Problem 4 Suppose that $B_n, n \geq 1$ is a countable partition of Ω and define $\mathcal{B} = \sigma(B_n, n \geq 1)$. Show a function $X : \Omega \mapsto (-\infty, \infty]$ is $\mathcal{B}$-meaurable iff X is the form

$$X = \sum_{i=1}^{\infty} c_i 1_{B_i},$$

for constants $\{c_i\}$.(What is $\mathcal{B}$?)

Proof.

(a) Assume $X = \sum_{i=1}^{\infty} c_i 1_{B_i}$.
Note the range of X is $\{c_i : i \in \mathbb{N}\}$.
$\forall i \in \mathbb{N}, X^{-1}(c_i) = B_i \in \sigma(B_n, n \geq 1)$.
So X is $\mathcal{B}$-measurable.

(b) Assume X is $\mathcal{B}$-measurable.
We claim $\mathcal{C} = \{\bigsqcup_{i \in I} B_i : I \subset \mathbb{N}\} = \mathcal{B} = \sigma(B_n, n \geq 1)$.
First we show $\mathcal{C}$ is a σ-algebra.

(a) $\Omega = \bigsqcup_{\mathbb{N}} B_i \in \mathcal{C}$.

(b) $\forall I \subset \mathbb{N}, \left(\bigsqcup_{i \in I} B_i\right)^c = \bigsqcup_{i \in N \setminus I} B_i \in \mathcal{C}$ since $\Omega = \bigsqcup_{i \in \mathbb{N}} B_i$.

(c) $\forall \{I_i\} \subset \mathbb{N}, I_i \cap I_j = \emptyset, \bigsqcup_{j \in I_i} B_j \in \mathcal{C}$.
Then $\bigsqcup_{i=1}^{\infty} \bigsqcup_{j \in I_i} B_j = \bigsqcup_{j \in \sqcup I_i} B_j \in \mathcal{C}$.

Then we can find that $\mathcal{C} \subset \mathcal{B}$ and $\{B_n, n \geq 1\} \subset \mathcal{C}$.
So $\mathcal{B} = \sigma(B_n, n \geq 1) \subset \mathcal{C}$.
Thus, $\mathcal{B} = \mathcal{C} = \{\bigsqcup_{i \in I} B_i : I \subset \mathbb{N}\}$.
Next, $\forall i \in \mathbb{N}$, let $b_i \in B_i \subset \Omega$, suppose $X(b_i) = c_i \in \mathbb{R}$.
Then $X_i^{-1}(c_i) \in \mathcal{B}$ since $c_i = [c_i, c_i] \subset \mathcal{B}(\mathbb{R})$ and X is $\mathcal{B}$-measurable.
So $X_i^{-1}(c_i) = \bigsqcup_{i \in I_i} B_j$, where $I_i \subset \mathbb{N}$.
Then $B_i \in \bigsqcup_{i \in I_i} B_j$ since $b_i \in B_i$ and $X(b_i) = c_i$.
Namely $B_i \subset X_i^{-1}(c_i)$.
Thus, we have $X(b) = c_i, \forall b \in B_i$.
As a result, we can write X as $X = \sum_{i=1}^{\infty} c_i 1_{B_i}$.

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