The Conditional Diagnosability of BCDC under the Comparison Diagnosis Model

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Abstract

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1 Introduction

introducion

2 Notations and Definitions

Definition 1. *[1] Two binary strings $x = x_1x_0$ and $y = y_1y_0$ of length 2 are said to be pair-related ($denoted$ by $x \sim y$) if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.*

$E(R_0, R_2)$

$B_n - F - V(A)$

$S_n \setminus F_2 \neq \emptyset$

$|F_0|$

Definition 2. *[1] The n-dimensional crossed cube, CQ_n, is recursively defined as follows. CQ_1 is the complete (undirected) graph on two nodes whose addresses are 0 and 1. CQ_n consists of CQ^0_{n-1} and CQ^1_{n-1}. The most significant bits of the addresses of the nodes in CQ^0_{n-1} and CQ^1_{n-1} are 0 and 1,*

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respectively. The nodes $u = u_{n-1}u_{n-2}\cdots u_0 \in V(CQ^0_{n-1})$ *and* $v = v_{n-1}v_{n-2}\cdots v_0 \in V(CQ^1_{n-1})$, *where* $u_{n-1} = 0$ *and* $v_{n-1} = 1$, *are joined by an edge in* CQ_n *if and only if*

1) $u_{n-2} = v_{n-2}$ *if* n *is even, and*

2) $u_{2i+1}u_{2i} \sim v_{2i+1}v_{2i}$ *(see Definition* 1*), for* $\lfloor \frac{n-1}{2} \rfloor > i \geq 0$.

Definition 3. *[1] The* n*-dimensional BCDC network,* B_n*, is recursively defined as follows.* B_2 *is a cycle with* 4 *nodes* $[00, 01]$, $[00, 10]$, $[01, 11]$, *and* $[10, 11]$. *For* $n \geq 3$, *we use* B^0_{n-1} *(resp.* B^1_{n-1}*) to denote the graph obtained by* B_{n-1} *with changing each node* $[x, y]$ *of* B_{n-1} *to* $[0x, 0y]$ *(resp.* $[1x, 1y]$*).* B_n *consists of* B^0_{n-1}, B^1_{n-1}, *and a node set* $S_n = \{[a, b] | a \in V(CQ^0_{n-1}),\ b \in V(CQ^1_{n-1}),\ \text{and } (a, b) \in E(CQ_n)\}$ *according to the following rules. For nodes* $u = [a, b] \in V(B^0_{n-1})$, $v = [c, d] \in S_n$, *and* $w = [e, f] \in V(B^1_{n-1})$:

1) $(u, v) \in E(B_n)$ *if and only if* $a = c$ *or* $b = c$.

2) $(v, w) \in E(B_n)$ *if and only if* $e = d$ *or* $f = d$.

Lemma 1. Let $G = (V, E)$ be a system. For any two distinct subsets F_1 and F_2 of $V(G)$, (F_1, F_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied:

1)$\exists u, v \in F_1 \setminus F_2$ and $\exists w \in V(G) \setminus (F_1 \cup F_2)$ such that $(u, v)_w \in C$.

2)$\exists u, v \in F_2 \setminus F_1$ and $\exists w \in V(G) \setminus (F_1 \cup F_2)$ such that $(u, v)_w \in C$.

3)$\exists u, w \in V(G) \setminus (F_1 \cup F_2)$ and $\exists v \in F_1 \triangle F_2$ such that $(u, v)_w \in C$.

We use $cn(G)$ to represent the maximum number of common nodes between any two nodes in graph G.

Lemma 2. For an $n-$dimensional BCDC, B_n has the following properties:

(1) B_n has $n2^{n-1}$ nodes and $n(n-1)2^{n-1}$ edges.

(1) B_n is $(2n-2)-$regular, $\kappa(B_n) = 2n - 2$.

Lemma 3. $cn(B_n, x, y) = n - 2$ if $(x, y) \in E(B_n)$ and $cn(B_n, x, y) = 2$ or 0 if if $(x, y) \notin E(B_n)$ for $n >= 3$.

