Vector Space – Rules how vectors can behave.

Vector spaces must have closure properties: $(+)$ & $(*)$ operations on vectors must remain in the space.

Rules:

① $\underline{u}+\underline{v}=\underline{v}+\underline{u}$ ② $\underline{u}+(\underline{v}+\underline{w})=(\underline{u}+\underline{v})+\underline{w}$

③ $\underline{0}$ exists such that $\underline{0}+\underline{u}=\underline{u}+\underline{u}+\underline{0}$

④ $-\underline{u}$ exists such that $\underline{u}+(-\underline{u})=\underline{0}=(-\underline{u})+\underline{u}$

⑤ $a(\underline{u}+\underline{v})=a\underline{u}+a\underline{v}$ ⑥ $(a+b)\underline{u}=a\underline{u}+b\underline{u}$

⑦ $ab(\underline{u})=a(b\underline{u})$ ⑧ $1\underline{u}=\underline{u}$

Also, $a\underline{0}=\underline{0}$, $0\underline{v}=\underline{0}$, $(-1)\underline{v}=-\underline{v}$

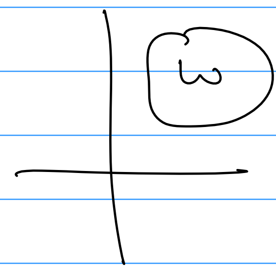
if $a\underline{v}=\underline{0}$, either $a=0$ or $\underline{v}=0$

Subspaces are subsets of a Vector Space V that flow the rules.

①, ②, ⑤, ⑥, ⑦ & ⑧ are trivial to prove for subspaces, W

ex.) ①: $\underbrace{\underline{u}+\underline{v}}_{\text{in } W}=\underbrace{\underline{u}+\underline{v}}_{V}=\underbrace{\underline{v}+\underline{u}}_{\text{in } V}=\underbrace{\underline{v}+\underline{u}}_{W}$

③ & ④ are trickier:



V is $\mathbb{R}^2$

③ Let W be a non-empty subset of V

Pick $\underline{w}_1 \in W$. As $W \subset V$, it must obey scalar multiplication, so $0\underline{w}_1$ must be in W:

$0\underline{w}_1 = \underline{0}$, $\underline{0} \in W$

Because V is a vector space, then $\underline{0} + \underline{v} = \underline{v}$ for any $\underline{v}$ in V. As $\underline{w}_1$ is in V, it also holds for $\underline{w}_1$: $\underline{0} + \underline{w}_1 = \underline{w}_1$.

④ Let $\underline{w}_1 \in W$, as $W \subset V$, $\underline{w}_1 \in V$

By rule ④ for V, $\underline{w}_1$ has an additive inverse in V: $(-\underline{w}_1)$. We need to show that $(-\underline{w}_1)$ also exists in W.
In V, we have $-\underline{w}_1 = (-1)\underline{w}_1$.
Since W is closed under scalar multiplication, $\underline{w}_1 \in W$ implies that $-\underline{w}_1 \in W$.

Function Introduction

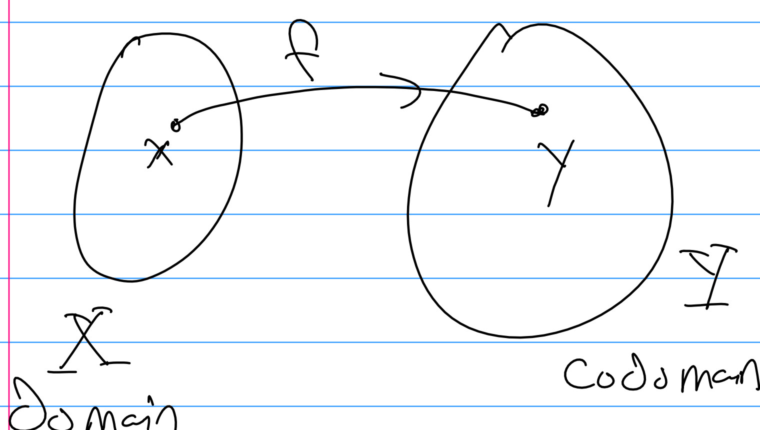
A function is the assignment of an element in the domain X into the codomain Y.

Write as $f : X \to Y$ or, $f(x) = y$

element in X element in Y.

