**Problem 2** Consider the semialgebra  $S := \{(a, b]; 0 \le a \le b \le 1\}$  of  $\Omega = (0, 1]$ . Show that if  $\{a_k\}_{k \ge 1}$  and  $\{b_k\}_{k \ge 1}$  satisfy

$$0 \le a_n < b_n = a_{n+1} < b_{n+1} < 1$$

for each  $n \geq 1$ , then

$$\bigcup_{n=1}^{\infty} (a_n, b_n]$$

cannot be a member of S. **Note:** This will be much more instructive after you read the construction of Lebesgue measure on (0,1] in Section 2.5 of Resnick's text.

*Proof.* Assume  $\bigsqcup_{n=1}^{\infty} (a_n, b_n] \in \mathcal{S}$ , then

$$\bigsqcup_{n=1}^{\infty} (a_n, b_n] = (c, d], \tag{1}$$

where  $0 \le c < d < 1$  since  $0 < a_n < b_n < 1$ . By De Morgan's laws,

$$\left\{ \bigsqcup_{n=1}^{\infty} (a_n, b_n) \right\}^c = \bigcap_{n=1}^{\infty} (0, a_n) \sqcup (b_n, 1)$$
$$= \left\{ \bigcap_{n=1}^{\infty} (0, a_n) \right\} \bigcup \left\{ \bigcap_{n=1}^{\infty} (b_n, 1) \right\}$$

since for each  $n \ge 1$ ,  $(0, a_n] \cap (b_n, 1] = \emptyset$ . Since  $a_1 < a_2 < ..., \bigcap_{n=1}^{\infty} (0, a_n] = (0, a_1]$ .

$$\left\{ \bigsqcup_{n=1}^{\infty} (a_n, b_n) \right\}^c = (0, a_1) \bigsqcup \left\{ \bigcap_{n=1}^{\infty} (b_n, 1) \right\}. \tag{2}$$

Also by (1), we have

$$\left\{ \bigsqcup_{n=1}^{\infty} (a_n, b_n) \right\}^c = (0, c] \sqcup (d, 1]. \tag{3}$$

Compare (2) and (3), we have

$$(0, a_1] = (0, c]$$

and

$$\bigcap_{n=1}^{\infty} (b_n, 1] = (d, 1] \tag{4}$$

since both of them are the disjoint union of two sets.

We claim for any  $n \ge 1, b_n < d$ . Otherwise, suppose there exist  $b_n \ge d$ , then  $b_{n+1} > b_n \ge d$ , and then  $\bigcap_{n=1}^{\infty} (b_n, 1] \subset (b_{n+1}, 1] \subsetneq (d, 1]$ , so it is contradicted by (4).

Thus, for any  $n \ge 1, b_n < d$ . Then

$$d \in (b_n, 1], n \ge 1.$$

So

$$d \in \bigcap_{n=1}^{\infty} (b_n, 1].$$

However  $d \notin (d, 1]$ , which is contradicted by (4). Therefore, the assumption  $\bigsqcup_{n=1}^{\infty} (a_n, b_n] \in \mathcal{S}$  does not hold. Namely,  $\bigsqcup_{n=1}^{\infty} (a_n, b_n] \notin \mathcal{S}$ .