Problem 5 Suppose $\infty < a \leq b < \infty$. Show that the indicator function $1_{(a,b]}(x)$ can be approximated by bounded and continuous functions; that is, show that there exist a sequence of contious functions $0 \leq f_n \leq 1$ such that $f_n \to 1_{(a,b]}$ pointwise.

Proof.
Let

$$\begin{aligned} f_n(x) &= n(x-a)1_{(a,a+1/n]}(x) + 1_{(a+1/n,b]}(x) - n(x-b-1/n)\,(b,b+1/n]\,(x) \\ &= \begin{cases} n(x-a), & \text{if } x \in \left(a, a+1/n\right], \\ 1, & \text{if } x \in \left(a+1/n, b\right], \\ -n(x-b-1/n), & \text{if } x \in \left(b, b+1/n\right]. \end{cases} \end{aligned}$$

Then we can find $0 \leq f_n \leq 1$ given $\infty < a \leq b < \infty$.
$f_n((a+1/n)^-) = 1 = f_n((a+1/n)^+) = 1 = f_n(a+1/n)$ and $f_n(b^-) = 1 = f_n(b^+) = 1 = f_n(b)$.
Namely, f is continuous at point $a+1/n$ and b.
It is obvious that f is also continuous on other points of $(a,b]$.
Thus, f_n is continuous on $(a,b]$.
Let $x_0 \in (a,b]$. Then there exists $N \in \mathbb{N}$ such that $a + 1/n < x_0 \leq b$ as $n \geq N$.
So when $n > N$, $f_n(x_0) = 1$ since $x_0 \in (a+1/n, b]$ as $n \geq N$.
So $|f_n(x_0) - 1_{(a,b]}(x_0)| = 0$ as $n \geq N$.
Hence, $f_n \to 1_{(a,b]}$ pointwise.

$\square$