Also called a mapping.

Each element in X is assigned to a single element in Y.



ex.) Let $X = \mathbb{Z}$, $X = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

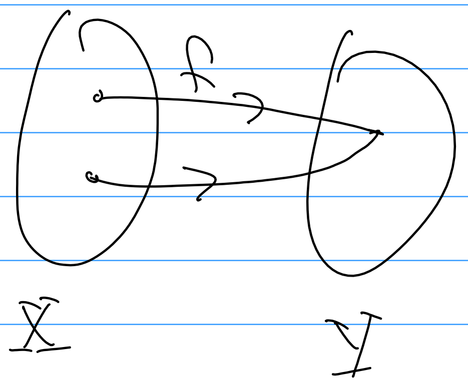
Define $f(x) = x^2$

$f(-2) = 4$

in domain   in codomain.

Note! Multiple elements in X can be assigned to the same element in Y!

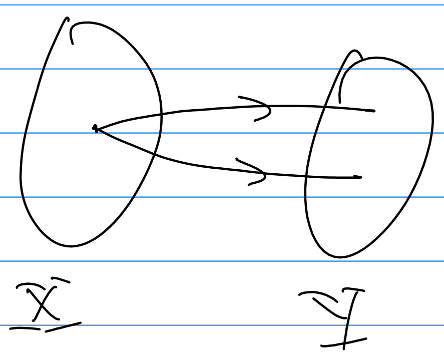
$f(-2) = +4$ $\quad$ $f(+2) = +4$,



Also, there must be at most one element in the codomain for every element in the domain.

ex.) $f(x) = \sqrt{x}$ is not a function,

$f(9) = +3$, or -3



Not a function.

The image of a domain element is the unique codomain element.

If $f(x) = x^2$, image of 2 is
$f(2) = 4$
← Image of 2.

The pre-image of a codomain element are the domain elements that map to it.

ex.) $f(x) = x^2$
$f(-2) = 4$ & $f(+2) = 4$,
+2 & −2 are the pre image of 4.

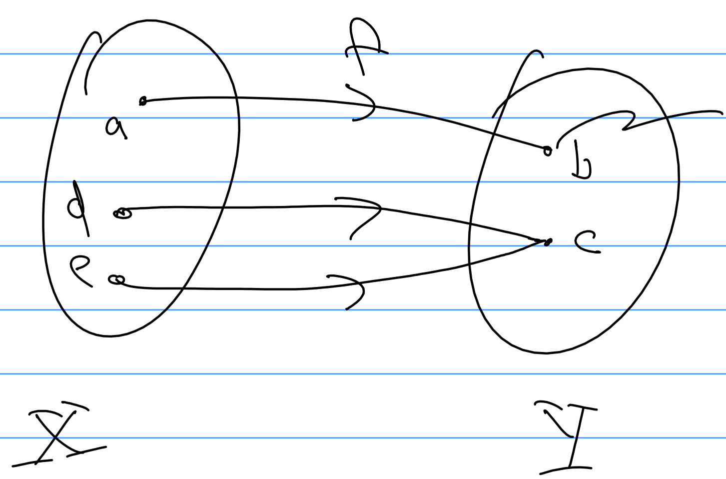


image of a
b = image of a,

d, e = pre image of c

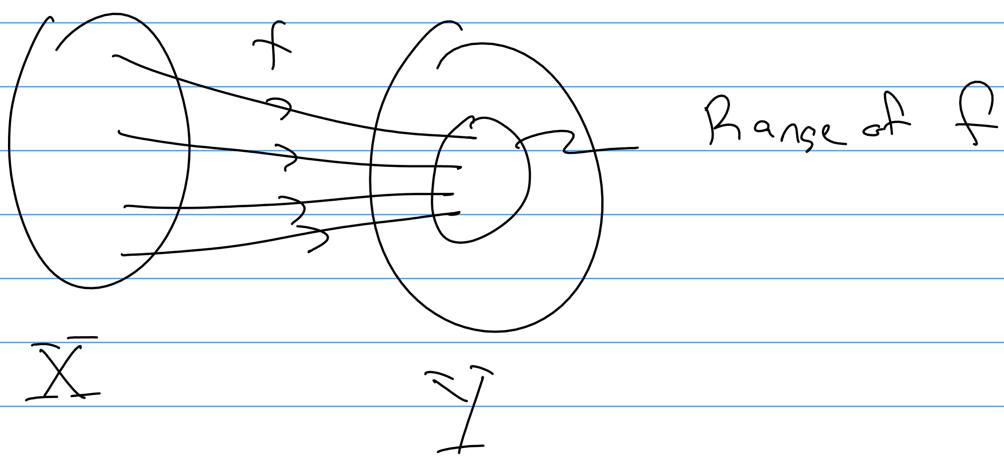
Not all elements in codomain might have a pre-image,

ex.) $f(x) = x^2$. 5 in Y, has no pre-image, if X = "Z"

