**Homework 4, MATH 9010**
Due on Thursday, September 15
Shuai Wei

**Problem 1** Let $X \in \mathcal{L}^1$ and let $A_n$ be a sequence of events such that $\lim_{n\to\infty} P(A_n) = 0$. Show that $\lim_{n\to\infty} E\{X\mathbb{1}_{A_n}\} = 0$.

*Proof.* Since $X$ is integrable, $|X|$ is also integrable.
Then
$$E(|X|) < \infty.$$
Let $n \in \mathbb{N}$, by Markov inequality,
$$P(|X| = \infty) \leq P(|X| > n) \leq \frac{E(|X|)}{n}$$
Since
$$\lim_{n\to\infty} \frac{E(|X|)}{n} = 0,$$
we have
$$P(|X| = \infty) = 0.$$
So
$$E(|X|\mathbb{1}_{\{|X|=\infty\}}) = 0$$
by what we have shown in class.
Besides,
$$\begin{aligned} E(|X|\mathbb{1}_{A_n}) &= E(|X|\mathbb{1}_{A_n\{|X|=\infty\}}) + E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \\ &\leq E(|X|\mathbb{1}_{\{|X|=\infty\}}) + E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \\ &= E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \end{aligned}$$
There exist $M > 0$ such that
$$|X|\mathbb{1}_{A_n\{|X|<\infty\}} \leq M\mathbb{1}_{A_n\{|X|<\infty\}} = M\mathbb{1}_{A_n}.$$
So
$$E(|X|\mathbb{1}_{A_n}) \leq E(|X|\mathbb{1}_{A_n\{|X|<\infty\}}) \leq MP(A_n).$$
Then
$$\lim_{n\to\infty} E\{|X|\mathbb{1}_{A_n}\} \leq M \lim_{n\to\infty} P(A_n) = 0.$$
Thus
$$\lim_{n\to\infty} E\{|X|\mathbb{1}_{A_n}\} = 0.$$
Since
$$\lim_{n\to\infty} E\{|X|\mathbb{1}_{A_n}\} = \lim_{n\to\infty} E\{X^+\mathbb{1}_{A_n}\} + \lim_{n\to\infty} E\{X^-\mathbb{1}_{A_n}\} = 0,$$

we have

$$\lim_{n\to\infty} E\{X^+ \mathbb{1}_{A_n}\} = \lim_{n\to\infty} E\{X^- \mathbb{1}_{A_n}\} = 0$$

Therefore,

$$\lim_{n\to\infty} E\{X \mathbb{1}_{A_n}\} = \lim_{n\to\infty} E\{X^+ \mathbb{1}_{A_n}\} - \lim_{n\to\infty} E\{X^- \mathbb{1}_{A_n}\} = 0$$

□

**Problem 2** Suppose $X_n, n \geq 1$ and $X$ are uniformly bounded random; i.e. there exists a constant $K$ such that

$$|X_n| \vee |X| \leq K.$$

If $X_n \to X$ as $n \to \infty$, show by means of dominated convergence that

$$E|X_n - X| \to 0.$$

*Proof.* For any $n \in \mathbb{N}$,

$$|X_n - X| \leq |X_n| + |X| \leq M + M = 2M,$$

and
Constant $M$ is integrable since

$$E(M) = E(M\mathbb{1}_\Omega) = M \times P(\Omega) = M < \infty.$$

Since $X_n \to X$, $X_n - X \to 0$.
By Dominated Convergence Theorem,

$$\lim_{n\to\infty} E(X_n - X) = E(0) = 0.$$

Namely,

$$E(X_n - X) \to 0.$$

□

**Problem 3** For $X \geq 0$, let

$$X_n^* = \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbb{1}_{[\frac{k-1}{2^n} \leq X < \frac{k}{2^n}]}.$$

Show

$$E(X_n^*) \downarrow E(X).$$

*Proof.* We first show $\{X_n^*\}_{n\geq 1}$ is monotone decreasing.
For $k, n \in \mathbb{N}$ and $k, n \geq 1$,

$$\begin{aligned}\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) &= \left[\frac{k-1}{2^n}, \frac{k-1/2}{2^n}\right) \bigcup \left[\frac{k-1/2}{2^n}, \frac{k}{2^n}\right) \\ &= \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right) \bigcup \left[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)\end{aligned}$$