Lemma 4. *[1] For any integer $n \geq 3$ and any two nodes $u = [a, b], v = [c, d] \in S_n$, we have*

$$|N_{B^0_{n-1}}(\{u, v\})| = \begin{cases} 2n-2, & \text{if } (a, c) \notin E(CQ_n), \\ 2n-3, & \text{if } (a, c) \in E(CQ_n), \end{cases}$$

and

$$|N_{B^1_{n-1}}(\{u, v\})| = \begin{cases} 2n-2, & \text{if } (b, d) \notin E(CQ_n), \\ 2n-3, & \text{if } (b, d) \in E(CQ_n). \end{cases}$$

Lemma 5. For any three nodes x, y and z in B^0_{n-1} where $(x, y), (y, z) \in E(B_n)$ and $n \geq 3$, $|N_{S_n}(N_{B^0_{n-1}}(x, y, z) \cup \{x, y, z\})| > |N_{B^0_{n-1}}(x, y, z)|$.

Proof. Let $x = [a, b]$, $y = [c, d]$ and $z = [e, f]$ and $H = \{a, b\} \cup \{c, d\} \cup \{e, f\}$. Since $(x, y), (y, z) \in E(B_n)$ and CO_n contains no circle of length 3, $|H| = 4$. Without loss of generality, suppose that $b = c$, $d = e$, If $(a, f) \in E(CQ^0_{n-1})$ or there exists a node m in CQ^0_{n-1}, which $(a, m), (m, f) \in E(CQ_{n-1})$, then $|N_{B^0_{n-1}}(x, y, z)| = |N_{CQ^0_{n-1}}(a, b, d, f)| + 1$. For each node in $N_{CQ^0_{n-1}}(H) \cup H$, there is a matching node in CQ^1_{n-1}. According to the construction method of BCDC, we have $|N_{CQ^0_{n-1}}(H) \cup H| = |N_{S_n}(N_{B^0_{n-1}}(x, y, z)) \cup \{x, y, z\}| > |N_{B^0_{n-1}}(u)|$. □

3 The Conditional Diagnosability of BCDC under the Comparison Diagnosis Model

Lemma 6. Let G be a graph $\delta(G) > 2$, and let F_1 and F_2 be any two distinct conditional faulty sets of $V(G)$ with $F_1 \subset F_2$. Then, (F_1, F_2) is a distinguishable conditional pair under the comparison diagnosis model.

Lemma 7. $t_c(B_n) \leq 4n - 6$, for $n \geq 6$.

Proof. Consider a path $P = \langle u, v, w\rangle$ where $u[0] = v[0] = w[0]$. Let $|F_1| = N_{B_n}(P) \cup \{u\}$ and $|F_2| = N_{B_n}(P) \cup \{w\}$. Since $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$, condition 1 and condition 2 of Lemma 1 are not satisfied; $F_1 \triangle F_2 = \{u, w\}$, for each node $x \in F_1 \triangle F_2$, there is only one node $v \in V(G) \setminus (F_1 \cup F_2)$ which is adjacent to x and v does not have a neighbour in $V(G) \setminus (F_1 \cup F_2)$, then condition 3 of Lemma 1 is not satisfied. Thus F_1 and F_2 are indistinguishable. By Lemma 3 and $\{(u, v), (u, w), (v, w)\} \subset E(B_n)$, $|F_1 \cap F_2| = |N_{B_n}(P)| = 4n - 6$ and $|F_2 \setminus F_1| = |F_1 \setminus F_2| = 1$. Hence, $|F_1| = |F_2| = 4n - 5$. Next, we show that F_1 and F_2 are conditional faulty sets. Without loss of generality, for each node $x \in V(B_n)$, we show $N_{B_n}(x) \subseteq F_1$, another situation $N_{B_n}x \subseteq F_2$ is similar.

Case 1. $x \in V(P)$. If $x = u$, then x has a neighbour $v \notin F_1$; If $x = W$, then x has a neighbour $v \notin F_1$; If $x = v$, then x has a neighbour $w \notin F_1$.

Case 2. $x \in N_{B_n}(P)$. If $x \in N_{B_n}(v)$, then x has a neighbour $v \notin F_1$; If $x \in N_{B_n}(w)$, then x has a neighbour $w \notin F_1$; If $x \in N_{B_n}(u)$, by Definition 3, x has two neighbours in S_n and u has two neighbours in S_n. Since $N_{S_n}(x) \cap N_{S_n}(u) \leq 1$, then x has a neighbour which does not belong to F_1;

Case 3. $x \notin N_{B_n}(P) \cup \{u, v, w\}$. x is not adjacent to any node in P, by Lemma 3, $\max(N_{B_n}(x, y)) = 2$ for each node y in P. Since $\kappa(x) = 2n - 2 > 6$ for $n \geq 5$, x has a neighbour which does not belong to F_1;

Therefore, $t_c(B_n) \leq 4n - 6$, for $n \geq 6$. □

Lemma 8. Let F be a faulty set of B_n with $|F| \leq 3n - 5$, where $n \geq 3$. Then, $B_n - F$ satisfies one of the following two conditions: 1) $B_n - F$ is connected. 2) $B_n - F$ has two components, one is K_1 and the other contains $n2^{n-1} - |F| - 1$ nodes.

