Solution to Scalar ODEs.

The solution of an ODE can be written as the sum of a homogeneous & particular solution

$$x(t) = x_h(t) + x_p(t)$$

Homogeneous solutions have forms of

$$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \qquad \text{2 real roots}$$

or

$$x_h(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t} \qquad \text{1 real root}$$

or

$$x_h(t) = e^{pt}(C_1 \cos qt + C_2 \sin qt) \qquad \lambda = p \pm iq$$

Higher order ODEs follow this pattern

ex.) if $\lambda = \lambda_1, \lambda_2, \lambda_2, \lambda_2, \lambda_5 = p + iq$

$$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 t e^{\lambda_2 t} + C_4 t^2 e^{\lambda_2 t}$$
$$+ C_5 e^{pt} \cos qt + C_6 e^{pt} \sin qt$$

Now turn to the particular solution.

This is trickier. In general, we can expect that the particular solution has the same form as the non-homogeneous part.

ex.) $a\ddot{x} + b\dot{x} + cx = f(t) = \sin(4t)$

$\Rightarrow$ we expect that

$$x_p(t) = A\sin(4t) + B\cos(4t)$$

<u>Inhomogeneous Part</u>	<u>Particular Solution</u>
$A\sin(\omega t)$	$B\sin(\omega t) + C\cos(\omega t)$
$A\cos(\omega t)$	$B\sin(\omega t) + C\cos(\omega t)$
$a_3t^3 + a_2t^2 + a_1t + a_0$	$b_3t^3 + b_2t^2 + b_1t + b_0$
Ce^{at}	de^{at}

ex.) let $a\ddot{x} + b\dot{x} + cx = \sin(2t) + t^2 - e^{3t}$

I expect that $x_p(t)$ has the form

$$x_p(t) = A\sin(2t) + B\cos(2t) + Ct^2 + Dt + E + Fe^{3t}$$

To get A, B, C, D, E & F, plug $x_p(t)$ into the ODE & match coefficients.

ex.) let $\dot{x} + 4x = 8t^2$ find $x_p(t)$

Try $x_p(t) = at^2 + bt + c$

$\dot{x}_p = 2at + b$

$\dot{x}_p + 4x_p = (2at + b) + 4(at^2 + bt + c) = 8t^2$

$= 4at^2 + (2a + 4b)t + (b + 4c) = 8t^2$

$\Rightarrow 4a = 8 \rightarrow a = 2$

$2a + 4b = 0 \rightarrow b = -1$

$b + 4c = 0 \rightarrow c = 1/4$

$x_p(t) = 2t^2 - t + 1/4$

Now bring it together

Find the complete solution to

$\ddot{x} + 2\dot{x} + x = \sin(2t)$ w/ $x(0) = 0$, $\dot{x}(0) = 0$

We need 2 homogenous solutions + 1 particular solution

(1) Homogeneous part: let $x_h(t) = e^{\alpha t}$

$$\ddot{x}_h + 2\dot{x}_h + x_h = \alpha^2 e^{\alpha t} + 2\alpha e^{\alpha t} + e^{\alpha t} = 0$$

$$\Rightarrow (\alpha^2 + 2\alpha + 1) e^{\alpha t} = 0$$

$$\Rightarrow \alpha^2 + 2\alpha + 1 \Rightarrow \alpha = -1, -1$$

$$\Rightarrow x_h(t) = C_1 e^{-t} + C_2 t e^{-t} = e^{-t}(C_1 + C_2 t)$$

Do <u>not</u> apply the initial conditions yet,

(2) Particular Solution!

expect that $x_p(t) = A\sin(2t) + B\cos(2t)$

$$\dot{x}_p = 2A\cos(2t) - 2B\sin(2t)$$

$$\ddot{x}_p = -4A\sin(2t) - 4B\cos(2t)$$

$$\begin{aligned}\ddot{x}_p + 2\dot{x}_p + x_p = & -4A\sin(2t) - 4B\cos(2t) \\ & + 4A\cos(2t) - 4B\sin(2t) \\ & + A\sin(2t) + B\cos(2t)\end{aligned}$$

$$= (-4A - 4B + A)\sin(2t) + (-4B + 4A + B)\cos(2t)$$
$$= \sin(2t)$$

$$\Rightarrow \quad \begin{aligned} -3A - 4B &= 1 \\ 4A - 3B &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A &= -3/25 \\ B &= -4/25 \end{aligned}$$