If S is a subset of X, the image of the subset is $f(S)$

If T is a subset of Y, the pre-image is $f^{-1}(T)$

The image of the entire domain X is called the Range of the function



$f(X)$ can never produce an answer outside of it's range

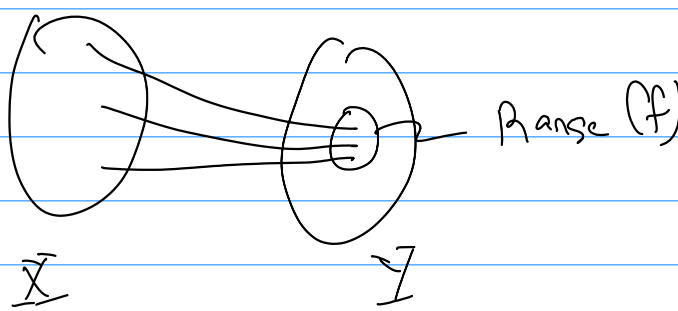
Def: Functions are one-to-one iff every point in X goes to a distinct & unique element in Y.

To show this, you must show that whenever $f(x_1) = f(x_2)$ for any x_1 or $x_2 \in X$, you <u>must</u> have

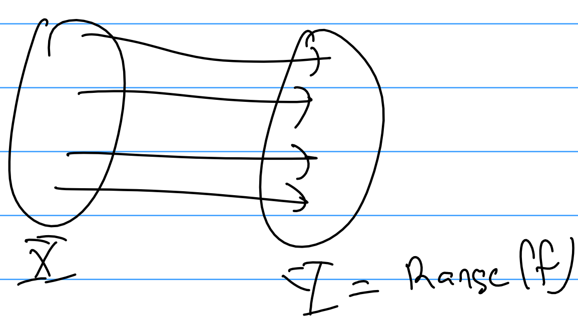
$x_1 = x_2$.

Def: Functions are onto iff every element of Y is the image of some element in X.

$\Rightarrow$ range(f) = Codomain(f)



Not Onto



Is onto

ex.) $f(x) = 2x$, $x \in \mathbb{R}^n$

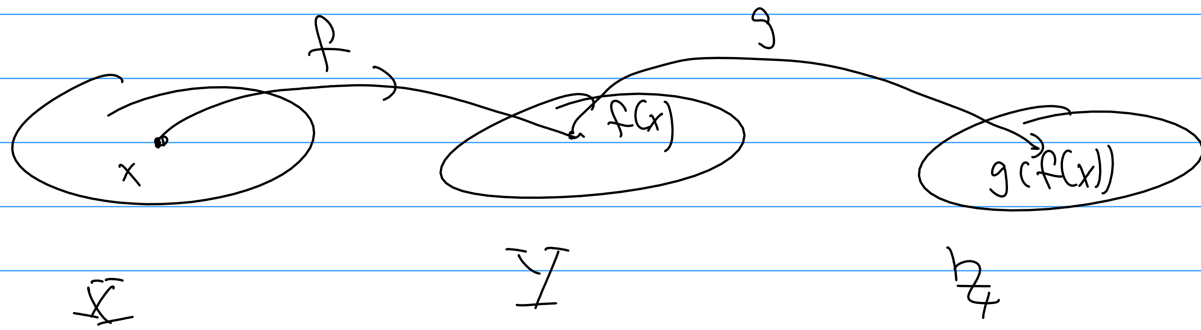
Onto because $x = \frac{1}{2} y$

Function Composition

Let $f: X \to Y$ and $g: Y \to Z$

The Composition is written as

$g \circ f : X \to Z$, or $(g \circ f)(x) = g(f(x))$



Theorem: 1) If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one, then $g \circ f : X \to Z$ is also one-to-one.

