Homework 4, MATH 9010
Due on Thursday, September 15
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Problem 1 Let $X \in \mathcal{L}^1$ and let A_n be a sequence of events such that $\lim_{n\to\infty} P(A_n) = 0$. Show that $\lim_{n\to\infty} E\{X\mathbb{1}_{A_n}\} = 0$.

Proof. Since X is integrable, $|X|$ is also integrable.
Then
$$E(|X|) < \infty.$$
Let $n \in \mathbb{N}$, by Markov inequality,
$$P(|X| = \infty) \le P(|X| > n) \le \frac{E(|X|)}{n}$$
Since
$$\lim_{n\to\infty} \frac{E(|X|)}{n} = 0,$$
we have
$$P(|X| = \infty) = 0.$$
So
$$E(|X|\mathbb{1}_{\{|X|=\infty\}}) = 0$$
by what we have shown in class.
Besides,
$$\begin{aligned} E(|X|\mathbb{1}_{A_n}) &= E(|X|\mathbb{1}_{A_n\{|X|=\infty\}}) + E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \\ &\le E(|X|\mathbb{1}_{\{|X|=\infty\}}) + E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \\ &= E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \end{aligned}$$

There exist $M > 0$ such that
$$|X|\mathbb{1}_{A_n\{|X|<\infty\}} \le M\mathbb{1}_{A_n\{|X|<\infty\}} = M\mathbb{1}_{A_n}.$$
So
$$E(|X|\mathbb{1}_{A_n}) \le E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \le MP(A_n).$$
Then
$$\lim_{n\to\infty} E\{|X|\mathbb{1}_{A_n}\} \le M \lim_{n\to\infty} P(A_n) = 0.$$
Thus
$$\lim_{n\to\infty} E\{|X|\mathbb{1}_{A_n}\} = 0.$$
Since
$$\lim_{n\to\infty} E\{|X|\mathbb{1}_{A_n}\} = \lim_{n\to\infty} E\{X^+\mathbb{1}_{A_n}\} + \lim_{n\to\infty} E\{X^-\mathbb{1}_{A_n}\} = 0,$$

we have

$$\lim_{n\to\infty} E\{X^+\mathbb{1}_{A_n}\} = \lim_{n\to\infty} E\{X^-\mathbb{1}_{A_n}\} = 0$$

Therefore,

$$\lim_{n\to\infty} E\{X\mathbb{1}_{A_n}\} = \lim_{n\to\infty} E\{X^+\mathbb{1}_{A_n}\} - \lim_{n\to\infty} E\{X^-\mathbb{1}_{A_n}\} = 0$$

□

Problem 2 Suppose $X_n, n \geq 1$ and X are uniformly bounded random; i.e. there exists a constant K such that

$$|X_n| \vee |X| \leq K.$$

If $X_n \to X$ as $n \to \infty$, show by means of dominated convergence that

$$E|X_n - X| \to 0.$$

Proof. For any $n \in \mathbb{N}$,

$$|X_n - X| \leq |X_n| + |X| \leq M + M = 2M,$$

and
Constant M is integrable since

$$E(M) = E(M\mathbb{1}_\Omega) = M \times P(\Omega) = M < \infty.$$

Since $X_n \to X$, $X_n - X \to 0$.
By Dominated Convergence Theorem,

$$\lim_{n\to\infty} E(X_n - X) = E(0) = 0.$$

Namely,

$$E(X_n - X) \to 0.$$

$\square$

Problem 3 For $X \geq 0$, let

$$X_n^* = \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{[\frac{k-1}{2^n} \leq X < \frac{k}{2^n}]}.$$

Show

$$E(X_n^*) \downarrow E(X).$$

Proof. We first show $\{X_n^*\}_{n \geq 1}$ is monotone decreasing.
For $k, n \in \mathbb{N}$ and $k, n \geq 1$,

$$\begin{aligned}\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) &= \left[\frac{k-1}{2^n}, \frac{k-1/2}{2^n}\right) \bigcup \left[\frac{k-1/2}{2^n}, \frac{k}{2^n}\right) \\ &= \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right) \bigcup \left[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)\end{aligned}$$

So

$$X_n^* = \sum_{k=1}^{\infty} \frac{2k}{2^{n+1}} \mathbb{1}_{[\frac{2k-2}{2^{n+1}} \leq X < \frac{2k-1}{2^{n+1}}]} + \frac{2k}{2^{n+1}} \mathbb{1}_{[\frac{2k-1}{2^{n+1}} \leq X < \frac{2k}{2^{n+1}}]}.$$

