Structure connectivity and substructure connectivity of the Augmented Cube

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July 19, 2019

Abstract

The augmented cube, denoted by AQ_n, is a variant of the hypercube. It retains many favorable properties of the hypercube and possesses several embeddable properties that the hypercube and its other variations do not possess. The connectivity of a graph is one of the most important parameters to analyze the reliability of a network. Structure and substructure connectivity are the two novel generalization of the connectivity, which provide a new way to evaluate the reliability and fault-tolerance of a network. In this paper, we investigate the structure connectivity and substructure connectivity of the augmented cube for $H \in \{K_{1,M}, P_L, C_N\}$, where $1 \leq M \leq 6, 1 \leq L \leq 2n-1$ and $3 \leq N \leq 2n-1$.

Keywords: structure connectivity; substructure connectivity; augmented cube

1 Introduction

It is well known that the topology of an interconnected network is crucial in designing the parallel and distributed systems. A good interconnection topology can offer higher speed and good communication performance, such as high fault-tolerant property. The connectivity is one of the most important indicators to evaluate the reliability and fault-tolerance of an interconnection network. The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. The larger the connectivity is, the more reliable the interconnection network is. However, this parameter has an obvious deficiency, that is, in the event of any random vertex failure, it is unlikely that all of the vertices adjacent to the same vertex can fail simultaneously in large-scale systems. To compensate the shortcomings of the connectivity stated above and measure the fault-tolerant ability of an interconnection network more accurately, Harary [?] proposed the concept of conditional connectivity by attaching some constrains on the remaining components of the resulting graph after deleting some faulty vertices. After that, several kinds of conditional connectivity were proposed and studied [?, ?, ?, ?, ?, ?, ?], such as g-extra connectivity, superconnectivity and R_g-connectivity. The *g-extra connectivity* of a graph G, denoted by $\kappa_g(G)$, is the minimum number of vertices of G whose deletion disconnects G and every remaining component with at least $g + 1$ vertices. The higher g-extra connectivity the network has, the more reliable the network is. Some recent results on the g-extra connectivity of graphs see [?, ?, ?, ?, ?]. g-extra connectivity is a generalization of the superconnectivity. The superconnectivity of a graph G corresponds to $\kappa_1(G)$ [?, ?]. R_g-connectivity of a graph G, denoted by $\kappa^g(G)$, is the minimum number of vertices of graph G whose deletion make G disconnected and every vertex of the remaining components has at least g fault-free neighbors [?, ?, ?].

As stated above, most works on the reliability and fault-tolerance of networks have concentrated on the effect of an individual vertex becoming faulty, that is, it assumes that the status of each individual vertex u, whether it is fault-free or faulty, is an event independent of the status of vertices around u. However, in reality, vertices that are linked could affect each other when one of them becomes faulty. The neighbors of a faulty vertex might become more vulnerable and have a higher probability of becoming faulty later. It should also be noted that with the development of technology, many networks and subnetworks are made into chips, that means if any vertex of vertices in a chip becomes faulty, the whole chip is regarded as faulty. All of these motivate us to study the fault-tolerance of networks not only from the perspective of an individual vertex, but also based on some structures of the network.

Lin et al. [?] introduced the concepts of structure connectivity and substructure connectivity, which measure the fault-tolerance of networks from the perspective of some special structures and studied $\kappa(Q_n, H)$, $\kappa^s(Q_n, H)$ of the hypercube Q_n for $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$, respectively. Let $F(G)$ be a set of elements and each element is a vertex subset of graph G, then $V(F(G)) = \cup_{v' \in F(G)} v'$. If $G - V(F(G))$ is a disconnected graph or a trivial graph, then $F(G)$ is called a *structure cut* of graph G. Let H be a connected subgraph of graph G. If the induced subgraph of each element in $F(G)$ is isomorphic to a spanning supergraph of H, then $F(G)$ is an *H-structure cut* of graph G. The *H-structure connectivity* of graph G, denoted by $\kappa(G, H)$, is defined as the minimum cardinality (number of elements) of all H-structure cuts of graph G. If the induced subgraph of each element in $F(G)$ is isomorphic to a spanning supergraph of a connected subgraph of H, then $F(G)$ is called an *H-substructure cut* of graph G. The *H-substructure connectivity* of graph G, denoted by $\kappa^s(G, H)$, is defined as the minimum cardinality of all H-substructure cuts of graph G. The complete graph on n vertices is denoted by K_n and K_1 is just an isolated vertex, thus K_1-structure connectivity and K_1-substructure connectivity are exactly the classical vertex connectivity. Later, Lv et al. [?] considered $\kappa(Q_n^k, H)$ and $\kappa^s(Q_n^k, H)$ for the k-ary n-cube Q_n^k and $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}\}$. For more works on structure connectivity and substructure connectivity see [?, ?, ?, ?].

In this paper, we focus on the structure and substructure connectivity of augmented cube. It is well known that the hypercube is one of the most popular multiprocessor systems for parallel computer and communication systems. As an improvement of the hypercube, the augmented cube, proposed by Choudum and Sunitha [?], not only retains many of the favorable properties of hypercube but also possesses some properties better than hypercube. For example, maximum connectivity (degree = connectivity = $2n-1$), best possible wide diameter ($= \lceil \frac{n}{2} \rceil + 1 =$ diameter $+ 1$), and several embeddable properties which are not shared by the hypercube and its other variations. In this paper, we establish H-structure and H-substructure connectivity of AQ_n for $H \in \{K_{1,M}, P_L, C_N\}$(shown in Figure **??**), respectively, where $1 \leq M \leq 6, 1 \leq L \leq 2n-1$ and $3 \leq N \leq 2n-1$.

The rest of the paper is structured as follows. In Section 2, the definition of augmented cube and some useful properties of it are presented. Then, Section 3 presents the main work of the paper, that is, we prove the results of $\kappa(AQ_n, H)$ and $\kappa^s(AQ_n, H)$ of augmented cube for each $H \in \{K_{1,M}, P_Q, C_N\}$. Conclusions are presented in Section 4.

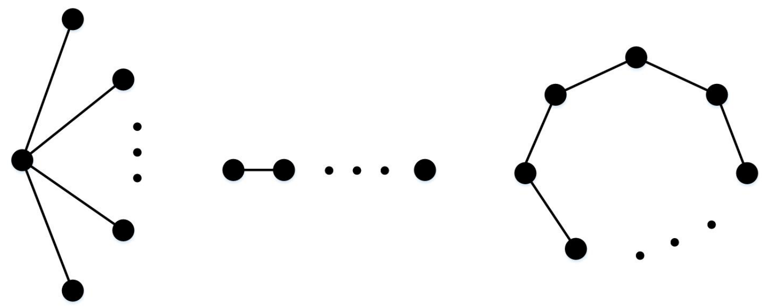


Figure 1: $K_{1,M}, P_Q, C_N$

2 Preliminaries

Generally, an interconnection network can be modeled by an undirected graph $G = (V, E)$, where V represents a set of all vertices of G and E represents a set of all edges of G. Each vertex in V corresponds to a processor and each edge (u, v) in E corresponds to a communication link between vertex u and vertex v. The neighborhood of a vertex u in G is the set of vertices adjacent to u, denoted by $N(u)$. Similarly, for $S \subset V$, the neighborhood of S in G is defined as $N(S) = \cup_{v\in S} N(v) - S$. A complete bipartite graph is a graph whose vertices can be partitioned into two disjoint and independent subsets U and V such that for every two vertices $u \in U$ and $v \in V$, (u, v) is an edge in E and no edge has both endpoints in the same subset. A complete bipartite graph with partitions of size $|U| = m$ and $|V| = n$, is denoted by $K_{m,n}$. For any n, $K_{1,n}$ is called a star. A path $P_k = \langle v_1, v_2, \ldots, v_k\rangle$ is a finite sequence of edges which connects a sequence of vertices which are all distinct from one another. A cycle $C_k = \langle v_1, v_2, \ldots, v_k, v_1\rangle$ for $k \geq 3$ is a path wherein a vertex is reachable from itself.

In the following, we shall introduce the definition of augmented cube and some properties of augmented cube.

Definition 1. *[?] Let an integer $n \geq 1$, an n-dimensional augmented cube AQ_n consists of 2^n vertices, each vertex in AQ_n is labeled by a unique n-bit binary string $u_n u_{n-1} \ldots u_2 u_1$. The augmented cube AQ_1 is a complete graph K_2 with two vertices 0 and 1. For $n > 1$, n-dimensional augmented cube AQ_n can be constructed from two copies of $(n-1)$-dimensional augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding $2 \times 2^{n-1}$ edges between them. A vertex $u = \{0a_{n-1} \ldots a_2 a_1 \mid a_i = 0 \text{ or } 1\}$ of AQ^0_{n-1} is adjacent to a vertex $v = \{1b_{n-1} \ldots b_2 b_1 \mid b_i = 0 \text{ or } 1\}$ of AQ^1_{n-1} if and only if, for $1 \leq i \leq n-1$, either (1) $a_i = b_i$, in this case, (u, v) is called a hypercube edge, or (2) $a_i = \bar{b}_i$, in this case, (u, v) is called a complement edge.*

For a n-bit binary string $u = a_n a_{n-1} \ldots a_1$ in augmented cube, we use u^i (respectively, $\bar{u}^i$) to denote the binary string $a_n \ldots a_{i+1}\bar{a}_i a_{i-1} \ldots a_1$ (respectively, $a_n \ldots a_{i+1}\bar{a}_i\bar{a}_{i-1} \ldots \bar{a}_1$), which differs with u in the ith bit position(respectively, from the first to the ith bit position). It is clear that $u^1 = \bar{u}^1$. we may mix these two notations whenever it is convenient. For example, if $u = 011001$, then $u^1 = \bar{u}^1 = 011000$, $u^4 = 010001$, $\bar{u}^4 = 010110$, $(u^4)^2 = 010011$, $(\bar{u}^4)^3 = 010010$ $(\overline{u^4})^2 = 010010$ and $(\overline{\bar{u}^4})^2 = 010101$.

