Review: Linear transformation is defined by the action on the basis of a vector space.

$\mathcal{B} = \{ \underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4 \}$ in $\mathbb{R}^4$

Any vector $\underline{v} \in \mathbb{R}^4$ is

$$\underline{v} = k_1 \underline{b}_1 + k_2 \underline{b}_2 + k_3 \underline{b}_3 + k_4 \underline{b}_4$$

$$L: V \to W : L(\underline{v}) = \underline{k_1} L(\underline{b}_1) + \underline{k_2} L(\underline{b}_2) + \underline{k_3} L(\underline{b}_3) + \underline{k_4} L(\underline{b}_4)$$

Write as a matrix - vector product:

$$\underline{A}_{BC} = [L(\underline{b}_1) \quad L(\underline{b}_2) \quad L(\underline{b}_3) \quad L(\underline{b}_4)]$$

$$= \begin{bmatrix} 3 & 2 & 4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} \leftarrow \text{from Last class}$$

$$\underline{A}_{BC} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = L(\underline{v})$$

$\underline{A}_{BC}$ is the matrix of a linear transformation

Thm: let B be an ordered basis for vector space V, & let C be an ordered basis for space W. For any linear transformation $L: V \to W$, there exists a matrix such that

$$\underline{A}_{BC} [\underline{v}]_B = [L(\underline{v})]_C$$

$\underline{v}$ written in basis B $L(\underline{v})$ written in basis C

Sample Geometric Linear Operators on $\mathbb{R}^3$,

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(1) Reflection (through xy-plane)

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \end{bmatrix}$$

$$L(\underline{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad L(\underline{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad L(\underline{e}_3) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \underline{A}_{BC} = [L(\underline{e}_1)\ L(\underline{e}_2)\ L(\underline{e}_3)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

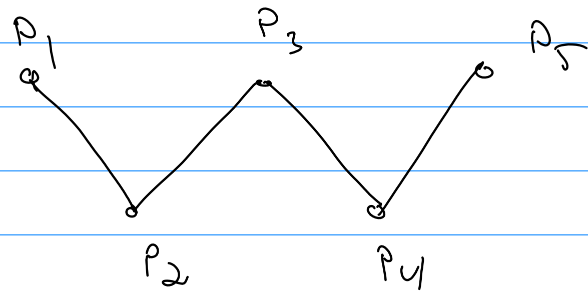
(2) Rotation

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \cos\theta - a_2 \sin\theta \\ a_1 \sin\theta + a_2 \cos\theta \\ a_3 \end{bmatrix}$$

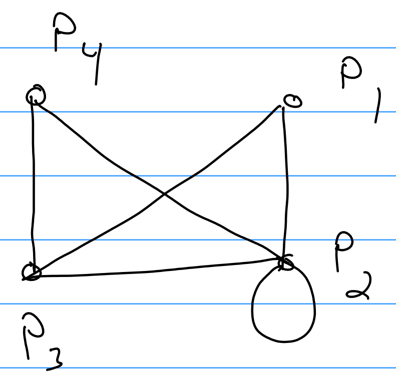
$$\underline{A}_{BC} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Graphs & Graph Theory