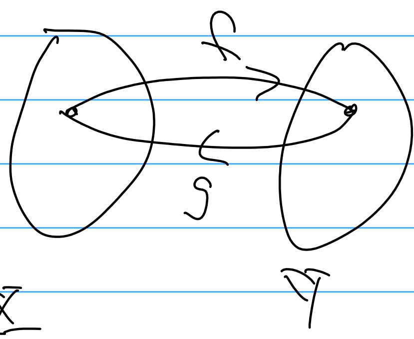
2) If $f: X \to Y$ & $g: Y \to Z$ are both onto, then $g \circ f : X \to Z$ is also onto.

Function Inverse

Functions $f: X \to Y$ and $g: Y \to X$ are inverses of each other if

$(g \circ f)x = y$ and $(f \circ g)y = x$ for any $x \in X$ and $y \in Y$.

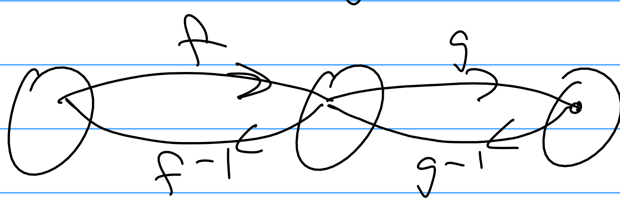
Theorem: $f: X \to Y$ has an inverse $g: Y \to X$ iff f is both one-to-one and onto.



Thm: If g is an inverse of f, then g is the <u>only</u> inverse of f.

Thm: If f & g both have inverses, the inverse of the function composition is

$g \circ f$ is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$



Linear Transformations

Linear transformations are functions on vector spaces that follow two rules:

Let V & W be vector spaces such that $f: V \to W$, f is a linear transformation iff

1) $f(\underline{v}_1 + \underline{v}_2) = f(\underline{v}_1) + f(\underline{v}_2)$

2) $f(a\underline{v}_1) = a f(\underline{v}_1)$

for any $\underline{v}_1, \underline{v}_2 \in V$ & $a \in \mathbb{R}$.

ex.) Look at $f: M_{mn} \to M_{nm}$

$f(\underline{A}) = \underline{A}^T$ One rule: $(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$

Is this linear?

1) $f(\underline{A}+\underline{B}) = (\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T = f(\underline{A}) + f(\underline{B})$

2) $f(c\underline{A}) = (c\underline{A})^T = c\underline{A}^T = c f(\underline{A})$

$\Rightarrow$ Yes, linear

Thm: Let V & W be vector spaces and let $L: V \to W$ be a linear transformation. Let $\underline{0}_V$ be the zero vector in V, $\underline{0}_W$ be the zero vector in W. Then:

1) $L(\underline{0}_V) = \underline{0}_W$

2) $L(-\underline{v}) = -L(\underline{v})$ for all $\underline{v} \in V$

3) $L(a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n) = a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) + \dots + a_n L(\underline{v}_n)$ for all $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n \in V$ & $a_1, a_2, \dots, a_n \in \mathbb{R}$

Proof

1) $L(\underline{0}_V) = L(0\,\underline{0}_V) \overset{\text{from def}}{=} 0\,L(\underline{0}_V) = \underline{0}_W$

2) $L(-\underline{v}) = L(-1\,\underline{v}) = -1\,L(\underline{v}) = -L(\underline{v})$

3) $L(a_1\underline{v}_1 + a_2\underline{v}_2) = L(a_1\underline{v}_1) + L(a_2\underline{v}_2)$

$= a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2)$ Same for higher n.

Thm: Let $L_1 : V_1 \to V_2$ & $L_2 : V_2 \to V_3$

be two linear transformations.

Then $L_2 \circ L_1 : V_1 \to V_3$

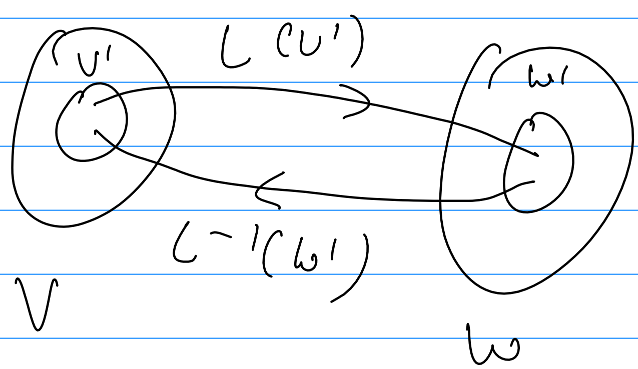
$\Rightarrow (L_2 \circ L_1)(v) = L_2(L_1(\underline{v}))$

is also a linear transformation.