The definition of augmented cube above is from a recursive perspective. As with hypercube or other graphs, augmented cube also have several definitions. An alternative definition of AQ_n is as follows:

Definition 2. *[?] An n-dimensional augmented cube with $n \geq 1$ contains 2^n vertices, each labeled by a unique n-bit binary string $u_n u_{n-1} \dots u_2 u_1$. There is an edge between two vertices $a = a_n a_{n-1} \dots a_2 a_1$ and $b = b_n b_{n-1} \dots b_2 b_1$ if and only if, there exists an integer k, $1 \leq k \leq n$ (1) $a_k = \bar{b}_k$ and $a_i = b_i$, for $1 \leq i \leq n$, $i \neq k$, or (2) $a_i = \bar{b}_i$ for $1 \leq i \leq k$ and $a_i = b_i$ for $k+1 \leq i \leq n$.*

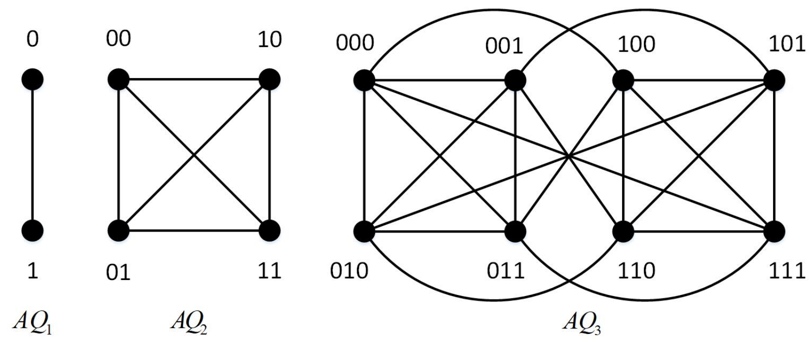


Figure 2: Augmented cubes of dimension 1, 2, and 3

The augmented cubes AQ_1, AQ_2, and AQ_3 are shown in Figure **??**.
Then, we give some properties of AQ_n

Theorem 1. *[?] $\kappa(AQ_1) = 1, \kappa(AQ_2) = 3, \kappa(AQ_3) = 4$, and for $n \geq 4$, $\kappa(AQ_n) = 2n - 1$.*

According to Theorem 1, we have the following result.

Theorem 2. *For $n \geq 4$, $\kappa(AQ_n, K_1) = \kappa^s(AQ_n, K_1) = 2n - 1$.*

Lemma 1. *[?] For $n \geq 6$, $\kappa_1(AQ_n) = 4n - 8$.*

Property 1. *[?] If (u, u^i) is a hypercube edge of dimension i $(1 \leq i \leq n)$, then*

$$N_{AQ_n}(u) \cap N_{AQ_n}(u^i) = \begin{cases} \{\bar{u}^i, \bar{u}^{i-1}\} & 2 \leq i \leq n, \\ \{\bar{u}^2, u^2\} & i = 1, \end{cases}$$

that is, u and u^i have exactly two common neighbors in AQ_n and $|N_{AQ_n}(\{u, u^i\})| = 4n - 6$.

Property 2. *[?] If $(u,\bar{u}^i)$ is a complement edge of dimension i $(2 \leq i \leq n)$,then*

$$N_{AQ_n}(u) \cap N_{AQ_n}(\bar{u}^i) = \begin{cases} \{u^i, u^{i+1}, \bar{u}^{i-1}, \bar{u}^{i+1}\} & 2 \leq i \leq n-1, \\ \{\bar{u}^{n-1}, u^n\} & i = n, \end{cases}$$

that is, u and $\bar{u}^i$ have exactly four common neighbors in AQ_n for $2 \leq i \leq n-1$ and $|N_{AQ_n}(\{u, \bar{u}^i\})| = 4n - 8$. Similarly, u and $\bar{u}^n$ have exactly two common neighbors in AQ_n and $|N_{AQ_n}(\{u, \bar{u}^n\})| = 4n - 6$.

Property 3. *[?] Any two vertices in AQ_n have at most four common neighbors for $n \geq 3$.*

Property 4. *[?] If two vertices u and w in AQ^0_{n-1}(respectively, AQ^1_{n-1}) have common neighbors in AQ^1_{n-1}(respectively, AQ^0_{n-1}), then $w = \bar{u}^{n-1}$ and they have exactly two common neighbors u^n and $\bar{u}^n$ in AQ^1_{n-1}.*

Property 5. *If (u, u^i) and (u, u^j) are two hypercube edges of dimension i and j $(1 \leq i \neq j \leq n)$. Without loss of generality, we set $i < j$, then*

$$N_{AQ_n}(u^i) \cap N_{AQ_n}(u^j) = \begin{cases} \{u, (u^i)^j, \bar{u}^i, (\overline{u}^j)^i\} & j = i+1 \quad \text{and} \quad i > 1, \\ \{u, (u^i)^j\} & j - i > 1 \quad \text{or} \quad i = 1, j = 2 \end{cases}$$

Property 6. *If $(u, \bar{u}^i)$ and $(u, \bar{u}^j)$ are two complement edges of dimension i and j $(1 \leq i \neq j \leq n)$. Without loss of generality, we set $i < j$, then*

$$N_{AQ_n}(\bar{u}^i) \cap N_{AQ_n}(\bar{u}^j) = \begin{cases} \{u, \bar{u}^{i+1}, (\overline{\bar{u}^j})^i, (\bar{u}^j)^{j-1}\} & j = i+2, \\ \{u, (\overline{\bar{u}^j})^i\} & j = i+1 \quad \text{or} \quad j - i > 2 \end{cases}$$

Property 7. *If (u, u^i) is a hypercube edges of dimension i and $(u, \bar{u}^j)$ is a complement edges of dimension j $(1 \leq i, j \leq n)$. then*

$$N_{AQ_n}(u^i) \cap N_{AQ_n}(\bar{u}^j) = \begin{cases} \{u, (u^i)^{i-1}, (\bar{u}^i)^{i-1}, \bar{u}^{i-1}\} & i = j, i > 1 \quad \text{and} \quad i = j+2, \\ \{u, \bar{u}^i, (\bar{u}^j)^{i+1}, (u^i)^{i+1}\} & i = j+1 \quad \text{and} \quad i < n, \\ \{u, \bar{u}^i\} & i = n, j = n-1, \\ \{u, (\bar{u}^j)^i\} & |i - j| > 2, \\ \{u, (u^j)^i, (\bar{u}^j)^i, \bar{u}^i\} & j = i+1, i > 1, \\ \{u, (\bar{u}^j)^i\} & j - i = 2 \quad \text{and} \quad i \neq 1, j \neq 3 \end{cases}$$

3 H-structure connectivity and H-substructure connectivity

In this section, we study the H-structure connectivity and H-substructure connectivity of AQ_n for each $H \in \{K_{1,M}, P_L, C_N\}$ where $1 \leq M \leq 6$, $1 \leq L \leq 2n-1$, $3 \leq N \leq 2n-1$. Let u be an arbitrary vertex in AQ_n. In order to make the representation of our proof more convenient, we introduce a set of tokens $v^{[1]}, v^{[2]}, \ldots, v^{[2n-1]}$, where $v^{[1]}, v^{[2]}, \ldots, v^{[2n-1]}$ and all the adjacent vertices of u: $u^1, u^2, \bar{u}^2, \ldots, u^n, \bar{u}^n$ form a one-to-one correspondence. The correspondence of u^j and $v^{[i]}$ is: (1)if i is even, then $j = \lfloor \frac{i}{2} \rfloor + 1$ and $u^j = v^{[i]}$. (2)if i is odd, then $j = \lfloor \frac{i}{2} \rfloor + 1$, and $\bar{u}^j = v^{[i]}$. For example, if $u = 000000$ is a vertex of AQ_6, then $u^4 = v^{[6]} = 001000$ and $\bar{u}^5 = v^{[9]} = 011111$. In this paper, we may mix these two notations whenever it is convenient.

