

## Homework 2, MATH 9010

Due on Thursday, September 8

Shuai Wei

**Problem 1** Suppose  $\mathcal{C}$  is a collection of subsets of  $\Omega$  that satisfies the following properties: (i)  $\Omega \in \mathcal{C}$ ; (ii) if  $A \in \mathcal{C}$ , then  $A^c \in \mathcal{C}$ ; (iii) If  $\{A_i\}_{i=1}^\infty$  is a collection of disjoint subsets in  $\mathcal{C}$ , then  $\cup_{i=1}^\infty A_i \in \mathcal{C}$ .

Show that  $\mathcal{C}$  is a  $\lambda$ -system.

**Problem 2** Consider the semialgebra  $\mathcal{S} := \{(a, b]; 0 \leq a \leq b \leq 1\}$  of  $\Omega = (0, 1]$ . Show that if  $\{a_k\}_{k \geq 1}$  and  $\{b_k\}_{k \geq 1}$  satisfy

$$0 \leq a_n < b_n = a_{n+1} < b_{n+1} < 1$$

for each  $n \geq 1$ , then

$$\bigcup_{n=1}^{\infty} (a_n, b_n]$$

cannot be a member of  $\mathcal{S}$ . **Note:** This will be much more instructive after you read the construction of Lebesgue measure on  $(0, 1]$  in Section 2.5 of Resnick's text.

**Problem 3** (Jacod and Protter, pg. 46 Problem 7.17) Suppose a distribution function  $F$  is given by

$$F(x) = \frac{1}{4}\mathbf{1}_{[0, \infty)}(x) + \frac{1}{2}\mathbf{1}_{[1, \infty)}(x) + \frac{1}{4}\mathbf{1}_{[2, \infty)}(x).$$

Let  $\mathbb{P}$  be given by

$$\mathbb{P}((-\infty, x]) = F(x), \quad x \in \mathbb{R}.$$

Find the probabilities of the following events:  $A = (-1/2, 1/2)$ ;  $B = (-1/2, 3/2)$ ;  $C = (2/3, 5/2)$ ;  $D = [0, 2)$ ;  $E = (3, \infty)$ .

**Problem 4** (Jacod and Protter, page 45 Problem 7.14) Let  $\{A_k\}_{k \geq 1}$  be a sequence of null events, i.e. events where  $\mathbb{P}(A_k) = 0$  for each  $k \geq 1$ . Show that  $\cup_{k=1}^\infty A_k$  is also a null event.

**Problem 5** (Resnick, pg. 63 Problem 1) Let  $\Omega$  be a nonempty set, and let  $\mathcal{F}_0$  be the collection of all subsets such that either  $A$  or  $A^c$  is finite.

Define, for each  $A \in \mathcal{F}_0$ , the set function  $\mathbb{P}$ , where

$$\mathbb{P}(A) = \begin{cases} 0, & \text{if } A \text{ is finite;} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

(a) If  $\Omega$  is countably infinite, show  $\mathbb{P}$  is additive on  $\mathcal{F}_0$ , but not countably additive.

(b) If  $\Omega$  is uncountable, show  $\mathbb{P}$  is countably additive on  $\mathcal{F}_0$ .

**Problem 6** (Resnick, page 25 Problem 34) Suppose  $\mathcal{B}$  is a  $\sigma$ -algebra of subsets of  $\Omega$ , and suppose  $A \notin \mathcal{B}$ . Show that  $\sigma(\mathcal{B} \cup \{A\})$ , the smallest  $\sigma$ -algebra containing both  $\mathcal{B}$  and  $A$ , consists of sets of the form

$$(A \cap B) \cup (A^c \cap B'), \quad B, B' \in \mathcal{B}.$$

**Problem 7** (Resnick, page 63 Problem 4) Suppose  $\mathbb{P}$  is a probability measure on a  $\sigma$ -algebra  $\mathcal{B}$  and suppose  $A \notin \mathcal{B}$ . Let  $\mathcal{B}_1 := \sigma(\mathcal{B} \cup \{A\})$  and show that  $\mathbb{P}$  has an extension to a probability measure  $\mathbb{P}_1$  on  $\mathcal{B}_1$  (Do this without applying an extension theorem).