$$\Rightarrow x_p(t) = -\frac{3}{25}\sin(2t) - \frac{4}{25}\cos(2t)$$

③ Complete Solution

$$x(t) = x_h(t) + x_p(t)$$

$$= e^{-t}(C_1 + C_2 t) - \frac{3}{25}\sin(2t) - \frac{4}{25}\cos(2t)$$

Now apply I.C to find C_1 & C_2

$$x(0) = C_1 - 4/25 = 0 \Rightarrow C_1 = 4/25$$

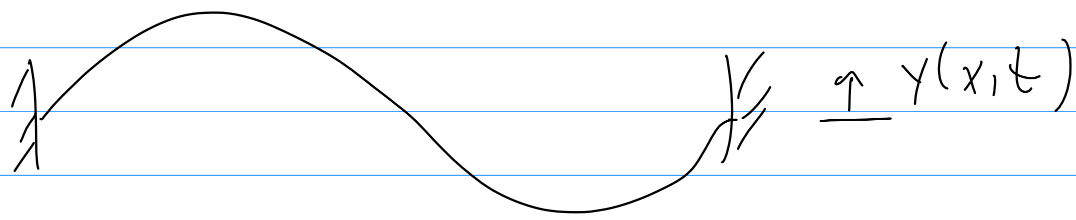
$$\dot{x}(0) = -C_1 + C_2 - \frac{6}{25} = 0 \Rightarrow C_2 = \frac{2}{5}$$

$$\Rightarrow x(t) = e^{-t}\left(\frac{4}{25} + \frac{2}{5}t\right) - \frac{3}{25}\sin(2t) - \frac{4}{25}\cos(2t)$$

Partial Differential Equations (PDE)

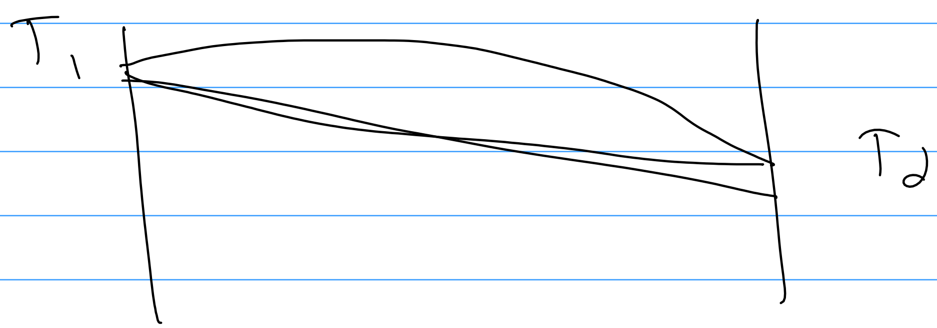
There are differential equations of 2 or more variables.

ex.) Wave Equation $\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$



ex.) Heat Equation $\frac{du}{dt} = \alpha \nabla^2 u$

u = Temperature α = thermal conductivity



Fourier Series Expansion (FSE)

A FSE states that a function $f(x)$ that is periodic on an interval $[-L, L]$ (period of $2L$) can be written as

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

for some set of coefficients

$$a_0, \ a_n, \ b_n \quad , \ n = 1, 2, \ldots \infty$$

This is true because $\sin\left(\frac{n\pi x}{L}\right)$ & $\cos\left(\frac{n\pi x}{L}\right)$ form an orthogonal basis

for the function (vector) space in $[-L, L]$

Let $u(x)$ & $v(x)$ exist in $x \in [a, b]$. u & v are mutually orthogonal if

$$\int_a^b u(x)\, v(x)\, dx = 0$$

You can show that:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \text{for any } m, n$$

Use this fact to use multiplication and integration to get:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots \infty$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, \ldots, \infty$$

You can also show that:

