Problem 6 Suppose $\mathcal{B}$ is a σ-field of subsets of $\mathbb{R}$. Show $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$ iff every real valued continuous function is measurable with respect to $\mathcal{B}$ and therefore $\mathcal{B}(\mathbb{R})$ is the smallest σ-field with respect to which all the continuous functions are measurable.

Proof.
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous funtion.
We claim f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.
Let $\mathcal{C}$ consists of the open sets in $\mathbb{R}$.
Let $B \in \mathcal{C}$, then $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$.
Since $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$, we know f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.

(a) Assume continuous function f is measurable with respect to $\mathcal{B}$.
We have shown $\forall B \in \mathcal{C}, f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ when f is a continuous function.
Then let f be an identity function, then $\forall B \in \mathcal{C},\ f^{-1}(B) = B \in \mathcal{B}(\mathbb{R})$.
So $B \in \mathcal{B}$ since f is measurable with respect to $\mathcal{B}$.
Thus, $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$.

(b) Assume $\mathcal{B}(\mathbb{R}) \subset \mathcal{B}$.
Use the symbols above, f is a continuous function , then $\forall B \in \mathcal{C}, f^{-1}(B) \in \mathcal{B}(\mathbb{R}) \subset \mathcal{B}$.
Since $\mathcal{B}$ is a σ-field of subsets of $\mathbb{R}$,
f is measurable with respect to $\mathcal{B}$.

Therefore $\mathcal{B}(\mathbb{R})$ is the smallest σ-field with respect to which all the continuous functions are measurable. $\square$