Proof. We prove the claim by induction on n. It is easy to verify that this Lemma holds when $n = 3$, so our proof starts from $n \geq 4$. Suppose that m_1 and m_2 are two K_1s in $B_n - F$. According to the Lemma 2, we have $|N(m_1)| = |N(m_2)| = 2n - 2$ and $|N(m_1) \cap N(m_2)| \leq 2$. Then $|S| \geq |V(m_1) \cup V(m_2)| = |V(m_1)| + |V(m_2)| - |N(m_1) \cap N(m_2)| \geq (2n - 2) + (2n - 2) - 2 = 4n - 6 > 3n - 5$ for $n \geq 4$. Hence, there exists at most one K_1 in $B_n - F$. Let A be a K_1 in $B_n - F$, then we will show that $R = B_n - F - V(A)$ is connected. Let $F_0 = F \cap V(B^0_{n-1})$, $F_1 = F \cap V(B^1_{n-1})$, $F_2 = F \cap V(S_n)$. Then we deal with the following cases.

Case 1. $F = F_0$ or $F = F_1$. If $F = F_0$, then $F_1 = F_2 = \emptyset$. We can easily verify that $B[V(B^1_{n-1}) \cup S_n]$ is connected. For each node x in $V(B^1_{n-1}) \setminus F_0$, by Definition 3, there are two nodes in S_n which are adjacent to x. Thus, $B_n - F - V(A)$ is connected; If $F = F_1$, this case is the same as $F = F_0$.

Case 2. $F = F_2$. If $F = F_2$, then $F_0 = F_1 = \emptyset$. We can easily verify that B^0_{n-1} and B^1_{n-1} are connected, respectively. For $n \geq 4$, $|S_n| = 2^{n-1} > 3n - 5$, then $S_n \setminus F_2 \neq \emptyset$. For each node u in

$S_n \setminus F_2$, by Definition 3, u has $n-1$ neighbours in R_0 and $n-1$ neighbours in R_1, respectively. Thus, $B_n - F - V(A)$ is connected.

Case 3. $F_0 \neq \emptyset$, $F_1 \neq \emptyset$, and $F_2 = \emptyset$. Then, we have the following two subcases.

Subcase 3.1. $n \leq |F_0| \leq 3n-6$ and $1 \leq |F_1| \leq 2n-5$. Since each node u in S_n has $(n-1)$ neighbours in $V(B^1_{n-1})$ and by Lemma 4, there exists at most a node $u_1 \in S_n$, where $N_{B^1_{n-1}}(u_1) \subset F_1$. For each node u in $S_n \setminus u_1$, there are $(n-1)$ nodes in $V(B^1_{n-1})$ which are adjacent to u and R_1 is connected due to $\kappa(B_{n-1}) = 2n-4$. Thus, $B_n[\{S_n \setminus u_1\} \cup V(B^1_{n-1})]$ is connected. Since $B_n - F - V(A)$ does not contain a K_1, there exists a node x_1 in $V(B^1_{n-1}) \setminus F_0$ where $(x_1, u_1) \in E(B_n)$. For each node x in $V(B^0_{n-1}) \setminus F_0$, there are two nodes in S_n which are adjacent to x. Then, $B_n - F - V(A)$ is connected.

Subcase 3.2. $n \leq |F_1| \leq 3n-6$ and $1 \leq |F_0| \leq 2n-5$. This argument is similar to Subcase 3.1.

Case 4. $F_0 \neq \emptyset$, $F_2 \neq \emptyset$, and $F_1 = \emptyset$. Then, we have the following three subcases.

