Solutions to Homework 1, MATH 9010

Problem 1 (Jacod and Protter, Problem 2.5) Let $\{A_n\}_{n\geq 1}$ be a sequence of sets. Show that

$$\limsup_{n\to\infty} \mathbf{1}_{A_n} - \liminf_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\{\limsup_{n\to\infty} A_n - \liminf_{n\to\infty} A_n\}}.$$

Solution First observe that the random variable $\limsup_{n\to\infty} \mathbf{1}_{A_n} - \liminf_{n\to\infty} \mathbf{1}_{A_n}$ can only take values in the set $\{0, 1\}$, since for each $\omega \in \Omega$,

$$\liminf_{n\to\infty} \mathbf{1}_{A_n}(\omega) \leq \limsup_{n\to\infty} \mathbf{1}_{A_n}(\omega).$$

Hence, to prove the claim it suffices to show

$$\limsup_{n\to\infty} \mathbf{1}_{A_n}(\omega) - \liminf_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1$$

if and only if

$$\omega \in \limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n.$$

Assume an outcome $\omega \in \Omega$ satisfies

$$\limsup_{n\to\infty} \mathbf{1}_{A_n}(\omega) - \liminf_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1.$$

This of course is equivalent to saying

$$\limsup_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1, \qquad \liminf_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 0.$$

Observe that the statement

$$\limsup_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1$$

is true if and only if there exists an increasing collection of integers $\{n_k\}_{k\geq 1}$ (i.e. $n_k < n_{k+1}$ for each $k \geq 1$) where

$$\mathbf{1}_{A_{n_k}}(\omega) = 1$$

for each $k \geq 1$. This is true if and only if $\omega \in \bigcup_{k=n}^{\infty} A_k$ for each $n \geq 1$, i.e. if and only if

$$\omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_{n\to\infty} A_n$$

Furthermore, the statement

$$\liminf_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 0$$

means there also exists an increasing collection of integers $\{m_k\}_{k\geq 1}$ satisfying

$$\mathbf{1}_{A_{m_k}}(\omega) = 0$$

which is true if and only if ω is not a member of $\liminf_{n\to\infty} A_n$, as membership in this set means there exists an integer k_0 such that $\omega \in A_k$ for each $k \geq k_0$. But finally, $w \in \limsup_{n\to\infty} A_n$, $w \notin \liminf_{n\to\infty} A_n$ is equivalent to

$$w \in \limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n$$

i.e.

$$\mathbf{1}_{\{\limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n\}}(\omega) = 1.$$

This argument proves

$$\limsup_{n\to\infty} \mathbf{1}_{A_n} - \liminf_{n\to\infty} \mathbf{1}_{A_n} = 1$$

if and only if

$$\mathbf{1}_{\{\limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n\}}(\omega) = 1$$

so clearly

$$\limsup_{n\to\infty} \mathbf{1}_{A_n} - \liminf_{n\to\infty} \mathbf{1}_{A_n} = 0$$

if and only if

$$\mathbf{1}_{\{\limsup_{n\to\infty} A_n \setminus \liminf_{n\to\infty} A_n\}}(\omega) = 0.$$

Problem 2 (Jacod and Protter, Problem 2.6) Let $\mathcal{A}$ be a σ-algebra of subsets of Ω and let $B \in \mathcal{A}$. Show that $\mathcal{F} = \{A \cap B : A \in \mathcal{A}\}$ is a σ-algebra of subsets of B. Is it still true when B is a subset of Ω that does not belong to $\mathcal{A}$?

Solution This problem may have been confusing to you, but here you are being asked to show $\mathcal{F}$ is a σ-algebra of subsets of B, not Ω. Hence, you are being asked to show (i) $B \in \mathcal{F}$; (ii) if $A \in \mathcal{F}$, then $B \setminus A \in \mathcal{F}$; and (iii) If $\{A_n\}_{n\geq 1} \subset \mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

(i) Observe that

$$B = \Omega \cap B \in \mathcal{F}$$

because $\Omega \in \mathcal{A}$.

