Homework 3, MATH 9010
Due on Thursday, September 15
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Problem 1 Let $(\Omega, \mathcal{B}, P) = \Big((0,1]\ , \mathcal{B}\big((0,1]\big), \lambda \Big)$ where λ is Lebesgue measure. Define

$$\begin{aligned} X_1(\omega) &= 0, \forall \omega \in \Omega, \\ X_2(\omega) &= 1_{\{1/2\}}(\omega), \\ X_3(\omega) &= 1_{\mathbb{Q}}(\omega) \end{aligned}$$

where $\mathbb{Q} \in (0,1]$ are rational numbers in $(0,1]$. Note

$$P[X_1 = X_2 = X_3 = 0] = 1$$

and give

$$\sigma(X_i), \quad i = 1, 2, 3.$$

(a) $X_1(\omega) = 0, \forall \omega \in \Omega$.
Note X_1 has range 0 , then $X_1^{-1}(\{0\}) = \Omega$.
So $\sigma(X_1) = \sigma(\emptyset, \Omega) = \{\emptyset, \Omega\}$.

(b) $X_2(\omega) = 1_{\{1/2\}}(\omega)$.
Note X_2 has range $\{0,1\}$, then $X_2^{-1}(\{0\}) = \{(0,1/2) \cup (1/2,1]\}$,$X_2^{-1}(\{1\}) = \{1/2\}$.
So $\sigma(X_2) = \{(0,1/2) \cup (1/2,1], 1/2, \emptyset, \Omega\}$.

(c) $X_3(\omega) = 1_{\mathbb{Q}}(\omega)$.
Note, X_3 has range $\{0,1\}$, then $X_1^{-1}(\{0\}) = \{\mathbb{Q}\}$, $X_1^{-1}(\{1\}) = \{\mathbb{Q}^c \cap (0,1]\}$.
So $\sigma(X_3) = \{\mathbb{Q}, \mathbb{Q}^c \cap (0,1], \emptyset, \Omega\}$.
where $\mathbb{Q} \in (0,1]$ are rational numbers in $(0,1]$.