On the other hand,

$$X_{n+1}^* = \sum_{k=1}^{\infty} \frac{2k-1}{2^{n+1}} \mathbb{1}_{[\frac{2k-2}{2^{n+1}} \leq X < \frac{2k-1}{2^{n+1}}]} + \frac{2k}{2^{n+1}} \mathbb{1}_{[\frac{2k-1}{2^{n+1}} \leq X < \frac{2k}{2^{n+1}}]}.$$

Thus, $X_n^* \geq X_{n+1}^*$.
We claim $X_n^* \downarrow X$.
For any $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that $\frac{1}{2^N} < \epsilon$.
Then by the definition of X_N^*, $X_N^* \geq X$ and so

$$|X_N^* - X| = X_N^* - X < \frac{1}{2^N} < \epsilon.$$

So when $n > N$, $X_n < X_N$ and

$$|X_N^* - X| = X_n^* - X < X_N^* - X < \epsilon.$$

Therefore, $X_n^* \downarrow X$.
Let $Y_n^* = X_1^* - X_n^* \geq 0$, then Y_n^* is monotone increasing.
Then

$$Y_n^* \uparrow X_1^* - X.$$

By the Monotone Convergence Theorem, we have

$$E(Y_n^*) \uparrow E(X_1^* - X).$$

Then

$$E(X_1^*) - E(X_n^*) \uparrow E(X_1^*) - E(X).$$

since $E(Y_n^*) = E(X_1^*) - E(X_n^*)$.
As a result,

$$E(X_n^*) \downarrow E(X).$$

$\square$

Problem 4 Suppose X is a non-negative random variable satisfying

$$P[0 \le X < \infty] = 1.$$

Show

(a)

$$\lim_{n\to\infty} nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) = 0,$$

(b)

$$\lim_{n\to\infty} n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\Big) = 0.$$

(a) *Proof.*

$$\begin{aligned} 0 \le nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) &= n\int_{\mathbb{1}_{[X>n]}} \frac{1}{X(\omega)}dP(\omega) \\ &= \int_{\mathbb{1}_{[X>n]}} \frac{n}{X(\omega)}dP(\omega) \\ &\le \int_{\mathbb{1}_{[X>n]}} dP(\omega) \\ &= P(\mathbb{1}_{[X>n]}) \end{aligned}$$

□

So

$$0 \le \lim_{n\to\infty} nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) \le \lim_{n\to\infty} P(\mathbb{1}_{[X>n]}) = P[X=\infty] = 1 - P[0 \le X < \infty] = 0.$$

Thus

$$\lim_{n\to\infty} nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) = 0.$$

(b) *Proof.*
When $X = 0, \mathbb{1}_{[X>n^{-1}]} = 0$, then $\frac{1}{X}\mathbb{1}_{[X>n^{-1}]} = 0$.
So $n^{-1}E\big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\big) = 0$.
When $X \neq 0,\ P(X=0) = 0$.

$$\begin{aligned} 0 \le n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\Big) &= n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[\frac{1}{X}<n]}\Big) \\ &\le n^{-1}E\Big(n\mathbb{1}_{[\frac{1}{X}<n]}\Big) \\ &= E\Big(\mathbb{1}_{[\frac{1}{X}<n]}\Big) \\ &= P(\frac{1}{X} < n) \end{aligned}$$

So

$$0 \le \lim_{n\to\infty} n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\Big) \le \lim_{n\to\infty} P(\frac{1}{X} < n) = P(X=0) = 0.$$

Thus,

$$\lim_{n\to\infty} n^{-1} E\left(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\right) = 0.$$

$\square$

Problem 5 Suppose $\{p_k, k \geq 0\}$ is a probability mass function on $\{0, 1, ...\}$ and define the generating function

$$P(s) = \sum_{k=0}^{\infty} p_k s^k, \quad 0 \leq s \leq 1.$$

Prove using dominated convergence that

$$\frac{d}{ds} P(s) = \sum_{k=1}^{\infty} p_k k s^{k-1}, \quad 0 \leq s \leq 1,$$

that is, prove differentiation and summation can be interchanged.