1. If $f(x)$ is an odd function $f(-x) = -f(x)$, then $a_0 = 0$, $a_n = 0$ for $n = 1, 2, \ldots$

(2) If $f(x)$ is an even function
$f(-x) = f(x)$ then

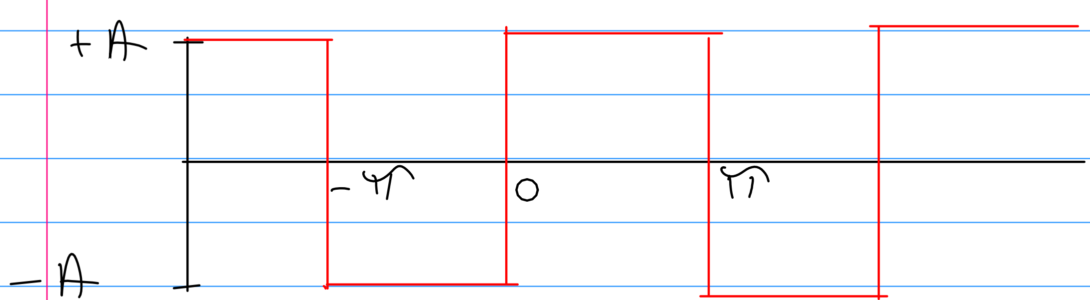
$b_n = 0$ for $n = 1, 2, \ldots$

Fourier Convergence Theorem

Let $f(x)$ & $f'(x)$ be piecewise continuous on $-L \le x \le L$ w/ a period of $2L$, Then $f(x)$ has a Fourier Representation and the FSE converges to $f(x)$ at every point x and to $\frac{1}{2}[f(x^+) + f(x^-)]$ at every discontinuous point x.

Notice that the value of the FSE converges to the average in the infinite limit.

ex.) Look at a square sine wave, w/ $L = \pi$



You can show that

$$a_0 = a_n = 0$$

$$b_n = \frac{2A}{n\pi}(1 - \cos n\pi) \qquad n = 1, 2, \ldots$$

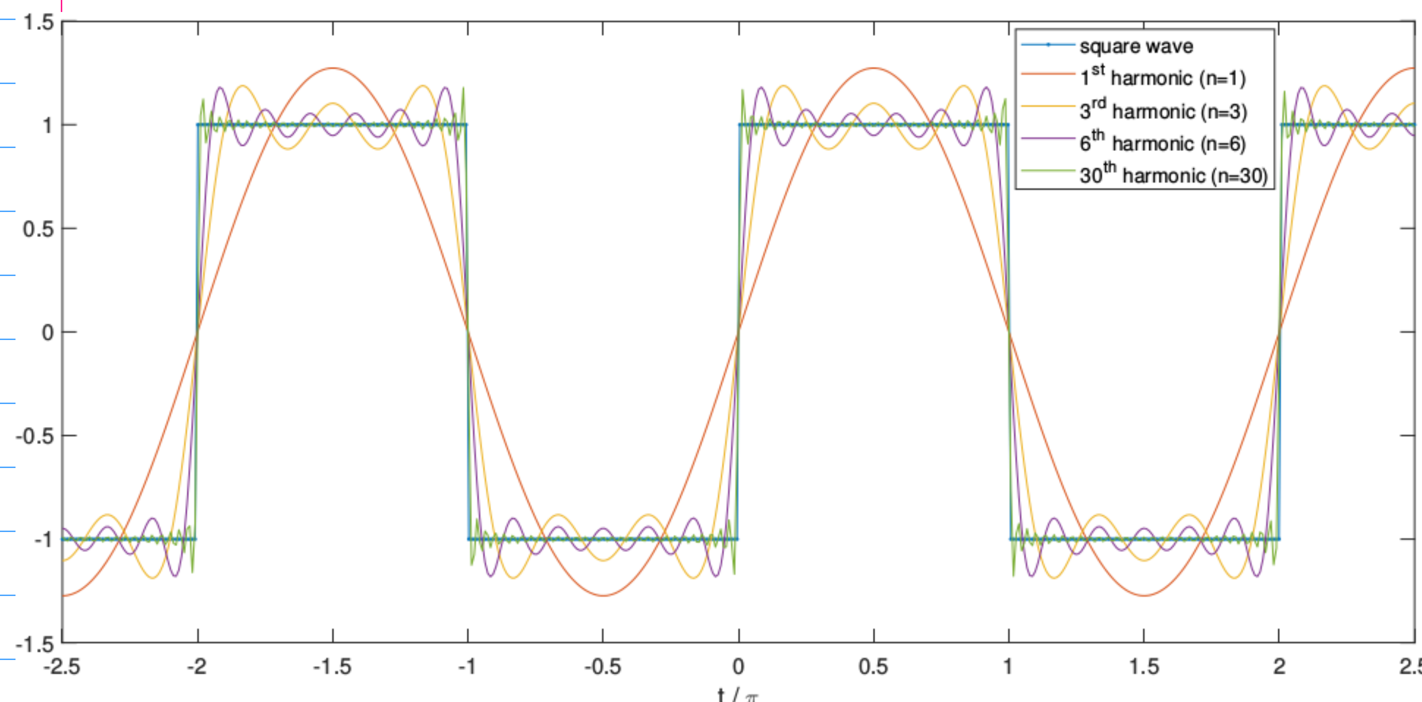
Since $\cos(n\pi) = (-1)^n$, all even

b_n values are zero. ($b_2 = 0$, $b_4 = 0$, ...)