Graph → A finite collection of vertices and edges

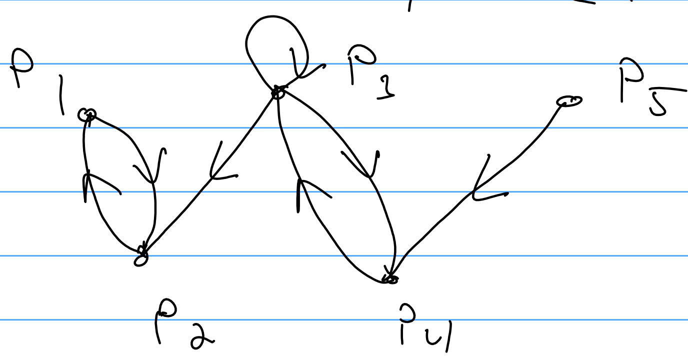


G_1

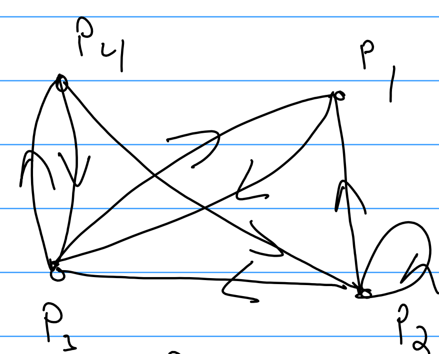


G_2

Directed Graphs: Indicate Direction



D_1



D_2

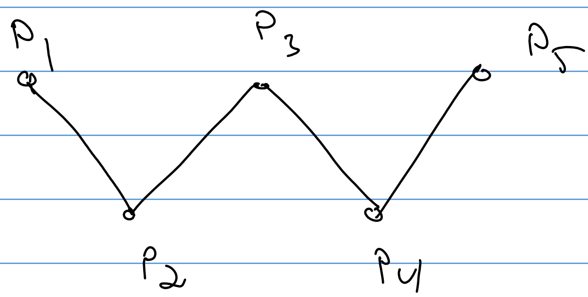
Digraphs

Applications:
- Linguistics → how languages relate
- Chemistry → represent molecules
- Machine learning → relationships
- Computer Science → networking, website, etc,
- Scientific Computing → Meshes

Uses
- Visibility problem
- Routing
- Decomposition / Partitioning
⋮

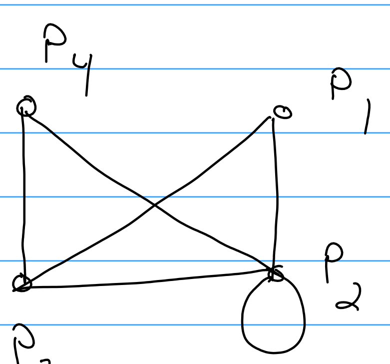
Adjacency Matrix : A matrix whos (i,j) is $\underline{1}$ if there is an edge between P_i & P_j, 0 otherwise,

For a Digraph, $\underline{1}$ if an arrow exists between P_i & P_j, 0 otherwise.



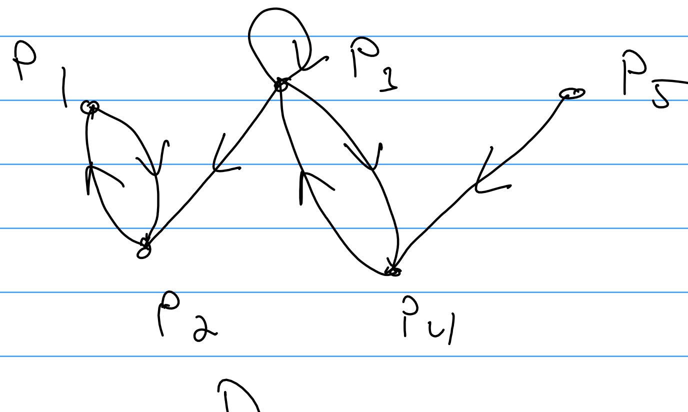
G_{T_1}

$$G_{T_1} : \begin{array}{c|c} & \begin{matrix} P_1 & P_2 & P_3 & P_4 & P_5 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$



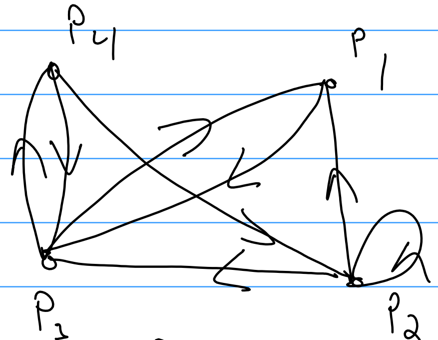
G_{T_2}

$$G_{T_2} : \begin{array}{c|c} & \begin{matrix} P_1 & P_2 & P_3 & P_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