Proof. Let X be the corresponding random variable. Then $P(X = k) = p_k$.

$$\begin{aligned} \frac{d}{ds} P(s) &= \frac{d}{ds} E\big(s^X\big) \\ &= \lim_{t \to s} \frac{E\big(s^X\big) - E\big(t^X\big)}{s - t} \\ &= \lim_{t \to s} \frac{E\big(s^X - t^X\big)}{s - t} \\ &= \lim_{t \to s} E\left(\frac{s^X - t^X}{s - t}\right). \end{aligned}$$

Then we choose a sequence of numbers $\{t_n\}_{n=1}^{\infty}$ such that $t_n \downarrow s$ and $t_n \neq s$. So

$$\begin{aligned} \frac{d}{ds} P(s) &= \lim_{t_n \to s} E\left(\frac{s^X - t_n^X}{s - t_n}\right) \\ &= \lim_{t_n \to s} E(Y_n) \end{aligned}$$

by letting $Y_n = \frac{s^X - t_n^X}{s - t_n}$.
Then

$$\begin{aligned} \lim_{n \to \infty} Y_n &= \lim_{n \to \infty} \frac{s^X - t_n^X}{s - t_n} \\ &= \lim_{n \to \infty} \frac{-(X-1)t_n^X}{-1} \\ &= \lim_{n \to \infty} (X-1)t_n^X \\ &= (X-1)s^X. \end{aligned}$$

Let $f(y) = y^X$, then $f'(y) = X y^{X-1}$.
By Mean Value Theorem, there exists $\theta \in [t_n,\ s]$ such that $f(s) - f(t_n) = (s - t_n)(f'(\theta))$.
So

$$
\begin{aligned}
|Y_n| = \left|\frac{s^X - t_n^X}{s - t_n}\right| &= |f'(\theta)| \\
&= \left|X\theta^{X-1}\right| \\
&\le Xs^{X-1}.
\end{aligned}
$$

Since Xs^{X-1} is an integrable random variable, by Dominated Convergence Theorem,

$$\lim_{t_n \to s} E(Y_n) = E(\lim_{n\to\infty} Y_n) = E\left((X-1)s^X\right)$$

Since $\lim_{t_n \to s} E(Y_n) = \frac{d}{ds}P(s)$ and $E\left((X-1)s^X\right) = \sum_{k=1}^{\infty} p_k k s^{k-1}$, we have

$$\frac{d}{ds}P(s) = \sum_{k=1}^{\infty} p_k k s^{k-1}.$$

$\square$

Problem 6 The Cauthy-Schwarz Inequality is as follows: give two random variables X and Y,

$$E[|XY|] \leq \sqrt{E\left[|X^2|\right]}\sqrt{E\left[|Y^2|\right]}.$$

Use this inequality to prove the following statement: given a random variable X satisfying $E\left[X^2\right] = 1$, and $E[|X|] \geq a$ for some $a > 0$, show that for each $\lambda \in [0, 1]$,

$$P(|X| \geq \lambda a) \geq (1-\lambda)^2 a^2.$$

Proof.

$$\begin{aligned} 0 \leq a(1-\lambda) &= a - a\lambda \\ &\leq E(|X|) - a\lambda \\ &= E(|X| - a\lambda) \\ &= E\left((|X| - a\lambda)\mathbb{1}_{|X|>\lambda a}\right) + E\left((|X| - \lambda a)\mathbb{1}_{|X|\leq\lambda a}\right) \\ &\leq E\left((|X| - a\lambda)\mathbb{1}_{|X|>\lambda a}\right) \end{aligned}$$

since $E\left((|X| - \lambda a)\mathbb{1}_{|X|<\lambda a}\right) \leq 0$.
Since $|X| - a\lambda \leq |X|$ when $a, \lambda \geq 0$,

$$\begin{aligned} 0 \leq a(1-\lambda) &\leq E[|X|\mathbb{1}_{|X|>\lambda a}] \\ &\leq \sqrt{E\left(|X|^2\right)}\sqrt{E\left(\left|\mathbb{1}_{|X|>\lambda a}\right|^2\right)} \end{aligned}$$

by Cauthy-Schwarz Inequality.
Since

$$P\left(\left|\mathbb{1}_{|X|>\lambda a}\right|^2 = 1\right) = P(|X| > \lambda a),$$

and

$$P\left(\left|\mathbb{1}_{|X|>\lambda a}\right|^2 = 0\right) = 1 - P(|X| > \lambda a),$$

$$E\left(\left|\mathbb{1}_{|X|>\lambda a}\right|^2\right) = P(|X| > \lambda a).$$

So

$$\sqrt{E\left(\left|\mathbb{1}^2_{|X|>\lambda a}\right|\right)} = \sqrt{P(|X| > \lambda a)}.$$

Besides,

$$E\left(X^2\right) = 1.$$

Thus,

$$0 \leq a(1-\lambda) \leq \sqrt{P(|X| > \lambda a)}.$$

Therefore,

$$P(|X| > \lambda a) \geq (1-\lambda)^2 a^2.$$

$\square$