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6.2 (a) P (Dave fails the quiz) $= p = 1/4$

Probability that Dave fails exactly two of the next six quizzes will be a Bernoulli process with the probability of one event (dave failing the quiz), $p = \frac{1}{4}$

$\therefore$ Probability that Dave fails exactly two of the next six quizzes is a binomial with parameters p and n. as $1/4$ & 6 respectively.

$$\therefore \; p_S(K) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{6}{2} p^2 (1-p)^{6-2} = \frac{6!}{4!\,2!}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4$$

$$= \frac{1215}{4096} = 0.2966$$

(b) Expected number of quizzes = Expected number of quizes failed + Expected number of quizzes passed. .. eq i)

Number of quizzes upto third failure is defined by Pascal random variable, Y_K of order three and $p = \frac{1}{4}$.

$$\therefore \; E[Y_K] = \frac{k}{p} = \frac{3.}{1/4} = 12$$

$\therefore$ From eq i),

Expected number of quizzes passed $= 12 - 3$

$= 9$ Ans.

(c) Given question says Dave fails exactly 1 quiz in first seven quizzes (A), quiz eight is a failure for Dave (B) and quiz nine is also a failure (C).

$\therefore$ Desired probability = $P(A \cap B \cap C)$

Since A, B and C are independent

$$= P(A) \cdot P(B) \cdot P(C)$$

$$= \binom{7}{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{7-1} \cdot \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right)$$

$$= \frac{7!}{6!}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{6}$$

$$= \frac{5103}{262144} = 0.0195$$

(d) If A is the event that Dave fails two quizzes in a row before he passes two quizzes in a row.

P: Dave passes the quiz $\quad P(P) = \frac{3}{4}$

F: Dave fails the quiz $\quad P(F) = \frac{1}{4}$

$\therefore P(A) = P(FF \cup PFF \cup FPFF \cup PFPFF \cup FPFPFF \cup PFPFPFF \cup \ldots)$

$= P(FF) + P(PFF) + P(FPFF) + P(PFPFF) + P(FPFPFF) + P(PFPFPFF) + \cdots$

~~(Since~~

$$= \left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \cdots$$

$$= \left[\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \cdots\right]$$

$$+\left[\frac{3}{4}\left(\frac{1}{4}\right)^2+\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2+\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2+\cdots\right]$$

Above are infinite Geometric series with common ratio $\frac{3}{16}$

$$\therefore P(A) = \frac{\left(\frac{1}{4}\right)^2}{1-\frac{3}{16}} + \frac{\frac{3}{4}\left(\frac{1}{4}\right)^2}{1-\frac{3}{16}} = \frac{7}{52} \text{ Ans.}$$

6.3 (a) Probability that the task is from user 1 in a particular slot $p_1 = P_{1/B} P_B$ [Here the trials are associated with busy slots) $= \frac{2}{5}\cdot\frac{5}{6} = \frac{1}{3}$

$\therefore$ P(task from user 1 is executed for the first time during the 4^{th} slot) $= p_1(1-p_1)^3 = \frac{1}{3}\left(\frac{2}{3}\right)^3 = 0.0988$

(b) This is a Bernoulli process which has a property of fresh start. We can consider a fresh start from slot 11 and ignore the conditioning information of the first 10 slots

Thus,

P(slot 11 is busy and slot 12 is idle) $= \frac{5}{6}\cdot\frac{1}{6} = 0.139$

(c) $P_1 = \frac{1}{3}$

Time of 5^{th} ~~slot~~ task from user 1 is a Pascal random variable, Y_k with $k=5$ and $p_1 = \frac{1}{3}$.

$$\therefore E[Y_k] = \frac{k}{P_1} = \frac{5}{1/3} = 15$$

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(d) In this case, we consider only busy slots upto and including 5^{th} task from user 1. Thus, here k^{th} arrival time will be defined by the Pascal's PMF Y_k with $P_{1|B} = \frac{2}{5}$

$$\therefore E[Y_5] = \frac{k}{P} = \frac{5}{2/5} = \frac{25}{2} = 12.5$$

(e) If we substract the number of busy slots occupied by user 1 (i.e 5) from the number of busy slots until the 5^{th} task from user 1 until the 5^{th} task from user 1, we get the number of busy slots occupied by user 2 until the 5^{th} task from user 1. This will be the number of tasks from user 2 until the 5^{th} task from user 1 (A)

