Problem 2 Consider the semialgebra $\mathcal{S} := \{(a,b]; 0 \le a \le b \le 1\}$ of $\Omega = (0,1]$. Show that if $\{a_k\}_{k\ge1}$ and $\{b_k\}_{k\ge1}$ satisfy

$$0 \le a_n < b_n = a_{n+1} < b_{n+1} < 1$$

for each $n \ge 1$, then

$$\bigcup_{n=1}^{\infty}(a_n, b_n]$$

cannot be a member of $\mathcal{S}$. **Note:** This will be much more instructive after you read the construction of Lebesgue measure on $(0,1]$ in Section 2.5 of Resnick's text.

Proof. Assume $\bigsqcup_{n=1}^{\infty}(a_n,b_n] \in \mathcal{S}$, then

$$\bigsqcup_{n=1}^{\infty}(a_n,b_n] = (c,d], \tag{1}$$

where $0 \le c < d < 1$ since $0 < a_n < b_n < 1$.
By De Morgan's laws,

$$\begin{aligned}\left\{\bigsqcup_{n=1}^{\infty}(a_n,b_n]\right\}^c &= \bigcap_{n=1}^{\infty}(0,a_n] \sqcup (b_n,1]\\ &= \left\{\bigcap_{n=1}^{\infty}(0,a_n]\right\}\bigcup\left\{\bigcap_{n=1}^{\infty}(b_n,1]\right\}\end{aligned}$$

since for each $n \ge 1$, $(0,a_n] \cap (b_n,1] = \emptyset$.
Since $a_1 < a_2 < \ldots, \bigcap_{n=1}^{\infty}(0,a_n] = (0,a_1]$.
So

$$\left\{\bigsqcup_{n=1}^{\infty}(a_n,b_n]\right\}^c = (0,a_1]\bigsqcup\left\{\bigcap_{n=1}^{\infty}(b_n,1]\right\}. \tag{2}$$

Also by (1), we have

$$\left\{\bigsqcup_{n=1}^{\infty}(a_n,b_n]\right\}^c = (0,c] \sqcup (d,1]. \tag{3}$$

Compare (2) and (3), we have

$$(0,a_1] = (0,c]$$

and

$$\bigcap_{n=1}^{\infty}(b_n,1] = (d,1] \tag{4}$$

since both of them are the disjoint union of two sets.
We claim for any $n \ge 1, b_n < d$. Otherwise, suppose there exist $b_n \ge d$, then $b_{n+1} > b_n \ge d$, and then $\bigcap_{n=1}^{\infty}(b_n,1] \subset (b_{n+1},1] \subsetneq (d,1]$, so it is contradicted by (4).

Thus, for any $n \geq 1, b_n < d$.
Then

$$d \in (b_n, 1], n \geq 1.$$

So

$$d \in \bigcap_{n=1}^{\infty} (b_n, 1].$$

However $d \notin (d, 1]$, which is contradicted by (4).
Therefore, the assumption $\bigsqcup_{n=1}^{\infty} (a_n, b_n] \in \mathcal{S}$ does not hold.
Namely, $\bigsqcup_{n=1}^{\infty} (a_n, b_n] \notin \mathcal{S}$.

$\square$