Subcase 4.1. $1 \leq |F_0| \leq 2n-5$ and $n \leq |F_2| \leq 3n-6$. Since $\kappa(B_{n-1}) = 2n-4$ and $F_1 = \emptyset$, $B_n[V(B^0_{n-1}) \setminus F_0]$ and B^1_{n-1} are connected, respectively. For $n \geq 4$, $|S_n| = 2^{n-1} > 3n-6$, then $S_n \setminus F_2 \neq \emptyset$. For each node u in $S_n \setminus F_2$, by Definition 3, u has $n-1$ neighbours in $V(B^1_{n-1})$. Thus, $B_n[\{S_n \setminus F_2\} \cup V(B^1_{n-1})]$ is connected. For each node $x \in V(B^0_{n-1})$, there are two nodes in S_n which are adjacent to x and for each node $u \in S_n$, u has $n-1$ neighbours in $V(B^0_{n-1})$. By Definition 3, $|E(R_0, R_2)| \geq (n-1)2^{n-1} - n(3n-6) - 2 > 0$ for $n \geq 4$. Thus, there exist two nodes $x \in V(B^0_{n-1}) \setminus F_0$, $u \in S_n$ and $(x, u) \in E(B_n)$. Since $B_n[V(B^0_{n-1}) \setminus F_0]$ and $B_n[\{S_n \setminus F_2\} \cup V(B^1_{n-1})]$ are connected, respectively, $B_n - F - V(A)$ is connected.

Subcase 4.2. $2n-4 \leq |F_0| \leq 3n-7$ and $2 \leq |F_2| \leq n-1$. We can easily verify that $B_n[\{S_n \setminus F_2\} \cup V(B^1_{n-1})]$ is connected. If $B_n[V(B^0_{n-1}) \setminus F_0]$ is connected, similar to Subcase 4.1, $B_n - F - V(A)$ is connected; Suppose that R_0 is disconnected, then R_0 consists of two components, one is K_1 and the other contains $|V(R_0)|-1$ nodes. Let x_1 be the K_1 and L be another component in R_0. Since there is only one K_1 in $B_n - F$, then x_1 has a neighbour in R_2. Thus $B_n[\{S_n \setminus F_2\} \cup V(B^1_{n-1}) \cup \{x_1\}]$ is connected. By Definition 3, $|E(L, R_2)| \geq (n-1)2^{n-1} - n(n-1) - 2(2n-4) > 0$ for $n \geq 4$. Thus, there exist two nodes $x \in V(L)$, $u \in V(S_n \setminus F_2)$ and $(x, u) \in E(B_n)$. Since L and $B_n[\{S_n \setminus F_2\} \cup V(B^1_{n-1}) \cup \{x_1\}]$ are connected, respectively, $B_n - F - V(A)$ is connected.

Subcase 4.3. $|F_0| = 3n-6$ and $|F_2| = 1$. We can easily verify that $B_n[\{S_n \setminus F_2\} \cup V(B^1_{n-1})]$ is connected. For each node x in $V(B^0_{n-1} \setminus F_0)$, by Definition 3, there are two nodes in S_n which are adjacent to x. Thus, $B_n - F - V(A)$ is connected;

Case 5. $F_1 \neq \emptyset$, $F_2 \neq \emptyset$, and $F_0 = \emptyset$. This argument is similar to Subcase 3.1.

Case 6. $F_0 \neq \emptyset$, $F_2 \neq \emptyset$, and $F_1 \neq \emptyset$. Then, we have the following four subcases.

Case 6.1. $1 \leq |F_0| \leq 2n-5$, $1 \leq |F_1| \leq 2n-5$, and $1 \leq |F_2| \leq 3n-7$. Since $\kappa(B_{n-1}) = 2n-4$, $B_n[B^0_{n-1} \setminus F_0]$, $B_n[B^1_{n-1} \setminus F_1]$ are connected respectively. By Lemma 4, there exists at most one node $u_1 \in S_n \setminus F_2$, where $N_{B^0_{n-1}}(u_1) \subset F_0$. For each $u \in S_n \setminus F_2 \setminus u_1$, u has $(n-1)$ neighbours in B^0_{n-1}. Thus, $B[\{R_2 \setminus u_1\} \cap R_0]$ is connected. Similarly, there exists at most one node $u_2 \in S_n$, where $N_{B^1_{n-1}}(u_2) \subset F_1$ and $u_1 \neq u_2$, and $B[\{R_2 \setminus u_2\} \cap R_1]$ is connected. For $n \geq 4$, $|S_n| = 2^{n-1} - (3n-7) > 2$, Thus, $B_n - F - V(A)$ is connected.