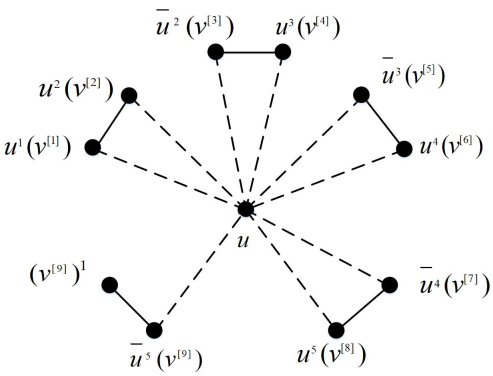
3.1 $\kappa(AQ_n, K_{1,M})$ and $\kappa^s(AQ_n, K_{1,M})$

According to the definition of AQ_n, property **??** and property **??**, if u is an arbitrary vertex of AQ_n and $(u, \bar{u}^i)$ is a complement edge of dimension i $(2 \leq i \leq n-1)$, then $N_{AQ_n}(u) \cap N_{AQ_n}(\bar{u}^i) = \{\bar{u}^{i-1}, u^i, u^{i+1}, \bar{u}^{i+1}\}$. The subgraph induced by $\{\bar{u}^i, \bar{u}^{i-1}, u^i, u^{i+1}, \bar{u}^{i+1}\}$ $(2 \leq i \leq n-1)$ is isomorphic to $K_{1,4}$. If $(u, \bar{u}^n)$ is a complement edge of dimension n, then $N_{AQ_n}(u) \cap N_{AQ_n}(\bar{u}^n) = \{\bar{u}^{n-1}, u^n\}$ and the subgraph induced by $\{\bar{u}^n, \bar{u}^{n-1}, u^n\}$ is isomorphic to $K_{1,2}$. Similarly, if (u, u^i) is a hypercube edge of dimension i $(2 \le i \le n)$, then $N_{AQ_n}(u) \cap N_{AQ_n}(u^i) = \{\bar{u}^i, \bar{u}^{i-1}\}$. The subgraph induced by $\{u^i, \bar{u}^i, \bar{u}^{i-1}\}$ is isomorphic to $K_{1,2}$.

Here, we discuss in the cases of $1 \le M \le 3$ and $4 \le M \le 6$.

3.1.1 $1 \le M \le 3$

Lemma 2. *For $n \ge 4$, $\kappa(AQ_n, K_{1,M}) \le \lceil \frac{2n-1}{1+M} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) \le \lceil \frac{2n-1}{1+M} \rceil$.*



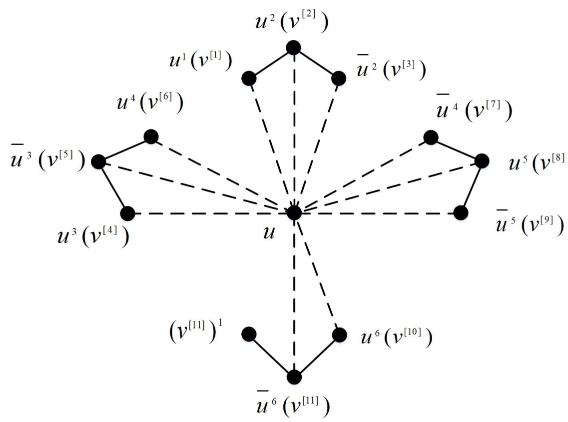


Figure 3: A $K_{1,1}$-structure-cut in AQ_5 and a $K_{1,2}$-structure-cut in AQ_6

Proof. Let u be an arbitrary vertex in AQ_n. In the following we distinguish cases pertaining to the value of M.

Case 1. $M = 1$. We set

$S_1 = \{v^{[(M+1)i+1]}, v^{[(M+1)i+2]} \mid 0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor\}$ and
$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1\}\}$

Case 2. $M = 2$.

Case 2.1. n mod $3 = 0$. We set

$S_1 = \{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]} \mid 0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor\}$

Case 2.2. n mod $3 = 1$. We set

$S_1 = \{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]} \mid 0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor\}$ and
$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2\}\}$

Case 2.3. n mod $3 = 2$. We set

$S_1 = \{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]} \mid 0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor\}$ and
$S_2 = \{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1\}\}$

Case 3. $M = 3$.
Case 3.1. n is odd. We set

$S_1 = \{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}, v^{[(M+1)i+4]} \mid 0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor\}$ and
$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3\}\}$

Case 3.2. n is even. We set

$S_1 = \{v^{[(M+1)i+1]}, v^{[(M+1)i+2]}, v^{[(M+1)i+3]}, v^{[(M+1)i+4]} \mid 0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor\}$ and
$S_2 = \{\{v^{[2n-3]}, v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1\}\}$

Clearly, if $M = 1$, the induced subgraph of each element in $S_1 \cup S_2$ is isomorphic to $K_{1,1}$; If $M = 2$, vertex $v^{[(M+1)i+2]}$ is adjacent to vertices $v^{[(M+1)i+1]}$ and $v^{[(M+1)i+3]}$ for $0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor$, vertex $v^{[2n-1]}$ is adjacent to vertices $(v^{[2n-1]})^1$ and $(v^{[2n-1]})^2(v^{[2n-2]}$ and $v^{[2n-1]})^1)$. Therefore, the subgraph induced by each element in $S_1 \cup S_2(S_1)$ is isomorphic to $K_{1,2}$; If $M = 3$, vertex $v^{[(M+1)i+3]}$ is adjacent to vertices $v^{[(M+1)i+1]}$, $v^{[(M+1)i+2]}$ and $v^{[(M+1)i+4]}$ for $0 \le i < \lfloor \frac{2n-1}{M+1} \rfloor$, vertex $v^{[2n-1]}$ is adjacent to vertices $(v^{[2n-1]})^1$, $(v^{[2n-1]})^2$ and $(v^{[2n-1]})^4(v^{[2n-3]}, v^{[2n-2]}$ and $(v^{[2n-1]})^1)$. Therefore, the subgraph induced by each element in $S_1 \cup S_2$ is isomorphic to $K_{1,3}$. Suppose $S = S_1 \cup S_2(S = S_1$, if $M = 2$ and $n \bmod 3 = 0)$, it is obvious that $|S| = \lceil \frac{2n-1}{1+M} \rceil$. Since $AQ_n - S$ is disconnected and one component of it is $\{u\}$. Then $\kappa(AQ_n, K_{1,M}) \le \lceil \frac{2n-1}{1+M} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) \le \lceil \frac{2n-1}{1+M} \rceil$. Figure **??** shows a $K_{1,1}$-structure-cut in AQ_5 and a $K_{1,2}$-structure-cut in AQ_6.

Lemma 3. *For $n \ge 4$,$\kappa^s(AQ_n, K_{1,M}) \ge \lceil \frac{2n-1}{1+M} \rceil$.*

Proof. Let F_n^* be a set of connected subgraphs in AQ_n, every element in the set is isomorphic to $K_{1,M}$ and $|F_n^*| \le \lceil \frac{2n-1}{1+M} \rceil - 1$. Thus $|V(F_n^*)| \le (1 + M) \times (\lceil \frac{2n-1}{1+M} \rceil - 1) < 2n - 1$. Since $\kappa(AQ_n) = 2n - 1$, then $AQ_n - F_n^*$ is connected, The lemma holds.

Since $\kappa(Q_n, K_{1,M}) \ge \kappa^s(Q_n, K_{1,M})$, we have $\kappa(Q_n, K_{1,M}) \ge \lceil \frac{2n-1}{1+M} \rceil$. By lemma **??** and lemma **??**, we have the following therorem.

Theorem 3. *For $n \ge 4$, $\kappa(AQ_n, K_{1,M}) = \lceil \frac{2n-1}{1+M} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) = \lceil \frac{2n-1}{1+M} \rceil$.*

3.1.2 $4 \le M \le 6$

Lemma 4. *For $n \ge 6$, $\kappa(AQ_n, K_{1,M}) \le \lceil \frac{n-1}{2} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) \le \lceil \frac{n-1}{2} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. In the following we distinguish cases pertaining to the value of M.
Case 1. $M = 4$.
Case 1.1. n is odd. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}\}\}$ and
$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\} \mid 0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1\}$

Case 1.2. n is even. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}\}\}$,
$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1\} \mid 0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1\}$ and
$S_3 = \{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3\}\}$.

Case 2. $M = 5$.
Case 2.1. n is odd. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n\}\}$ and
$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1, (v^{[5+4i+2]})^2\} \mid 0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1\}$

Case 2.2. n is even. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n\}\}$,
$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1, (v^{[5+4i+2]})^2\} \mid 0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1\}$ and
$S_3 = \{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3, (v^{[2n-1]})^4\}\}$.

Case 3. $M = 6$.
Case 3.1. n is odd. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n, (\overline{v^{[3]}})^n\}\}$ and
$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1, (v^{[5+4i+2]})^2, (v^{[5+4i+2]})^3\} \mid 0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1\}$

Case 3.2. n is even. We set

$S_1 = \{\{v^{[1]}, v^{[2]}, v^{[3]}, v^{[4]}, v^{[5]}, (v^{[3]})^n, (\overline{v^{[3]}})^n\}\}$,
$S_2 = \{\{v^{[5+4i+1]}, v^{[5+4i+2]}, v^{[5+4i+3]}, v^{[5+4i+4]}, (v^{[5+4i+2]})^1, (v^{[5+4i+2]})^2, (v^{[5+4i+2]})^3\} \mid 0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1\}$ and
$S_3 = \{\{v^{[2n-2]}, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (v^{[2n-1]})^3, (v^{[2n-1]})^4, (v^{[2n-1]})^5\}\}$.