We can write this as a Pascal Random variable, with $k = 5$ and $P_{1|B} = 2/5$

Thus, PMF is given by,

$$P_{Y_k}(t) = \binom{t-1}{k-1} P_{1|B}^{k} (1 - P_{1|B})^{t-k} \qquad t = k, k+1, \ldots$$

Putting $k = 5 \Rightarrow P_B(t) = \binom{t-1}{4} \left(\frac{2}{5}\right)^5 \left(1 - \frac{2}{5}\right)^{t-5}, \quad t = 5, 6 \ldots$

$A = B - 5 \quad \therefore P_A(t) = P_B(t+5)$

$$\therefore P_A(t) = \binom{t+4}{4} \left(\frac{2}{5}\right)^5 \left(1 - \frac{2}{5}\right)^t, \quad t = 0, 1 \ldots$$

Using the formulas for mean and variance of the Pascal random variable B, we have

$$E[A] = E[B] - 5 = \frac{25}{2} - 5 = 7.5$$

$$var[A] = var[B] = \frac{5(1 - 2/5)}{(2/5)^2} = \frac{3}{0.16} = 18.75$$

6.10 (a) This is a Poisson process with 0 arrivals in 2 hours

$\lambda = 0.6$ per hour.

$$\therefore \quad P_{N_\tau}(k) = P(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}$$

$\therefore k=0$, $\tau = 2$

$$= e^{-0.6\times 2}\frac{(0.6\times 2)^0}{0!}$$

$$= e^{-0.6\times 2} = 0.301$$

(b) For this to happen, he should not catch fish in first 2 hours (0 successes in first two hours) and at least one fish between 2 and 5 hours. These are disjoint events and the probability that the total time he spends fishing is between two and five hours is given by,

$$P(0,2)(1-P(0,3)) = e^{-0.6\times 2}(1-e^{-0.6\times 3}) = 0.251$$

(c) The only case when the fisherman catches at least 2 fish is when he fishes for exactly 2 hours. Thus, the probability that he catches at least two fish is same as the probability that the number of fish caught in first two hours is at least two, i.e.,

$$\sum_{k=2}^{\infty} P(k,2) = 1 - P(0,2) - P(1,2) = 1 - e^{-0.6\times 2} - (0.6\times 2)e^{-0.6\times 2}$$

$$= 0.337$$

(d) Expected number of fish caught = Expected number of fish caught in first two hours + Expected number of fish caught after first two hours if 0 fish is caught in first two hours.

Expected no. of fish caught in first two hours $= \lambda\tau$
$= 2\lambda = 1.2$

There are two possibilities after two hours -

1. he quits in which case 0 fish is caught after 2 hours.
2. he catches 1 fish in which case 0 fish is caught in first two hours.

$\therefore$ E[event 2 ~~hap~~] = P[0 fish caught in 1st two hour] = $P(0,2) = 0.301$

$\therefore$ Ans. $1.2 + 0.301 = 1.501$

(e) We know, Poisson process is memoryless
Thus, expected time spent to catch 1 fish after 4 hours, Can be found by using the exponential random variable with parameter λ.