Thm: Let $L : V \to W$ be a linear transformation

1) If V' is a subspace of V, then $L(V')$ is a subspace of W.

2) If W' is a subspace of W, then $L^{-1}(W')$ is a subspace of V.



ex.) Let $L : M_{22} \to \mathbb{R}^3$

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a, 0, c]$$

Linear?

1) $L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

$= L\left(\begin{bmatrix} a_1+a_2 , & b_1+b_2 \\ c_1+c_2 , & d_1+d_2 \end{bmatrix}\right) = [a_1+a_2 , 0, c_1+c_2]$

$= [a_1, 0, c_1] + [a_2, 0, c_2]$

$= L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

2) $L\left(e\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = L\left(\begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}\right) = [ea, 0, ec]$

$= e[a, 0, C] = e\, L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

The Range of L, given by $\{[a, 0, c],$ for $a, c \in \mathbb{R}\}$ forms a subspace of $\mathbb{R}^3$.

A linear transformation is determined by it's actions on the basis of a vector space.

A basis: The minimum set of unique and independent vectors that span the vector space.

ex.) let $\underline{b}_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$ $\underline{b}_2 = \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ $\underline{b}_3 = \begin{bmatrix} -3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$ $\underline{b}_4 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

$B = \{\underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4\}$ form a basis for $\mathbb{R}^4$.

Let $L: \mathbb{R}^4 \to \mathbb{R}^3$ such that

$L(\underline{b}_1) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ $L(\underline{b}_2) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $L(\underline{b}_3) = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ $L(\underline{b}_4) = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$

What is $L\left(\begin{bmatrix} -4 \\ 14 \\ 1 \\ 4 \end{bmatrix}\right) = ?$

Recall that if $\mathcal{B}$ is a basis for $\mathbb{R}^4$, then all vectors in $\mathbb{R}^4$ can be written as

$$k_1 \underline{b}_1 + k_2 \underline{b}_2 + k_3 \underline{b}_3 + k_4 \underline{b}_4 = \underline{v} \in \mathbb{R}^4$$

$$L(\underline{v}) = L(k_1 \underline{b}_1 + k_2 \underline{b}_2 + k_3 \underline{b}_3 + k_4 \underline{b}_4)$$

$$= k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + k_3 L(\underline{b}_3) + k_4 L(\underline{b}_4)$$

$$= k_1 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + k_4 \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

For $\begin{bmatrix} -4 \\ 14 \\ 1 \\ 4 \end{bmatrix} = \underset{\substack{\uparrow \\ k_1}}{2} \underline{b}_1 \underset{\substack{\uparrow \\ k_2}}{-} \underline{b}_2 \underset{\substack{\uparrow \\ k_3}}{+} \underline{b}_3 + \underset{\substack{\uparrow \\ k_4}}{3} \underline{b}_4$

$$\Rightarrow L\left(\begin{bmatrix} -4 \\ 14 \\ 1 \\ 4 \end{bmatrix} \right) = 2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix}$$

Thm: Let $B = \{\underline{b}_1, \underline{b}_2, \underline{b}_3, \ldots, \underline{b}_n\}$ be a basis for vector space V. Let $\underline{w}_1, \underline{w}_2, \underline{w}_3, \ldots$ be any n vectors in space W, there is a unique linear transformation such that $L: V \to W$ such that $L(\underline{b}_1) = \underline{w}_1$, $L(\underline{b}_2) = \underline{w}_2, \ldots,$ $L(\underline{b}_n) = \underline{w}_n$

Next time! This transformation can be written as a matrix-vector product!

$$k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + k_3 L(\underline{b}_3) + k_4 L(\underline{b}_4)$$

$$= \begin{bmatrix} L(\underline{b}_1) & L(\underline{b}_2) & L(\underline{b}_3) & L(\underline{b}_4) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

Column