Obviously, the subgraph induced by the element in S_1 is isomorphic to $K_{1,M}$. For $0 \le i < \lfloor \frac{n-1}{2} \rfloor - 1$, vertex $v^{[5+4i+2]}$ is adjacent to vertices $v^{[5+4i+j]}$ with $j = 1, 3$ or 4 and $(v^{[5+4i+2]})^l$ for $1 \le l \le 1 + M - 4$. Thus, the subgraph induced by each element in S_2 is isomorphic to $K_{1,M}$. Vertex $v^{[2n-1]}$ is adjacent to vertices $v^{[2n-2]}$ and $(v^{([2n-1]})^q$ for $1 \le q \le 1 + M - (2n - 2 - 4 \times \lfloor \frac{n-1}{2} \rfloor)$. Therefore, the subgraph induced by the element in S_3 is isomorphic to $K_{1,M}$. Suppose $S = S_1 \cup S_2 (S = S_1 \cup S_2 \cup S_3$, if n is even). Note that $|S| = \lceil \frac{n-1}{2} \rceil$. Since $AQ_n - S$ is disconnected and one component of it is $\{u\}$. Thus, $\kappa(AQ_n, K_{1,M}) \le \lceil \frac{n-1}{2} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) \le \lceil \frac{n-1}{2} \rceil$ with $4 \le M \le 6$. Figure **??** shows a $K_{1,5}$-structure-cut of AQ_6.

Lemma 5. *Let F_n be a $K_{1,M}$-substructure set of AQ_n and u be a vertex in $AQ_n - V(F_n)$. If u is an isolated vertex in $AQ_n - V(F_n)$, then $|F_n| \ge \lceil \frac{n-1}{2} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. We set $W = \{x \mid (x, u)$ is a hypercube edge$\}$ and $C = \{y \mid (y, u)$ is a complement edge$\}$. Obviously, $|W| = n - 1$ and $|C| = n$. According to the definition of AQ_n, property **??**, property **??** and property **??**, each element in F_n contains at most five distinct vertices in $N(u)$. Namely, $\{\bar{u}^i, \bar{u}^{i-1}, u^i, u^{i+1}, \bar{u}^{i+1}\}$ with $2 \le i \le n - 1$. Since $\{(\bar{u}^i, \bar{u}^{i-1}), (\bar{u}^i, u^i), (\bar{u}^i, u^{i+1}), (\bar{u}^i, \bar{u}^{i+1})\} \subseteq E(AQ_n)$, then the subgraph induced by $\{\bar{u}^i, \bar{u}^{i-1}, u^i, u^{i+1}, \bar{u}^{i+1}\}$ is isomorphic to $K_{1,4}$. We set $B = \{b_i \mid b_i \in F_n$ and $\{\bar{u}^{i-1}, u^i, \bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i) \cap N(u)\}$. Since each $V(b_i)$ contains three vertices in C and two vertices in H and $|C| > |W|$. Thus, $|B| < \lfloor \frac{2n-1}{5} \rfloor$. In the following we distinguish cases pertaining to the value of $|B|$.

Case 1. $|B| = 0$. Since each element in F_n contains at most four distinct vertices in $N(u)$. Thus, $|F_n| \geq \lceil \frac{2n-1}{4} \rceil$.

Case 2. $|B| = 1$. Suppose $\{\bar{u}^{i-1}, u^i, \bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i)$ with $2 \leq i \leq n-1$. S-ince each element in $F_n - B$ contains at most four distinct vertices in $N(u) - V(B)$. Thus, $|F_n| \geq 1 + \lceil \frac{2n-6}{4} \rceil = \lceil \frac{n-1}{2} \rceil$.

Case 3. $|B| = 2$. Suppose $\{\bar{u}^{i-1}, u^i, \bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i), \{\bar{u}^{j-1}, u^j, \bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} \subseteq V(b_j)$ with $2 \leq i, j \leq n-1$ and $|i-j| \geq 3$. Without loss of generality, we set $j > i$.

Case 3.1. $j-i = 3$. Without loss of generality, we set $j = i+3$. Then $\{\bar{u}^{j-1}, u^j, \bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} = \{\bar{u}^{i+2}, u^{i+3}, \bar{u}^{i+3}, u^{i+4}, \bar{u}^{i+4}\}$. Suppose $w_1, w_2 \in N(u) - V(B) - \{u^{i+2}\}$ and $w_1 \neq w_2$. According to the property **??**, **??** and **??**, $N(u^{i+2}) \cap N(w_1) \cap N(w_2) = \emptyset$. Besides, vertex u^{i+2} is not adjacent to the vertex in $N(u) - V(B)$. Therefore, there is an element a in $F_n - B$, $u^{i+2} \in V(a)$ and $V(a) \cap \{N(u) - V(B)\} \leq 2$. Since the element in $F_n - B - a$ contains at most four distinct vertices in $N(u) - V(B) - V(a)$. Thus, $|F_n| \geq 2 + 1 + \lceil \frac{2n-13}{4} \rceil = \lceil \frac{2n-1}{4} \rceil$.

Case 3.2. $j - i = 4$. Then $\{\bar{u}^{j-1}, u^j, \bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} = \{\bar{u}^{i+3}, u^{i+4}, \bar{u}^{i+4}, u^{i+5}, \bar{u}^{i+5}\}$. In the following we distinguish cases pertaining to the number of elements in $F_n - B$ containing three vertices $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3}. We denote the number of elements as S.

Case 3.2.1. $S = 3$. There are three distinct elements in $N(u) - V(B)$ that contain one of three vertices $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3}, respectively. Suppose $w_1, w_2 \in N(u) - V(B) - \{u^{i+2}, \bar{u}^{i+2}, u^{i+3}\}$ and $w_1 \neq w_2$. According to the property **??**, **??** and **??**, $N(u^{i+2}) \cap N(w_1) \cap N(w_2) = \emptyset$. Besides, vertex u^{i+2} is not adjacent to the vertex in $N(u) - V(B) - \{\bar{u}^{i+2}, u^{i+3}\}$. Therefore, there is an element a in $F_n - B$, $u^{i+2} \in V(a)$ and $V(a) \cap \{N(u) - V(B) - \{\bar{u}^{i+2}, u^{i+3}\}\} \leq 2$. Vertex $\bar{u}^{i+2}$ and u^{i+3} are same as u^{i+2}. Since the element in $F_n - B - a$ contains at most four distinct vertices in $N(u) - V(B) - V(a)$. Thus, $|F_n| \geq 2 + 3 + \lceil \frac{2n-17}{4} \rceil = \lceil \frac{2n+3}{4} \rceil$.

Case 3.2.2. $S = 2$. There are two distinct elements in $N(u) - V(B)$, one of which contains one vertex in $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3}, and the other contains the other two vertices. We assume that a_1 contains one vertex in $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3} and a_2 contains the other two vertices.

Case 3.2.2.1. $u^{i+2} \in V(a_1)$ and $\{\bar{u}^{i+2}, u^{i+3}\} \subseteq V(a_2)$. According to the discussion of Case 3.2.1, $V(a_1) \cap \{N(u) - V(B) - \{\bar{u}^{i+2}, u^{i+3}\}\} \leq 2$. Since $N(\bar{u}^{i+2}) \cap N(u^{i+3}) - \{u, \bar{u}^{i+3}\} = \{(\bar{u}^{i+2})^{i+4}, (u^{i+3})^{i+4}\}$ and each element in $\{(\bar{u}^{i+2})^{i+4}, (u^{i+3})^{i+4}\}$ is not adjacent to the vertices in $N(u) - V(B) - \{u^{i+2}, \bar{u}^{i+2}, u^{i+3}\}$. On the other hand, each element in $\{\bar{u}^{i+2}, u^{i+3}\}$ is not adjacent to the vertex in $N(u) - V(B) - \{u^{i+2}\}$. Therefore, $V(a_2) \cap \{N(u) - V(B) - \{u^{i+2}\}\} = 2$ and $\{\bar{u}^{i+2}, u^{i+3}\} \subseteq V(a_2)$. Since the element in $F_n - B - a_1 - a_2$ contains at most four distinct vertices in $N(u) - V(B) - V(a_1) - V(a_2)$. Thus, $|F_n| \geq 2 + 1 + 1 + \lceil \frac{2n-15}{4} \rceil = \lceil \frac{2n+1}{4} \rceil$. $u^{i+3} \in V(a_1)$ and $\{\bar{u}^{i+2}\ u^{i+2}\} \subseteq V(a_2)$ is similar.

Case 3.2.2.2. $\bar{u}^{i+2} \in V(a_1)$ and $\{u^{i+2}, u^{i+3}\} \subseteq V(a_2)$. According to the discussion of Case 3.2.1, $\underline{V(a_1) \cap \{N(u) - V(B) - \{u^{i+2}, u^{i+3}\}\} \leq 2}$. Since $N(u^{i+2}) \cap N(u^{i+3}) - \{u, \bar{u}^{i+2}\} = \{(u^{i+2})^{i+3}, (\overline{u^{i+3}})^{i+1}\}$ and each element in $\{(u^{i+2})^{i+3}, (\overline{u^{i+3}})^{i+1}\}$ is not adjacent to the vertices in

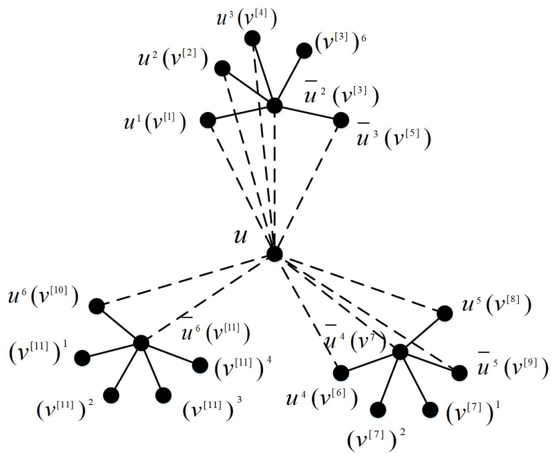


Figure 4: A $K_{1,5}$-structure-cut in AQ_6

$N(u) - V(B) - \{\bar{u}^{i+2}\}$. On the other hand, $(u^{i+2}, u^{i+3}) \notin E(AQ_n)$. Therefore, $V(a_2) \cap \{N(u) - V(B) - \{\bar{u}^{i+2}\}\} = 2$ and $\{u^{i+2}, u^{i+3}\} \subseteq V(a_2)$. Since the element in $F_n - B - a_1 - a_2$ contains at most four distinct vertices in $N(u) - V(B) - V(a_1) - V(a_2)$. Thus, $|F_n| \geq 2+1+1+\lceil \frac{2n-15}{4} \rceil = \lceil \frac{2n+1}{4} \rceil$.

