solution We follow the hint, and define, for each integer $n \geq 1$, the function $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} 0, & x \leq a; \\ n(x-a), & a < x \leq a + 1/n; \\ 1, & a + 1/n \leq x < b; \\ 1 - n(x-b), & b \leq x < b + 1/n; \\ 0, & x \geq b + 1/n. \end{cases}$$

You can easily show that each function f_n is bounded on $\mathbb{R}$, as well as continuous at each point of $\mathbb{R}$. Furthermore,

$$\lim_{n\to\infty} f_n(x) = \mathbf{1}_{(a,b]}(x)$$

for each $x \in \mathbb{R}$.

Problem 6 Suppose $\mathcal{B}$ is a σ-algebra of subsets of $\mathbb{R}$. Show $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$ if and only if every real-valued continuous function is measurable with respect to $\mathcal{B}$ and therefore $\mathcal{B}(\mathbb{R})$ is the smallest σ-field with respect to which all the continuous functions are measurable.

Solution Assume $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$. Then for each open set $O \subset \mathbb{R}$, we have

$$f^{-1}(O) \in \mathcal{B}(\mathbb{R}) \subset \mathcal{B}$$

for each continuous function $f : \mathbb{R} \to \mathbb{R}$, since $f^{-1}(O)$ is open whenever O is open and f is continuous on $\mathbb{R}$. Thus, any continuous function f is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.

Next, suppose $\mathcal{B}$ is a σ-algebra consisting of subsets of $\mathbb{R}$, where every continuous function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable. To show $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$, it suffices to show that $(a, b] \in \mathcal{B}$ for each set $(a, b] \subset \mathbb{R}$ satisfying $a < b$, since these sets generate $\mathcal{B}(\mathbb{R})$.

Notice though that the set $(a, b] \in \mathcal{B}$ if and only if the indicator function $\mathbf{1}_{(a,b]} : \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable. Using the solution to Problem 5, we can construct a sequence of functions $\{f_n\}_{n\geq 1}$, where each function is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable, and

$$\mathbf{1}_{(a,b]}(x) = \lim_{n\to\infty} f_n(x)$$

for each $x \in \mathbb{R}$. Since measurability is preserved under limits, we conclude that $\mathbf{1}_{(a,b]}$ is $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable, which means

$$(a, b] \in \mathcal{B}.$$

This proves $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$, which also implies that $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that makes all continuous functions $f : \mathbb{R} \to \mathbb{R}$ measurable, since each such σ-algebra must contain $\mathcal{B}(\mathbb{R})$.