Case 6.2. $2n-4 \leq |F_0| \leq 3n-7$, $1 \leq |F_1| \leq n-2$, and $1 \leq |F_2| \leq n-2$. We can easily verify that $B[V(R_1) \cup V(R_2)]$ is connected. If R_0 is connected, similar to Subcase 6.1, $B_n - F - V(A)$ is connected; Suppose that R_0 is disconnected, then R_0 consists of two components, one is K_1 and the other contains $V(R_0) - 1$ nodes. Let x_1 be the K_1 and L be another component in R_0. Since there is only one K_1 in $B_n - F$, then x_1 has a neighbour in R_0. Thus $B[V(R_1) \cup V(R_2) \cup \{x_1\}]$ is connected. By Definition 3, $|E(L, R_2)| \geq (n-1)2^{n-1} - n(n-2) - 2(2n-4) > 0$ for $n \geq 4$. Thus, there exist two nodes $x \in V(L)$, $u \in V(R_2)$ and $(x, u) \in E(B_n)$. Since L and $B[V(R_1) \cup V(R_2) \cup \{x_1\}]$ are connected, respectively, $B_n - F - V(A)$ is connected.

Case 6.3. $2n-4 \leq |F_1| \leq 3n-7$, $1 \leq |F_0| \leq n-2$, and $1 \leq |F_2| \leq n-2$. This argument is similar to Subcase 6.2.

The above arguments indicate that if there is no K_1 in $B_n - F$, then $B_n - F$ is connected. Therefore, the Lemma holds. □

Lemma 9. Let F be a conditional faulty set of B_n with $|F| \leq 4n-7$, where $n \geq 5$. Then, $B_n - F$ satisfies one of the following two conditions: 1) $B_n - F$ is connected. 2) $B_n - F$ has two components, one is K_2 and the other contains $n2^{n-1} - |F| - 2$ nodes.

Proof. We first prove the second case. Since F is a conditional faulty set, there is no an isolated node in $B_n - F$. Suppose that m_1 and m_2 are two K_2s in $B_n - F$. According to the Lemma 3, we have $|V(m_1)| = |V(m_2)| = (2n-3)+(2n-3)-(n-2) = 3n-4$ and $|N(m_1) \cap N(m_2)| \leq 8$. Then $|S| \geq |V(m_1) \cup V(m_2)| = |V(m_1)| + |V(m_2)| - |N(m_1) \cap N(m_2)| \geq (3n-4)+(3n-4)-8 = 6n-16 > 4n-7$ for $n \geq 5$. Hence, there exists at most one K_2 in $B_n - F$. Let A be a K_2 in $B_n - F$, then we will show that $R = B_n - F - V(A)$ is connected. Let $F_0 = F \cap V(B^0_{n-1})$, $F_1 = F \cap V(B^1_{n-1})$, $F_2 = F \cap V(S_n)$, $R_0 = B[V(R) \cap V(B^0_{n-1})]$, $R_1 = B[V(R) \cap V(B^1_{n-1})]$, and $R_2 = B[V(R) \cap V(S_n)]$. Then we deal with the following cases.

Case 1. $F = F_0$ or $F = F_1$. If $F = F_0$, then $F_1 = F_2 = \emptyset$. We can easily verify that $B[V(R_1) \cup V(R_2)]$ is connected. For each node x in $V(R_0)$, by Definition 3, there are two nodes in S_n which are adjacent to x. Thus, $B_n - F - V(A)$ is connected; If $F = F_1$, this argument is similar to $F = F_0$.

Case 2. $F = F_2$. If $F = F_2$, then $F_0 = F_1 = \emptyset$. We can easily verify that R_0 and R_1 are connected, respectively. For $n \geq 5$, $|S_n| = 2^{n-1} > 4n-7$, then $S_n \setminus F_2 \neq \emptyset$. For each node u in R_2, by Definition 3, u has $n-1$ neighbours in R_0 and $n-1$ neighbours in R_1, respectively. Thus, $B_n - F - V(A)$ is connected.

Case 3. $F_0 \neq \emptyset$, $F_1 \neq \emptyset$, and $F_2 = \emptyset$. Then, we have the following four subcases.