Case 3.2.3. $S = 1$. According to the discussion of Case 3.2.1 and Case 3.2.2, there is an element a in $F_n - B$, $V(a) \cap \{N(u) - V(B)\} = 3$ and $\{u^{i+2}, \bar{u}^{i+2}, u^{i+3}\} \subseteq V(a)$. Since the element in $F_n - B - a$ contains at most four distinct vertices in $N(u) - V(B) - V(a)$. Thus, $|F_n| \geq 2 + 1 + \lceil \frac{2n-14}{4} \rceil = \lceil \frac{n-1}{2} \rceil$.

Case 3.3. $j - i \geq 5$. We set $U_{ij} = \{u^{i+2}, \bar{u}^{i+2}, \ldots, u^{j-1}\}$. In the following we will calculate the number of elements in $F_n - B$ containing all of vertices in U_{ij}. Since $5 \leq |U_{ij}| \leq 2n - 11$ and the element in $F_n - B$ contains at most four distinct vertices in $N(u) - V(B)$. We will distinguish cases pertaining to the value of $|U_{ij}|$ mod 4.

Case 3.3.1. $|U_{ij}|$ mod $4 = 1$. Similar to the discussion in Case 3.1. $|F_n| \geq 2 + 1 + \lceil \frac{2n-13}{4} \rceil = \lceil \frac{2n-1}{4} \rceil$.

Case 3.3.2. $|U_{ij}|$ mod $4 = 3$. Similar to the discussion in Case 3.2. $|F_n| \geq 2 + 1 + \lceil \frac{2n-14}{4} \rceil = \lceil \frac{n-1}{2} \rceil$.

Case 4. $|B| \geq 3$. If $b_i, b_j \in B$ and no $b_k \in B$ with $i < k < j$, we set $U_{ij} = \{u^{i+2}, \bar{u}^{i+2}, \ldots, u^{j-1}\}$. According to the discussion of Case 3. If $|U_{ij}|$ mod $4 = 3$. The value of F_n will be the smallest. Thus, $|F_n| \geq |B| + (|B| - 1) + \lceil \frac{2n-1-5\times|B|-3\times(|B|-1)}{4} \rceil = \lceil \frac{n-1}{2} \rceil$.

In summary, the lemma holds.

Lemma 6. *For $n \geq 6$, $\kappa(AQ_n, K_{1,M}) \geq \lceil \frac{n-1}{2} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) \geq \lceil \frac{n-1}{2} \rceil$.*

Proof. Let F_n^* be a $K_{1,M}$-substructure set of AQ_n and we will show that $AQ_n - V(F_n^*)$ is connected if $F_n^* \leq \lceil \frac{n-1}{2} \rceil - 1$. Suppose not, Let R be the smallest component of $AQ_n - V(F_n^*)$. Note that

$|V(F_n^*)| \leq (1+M) \times (\lceil \frac{n-1}{2} \rceil - 1) \leq 7 \times (\lceil \frac{n-1}{2} \rceil - 1)$. By lemma **??**, we have $7 \times (\lceil \frac{n-1}{2} \rceil - 1) \leq 4n-8$ for $n \geq 6$. So $|V(R)| = 1$. Therefore, we assume that vertex $u \in V(R)$. By lemma **??**, $|N(u) \cap V(F_n^*)| \leq 2n-2 < 2n-1$, which implies that there exists a neighbor of u in $AQ_n - V(F_n^*)$. Therefore, we have $|V(R)| \geq 2$, a contradiction. Thus, $AQ_n - V(F_n^*)$ is connected. The lemma holds.

Combining lemma **??**, we have $\kappa^s(AQ_n, K_{1,M}) = \lceil \frac{n-1}{2} \rceil$. Since $\kappa(Q_n, K_{1,M}) \geq \kappa^s(Q_n, K_{1,M})$, we have $\kappa(Q_n, K_{1,M}) \geq \lceil \frac{2n-1}{1+M} \rceil$. By Lemma **??** and Lemma **??**, we have the following theorem.

Theorem 4. *For $n \geq 6$, $\kappa(AQ_n, K_{1,M}) = \lceil \frac{n-1}{2} \rceil$ and $\kappa^s(AQ_n, K_{1,M}) = \lceil \frac{n-1}{2} \rceil$.*

3.2 $\kappa(AQ_n, P_L)$ and $\kappa^s(AQ_n, P_L)$

Let u be a vertex in AQ_n, according to the definition of AQ_n, $(u^i, \bar{u}^{i-1})$ and $(u^i, \bar{u}^i)$ $(2 \leq i \leq n)$ are complement edges which differ from the first to the $(i-1)$th and ith bit positions. Then all neighbors of u can form a path with length of $2n-1$.

Since $P_2(P_3)$ is isomorphic to $K_{1,1}(K_{1,2})$ and we have proved $\kappa(AQ_n, K_{1,1})(\kappa^s(AQ_n, K_{1,1}))$ and $\kappa(AQ_n, K_{1,2})(\kappa^s(AQ_n, K_{1,2}))$ before, so we start with $L \geq 4$.

Lemma 7. *For $n \geq 3$, $\kappa(AQ_n, P_L) \leq \lceil \frac{2n-1}{L} \rceil$ and $\kappa^s(AQ_n, P_L) \leq \lceil \frac{2n-1}{L} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. We set

$S_1 = \{\{v^{[i \times L+1)]}, v^{[i \times L+2]}, \ldots, v^{[(i+1) \times L]}\} \mid 0 \leq i < \lfloor \frac{2n-1}{L} \rfloor\}$,

If $(2n-1) \bmod L = 0$, then $S_2 = \emptyset$. Otherwise, according to the values of $\lceil \frac{2n-1}{L} \rceil \times L - 2n + 1$ and L, we will discuss the following several cases,

Case 1. $\lceil \frac{2n-1}{L} \rceil \times L - 2n + 1$ is even and $L \leq 2n-4$.

$S_2 = \{\{v^{[\lfloor \frac{2n-1}{L} \rfloor \times L+1]}, \ldots, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, \ldots, (v^{[2n-1]})^{\frac{\lceil \frac{2n-1}{L} \rceil \times L - 2n+1}{2}+1}\}\}$.

Case 2. $\lceil \frac{2n-1}{L} \rceil \times L - 2n + 1$ is odd and $L \leq 2n-3$.

$S_2 = \{\{v^{[\lfloor \frac{2n-1}{L} \rfloor \times L+1]}, \ldots, v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, \ldots, (\overline{v^{[2n-1]}})^{\frac{\lceil \frac{2n-1}{L} \rceil \times L - 2n}{2}+1}\}\}$.

Case 3. $L = 2n-2$.

$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, \ldots, (v^{[2n-1]})^{n-1}, ((v^{[2n-1]})^{n-1})^1\}\}$.

Obviously, the subgraph induced by each element in $S_1 \cup S_2$ is isomorphic to P_L and $|S_1 \cup S_2| = \lceil \frac{2n-1}{L} \rceil$. Since $AQ_n - (S_1 \cup S_2)$ is disconnected and one component of it is $\{u\}$. Then $\kappa(AQ_n, P_L) \leq \lceil \frac{2n-1}{L} \rceil$ and $\kappa^s(AQ_n, P_L) \leq \lceil \frac{2n-1}{L} \rceil$.

Lemma 8. *For $n \geq 3$,$\kappa^s(AQ_n, P_L) \geq \lceil \frac{2n-1}{L} \rceil$.*

Proof. Let F_n^* be a set of connected subgraphs in AQ_n, every element in the set is isomorphic to a connected subgraph of P_L and $|F_n^*| \leq \lceil \frac{2n-1}{L} \rceil - 1$. Thus $|V(F_n^*)| \leq L \times (\lceil \frac{2n-1}{L} \rceil - 1) < 2n-1$. Since $\kappa(AQ_n) = 2n-1$, $AQ_n - F_n^*$ is connected, The lemma holds.

By Lemma **??** and Lemma **??**, we have the following theorem.

Theorem 5. *For $n \geq 3$, $\kappa(AQ_n, P_L) = \kappa^s(AQ_n, P_L) = \lceil \frac{2n-1}{L} \rceil$.*

3.3 $\kappa(AQ_n, C_N)$ and $\kappa^s(AQ_n, C_N)$

At first, we discuss $\kappa^s(AQ_n, C_N)$. Then we discuss $\kappa(AQ_n, C_N)$.