D_1

$$D_1: \begin{array}{c|ccccc} & P_1 & P_2 & P_3 & P_4 & P_5 \\ \hline P_1 & 0 & 1 & 0 & 0 & 0 \\ P_2 & 1 & 0 & 0 & 0 & 0 \\ P_3 & 0 & 1 & 1 & 1 & 0 \\ P_4 & 0 & 0 & 1 & 0 & 0 \\ P_5 & 0 & 0 & 0 & 1 & 0 \end{array}$$



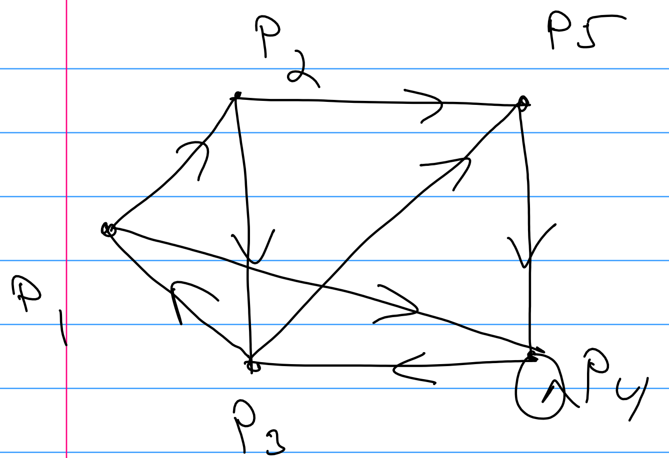
D_2

$$D_2 \quad \begin{array}{c|cccc} & P_1 & P_2 & P_3 & P_4 \\ \hline P_1 & 0 & 0 & 1 & 0 \\ P_2 & 1 & 1 & 1 & 0 \\ P_3 & 1 & 0 & 0 & 1 \\ P_4 & 0 & 1 & 1 & 0 \end{array}$$

Path: A path between P_i & P_j in a graph is a finite sequence of edges:

1.) First edge starts at P_i
2.) Last edge ends at P_j
3.) Each edge after the first begins at the vertex where the prior edge ends.

Length: # of edges to go from P_i to P_j.



$P_1 \to P_2 \to P_5$ length $= 2$

$P_1 \to P_2 \to P_3 \to P_5$ length $= 3$

C_{T_3}

Counting Paths: How many paths of a given length exist between two vertices?

Thm: let $\underline{A}$ be the adjacency matrix for vertices $P_1, P_2, \ldots, P_n$

The # of paths of length K is given by the (i,j) value in

$\underline{A}^K$. (P_i to P_j)

Cullorary: The # of path $\leq K$ from P_i to P_j is given by the sum

$$\underline{A} + \underline{A}^2 + \ldots + \underline{A}^K$$

For G_3 :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

P_4 to P_2 $\qquad$ P_2 to P_4

$$A^2 = \begin{bmatrix} 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

← Length = 2 is minimum for P_2 to P_4

$$A^3 = \begin{bmatrix} 2 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Length = 3 is minimum for P_4 to P_2.

Length = 3 for P_2 to P_4

$P_2 \to P_3 \to P_5 \to P_4$

$P_2 \to P_5 \to P_4 \to P_4$

$P_2 \to P_3 \to P_1 \to P_4$

Markov Chains

How will future states of a system vary over time?

Ex.) 3 banks: A, B, C

Initially, A has 40%, B has 10%, C has 50%

$$P = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.5 \end{bmatrix}$$

In a given year, A retains 50%, 25% each go to B & C. $\begin{bmatrix} 0.5 \\ 0.25 \\ 0.25 \end{bmatrix}$

B retains 66.7%, while 16.7% go to A & C

$$\begin{bmatrix} 0.167 \\ 0.667 \\ 0.167 \end{bmatrix}$$

C: Retain 50%, 25% each to A & B

$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$$

Transition matrix, $\underline{M}$:

$$\underline{M} = \begin{bmatrix} 0.5 & 0.167 & 0.25 \\ 0.25 & 0.667 & 0.25 \\ 0.25 & 0.167 & 0.5 \end{bmatrix}$$

To find the results after 1 year:

$$\underline{P}_1 = \underline{M}\,\underline{P}_0 = \begin{bmatrix} 0.342 \\ 0.292 \\ 0.367 \end{bmatrix} \quad \left(P_0 = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.5 \end{bmatrix} \right)$$

$$\underline{P}_2 = \underline{M}\,\underline{P}_1 = \underline{M}^2\,\underline{P}_0 = \begin{bmatrix} 0.312 \\ 0.372 \\ 0.318 \end{bmatrix}$$

$$\underline{P}_3 = \underline{M}\,\underline{P}_2 = \underline{M}^3\,\underline{P}_0$$

Stochastic Matrix: Square matrix w/ entries that are non-zero and each column sums to 1.

