**Homework 3, MATH 9010**
Due on Thursday, September 15
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**Problem 1** Let $(\Omega, \mathcal{B}, P) = \Big( (0,1] \,, \mathcal{B}\big((0,1]\big), \lambda \Big)$ where $\lambda$ is Lebesgue measure. Define

$$\begin{aligned} X_1(\omega) &= 0, \forall \omega \in \Omega, \\ X_2(\omega) &= 1_{\{1/2\}}(\omega), \\ X_3(\omega) &= 1_{\mathbb{Q}}(\omega) \end{aligned}$$

where $\mathbb{Q} \in (0,1]$ are rational numbers in $(0,1]$. Note

$$P[X_1 = X_2 = X_3 = 0] = 1$$

and give

$$\sigma(X_i), \qquad i = 1, 2, 3.$$

1. $X_1(\omega) = 0, \forall \omega \in \Omega$.
   Note $X_1$ has range $0$ , then $X_1^{-1}(\{0\}) = \emptyset$.
   So $\sigma(X_1) = \sigma(\emptyset, \Omega) = \{\emptyset, \Omega\}$.

2. $X_2(\omega) = 1_{\{1/2\}}(\omega)$.
   Note $X_2$ has range $\{0,1\}$, then $X_1^{-1}(\{0\}) = \{(0,1/2) \cup (1/2,1]\}$,$X_2^{-1}(\{1\}) = \{1/2\}$.
   So $\sigma(X_2) = \{(0,1/2) \cup (1/2,1], 1/2, \emptyset, \Omega\}$.

3. $X_3(\omega) = 1_{\mathbb{Q}}(\omega)$.
   Note, $X_3$ has range $\{0,1\}$, then $X_1^{-1}(\{0\}) = \{\mathbb{Q}\}$, $X_1^{-1}(\{1\}) = \{\mathbb{Q}^c \cap (0,1]\}$.
   So $\sigma(X_3) = \{\mathbb{Q}, \mathbb{Q}^c \cap (0,1], \emptyset, \Omega\}$.
   where $\mathbb{Q} \in (0,1]$ are rational numbers in $(0,1]$.