After some manipulation, you get

$$f(x) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

Now look at truncations of the FSE
(the modes)



The oscillation at the discontinuity is called Gibbs Phenomena.

Now, what if $f(x)$ is <u>not</u> periodic.

Let $f(x)$ be some function defined over $0 \le x \le L$.

Make $f(x)$ periodic over $-L \le x \le L$ by choosing what $f(-x)$ is.

① Set $f(-x) = f(x) \rightarrow$ Even function

Only Cosine Terms $\rightarrow$ Cosine Series

You can show that

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = 0$$

② Set $f(-x) = -f(x) \rightarrow$ Odd function
$\rightarrow$ Only sin terms, Sine Series

$$a_n = 0$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

PDEs

we will be using separation of variables.

Look at the wave equation:

$$\frac{d^2y}{dt^2} = \alpha^2 \frac{d^2y}{dx^2} \rightarrow y(x,t)$$

Assume that you can separate the variables:

$$y(x,t) = X(x)\,T(t)$$

$$y_t = X(x)\,T_t \qquad y_{tt} = X(x)\,T_{tt}$$

$$y_x = X_x\,T(t) \qquad y_{xx} = X_{xx}\,T(t)$$

From $y_{tt} = \alpha^2 y_{xx}$

$$X(x)\,T_{tt} = \alpha^2 X_{xx}\,T(t)$$

$$\underbrace{\frac{1}{X(x)} X_{xx}}_{\text{only a fcn of } x} = \underbrace{\frac{1}{\alpha^2}\frac{1}{T(t)} T_{tt}}_{\text{only a fcn of } t} = -k^2$$

only true if both equal a constant.

Now I have 2 ODEs:

ODE 1: $\frac{1}{X(x)} X_{xx} = -k^2 \Rightarrow X_{xx} + k^2 X = 0$

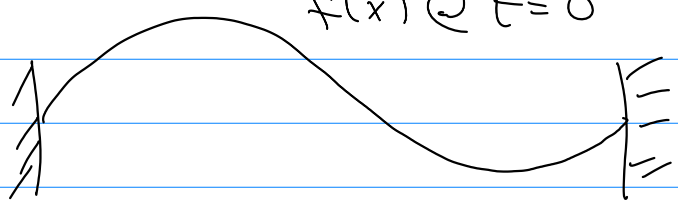
ODE 2: $\frac{1}{\alpha^2 T(t)} T_{tt} = -k^2 \Rightarrow T_{tt} + \alpha^2 k^2 T = 0$

Need 2 conditions for each ODE
$\rightarrow$ Need 4 Boundary or initial conditions.

Let $y(0,t) = 0 \qquad y(L,t) = 0 \qquad$ B.C.

$y(x,0) = f(x) \qquad y_t(x,0) = 0 \qquad$ I.C.

$f(x)$ @ $t=0$



$y=0$ $\updownarrow$

Convert to ODE condition

$y(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$
$y(L,t) = X(L)\,T(t) = 0 \Rightarrow X(L) = 0$

$y(x,0) = X(x)\,T(0) = f(x)$
$y_t(x,0) = X(x)\,T_t(0) = 0 \Rightarrow T_t(0) = 0$

Solve ODE 1:

$X_{xx} + k^2 X = 0$ w/ $X(0) = X(L) = 0$

roots of $\alpha = \pm ik$

$\Rightarrow X(x) = e^{0x}(C_1 \cos(kx) + C_2 \sin(kx))$

$= C_1 \cos(kx) + C_2 \sin(kx)$

$X(0) = C_1 = 0$

$X(L) = C_2 \sin(kL) = 0$

If $C_2 = 0$, then $X(x) = 0$

Instead, find all points where $\sin(kL) = 0$

$\Rightarrow k = \frac{n\pi}{L}$, $n = 1, 2, \ldots, \infty$

$\Rightarrow X(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, \ldots, \infty$