Subcase 3.1. $2n-2 \leq |F_0| \leq 4n-8$ and $1 \leq |F_1| \leq 2n-5$. By Lemma 4, there exists at most a node $u_1 \in S_n$, where $N_{R_1}(u_1) \subset F_1$. Since $\kappa(B_{n-1}) = 2n-4$, R_1 is connected. For each node u in R_2, there are $(n-1)$ nodes in $V(B^1_{n-1})$ which are adjacent to u. Thus, $B[\{R_2 \setminus u_1\} \cup V(R_1)]$ is connected. Since F is a conditional faulty set, there exists a node x in $V(R_0)$ where $(u_1, x) \in E(B_n)$. For each node x_1 in $V(R_0)$, there are two nodes in S_n which are adjacent to x_1. Then, $B_n - F - V(A)$ is connected.

Subcase 3.2. $|F_1| \geq 2n-2$ and $|F_0| \leq 2n-5$. This argument is similar to Subcase 3.1.

Subcase 3.3. $|F_0| = 2n-3$ and $|F_1| = 2n-4$. If R_1 is connected, the argument is similar to Subcase 3.1; Suppose that R_1 is disconnected, by Lemma 8, there exists only one isolated node y_1 and another connected component L. By Definition 3, y_1 has two neighbours u_1 and u_2 in S_n. By Lemma 4, $\min(|N_{B^1_{n-1}}(\{u_1, u_2\})|) = 2n-3 > N_{B^1_{n-1}(y_1)} = |F_1| = 2n-4$, so there exists a node y_2 in L which is adjacent to u_1 or u_2. Without loss of generality, we set $(y_2, u_1) \in E(B_n)$. Since L is connected and

$B[\{y_1, u_1, u_2\}]$ is connected, $B[S_n \cap V(R_1)]$ is connected. For each node x in $V(R_0)$, there are two nodes in S_n which are adjacent to x. Thus, $B_n - F - V(A)$ is connected.

Subcase 3.4. $|F_1| = 2n-3$ and $|F_0| = 2n-4$. This argument is similar to Subcase 3.3.

Case 4. $F_0 \neq \emptyset$, $F_2 \neq \emptyset$, and $F_1 = \emptyset$. Then, we have the following two subcases.

Subcase 4.1. $1 \leq |F_0| \leq 2n-5$ and $2n-2 \leq |F_2| \leq 4n-8$. Since $\kappa(B_{n-1}) = 2n-4$ and $F_1 = \emptyset$, R_0 and R_1 are connected, respectively. For $n \geq 5$, $|S_n| = 2^{n-1} > 4n-8$, then $S_n \setminus F_2 \neq \emptyset$. For each node u in $S_n \setminus F_2$, by Definition 3, u has $n-1$ neighbours in R_1. Thus, $B[V(R_1) \cup V(R_2)]$ is connected. For each node $x \in V(R_0)$, there are two nodes in S_n which are adjacent to x and for each node $u \in V(R_2)$, u has $n-1$ neighbours in R_0. By Definition 3, $|E(R_0, R_2)| \geq (n-1)2^{n-1} - (n-1)(2n-3) - 2(2n-4) > 0$ for $n \geq 5$. Thus, there exist two nodes $x \in V(R_0)$, $u \in V(R_2)$ and $(x, u) \in E(B_n)$. Since R_0 and $B[V(R_1) \cup V(R_2)]$ are connected, respectively, $B_n - F - V(A)$ is connected.

Subcase 4.2. $2n-4 \leq |F_0| \leq 4n-8$ and $1 \leq |F_2| \leq 2n-3$. We can easily verify that $B[V(R_1) \cup V(R_2)]$ is connected. If R_0 is connected, similar to Subcase 4.1, $B_n - F - V(A)$ is connected; Suppose that R_0 is disconnected. If for each node $x \in V(R_0)$, there exists a neighbour u in R_2, then $B_n - F - V(A)$ is connected; Otherwise, since F is a conditional faulty set, there is no an isolated node in $B_n - F$, for each node $x \in V(R_0)$, there exists a node v, where $v \in N_{R_0}(x)$. According to the proof at the beginning, R does not contain a K_2. Then there exists a node w, where $w \in N_{R_0}(x, v)$. If $w \in V(R_2)$, then $B_n - F - V(A)$ is connected; Otherwise, $w \in N_{R_0}(x, v)$. Let $X = N_{R_0}(x, v, w)$, if there exists a node $z \in X$ and z has a neighbour in $V(R_2)$, then $B_n - F - V(A)$ is connected; Otherwise, by Lemma 5, $N_{S_n}(N_{B_n - F}(x, v, w) \cup \{x, v, w\}) > 4n-6 > |F| = 4n-7$, a contradiction.