Thus, $E[T] = \frac{1}{\lambda} = \frac{1}{0.6} = 1.667$

$\therefore$ Ans. $4 + 1.667 = 5.667$

6.14 (a) T : time until the first bulb failure
A : First bulb is of type A $\quad P(A) = \frac{1}{2}$
B : First bulb is of type B $\quad P(B) = \frac{1}{2}$

using total expectation theorem,

$$E[T] = E[T|A]\,P[A] + E[T|B]\,P[B] = \frac{1}{\lambda_A}\cdot\frac{1}{2} + \frac{1}{\lambda_B}\cdot\frac{1}{2}$$

$$= 1\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{2}{3} \text{ Ans.}$$

(b) If C is the event of no bulb failure before time t. We can use total probability theorem as,

$$P(C) = P(C|A)P(A) + P(C|B)P(B)$$

$$P(C|A) = e^{-t}$$

$$P(C|B) = e^{-3t}$$

$\therefore$ Ans. $\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$

(c) Using Bayes' Rule,

$$P(A|C) = \frac{P(C|A)P(A)}{P(C)}$$

$$= \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}} = \frac{1}{1+e^{-2t}}$$

(d) $E[T^2] = E[T]^2 + \mathrm{var}[T]$. Because this is second moment of an exponential random variable with parameter λ.

$$= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$$

Conditional form of total expectation theorem,

$$E[T^2] = E[T^2|A]P(A) + E[T^2|B]P(B) = 2\cdot\frac{1}{2} + \frac{2}{9}\cdot\frac{1}{2} = \frac{10}{9}$$

$$\therefore \mathrm{var}(T) = E[T^2] - (E[T])^2$$

$$= \frac{10}{9} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} \quad \text{Ans.}$$

(e) This can be understood as the event that out of first 11 bulbs, exactly 3 were of type A as well as 12th bulb was of type A.

Thus, $P = \binom{11}{3}\left(\frac{1}{2}\right)^{11} \cdot \left(\frac{1}{2}\right)$

$= \binom{11}{3}\left(\frac{1}{2}\right)^{12}$

(f) Probability that exactly 4 type-A bulbs have failed upto and including the 12^{th} bulb failure = probability that out of first 12 bulbs, exactly 4 were type A $= \binom{12}{4}\left(\frac{1}{2}\right)^{12}$

(g) PDF of the time between failures is $\frac{e^{-x} + 3e^{-3x}}{2}, x \geq 0$

Associated transform is given by,

$$\frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right)$$

The times between successive failures are independent with the time until the

$\therefore$ the transform associated with the time until the 12^{th} failure is given by

$$\left[\frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right)\right]^{12}$$

(h) Y: total period of illumination provided by the first two type-B bulbs.

This has an Erlang distribution of order 2.

PDF:
$$f_Y(y) = 9ye^{-3y}, \quad y \geq 0$$

If T is the period of illumination provided by first type-A bulb. Its PDF is

$$f_T(t) = e^{-t}, \quad t \geq 0$$

For $T<4$

$P(T<4 \mid Y=y) = 1-e^{-y}, \quad y \geq 0.$

$\therefore P(T<4) = \int_0^\infty f_Y(y)\, P(T<4 \mid Y=y)\, dy$

$= \int_0^\infty 9ye^{-3y}(1-e^{-y})\, dy = \frac{7}{16}$

Let T_1^A be period of illumination of first Type A bulb

Let T_1^B & T_2^B be period of illumination of 1^{st} and 2^{nd} type B bulb respectively.

for $\{T_1^A < T_1^B + T_2^B\}$

$P(T_1^A < T_1^B + T_2^B) = P(T_1^A < T_1^B) + P(T_1^A \geq T_1^B)\, P(T_1^A < T_1^B + T_2^B \mid T_1^A \geq T_1^B)$

$= \frac{1}{1+3} + P(T_1^A \geq T_1^B)\, P(T_1^A - T_1^B < T_2^B \mid T_1^A \geq T_1^B)$

$= \frac{1}{4} + \frac{3}{4} P(T_1^A - T_1^B < T_2^B \mid T_1^A \geq T_1^B)$

Given event, $T_1^A \geq T_1^B$ and using memoryless ness remaining time $T_1^A - T_1^B$ is exponentially distributed