Thm: The product of a finite number of stochastic matrices is a stochastic matrix.

Markov Chain: Distinct States $S_1, \ldots, S_n$:
1.) each element resides in one of the states
2.) elements can move from one state to another
3.) Probabilities of move is fixed.

In ex.) States = banks A, B, C
elements = Investors

Thm: After n steps, $n \geq 1$, the probability is given by

$$\underline{P}_n = \underline{M}^n \underline{P}_0$$

$\to$ Given $\underline{M}$ & $\underline{P}_0$, all future steps are determined.

What happens if $n \to \infty$?

$$\lim_{k \to \infty} \underline{P}_k = \underline{P}_\infty = \lim_{k \to \infty} \underline{M}^k \underline{P}_0$$

In our example:

$$\lim_{k \to \infty} \underline{M}^k = \underline{M}_\infty = \begin{bmatrix} 0.286 & 0.286 & 0.286 \\ 0.429 & 0.429 & 0.429 \\ 0.286 & 0.286 & 0.286 \end{bmatrix}$$

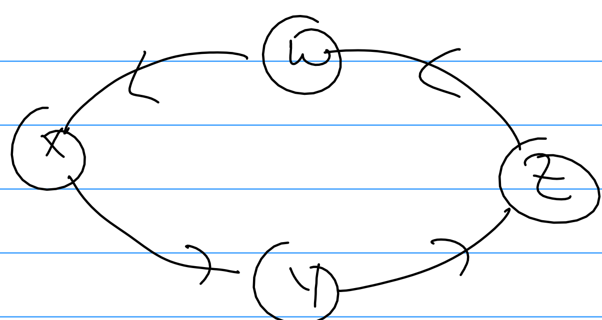
Then, $\underline{M}_\infty \underline{P}_0 = \begin{bmatrix} 0.286 \\ 0.429 \\ 0.286 \end{bmatrix}$ ← equilibrium probabilities

Written as $\underline{P}_\infty = \underline{M}\,\underline{P}_\infty$ ← Called a fixed point $(f(x) = x)$

If I know $\underline{M}$, I don't need $\underline{M}_\infty$

Solve $\underline{M}\,\underline{x} = \underline{x}$ ← Eigen problem
w/ $\|\underline{x}\| = 1$
if a unique solution exists.

Ex.)



Train Station

12 initial trains: w → 6, x → 3, y → 2, z → 1

$\underline{P}_0 = \begin{bmatrix} 0.5 \\ 0.25 \\ 0.167 \\ 0.083 \end{bmatrix}$

$$\underline{M} = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} w \\ x \\ y \\ z \end{array}$$

$\underline{M}^2, \underline{M}^3, \ldots,$ results in 4 states,

$$\begin{bmatrix} 0.5 \\ 0.25 \\ 0.167 \\ 0.083 \end{bmatrix} \text{ or } \begin{bmatrix} 0.083 \\ 0.5 \\ 0.25 \\ 0.167 \end{bmatrix} \text{ or } \begin{bmatrix} 0.167 \\ 0.083 \\ 0.5 \\ 0.25 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 0.25 \\ 0.167 \\ 0.083 \\ 0.5 \end{bmatrix} \ldots$$

To have an equilibrium state, need a Regular Matrix, $\underline{R}$,

$\underline{R}$ is regular iff $\underline{R}$ is a stochastic matrix and for some $k \geq 1$,

$\underline{R}^k$ has all non-zero entries,

Thm: If transition matrix $\underline{M}$ is regular

1) $\lim_{k \to \infty} \underline{M}^k = \underline{M}_\infty$

2.) $\underline{M}_\infty$ has all positive entries & every column is identical.