3.3.1 $\kappa^s(AQ_n, C_N)$ with $3 \leq N \leq 2n-1$

Since P_N is a connected subgraph of C_N, we have the following lemma.

Lemma 9. *For $n \geq 3$, $\kappa^s(AQ_n, C_N) \leq \lceil \frac{2n-1}{N} \rceil$.*

Lemma 10. *For $n \geq 3$, $\kappa^s(AQ_n, C_N) \geq \lceil \frac{2n-1}{N} \rceil$.*

Proof. Let F_n^* be a set of connected subgraphs in AQ_n, every element in the set is isomorphic to a connected subgraph of C_N and $|F_n^*| \leq \lceil \frac{2n-1}{N} \rceil - 1$. Thus $V(F_n^*) \leq N \times (\lceil \frac{2n-1}{N} \rceil - 1) < 2n-1$. Since $\kappa(AQ_n) = 2n-1$, $AQ_n - F_n^*$ is connected, The lemma holds.

By Lemma **??** and Lemma **??**, we have the following theorem.

Theorem 6. *For $n \geq 3$, $\kappa^s(AQ_n, C_N) = \lceil \frac{2n-1}{N} \rceil$.*

Now, we discuss $\kappa(AQ_n, C_3)$,$\kappa(AQ_n, C_4)$ and $\kappa(AQ_n, C_N)$ with $4 < N \leq 2n-1$.

3.3.2 $\kappa(AQ_n, C_3)$

Lemma 11. *For $n \geq 6$, $\kappa(AQ_n, C_3) \leq n-1$.*

Proof. Let u be an arbitrary vertex in AQ_n. We set

$$S_1 = \{\{u^1, u^2, \bar{u}^2, u^1\}\},$$
$$S_2 = \{\{u^i, \bar{u}^i, (u^i)^{i-1}, u^i\} \mid 3 \leq i \leq n\}$$

Obviously, the subgraph induced by element in S_1 is isomorphic to C_3. For $3 \leq i \leq n$, $\{\{u^i, \bar{u}^i\}, \{\bar{u}^i, (u^i)^{i-1}\}, \{(u^i)^{i-1}, u^i\}\} \subseteq E(AQ_n)$. Thus, the subgraph induced by each element in $S_1 \cup S_2$ is isomorphic to C_3 and $|S_1 \cup S_2| = n-1$. Since $AQ_n - (S_1 \cup S_2)$ is disconnected and one component of it is $\{u\}$. Then $\kappa(AQ_n, C_3) \leq n-1$.

Lemma 12. *Let F_n be a C_3-structure set of AQ_n and u be a vertex in $AQ_n - V(F_n)$. If u is an isolated vertex in $AQ_n - V(F_n)$, then $|F_n| \geq n-1$.*

Proof. Let u be an arbitrary vertex in AQ_n. We set $H = \{x \mid (x,u)$ is a hypercube edge$\}$ and $E = \{y \mid (y,u)$ is a complement edge$\}$. Obviously, $|H| = n-1$ and $|E| = n$. According to the definition of AQ_n, property **??** and property **??**, each element in F_n contains at most three distinct vertices in $N(u)$. Namely, $\{\bar{u}^i, u^{i+1}, \bar{u}^{i+1}\}$ with $1 \leq i \leq n-1$ and $\{(\bar{u}^i, u^{i+1}), (\bar{u}^i, \bar{u}^{i+1}), (u^{i+1}, \bar{u}^{i+1})\} \subseteq E(AQ_n)$. Thus, the subgraph induced by $\{\bar{u}^i, u^{i+1}, \bar{u}^{i+1}\}$ is isomorphic to C_3. We set $B = \{b_i \mid b_i \in F_n$ and $\{\bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i) \cap N(u)\}$. Since each $V(b_i)$ contains two vertices in E and one vertices in H and $|E| > |H|$. Thus, $|B| < \lfloor \frac{2n-1}{3} \rfloor$. In the following we distinguish cases pertaining to the value of $|B|$.

Case 1. $|B| = 0$. Since each element in F_n contains at most two distinct vertices in $N(u)$. Thus, $|F_n| \geq \lceil \frac{2n-1}{2} \rceil$.

Case 2. $|B| = 1$. Suppose $\{\bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i)$ with $2 \leq i \leq n-1$. Since each element in $F_n - B$ contains at most two distinct vertices in $N(u) - V(B)$. Thus, $|F_n| \geq 1 + \lceil \frac{2n-4}{2} \rceil = n-1$.

Case 3. $|B| = 2$. Suppose $\{\bar{u}^i, u^{i+1}, \bar{u}^{i+1}\} \subseteq V(b_i), \{\bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} \subseteq V(b_j)$ with $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$. Without loss of generality, we set $j > i$.

Case 3.1. $j - i = 2$. Then $\{\bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} = \{\bar{u}^{i+2}, u^{i+3}, \bar{u}^{i+3}\}$. Suppose $w_1, w_2 \in N(u) - V(B) - \{u^{i+2}\}$ and $w_1 \neq w_2$. According to the definition of AQ_n, vertex u^{i+2} is not adjacent to the vertex in $N(u) - V(B)$. Then there is an element a in $F_n - B$, $u^{i+2} \in V(a)$ and $V(a) \cap \{N(u) - V(B)\} = 1$. Since the element in $F - B - a$ contains at most two distinct vertices in $N(u) - V(B) - V(a)$. Thus, $|F_n| \geq 2 + 1 + \lceil \frac{2n-8}{2} \rceil = n-1$.

Case 3.2. $j - i = 3$. Then $\{\bar{u}^j, u^{j+1}, \bar{u}^{j+1}\} = \{\bar{u}^{i+3}, u^{i+4}, \bar{u}^{i+4}\}$. In the following we distinguish cases pertaining to the number of elements in $F_n - B$ containing three vertices $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3}. We denote the number of elements as N.

Case 3.2.1. $N = 3$. There are three distinct elements in $N(u) - V(B)$ that contain one of three vertices $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3} respectively. According to the definition of AQ_n, vertex u^{i+2} is not adjacent to the vertex in $N(u) - V(B) - \{\bar{u}^{i+2}, u^{i+3}\}$. Then there is an element a in $F_n - B$, $u^{i+2} \in V(a)$ and $V(a) \cap \{N(u) - V(B) - \{\bar{u}^{i+2}, u^{i+3}\} = 1$. Vertex $\bar{u}^{i+2}$ and u^{i+3} are same as u^{i+2}. Since the element in $F_n - B - a$ contains at most two distinct vertices in $N(u) - V(B) - V(a)$. Thus, $|F_n| \geq 2 + 3 + \lceil \frac{2n-10}{2} \rceil = n$.

Case 3.2.2. $N = 2$. There are two distinct elements in $N(u) - V(B)$, one of which contains one vertex in $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3}, and the other contains the other two vertices. We assume that a_1 contains one vertex in $u^{i+2}, \bar{u}^{i+2}$ and u^{i+3} and a_2 contains the other two vertices.

Case 3.2.2.1. $u^{i+2} \in V(a_1)$ and $\{\bar{u}^{i+2}, u^{i+3}\} \subseteq V(a_2)$. According to the discussion of Case 3.2.1, $V(a_1) \cap \{N(u) - V(B) - \{\bar{u}^{i+2}, u^{i+3}\}\} = 1$. Vertex $\bar{u}^{i+2}$ or u^{i+3} is not adjacent to the vertex in $N(u) - V(B) - \{u^{i+2}\}$ and $\{\bar{u}^{i+2}, u^{i+3}\} \in E(AQ_n)$. Then $\{\bar{u}^{i+2}, u^{i+3}\} \subseteq a_2$ and $V(a_2) \cap \{N(u) - V(B) - \{u^{i+2}\}\} = 2$. Since the element in $F_n - B - a_1 - a_2$ contains at most two distinct vertices in $N(u) - V(B) - V(a_1) - V(a_2)$. Thus, $|F_n| \geq 2 + 1 + 1 + \lceil \frac{2n-10}{2} \rceil = n-1$. $u^{i+3} \in V(a_1)$ and $\{\bar{u}^{i+2}\ u^{i+2}\} \subseteq V(a_2)$ is similar.

Case 3.2.2.2. $\bar{u}^{i+2} \in V(a_1)$ and $\{u^{i+2}, u^{i+3}\} \subseteq V(a_2)$. Since $\{u^{i+2}, u^{i+3}\} \notin E(AQ_n)$. So this situation does not exist.

Case 3.2.3. $N = 1$. Since $\{u^{i+2}, u^{i+3}\} \notin E(AQ_n)$. So this situation does not exist.

Case 3.3. $j - i \geq 4$. we set $U_{ij} = \{u^{i+2}, \bar{u}^{i+2}, \ldots, u^{j-1}\}$. We will calculate the number of elements in $F_n - B$ containing all of vertices in U. Obviously, $5 \leq |U| \leq 2n - 7$. Since the element in $F_n - B$ contains at most two distinct vertices in $N(u) - V(B)$ and U mod $2 = 1$. Similar to the discussion in Case 3.2.1. $|F_n| \geq 2 + 1 + \lceil \frac{2n-8}{2} \rceil = n - 1$.