(ii) Next, suppose $A \in \mathcal{F}$. Then A can be expressed as $C \cap B$ for some $C \in \mathcal{A}$. Furthermore,

$$B \setminus A = B \setminus (C \cap B) = B \cap (C \cap B)^c = B \cap (C^c \cup B^c) = (B \cap C^c) \cup (B \cap B^c) = B \cap C^c \in \mathcal{F}$$

since $C^c \in \mathcal{A}$. Hence, $\mathcal{F}$ is closed under complementation.

(iii) Finally, assume $\{A_n\}_{n\geq 1} \subset \mathcal{F}$. Then for each $n \geq 1$, $A_n = C_n \cap B$ for some set $C_n \in \mathcal{A}$. Furthermore,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (C_n \cap B) = \left(\bigcup_{n=1}^{\infty} C_n \right) \cap B \in \mathcal{F}$$

since $\cup_{n=1}^{\infty} C_n \in \mathcal{A}$. Hence, $\mathcal{F}$ is a σ-algebra of subsets of B.

Finally, notice that our argument remains valid even if B is not a member of $\mathcal{A}$.

Problem 3 (Jacod and Protter, Problem 2.7) Let $f : \Omega \to E$, where E is an arbitrary space equipped with a σ-algebra $\mathcal{E}$. Let $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{E}\}$. Show that $\mathcal{A}$ is a σ-algebra on Ω.

Solution Again, we need to show $\mathcal{A}$ contains Ω, and is closed under both complements and countable unions.

(i) Here $\Omega = f^{-1}(E) \in \mathcal{A}$, since $E \in \mathcal{E}$, due to $\mathcal{E}$ being a σ-algebra of subsets of $\mathcal{E}$.

(ii) Next, suppose $A \in \mathcal{A}$. This is true if and only if $A = f^{-1}(B)$ for some set $B \in \mathcal{E}$. Furthermore,

$$A^c = (f^{-1}(B))^c = f^{-1}(B^c) \in \mathcal{A}$$

since $B^c \in \mathcal{E}$.

(iii) Finally, suppose $\{A_n\}_{n\geq 1} \subset \mathcal{A}$. Then for each $n \geq 1$, there exists a set $B_n \in \mathcal{E}$ such that $A_n = f^{-1}(B_n)$. This further implies

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in \mathcal{A}$$

since $\cup_{n\geq 1} B_n \in \mathcal{E}$.

This proves $\mathcal{A}$ is a σ-algebra of subsets of Ω.

Problem 4 (Jacod and Protter, Problem 2.17) Suppose that Ω is an infinite set (countable or not), and let $\mathcal{A}$ be the family of all subsets which are either finite or have a finite complement. Show that $\mathcal{A}$ is an algebra, but not a σ-algebra.

Solution To show $\mathcal{A}$ is an algebra, we need to show it (i) contains Ω; (ii) it is closed under complementation; and (iii) it is closed under finite unions.

(i) Observe that $\Omega^c = \emptyset$, the empty set, which contains a finite number (zero) of elements. Hence, $\Omega \in \mathcal{A}$.

(ii) Suppose $A \in \mathcal{A}$. Then for A^c to be a member of $\mathcal{A}$, either A^c or $(A^c)^c = A$ must be finite, but one of these statements is true since $A \in \mathcal{A}$. Hence, $A^c \in \mathcal{A}$.

(iii) Finally, suppose $A, B \in \mathcal{A}$. There are two cases to consider:

(a) Suppose A is infinite. Then A^c must be a finite set. Furthermore,

$$(A \cup B)^c = A^c \cap B^c \subset A^c$$

and this proves $(A \cup B)^c$ is also a finite set. Hence, $A \cup B \in \mathcal{A}$, as we have already shown that $\mathcal{A}$ is closed under forming complements. The same type of proof still works if B is infinite, and A is finite.

(b) If A and B are both finite, then $A \cup B$ must be as well, so $A \cup B \in \mathcal{A}$.

These statements prove $\mathcal{A}$ is an algebra of subsets of Ω.