3.) For any initial state vector $\underline{P}_0$, the Markov chain limit is $\underline{P}_\infty$

4.) $\underline{P}_\infty$ equals any column of $\underline{M}_\infty$

5.) $\underline{P}_\infty$ is the unique fixed point $\underline{P}_\infty = \underline{M}\,\underline{P}_\infty$

Linear Equations & Systems

Many physical systems can be represented as linear transformations,

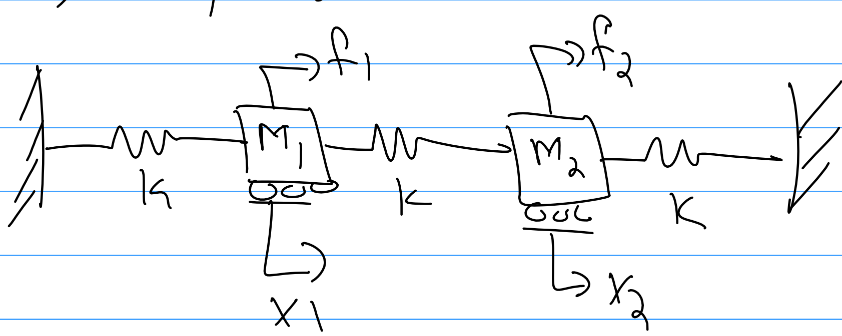
written as $\underline{A}\,\underline{x} = \underline{b}$

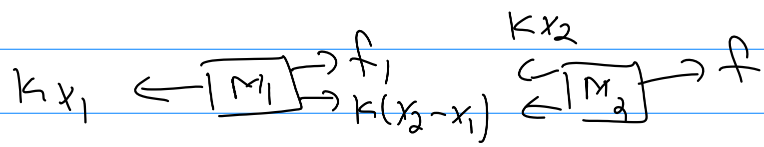
Generic 2x2 system:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Usually unknown}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\text{Usually known}}$$

Ex1) Spring-Mass





$\Sigma F = 0$

$$f_1 + k(x_2 - x_1) - kx_1 = 0$$

$$f_2 - kx_2 - k(x_2 - x_1) = 0$$

$$\Rightarrow \quad +2kx_1 - kx_2 = f_1$$
$$-kx_1 + 2kx_2 = f_2$$

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

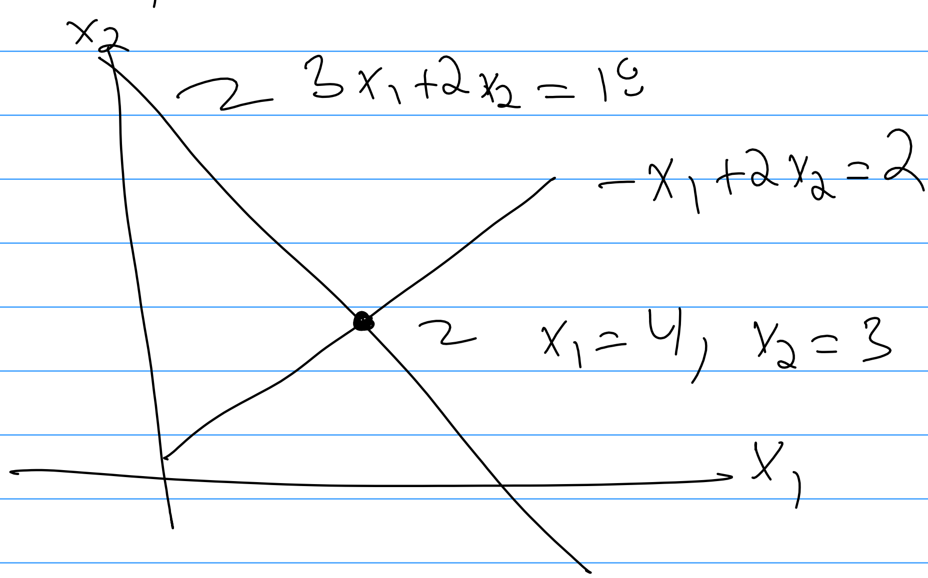
Given f_1, f_2, determine equilibrium,

Solution looks like $\underline{x} = \underline{A}^{-1}\underline{b}$

Existance & Uniquness.