Case 4. $|B| \geq 3$. If $b_i, b_j \in B$ and no $b_k \in B$ with $i < k < j$, we set $U_{ij} = \{u^{i+2}, \bar{u}^{i+2}, \ldots, u^{j-1}\}$. Obviously, $1 \leq |U| \leq 2n - 7$. According to the discussion of Case 3. the minimum number of elements in $F_n - B$ that contain all vertices in U_{ij} is $\lceil \frac{|U_{ij}|}{2} \rceil$. Thus, $|F_n| \geq |B| + (|B| - 1) + \lceil \frac{2n-1-3\times|B|-(|B|-1)}{2} \rceil = n - 1$

In summary, the lemma holds.

Lemma 13. *For $n \geq 6$, $\kappa(AQ_n, C_3) \geq n - 1$.*

Proof. Let F_n^* be a C_3-structure set of AQ_n and we will show that $AQ_n - V(F_n^*)$ is connected if $F_n^* \leq n - 2$. Suppose not, Let R be the smallest component of $AQ_n - V(F_n^*)$. Note that $|V(F_n^*)| \leq 3 \times (n-2) = 3n - 6$. By lemma **??**, we have $3n - 6 \leq 4n - 8$ for $n \geq 6$. So $|V(R)| = 1$. Therefore, we assume that vertex $u \in V(R)$. By lemma **??**, $|N(u) \cap V(F_n^*)| \leq 2n - 2 < 2n - 1$, which implies that there exists a neighbor of u in $AQ_n - V(F_n^*)$. Therefore, we have $|V(R)| \geq 2$, a contradiction. Thus, $AQ_n - V(F_n^*)$ is connected. The lemma holds.

By Lemma **??** and Lemma **??**, we have the following theorem.

Theorem 7. *For $n \geq 6$, $\kappa(AQ_n, C_3) = n - 1$.*

3.3.3 $\kappa(AQ_n, C_4)$

Lemma 14. *For $n \geq 4$, $\kappa(AQ_n, C_4) \leq \lceil \frac{2n-1}{3} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. According to the value of $(2n-1)$ mod 3, we will discuss the following two cases.

Case 1. $(2n-1)$ mod $3 \neq 0$. We set

$$S_1 = \{\{v^{[3i+1]}, v^{[3(i+1)]}, v^{[3i+2)]}, (v^{[3i+1]})^{\lfloor \frac{3i+1}{2} \rfloor + 3}, v^{[3i+1]}\} \mid 0 \leq i < \lfloor \frac{2n-1}{3} \rfloor \text{ and } i \bmod 2 = 0\},$$
$$S_2 = \{\{v^{[3i+1]}, v^{[3i+2]}, v^{[3(i+1)]}, (v^{[3i+1]})^{\lfloor \frac{3i+1}{2} \rfloor + 2}, v^{[3i+1]}\} \mid 0 \leq i < \lfloor \frac{2n-1}{3} \rfloor \text{ and } i \bmod 2 = 1\},$$
$$S_3 = \begin{cases} \{\{v^{[2n-1]}, (\overline{v^{[2n-1]}})^1, (\overline{v^{[2n-1]}})^2, (\overline{v^{[2n-1]}})^3, v^{[2n-1]}\}\} & (2n-1) \bmod 3 = 1, \\ \{\{v^{[2n-2]}, v^{[2n-1]}, (\overline{v^{[2n-1]}})^{n-3}, (\overline{v^{[2n-1]}})^{n-2}, v^{[2n-2]}\}\} & (2n-1) \bmod 3 = 2. \end{cases}$$

Case 2. $(2n-1)$ mod $3 = 0$. We set

$S_1 = \{(v^{[3i+1]}, v^{[3(i+1)]}, v^{[3i+2]}, (v^{[3i+1]})^{\lfloor \frac{3i+1}{2} \rfloor+3}, v^{[3i+1]}) \mid 0 \leq i < \lfloor \frac{2n-1}{3} \rfloor - 1 \text{ and } i \bmod 2 = 0\}$,
$S_2 = \{(v^{[3i+1]}, v^{[3i+2]}, v^{[3(i+1)]}, (v^{[3i+1)]})^{\lfloor \frac{3i+1}{2} \rfloor+2}, v^{[3i+1]}) \mid 0 \leq i < \lfloor \frac{2n-1}{3} \rfloor - 1 \text{ and } i \bmod 2 = 1\}$,
$S_3 = \{(v^{[2n-2]}, v^{[2n-3]}, v^{[2n-1]}, (v^{[2n-1]})^{n-1}, v^{[2n-2]})\}$.

According to the definition of AQ_n, the subgraph induced by each elements in $S_1 \cup S_2 \cup S_3$ is isomorphic to C_4 and $|S_1 \cup S_2 \cup S_3| = \lceil \frac{2n-1}{3} \rceil$. Since $AQ_n - (S_1 \cup S_2 \cup S_3)$ is disconnected and one component of it is $\{u\}$. Then $\kappa(AQ_n, C_4) \leq \lceil \frac{2n-1}{3} \rceil$.

Lemma 15. *Let F_n be a C_4-structure set of AQ_n and u be a vertex in $AQ_n - V(F_n)$. If u is an isolated vertex in $AQ_n - V(F_n)$, then $|F_n| \geq \lceil \frac{2n-1}{3} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. According to the definition of AQ_n, property **??** and property **??**, any four neighbors of u can't form a C_4 and each element in F contains at most three distinct vertices in $N(u)$. Thus, $|F_n| \geq \lceil \frac{2n-1}{3} \rceil$.

Lemma 16. *For $n \geq 6$, $\kappa(AQ_n, C_4) \geq \lceil \frac{2n-1}{3} \rceil$.*

Proof. Let F_n^* be a C_4-structure set of AQ_n and we will show that $AQ_n - V(F_n^*)$ is connected if $F_n^* \leq \lceil \frac{2n-1}{3} \rceil - 1$. Suppose not, Let R be the smallest component of $AQ_n - V(F_n^*)$. Note that $|V(F_n^*)| \leq 4 \times (\lceil \frac{2n-1}{3} \rceil - 1)$. By lemma **??**, we have $4 \times (\lceil \frac{2n-1}{3} \rceil - 1) \leq 4n-8$ for $n \geq 6$. So $|V(R)| = 1$. Therefore, we assume that vertex $u \in V(R)$. By lemma **??**, $|N(u) \cap V(F_n^*)| \leq 2n-2 < 2n-1$, which implies that there exists a neighbor of u in $AQ_n - V(F_n^*)$. Therefore, we have $|V(R)| \geq 2$, a contradiction. Thus, $AQ_n - V(F_n^*)$ is connected. The lemma holds.

By Lemma **??** and Lemma **??**, we have the following theorem.

Theorem 8. *For $n \geq 6$, $\kappa(AQ_n, C_4) = \lceil \frac{2n-1}{3} \rceil$.*

3.3.4 $\kappa(AQ_n, C_N)$ with $4 < N \leq 2n-1$

Lemma 17. *For $n \geq 6$, $\kappa(AQ_n, C_N) \leq \lceil \frac{2n-1}{N-1} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. According to the parity of N, we will discuss the following two cases.

Case 1. N is odd. We set

$$S_1 = \{\{v^{[(N-1)i+1]}, v^{[(N-1)i+2]}, \ldots, v^{[(i+1)(N-1)]}, (\overline{v^{[(i+1)(N-1)]}})^{\lfloor \frac{(N-1)i+1}{2} \rfloor+1}, v^{[i*(N-1)+1]}\} \mid 0 \leq i < \lfloor \frac{2n-1}{N-1} \rfloor\}$$

Case 1.1. $(2n-1) \bmod (N-1) = 1$

Case 1.1.1. $N \leq 2n-3$

$$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, \ldots, (v^{[2n-1]})^{\lfloor \frac{N}{2} \rfloor}, (\overline{v^{[2n-1]}})^{\lfloor \frac{N}{2} \rfloor}, (v^{[2n-1]})^{\lceil \frac{N}{2} \rceil}, v^{[2n-1]}\}\}$$

Case 1.1.2. $N = 2n-1$

$$S_2 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, \ldots, (v^{[2n-1]})^{n-3}, (\overline{v^{[2n-1]}})^{n-3}, (\overline{v^{[2n-1]}})^{n-2},$$

$((\overline{v^{[2n-1]}})^{n-2})^{n-4}, (\overline{(\overline{v^{[2n-1]}})^{n-2}})^{n-4}, ((\overline{v^{[2n-1]}})^{n-2})^{n-3}, (\overline{(\overline{v^{[2n-1]}})^{n-2}})^{n-3}, v^{[2n-1]}\}\}$

Case 1.2. $(2n-1) \bmod (N-1) = 3$

$$S_2 = \{\{v^{[2n-1]}, v^{[2n-3]}, v^{[2n-2]}, (v^{[2n-2]})^{n-1-\frac{N-3}{2}}, (\overline{v^{[2n-2]}})^{n-1-\frac{N-3}{2}}, (v^{[2n-2]})^{n-\frac{N-3}{2}}, \dots, (v^{[2n-2]})^{n-2}, (\overline{v^{[2n-2]}})^{n-2}, v^{[2n-1]}\}\}$$