Case 5. $F_1 \neq \emptyset$, $F_2 \neq \emptyset$, and $F_0 = \emptyset$. This argument is similar to Subcase 3.3.

Case 6. $F_0 \neq \emptyset$, $F_2 \neq \emptyset$, and $F_1 \neq \emptyset$. Then, we have the following four subcases.

Case 6.1. $1 \leq |F_0| \leq 2n-5$, $1 \leq |F_1| \leq 2n-5$, and $3 \leq |F_2| \leq 4n-9$. Since $\kappa(B_{n-1}) = 2n-4$, R_0 and R_1 are connected, respectively. By Lemma 4, there exists at most a node $u_1 \in R_2$, where $N_{B^0_{n-1}}(u_1) \subset F_1$. For each node u in $R_2 \setminus \{u_1\}$, there are $(n-1)$ nodes in $V(B^1_{n-1})$ which are adjacent to u. Thus, $B[\{R_2 \setminus u_1\} \cup V(R_1)]$ is connected. For R_0 and R_2, we have $|E(R_0, R_2)| \geq (n-1)2^{n-1} - (n-1)(2n-3) - 2(2n-3) > 0$ for $n \geq 5$. Then $B[V(R_0) \cup V(R_2)]$ is connected. Similarly, there exists at most one node $u_2 \in S_n$, where $N_{B^1_{n-1}}(u_2) \subset F_1$ and $u_1 \neq u_2$, and $B[\{R_2 \setminus u_2\} \cap R_1]$ is connected. For $n \geq 5$, $|S_n| = 2^{n-1} - (4n-9) > 2$. Thus, $B_n - F - V(A)$ is connected.

Subcase 6.2. $2n-4 \leq |F_0| \leq 4n-9$, and $1 \leq |F_2| \leq 2n-5$ and $1 \leq |F_2| \leq 2n-5$. We can easily verify that $B[V(R_1) \cup V(R_2)]$ is connected. If R_0 is connected, similar to Subcase 4.1, $B_n - F - V(A)$ is connected; Suppose that R_0 is disconnected. If for each node $x \in V(R_0)$, there exists a neighbour u in R_2, then $B_n - F - V(A)$ is connected; Otherwise, since F is a conditional faulty set, there is no an isolated node in $B_n - F$, for each node $x \in V(R_0)$, there exists a node v, where $v \in N_{R_0}(x)$. According to the proof at the beginning, R does not contain a K_2. Then there exists a node w, where $w \in N_{R_0}(x, v)$. If $w \in V(R_2)$, then $B_n - F - V(A)$ is connected; Otherwise, $w \in N_{R_0}(x, v)$. Let $X = N_{R_0}(x, v, w)$, if there exists a node $z \in X$ and z has a neighbour in $V(R_2)$, then $B_n - F - V(A)$ is connected; Otherwise, by Lemma 5, $N_{S_n}(N_{B_n - F}(x, v, w) \cup \{x, v, w\}) > 4n-6 > |F| = 4n-7$, a contradiction.

The above arguments indicate that if there is no K_2 in $B_n - F$, then $B_n - F$ is connected. Therefore, the Lemma holds. □

Lemma 10. Let F_1 and F_2 be two different conditional faulty node sets in B_n, where $F_1 \leq 4n-6$ and $F_2 \leq 4n-6$, and let L be the largest component in $B_n - (F_1 \cap F_2)$, then for each node $x \in F_1 \triangle F_2$, $x \in L$.

Proof. Without loss of generality, assume that $x \in F_2 - F_1$. Since F_2 is a conditional fault node set in B_n, x has a neighbour $y \in V(B_n) - F_2 - x$. Suppose that $x \notin L$, then $B_n - (F_1 \cap F_2)$ is disconnected. Since $F_1 \triangle F_2 \leq 4n-7$, by Lemma 9, $\{x, y\}$ belong to the smaller component K_2. Then, all the neighbors of y belong to F_1, which contradicts the fact that F_2 is a conditional faulty set. □

Lemma 11. $t_c(B_n) \leq 4n-6$, for $n \leq 6$.

