Problem 3 (Jacod and Protter, pg. 46 Problem 7.17) Suppose a distribution function F is given by

$$F(x) = \frac{1}{4}\mathbf{1}_{[0,\infty)}(x) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(x) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(x).$$

Let $\mathbb{P}$ be given by

$$\mathbb{P}((-\infty, x]) = F(x), \qquad x \in \mathbb{R}.$$

Find the probabilities of the following events: $A = (-1/2, 1/2)$; $B = (-1/2, 3/2)$; $C = (2/3, 5/2)$; $D = [0, 2)$; $E = (3, \infty)$.

Solution.
Based on the Corollary 7.1 from Jacod's textbook,

(1)

$$\mathbb{P}(A) = \mathbb{P}((-1/2, 1/2)) = F((1/2)^-) - F(-1/2).$$

$$\begin{aligned} F((1/2)^-) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}((1/2)^-) + \frac{1}{2}\mathbf{1}_{[1,\infty)}((1/2)^-) + \frac{1}{4}\mathbf{1}_{[2,\infty)}((1/2)^-) \\ &= 1/4 + 0 + 0 = 1/4, \end{aligned}$$

and

$$\begin{aligned} F(-1/2) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}(-1/2) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(-1/2) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(-1/2) \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

So

$$\mathbb{P}(A) = 1/4 - 0 = 1/4.$$

(2)

$$\mathbb{P}(B) = \mathbb{P}((-1/2, 3/2)) = F((3/2)^-) - F(-1/2).$$

$$\begin{aligned} F((3/2)^-) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}((3/2)^-) + \frac{1}{2}\mathbf{1}_{[1,\infty)}((3/2)^-) + \frac{1}{4}\mathbf{1}_{[2,\infty)}((3/2)^-)- \\ &= 1/4 + 1/2 + 0 = 3/4, \end{aligned}$$

and

$$\begin{aligned} F(-1/2) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}(-1/2) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(-1/2) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(-1/2) \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

So

$$\mathbb{P}(B) = 3/4 - 0 = 3/4.$$

(3)

$$\mathbb{P}(C) = \mathbb{P}((3/2, 5/2)) = F((5/2)^-) - F(2/3).$$

$$\begin{aligned} F((5/2)^-) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}((5/2)^-) + \frac{1}{2}\mathbf{1}_{[1,\infty)}((5/2)^-) + \frac{1}{4}\mathbf{1}_{[2,\infty)}((5/2)^-) \\ &= 1/4 + 1/2 + 1/4 = 1, \end{aligned}$$

and

$$\begin{aligned} F(3/2) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}(2/3) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(2/3) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(2/3) \\ &= 1/4 + 0 + 0 = 1/4. \end{aligned}$$

So

$$\mathbb{P}(C) = 1 - 3/4 = 3/4.$$

(4)

$$\mathbb{P}(D) = \mathbb{P}([0, 2)) = F(2^-) - F(0^-).$$

$$\begin{aligned} F(2^-) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}(2^-) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(2^-) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(2^-) \\ &= 1/4 + 1/2 + 0 = 3/4, \end{aligned}$$

and

$$\begin{aligned} F(0^-) &= \frac{1}{4}\mathbf{1}_{[0,\infty)}(0^-) + \frac{1}{2}\mathbf{1}_{[1,\infty)}(0^-) + \frac{1}{4}\mathbf{1}_{[2,\infty)}(0^-) \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

So

$$\mathbb{P}(D) = 3/4 - 0 = 3/4.$$

(5)

$$\mathbb{P}(E) = \mathbb{P}((3, \infty)) = F(\infty) - F(3) = 1 - F(3).$$

$$F(3) \geq F(5/2) = 1,$$

since F is monotone increasing and

$$F(3) \leq 1$$

since $0 \leq F(x) \leq 1, x \in R$.
So

$$F(3) = 1.$$

Thus

$$\mathbb{P}(E) = 1 - F(3) = 0.$$