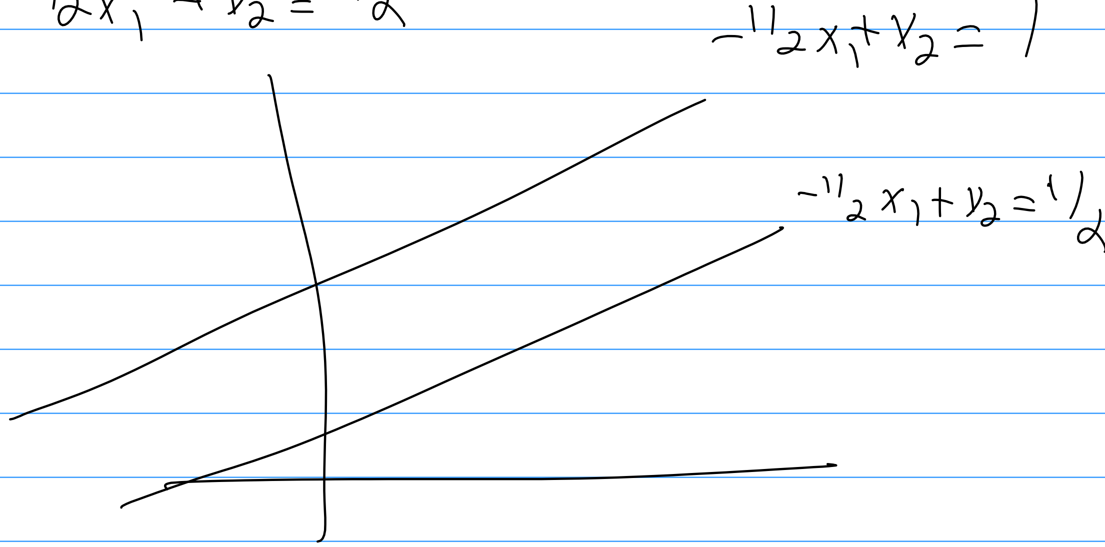
- Does a solution exist?
- Is it unique?

ex) $-x_1 + 2x_2 = 2$
$3x_1 + 2x_2 = 18$



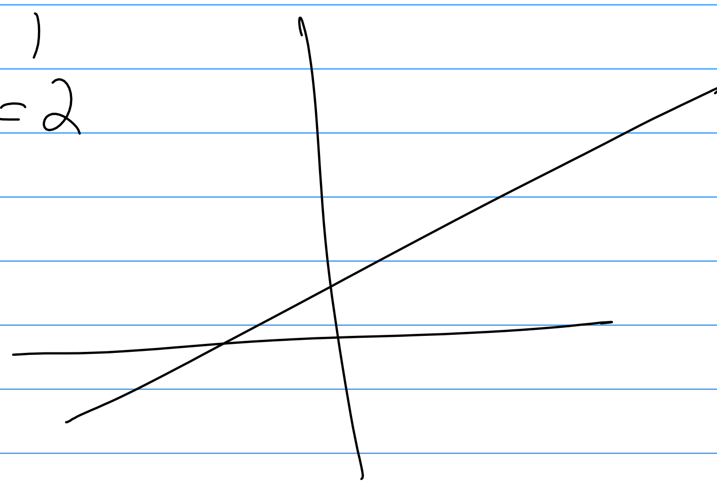
Unique Solution exists.

ex1) $-\frac{1}{2}x_1 + x_2 = 1$
$-\frac{1}{2}x_1 + x_2 = \frac{1}{2}$



parallel lines → no solutions

ex) $-\frac{1}{2}x_1 + x_2 = 1$
$-x_1 + 2x_2 = 2$



Two overlapping lines → ∞ solutions

Only possibilities are: 1, 0, or ∞
solutions.