Problem 6 (Resnick, page 25 Problem 34) Suppose $\mathcal{B}$ is a σ-algebra of subsets of Ω, and suppose $A \notin \mathcal{B}$. Show that $\sigma(\mathcal{B} \cup \{A\})$, the smallest σ-algebra containing both $\mathcal{B}$ and A, consists of sets of the form

$$(A \cap B) \cup (A^c \cap B^{'}), \qquad B, B^{'} \in \mathcal{B}.$$

Proof.

Define $\mathcal{C} = \mathcal{B} \cup \{A\}$ and $\mathcal{D} = (A \cap B) \cup (A^c \cap B^{'})$, then we need to show $\sigma(\mathcal{C}) = \sigma(\mathcal{D})$.
It suffices to show $\mathcal{C} \subset \sigma(\mathcal{D})$ and $\mathcal{D} \subset \sigma(\mathcal{C})$.
$A \cap \Omega = A$ and $A^c \cap \emptyset = \emptyset$.
So $A = A \cup \emptyset = (A \cap \Omega) \cup (A^c \cap \emptyset)$.
Since $\mathcal{B}$ is a σ-algebra of subses of Ω, $\emptyset, \Omega \in \mathcal{B}$.
Then $A \in \mathcal{D} \subset \sigma(\mathcal{D})$.
For any set $B \in \mathcal{B}$, $B = \Omega \cap B = (A \cup A^c) \cap B = (A \cap B) \cup (A^c \cap B)$.
Then $B \in \mathcal{D} \subset \sigma(\mathcal{D})$. So $\mathcal{B} \subset \sigma(\mathcal{D})$.
Thus $\mathcal{C} \subset \sigma(\mathcal{D})$.
On the other hand, we show $\mathcal{D} \subset \sigma(\mathcal{C})$.
$A \in \mathcal{C} \subset \sigma(\mathcal{C})$, then $A^c \in \sigma(\mathcal{C})$.
For any two sets $B, B^{'} \in \mathcal{B}$, $B, B^{'} \in \mathcal{C} \subset \sigma(\mathcal{C})$.
So $A \cap B, A^c \cap B^{'} \in \sigma(\mathcal{C})$ since σ-algebras are closed under finite intersections.
Then $(A \cap B) \cup (A^c \cap B^{'}) \in \sigma(\mathcal{C})$ since σ-algebras are closed under finite unions.
Thus, $\mathcal{D} \subset \sigma(\mathcal{C})$.

$\square$