Problem 5 (Resnick, pg. 63 Problem 1) Let Ω be a nonempty set, and let $\mathcal{F}_0$ be the collection of all subsets such that either A or A^c is finite.

Define, for each $A \in \mathcal{F}_0$, the set function $\mathbb{P}$, where

$$\mathbb{P}(A) = \begin{cases} 0, & \text{if } A \text{ is finite;} \\ 1, & \text{if } A^c \text{ is finite.} \end{cases}$$

(a) If Ω is countably infinite, show $\mathbb{P}$ is additive on $\mathcal{F}_0$, but not countably additive.

(b) If Ω is uncountable, show $\mathbb{P}$ is countably additive on $\mathcal{F}_0$.

Proof.
Let $\{A_k\}_{k\geq 1} \subset \mathcal{F}_0$ be pairwise disjoint.
We claim there is at most one A_k in $\{A_k\}_{k=1}^{\infty}$ satisfying that A_k^c is finite.
Suppose there are disjoint sets A_k and A_j satisfying that both of A_k^c and A_j^c are finite.
Then $A_k \cap A_j = \emptyset$ and

$$\mathbb{P}(A_k \cap A_j) = 0. \tag{1}$$

We also have $(A_k)^c \cup (A_j)^c = (A_k \cap A_j)^c$ is finite by assumption.
So $\mathbb{P}(A_k \cap A_j) = 1$, which is contradicted by (1).
Thus, there is at most one A_k in $\{A_k\}_{k=1}^{n}$ satisfying that A_k^c is finite.

(a) (1) Consider finite subsets $\{A_k\}_{k=1}^{n}$ of $\mathcal{F}_0$.

(i) If A_k is finite for $1 \leq k \leq n$, $\cup_{k=1}^{n} A_k$ is finite.
Then $\mathbb{P}(A_k) = 0$, for $1 \leq k \leq n$ and $\mathbb{P}(\cup_{k=1}^{n} A_k) = 0$.
So $\sum_{k=1}^{n} \mathbb{P}(A_k) = 0 = \mathbb{P}(\cup_{k=1}^{n} A_k)$.

(ii) Assume there exists the unique $i \in \mathbb{N}, 1 \leq i \leq n$ such that A_i^c is finite.
Then $\mathbb{P}(A_i) = 1$ and $\mathbb{P}(A_j) = 0$ for $j \neq i, 1 \leq j \leq n$.
So $\sum_{i=1}^{n} \mathbb{P}(A_i) = 1$.
Morevoer, $(\cup_{k=1}^{n} A_k)^c = \cap_{k=1}^{n} A_k^c \subset A_i^c$.
Then $(\cup_{k=1}^{n} A_k)^c$ is finite, and $\mathbb{P}(\cup_{k=1}^{n} A_k) = 1 = \sum_{i=1}^{n} \mathbb{P}(A_i)$.

Thus, $\mathbb{P}$ is additive on $\mathcal{F}_0$ when Ω is countably infinite.

(2) Consider countable many subsets $\{A_k\}_{k\geq 1}$ of $\mathcal{F}_0$.
If A_k is finite for all $k \geq 1$, then $\mathbb{P}(A_k) = 0$.
So $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = 0$.
However, $(\cup_{k=1}^{\infty} A_k)^c$ can be finite.
For example, let $\Omega = \mathbb{N}$ and $A_k = k$ for $k \geq 1$, then A_k's are disjoint, and $(\cup_{k=1}^{\infty} A_k)^c = (\cup_{k=1}^{\infty} k)^c = (\mathbb{N})^c = \emptyset$. So $\cup_{k=1}^{\infty} A_k \in \mathcal{F}_o$, but $\mathbb{P}(\cup_{k=1}^{\infty} A_k) = 1 \neq \sum_{k=1}^{\infty} \mathbb{P}(A_k)$.
Thus, $\mathbb{P}$ is not countably additive when Ω is countably infinite.

(b) Just consider countable many subsets $\{A_k\}_{k\geq 1}$ of $\mathcal{F}_0$.

(1) If A_k is finite for all $k \geq 1$, $\cup_{k=1}^{\infty} A_k$ is countably infinite.
Then $\mathbb{P}(A_k) = 0, k \geq 1$.
So $\sum_{k=1}^{n} \mathbb{P}(A_k) = 0$.
Besides, $(\cup_{k=1}^{\infty} A_k)^c = \Omega \setminus (\cup_{k=1}^{\infty} A_k)$ is uncountably infinite since Ω is uncountable.
So both $\cup_{k=1}^{\infty} A_k$ and $(\cup_{k=1}^{\infty} A_k)^c$ are infinite.
As a result, when A_k is finite for all $k \geq 1$, $\cup_{k=1}^{\infty} A_k \notin \mathcal{F}_0$.

(2) Assume there exists the unique $i \in \mathbb{N}, i \geq 1$ such that A_i^c is finite.
Then $\mathbb{P}(A_i) = 1$ and $\mathbb{P}(A_j) = 0$ for $j \neq i, j \geq 1$.
So $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = 1$.
Morevoer, $(\cup_{k=1}^{\infty} A_k)^c = \cap_{k=1}^{\infty} A_k^c \subset A_i^c$.
Then $(\cup_{k=1}^{\infty} A_k)^c$ is finite, and $\mathbb{P}(\cup_{k=1}^{\infty} A_k) = 1 = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

Thus, $\mathbb{P}$ is countably additive on $\mathcal{F}_0$ when Ω is uncountable.

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