So

$$X_n^* = \sum_{k=1}^{\infty} \frac{2k}{2^{n+1}} \mathbb{1}_{[\frac{2k-2}{2^{n+1}} \leq X < \frac{2k-1}{2^{n+1}}]} + \frac{2k}{2^{n+1}} \mathbb{1}_{[\frac{2k-1}{2^{n+1}} \leq X < \frac{2k}{2^{n+1}}]}.$$

On the other hand,

$$X_{n+1}^* = \sum_{k=1}^{\infty} \frac{2k-1}{2^{n+1}} \mathbb{1}_{[\frac{2k-2}{2^{n+1}} \leq X < \frac{2k-1}{2^{n+1}}]} + \frac{2k}{2^{n+1}} \mathbb{1}_{[\frac{2k-1}{2^{n+1}} \leq X < \frac{2k}{2^{n+1}}]}.$$

Thus, $X_n^* \geq X_{n+1}^*$.
We claim $X_n^* \downarrow X$.
For any $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that $\frac{1}{2^N} < \epsilon$.
Then by the definition of $X_N^*$, $X_N^* \geq X$ and so

$$|X_N^* - X| = X_N^* - X < \frac{1}{2^N} < \epsilon.$$

So when $n > N$, $X_n < X_N$ and

$$|X_N^* - X| = X_n^* - X < X_N^* - X < \epsilon.$$

Therefore, $X_n^* \downarrow X$.
Let $Y_n^* = X_1^* - X_n^* \geq 0$, then $Y_n^*$ is monotone increasing.
Then

$$Y_n^* \uparrow X_1^* - X.$$

By the Monotone Convergence Theorem, we have

$$E(Y_n^*) \uparrow E(X_1^* - X).$$

Then

$$E(X_1^*) - E(X_n^*) \uparrow E(X_1^*) - E(X).$$

since $E(Y_n^*) = E(X_1^*) - E(X_n^*)$.
As a result,

$$E(X_n^*) \downarrow E(X).$$

$\square$

**Problem 4** Suppose $X$ is a non-negative random variable satisfying

$$P[0 \le X < \infty] = 1.$$

Show

(a)

$$\lim_{n\to\infty} nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) = 0,$$

(b)

$$\lim_{n\to\infty} n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\Big) = 0.$$

(a) *Proof.*

$$\begin{aligned} 0 \le nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) &= n\int_{\mathbb{1}_{[X>n]}} \frac{1}{X(\omega)}dP(\omega) \\ &= \int_{\mathbb{1}_{[X>n]}} \frac{n}{X(\omega)}dP(\omega) \\ &\le \int_{\mathbb{1}_{[X>n]}} dP(\omega) \\ &= P(\mathbb{1}_{[X>n]}) \end{aligned}$$

□

So

$$0 \le \lim_{n\to\infty} nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) \le \lim_{n\to\infty} P(\mathbb{1}_{[X>n]}) = P[X=\infty] = 1 - P[0 \le X < \infty] = 0.$$

Thus

$$\lim_{n\to\infty} nE\Big(\frac{1}{X}\mathbb{1}_{[X>n]}\Big) = 0.$$

(b) *Proof.*
When $X = 0, \mathbb{1}_{[X>n^{-1}]} = 0$, then $\frac{1}{X}\mathbb{1}_{[X>n^{-1}]} = 0$.
So $n^{-1}E\big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\big) = 0$.
When $X \ne 0,\ P(X=0) = 0$.

$$\begin{aligned} 0 \le n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\Big) &= n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[\frac{1}{X}<n]}\Big) \\ &\le n^{-1}E\Big(n\mathbb{1}_{[\frac{1}{X}<n]}\Big) \\ &= E\Big(\mathbb{1}_{[\frac{1}{X}<n]}\Big) \\ &= P(\frac{1}{X} < n) \end{aligned}$$

So

$$0 \le \lim_{n\to\infty} n^{-1}E\Big(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\Big) \le \lim_{n\to\infty} P(\frac{1}{X} < n) = P(X=0) = 0.$$

Thus,

$$\lim_{n\to\infty} n^{-1} E\left(\frac{1}{X}\mathbb{1}_{[X>n^{-1}]}\right) = 0.$$

$\square$