Proof. If $F_1 \subset F_2$ or $F_2 \subset F_1$, by Lemmas 2 and 6, F_1 and F_1 are distinguishable. Then we consider that $F_1 - F_2 \neq \emptyset$ and $F_2 - F_1 \neq \emptyset$. Let $S = F_1 \cap F_2$ with $|S| \leq 4n-7$, and L be the maximum component of $B_n - S$. From Lemma 10, we know that for each node $u \in F_1 \triangle F_2$, $u \in L$. Next, we will prove that there exists a node $x \in L \setminus \{F_1 \cup F_2\}$, which has no neighbour in S. Then we will complete our proof by u and x.

By Lemmas 2 and 9, we have $N_{B_n}(S) \leq (2n-2)|S|$ and $|V(L)| \geq V(B_n)-|S|-2 = n2^{n-1}-|S|-2$. Then, $|V(L)|-|F_1 \triangle F_2|-(2n-2)|S| \geq n2^{n-1}-|S|-2-(2(4n-6)-2|S|)-(2n-2)|S| \geq n2^{n-1}-8n+10-(2n-3)|S| \geq n2^{n-1}-8n+10-(2n-3)(4n-7) \geq 6\times 2^{6-1}-8\times 6+10-(2\times 6-3)(4\times 6-7) = 1$.

For each node x in $|V(L)| - |F_1 \triangle F_2| - (2n-2)|S|$, we have the following two cases:

Case 1. $N_{B_n}(x) \cap (F_1 \triangle F_2) \neq \emptyset$. If x has two or more neighbours in $F_1 - F_2$ ($F_2 - F_1$), condition 1 (condition 2) of Lemma 1 is satisfied; If x has only one neighbour in $F_1 - F_2$ ($F_2 - F_1$), since x has no neighbours in S, there exists a neighbour which belongs to $V(G) \setminus (F_1 \cup F_2)$. Thus, condition 3 of Lemma 1 is satisfied.

Case 2. $N_{B_n}(x) \cap (F_1 \triangle F_2) = \emptyset$. Since for each node $u \in F_1 \triangle F_2$, $u \in L$ and L is connected, there exists a path $P = \langle x(=x_0), x_1, \ldots, x_j, u\rangle$ where $|P| \geq 2$, and $u \in F_1 \triangle F_2$, and $x_j \in V(L) - F_1 \triangle F_2$ with $j \geq 1$. Then, x_j has a neighbour in $F_1 \triangle F_2$ (u) and a neighbour x_{j-1} in $L \setminus (F_1 \cup F_2)$. Thus, condition 3 of Lemma 1 is satisfied. □

4 The Conditional Diagnosability of BCDC under the PMC Model

Lemma 12. Let F be a conditional faulty set of B_n with $n \geq 6$. $B_n - F$ is disconnected and there exists a component P with $N_P(u) \geq 2$ for $u \in V(P)$. Then one of two following conditions holds: 1) $|F| \geq 5n-8$; 2) $|V(P)| \geq 4n-5$.

Proof. Let $F_0 = F \cap V(B^0_{n-1})$, $F_1 = F \cap V(B^1_{n-1})$, $F_2 = F \cap V(S_n)$. According to the distribution of P, we have the following cases:

Case 1. $V(P) \subset V(B^0_{n-1})$ or $V(P) \subset V(B^1_{n-1})$.

Without loss of generality, suppose that $V(P) \subseteq V(B^0_{n-1})$. Then, either $V(B^0_{n-1}) = V(P) \cup V(F_0)$ or $B^0_{n-1} - F_0$ is disconnected. If $V(B^0_{n-1}) = V(P)$, since $|F| \leq 6n-8$, $V(P) = V(B^0_{n-1}) - (6n-8) \geq 4n-4$. If $B^0_{n-1} - F_0$ is disconnected and $|P| \geq 4n-4$, then condition 2 holds. Otherwise, we consider $|P| \leq 4n-5$. We choose a node set C in $V(P)$ with $|V(C)| = 4$. Since $V(C) \subseteq V(P)$, $N_{B^0_{n-1}}(C) \subseteq F_0 \cup (V(P) - V(C))$. Then we have $N_{B^0_{n-1}}(C) - (V(P) - V(C)) \subseteq F_0$. Thus, $|F_0| \geq |N_{B^0_{n-1}}(C)| - |V(P) - V(C)| = |N_{B^0_{n-1}}(C)| - |V(P)| + |V(C)| = |N_{B^0_{n-1}}(C)| - |V(P)| + 4$. On the other hand, we have $|F_2| \geq |N_{S_n}(P)|$, □

5 Conclusions

conclusions

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References

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