Case 1.3. $(2n-1) \bmod (N-1) > 3$

$$S_2 = \{\{v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+2]}, \dots, v^{[2n-3]}, v^{[2n-1]}, v^{[2n-2]},$$
$$(v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+2-\frac{N-(2n-1-\lfloor\frac{2n-1}{N-1}\rfloor(N-1))}{2}},$$
$$(\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+2-\frac{N-(2n-1-\lfloor\frac{2n-1}{N-1}\rfloor(N-1))}{2}},$$
$$(v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+3-\frac{N-(2n-1-\lfloor\frac{2n-1}{N-1}\rfloor(N-1))}{2}},$$
$$\dots,$$
$$(\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]}\}\}$$

Case 2. N is even. We set

$S_1 = \{\{v^{[(N-1)i+1]}, v^{[(N-1)i+2]}, \dots, v^{[(i+1)(N-1)]}, (\overline{v^{[(i+1)(N-1)]}})^{\lfloor\frac{(N-1)i+1}{2}\rfloor+1}, v^{[(N-1)i+1]}\} \mid 0 \le i < \lfloor\frac{2n-1}{N-1}\rfloor - 1$ and $i \bmod 2 = 0\}$
$S_2 = \{\{v^{[(N-1)i+1]}, v^{[(N-1)i+2]}, \dots, v^{[(i+1)(N-1)]}, (v^{[(i+1)(N-1)]})^{\lfloor\frac{(N-1)i+1}{2}\rfloor+1}, v^{[(N-1)i+1]}\} \mid 0 \le i < \lfloor\frac{2n-1}{N-1}\rfloor - 1$ and $i \bmod 2 = 1\}$

If $(2n-1) \bmod (N-1) = 0$, then $S_3 = \emptyset$, otherwise, we will discuss the following several cases,

Case 2.1. $(2n-1) \bmod (N-1) = 1$.

$$S_3 = \{\{v^{[2n-1]}, (v^{[2n-1]})^1, (v^{[2n-1]})^2, (\overline{v^{[2n-1]}})^2, \dots, (v^{[2n-1]})^{\frac{N}{2}}, (\overline{v^{[2n-1]}})^{\frac{N}{2}}, v^{[2n-1]}\}\}$$

Case 2.2. $(2n-1) \bmod (N-1) = 2$.

Case 2.2.1. $N < n$.

$$S_3 = \{\{v^{[2n-1]}, v^{[2n-2]}, (v^{[2n-2]})^{n-1}, ((v^{[2n-2]})^{n-1})^{n-2}, \dots, (((v^{[2n-2]})^{n-1})\dots)^{n-(N-2)}, v^{[2n-1]}\}\}$$

Case 2.2.2. $N = 2n-2$.

$$S_3 = \{\{v^{[2n-1]}, v^{[2n-2]}, (v^{[2n-2]})^1, (v^{[2n-2]})^2, (\overline{(v^{[2n-2]}})^2, \dots, (\overline{(v^{[2n-2]}})^{n-2}, (v^{[2n-2]})^{n-1}, v^{[2n-1]}\}\}$$

Case 2.3. $(2n-1) \bmod (N-1) = 3$.

$S_3 = \{\{v^{[2n-2)]}, v^{[2n-3]}, v^{[2n-1]}, (\overline{v^{[2n-1]}})^{2+n-N}, (\overline{v^{[2n-1]}})^{3+n-N}, \ldots, (\overline{v^{[2n-1]}})^{n-2}, v^{2n-2}\}\}$

Case 2.4. $(2n-1) \bmod (N-1) = 4$.

Case 2.4.1. $N < n$.

$S_3 = \{\{v^{[2n-4]}, v^{[2n-3]}, v^{[2n-1]}, v^{[2n-2]}, (v^{[2n-2)]})^{4+n-N}, (v^{[2n-2]})^{5+n-N}, \ldots, (v^{[2n-2]})^{n-1}, v^{[2n-4]}\}\}$

Case 2.4.2. $N = 2n-4$.

$S_3 = \{\{v^{[2n-4]}, v^{[2n-3]}, v^{[2n-1]}, v^{[2n-2]}, (\overline{v^{[2n-2]}})^{3}, (v^{[2n-2]})^{4}, (\overline{v^{[2n-2]}})^{4}, \ldots, ((v^{[2n-2]})^{n-1}, v^{[2n-1]}\}\}$

Case 2.5. $(2n-1) \bmod (N-1)$ is odd(except 1 and 3).

$S_3 = \{\{v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+2]}, \ldots, v^{[2n-3]}, v^{[2n-1]}, v^{[2n-2]},$
$(\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+2n-\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}, (\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+2n-\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+2}, \ldots,$
$(\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]}\}\}$

Case 2.6. $(2n-1) \bmod (N-1)$ is even(except 0, 2 and 4).

Case 2.6.1. $N < n$.

$S_3 = \{\{v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+2]}, \ldots, v^{[2n-3]}, v^{[2n-1]}, v^{[2n-2]},$
$(\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1},$
$\overline{((\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})}^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+2n-\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1},$
$\overline{((\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})}^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+2n-\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+2},$
$\ldots,$
$\overline{((\overline{v^{[2n-2]}})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})}^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]}\}\}$

Case 2.6.2. $n \leq N \leq 2n-6$.

$S_3 = \{\{v^{[2n-2]}, v^{[2n-1]}, v^{[2n-3]}, \ldots, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+2]}, v^{[\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1]},$
$(v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1},$
$\overline{((v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})}^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+n+1},$
$((v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+n+2},$
$\overline{((v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})}^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor-N+n+2},$
$\ldots,$
$\overline{((v^{[2n-2]})^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor+1})}^{\lfloor\frac{\lfloor\frac{2n-1}{N-1}\rfloor(N-1)+1}{2}\rfloor},$

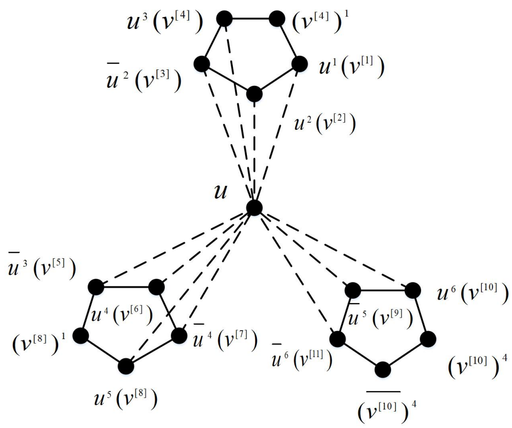


Figure 5: A C_5-structure-cut in AQ_6

$v^{[2n-2]}\}\}$

We set $F = S_1 \cup S_2$ when N is odd(respectively. $F = S_1 \cup S_2 \cup S_3$ when N is even). According to the definition of AQ_n, the subgraph induced by each element in F is isomorphic to C_N and $|F| = \lceil \frac{2n-1}{N-1} \rceil$. Since $AQ_n - F$ is disconnected and one component of it is $\{u\}$. Then $\kappa(AQ_n, C_N) \leq \lceil \frac{2n-1}{N-1} \rceil$. Figure **??** shows a C_5-structure-cut in AQ_6.

Lemma 18. *Let F_n be a C_N-structure set of AQ_n and u be a vertex in $AQ_n - V(F_n)$. If u is an isolated vertex in $AQ_n - V(F_n)$, then $|F_n| \geq \lceil \frac{2n-1}{N-1} \rceil$.*

Proof. Let u be an arbitrary vertex in AQ_n. According to the definition of AQ_n, property **??** and property **??**, any N neighbors of u can't form a C_N and each element in F contains at most $N-1$ distinct vertices in $N(u)$. Thus, $|F_n| \geq \lceil \frac{2n-1}{N-1} \rceil$.

Lemma 19. *For $n \geq 5$, $\kappa(AQ_n, C_N) \geq \lceil \frac{2n-1}{N-1} \rceil$.*

Proof. Let F_n^* be a C_N-structure set of AQ_n and we will show that $AQ_n - V(F_n^*)$ is connected if $F_n^* \leq \lceil \frac{2n-1}{N-1} \rceil - 1$. Suppose not, Let R be the smallest component of $AQ_n - V(F_n^*)$. Not that $|V(F_n^*)| \leq N \times (\lceil \frac{2n-1}{N-1} \rceil - 1)$. By lemma **??**, we have $N \times (\lceil \frac{2n-1}{N-1} \rceil - 1) \leq 4n - 8$ for $n \geq 6$. So $|V(R)| = 1$. Therefore, we assume that vertex $u \in V(R)$. By lemma **??**, $|N(u) \cap V(F_n^*)| \leq 2n - 2 < 2n - 1$, which implies that there exists a neighbor of u in $AQ_n - V(F_n^*)$. Therefore, we have $|V(R)| \geq 2$, a contradiction. Thus, $AQ_n - V(F_n^*)$ is connected. The lemma holds.

By Lemma **??** and Lemma **??**, we have the following theorem.

Theorem 9. *For $n \geq 5$, $\kappa(AQ_n, C_N) = \lceil \frac{2n-1}{N-1} \rceil$.*

4 Conclusions

In this paper, we study the H-structure and H-substructure connectivity results of augmented cube , two novel measures to estimate the reliability and fault-tolerability of the interconnection networks, The structure and substructure connectivity evaluate the fault-tolerance not only from the perspective of individual vertex, but also some structure of the network, which can be to some extent more precisely and practically than classical connectivity. Through the approach used in this paper, one can explore other interconnection networks's structure and substructure connectivity and more complex structures remain open.

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