$\mathcal{A}$ is not necessarily a σ-algebra. To see why, suppose $\Omega = \{1, 2, 3, 4, 5, \ldots\}$, and for each $n \geq 1$, define $A_n = \{2n\}$. Each A_n is a member of $\mathcal{A}$, yet

$$\bigcup_{n=1}^{\infty} A_n = \{2, 4, 6, 8, 10, \ldots\}, \qquad \left(\bigcup_{n=1}^{\infty} A_n\right)^c = \{1, 3, 5, 7, 9, \ldots\}$$

which means $\cup_{n \geq 1} A_n$ cannot be a member of $\mathcal{A}$.

Problem 5 (Resnick, page 22 Problem 14) Suppose $\{\mathcal{A}_n\}_{n\geq 1}$ are a collection of algebras satisfying $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, $n \geq 1$. Show that $\cup_{n\geq 1}\mathcal{A}_n$ is also an algebra.

Solution Define $\mathcal{A} := \cup_{n=1}^{\infty}\mathcal{A}_n$. To show $\mathcal{A}$ is an algebra, we need to show (i) $\Omega \in \mathcal{A}$; (ii) $\mathcal{A}$ is closed under forming complements; (iii) $\mathcal{A}$ is closed under forming countable unions.

(i) Here $\Omega \in \mathcal{A}_1 \subset \mathcal{A}$, so $\Omega \in \mathcal{A}$.

(ii) Suppose $A \in \mathcal{A}$. Then there exists an integer $j \geq 1$ where $A \in \mathcal{A}_j$, and since $\mathcal{A}_j$ is an algebra, $A^c \in \mathcal{A}_j$ too, implying $A^c \in \mathcal{A}$.

(iii) Finally, suppose $\{A_k\}_{k=1}^m \subset \mathcal{A}$. Then for each $k = 1, 2, \ldots, m$, there exists an integer n_k such that $A_k \in \mathcal{A}_n$ for $n \geq n_k$: this is true since $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for each $n \geq 1$. Next, choose $n_0 := \max(n_1, n_2, \ldots, n_m)$: then $A_k \in \mathcal{A}_{n_0}$ for each $k \in \{1, 2, \ldots, m\}$, and so

$$\bigcup_{k=1}^{m} A_k \in \mathcal{A}_{n_0} \subset \mathcal{A}$$

which proves $\mathcal{A}$ is an algebra.

Problem 6 (Resnick, page 24 Problem 30) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two σ-algebras of Ω, and define $\mathcal{B}_1 \vee \mathcal{B}_2$ as the smallest σ-algebra containing both $\mathcal{B}_1$ and $\mathcal{B}_2$. Show that $\mathcal{B}_1 \vee \mathcal{B}_2$ is generated by the collection of sets of the form $B_1 \cap B_2$, where $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$.

Solution We define two collections of subsets $\mathcal{C}, \mathcal{D}$, where

$$\mathcal{C} = \mathcal{B}_1 \cup \mathcal{B}_2, \qquad \mathcal{D} := \{B_1 \cap B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}.$$

The problem is asking you to show $\sigma(\mathcal{C}) = \sigma(\mathcal{D})$. In order to show these two σ-algebras are the same, it suffices to show $\mathcal{C} \subset \sigma(\mathcal{D})$, and $\mathcal{D} \subset \sigma(\mathcal{C})$.

First, notice that since each set $B \in \mathcal{B}_1$ can be written as $B \cap \Omega$, and $\Omega \in \mathcal{B}_2$, we clearly see that $\mathcal{B}_1 \subset \mathcal{D}$. A similar argument also proves $\mathcal{B}_2 \subset \mathcal{D}$, since $\Omega \in \mathcal{B}_1$ as well. Hence,

$$\mathcal{C} \subset \mathcal{D} \subset \sigma(\mathcal{D}).$$

It remains to show $\mathcal{D} \subset \sigma(\mathcal{C})$. A typical set within $\mathcal{D}$ is of the form $B_1 \cap B_2$, for $B_1 \in \mathcal{B}_1$, and $B_2 \in \mathcal{B}_2$. But σ-algebras are closed under finite intersections, and since $B_1, B_2 \in \mathcal{C}$, it must be the case that

$$B_1 \cap B_2 \in \sigma(\mathcal{C}).$$

This proves $\mathcal{D} \subset \sigma(\mathcal{C})$.