Problem 2 Suppose

$$f : \mathbb{R}^k \mapsto \mathbb{R} \text{ and } f \in \mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R}).$$

Let $X_1, ..., X_k$ be random variables on $(\Omega, \mathcal{B})$. Then

$$f(X_1, ...X_k) \in \sigma(X_1, ..., X_k).$$

Proof.
Let $X = (X_1, X_2, ..., X_k)$ be a random vector, then X is $\mathcal{B}/\mathcal{B}(\mathbb{R}^k)$ measurable.

$$\sigma(X) = [X^{-1}(A), A \in \mathcal{B}(\mathbb{R}^k)].$$

Since f is $\mathcal{B}(\mathbb{R}^k)/\mathcal{B}(\mathbb{R})$ measurable, $f^{-1}(B) \in \mathcal{B}(\mathbb{R}^k)$.
Assume the range of X is $\mathcal{C}$.
$\forall B \in \mathcal{B}(\mathbb{R})$,
if $f^{-1}(B) \notin \mathcal{C}$, $X^{-1}f^{-1}(B) = \emptyset \in \sigma(X)$,
else, $f^{-1}(B) \in \mathcal{C}$, then $X^{-1}f^{-1}(B) \in \sigma(X)$.
So

$$\forall B \in \mathcal{B}(\mathbb{R}), X^{-1}f^{-1}(B) = (f(X))^{-1}(B) \in \sigma(X).$$

Thus, $f(X) \in \sigma(X)$.
Namely, $f(X_1, ...X_k) \in \sigma(X_1, ..., X_k)$.

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