$P(T_1^A - T_1^B < T_2^B \mid T_1^A \geq T_1^B) = P(T_1^A < T_2^B)$

$= P(T_1^A < T_2^B) = \frac{1}{4}$

Ans. $\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16}$

(i) V : total period of illumination provided by type B bulbs

N : No. of light bulbs

x_i : period of illumination from the i^{th} type B bulb

$V = Y_1 + \dots + Y_N$

N - binomial Random variable with $n = 12$, $p = \frac{1}{2}$

$E[N] = 6$, $var(N) = 12 \cdot \frac{1}{2} \cdot \frac{1}{2} = 3$

$E[X_i] = \frac{1}{3}$ $\quad var(X_i) = \frac{1}{9}$

$E[V] = E[N] \cdot E[X_i] = 2$

$var(V) = var(X_i) \cdot E[N] + E[X_i]^2 \, var(N)$

$= \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1$

(j) $E[T|D] = t + E[T-t \,|\, D \cap A] P[A|D) + E[T-t \,|\, D \cap B] P(B|D)$

$= t + 1 \cdot \frac{1}{1+e^{-2t}} + \frac{1}{3}\left(1 - \frac{1}{1+e^{-2t}}\right)$

$= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1+e^{-2t}}$

<u>other Problems</u>

1. $\lambda = 0.4$

$\{N(t), t \in [0, \infty)\}$

(a) P (No of arrival in (2,4])

$= P(N(4) - N(2) = 0)$

$= P(N(2) = 0) = \frac{e^{-0.4 \times 2} \times (0.4 \times 2)^0}{0!} = e^{-0.8} = 0.449$

(b) P(1 arrival in (0,1] , (3,5])

$P(N(1) - N(0) = 1) \cdot P(N(5) - N(3) = 1)$

$= \dfrac{e^{-0.4} \times (0.4 \times 1)^1}{1!} \times \dfrac{e^{-0.8} \times 0.8}{1}$

$= 0.26812 \times 0.35948 = 0.09638$

(c) P(1 arrival in (0,1] & 3 arrival in (0,5])

$(0,1] \cap (0,5] = (0,5]$

Let X, Y be no. of arrival in $(0,1]$ & $(1,5]$ respectively

$X \sim \text{Poisson}(\lambda, 1)$

$Y \sim \text{Poisson}(\lambda, 4)$

$P(A) = P(X = 1 \ \& \ X + Y = 3)$

$= P(X = 1) \ \& \ P(Y = 2)$

$= P(N(1) = 1) \cdot P(N(4) = 2)$

$= \dfrac{e^{-0.4 \times 1} \times 0.4}{1} \times \dfrac{e^{-0.4 \times 4} \times (0.4 \times 4)^2}{2!}$

$= 0.06925$

2. $N(t) = N_1(t) + N_2(t)$

$N(t) \sim \text{Poisson}(\lambda_1 t + \lambda_2 t)$

$N(t) \sim \text{Poisson}(3t)$

$\therefore \lambda = 3$

P(2 arrivals in [0,1] and 3 arrivals in (1,2]

$$= \frac{(3\times1)^2 e^{-3\times1}}{2!} \times \frac{(3\times1)^3 e^{-3\times1}}{3!}$$

$$= \frac{9}{2} e^{-3} \times \frac{9}{2} e^{-3} = \frac{81}{4} e^{-6} = 0.0501 \text{ Ans.}$$

(b) $N(1) = 2$

$$P(N_1(1) = 1 \mid N(1) = 2)$$

$$= \frac{P(N_1(1) = 1,\ N(1) = 2)}{P(N(1) = 2)}$$ (By conditional probability)

$$= \frac{P(N_1(1) = 1,\ N_2(1) = 1)}{(P(N(1)) = 2)}$$ (if $N_1(1) = 1$, & $N(1) = 2$ then $N_2(1) = 1$)

$$= \frac{P(N_1(1) = 1) \times P(N_2(1) = 1)}{P(N(1) = 2)}$$ N_1 & N_2 are independent

$$= \frac{\frac{(1\times1)^1 e^{-1}}{1!} \times \frac{(2\times1)^1 e^{-2}}{1!}}{\frac{(3\times1)^2 \times e^{-3}}{2!}} = \frac{4}{9} \text{ Ans.}$$