Problem 4 (Jacod and Protter, page 45 Problem 7.14) Let $\{A_k\}_{k\geq 1}$ be a sequence of null events, i.e. events where $\mathbb{P}(A_k) = 0$ for each $k \geq 1$. Show that $\cup_{k=1}^{\infty} A_k$ is also a null event.

Proof.
Let Ω be sample space and $\mathcal{A}$ be the σ-algebra containg $\{A_k\}_{k\geq 1}$.
Since $\{A_k\}_{k\geq 1}$ be a sequence of null events,
there exists $\{B_k\}_{k\geq 1} \subset \mathcal{A}$ such that $A_k \subset B_k$ and $\mathbb{P}(B_k) = 0$ for $k \geq 1$.
Then $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} B_k \in \mathcal{A}$.
Let $C_1 = B_1, C_k = B_k \setminus \bigcup_{i=1}^{k-1} B_i \in \mathcal{A}, k \geq 2$, then
$C_k's$ are disjoint and $\mathbb{P}(C_k) \leq \mathbb{P}(B_k)$by the monotonicity of probability measure.
So $\mathbb{P}(\bigcup_{k=1}^{\infty} B_k) = \mathbb{P}(\bigsqcup_{k=1}^{\infty} C_k) = \sum_{k=1}^{\infty} \mathbb{P}(C_k) \leq \sum_{k=1}^{\infty} \mathbb{P}(B_k) = \sum_{k=1}^{\infty} 0 = 0$.
Since we find $\bigcup_{k=1}^{\infty} B_k$ such that $\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty} B_k$ and $\mathbb{P}(\bigcup_{k=1}^{\infty} B_k) = 0$,
$\bigcup_{k=1}^{\infty} A_k$ is also a null event. □