Solution Methods for $\underline{A}\,\underline{x} = \underline{b}$

$\underline{A}\,\underline{x} = \underline{b}$ might arise from linear system of equations, regression, etc.

① LU-Decomposition : $\underline{A} = \underline{L}\,\underline{U}$ $\quad$ $\underline{L}$ = lower triangular
$\underline{U}$ = Upper triangular

To get $\underline{A} = \underline{L}\,\underline{U}$, used Gaussian elimination

$\underline{A}\,\underline{x} = \underline{b}$	
$\underline{L}\,\underline{U}\,\underline{x} = \underline{b}$	Get LU Decomp
$\underline{U}\,\underline{x} = \underline{L}^{-1}\underline{b}$	Forward Substitution
$\underline{x} = \underline{U}^{-1}\underline{L}^{-1}\underline{b}$	Backward Sub.

② QR Decomp : $\underline{A} = \underline{Q}\,\underline{R}$ $\quad$ $\underline{Q}^T\underline{Q} = \underline{I}$
$\underline{R}$ = Upper triangular

To get QR : Classical Gram-Schmidt
Modified G-S
Householder

$\downarrow$ more stable & expensive

$\underline{A}\,\underline{x} = \underline{b}$	
$\underline{Q}\,\underline{R}\,\underline{x} = \underline{b}$	Get QR Decomp.
$\underline{R}\,\underline{x} = \underline{Q}^T\underline{b}$	Orthogonalize
$\underline{x} = \underline{R}^{-1}\underline{Q}^T\underline{b}$	Forward Sub

(3) Singular Value Decomp.:

$$\underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$$

$\underline{U}^T \underline{U} = \underline{U}\,\underline{U}^T = \underline{I}$

$\underline{V}^T \underline{V} = \underline{V}\,\underline{V}^T = \underline{I}$

$\underline{\Sigma}$ = Diagonal matrix

Methods: Golub - Kahan, Lawson - Hanson - Chan

$\underline{A}\,\underline{x} = \underline{b}$

$\underline{U}\,\underline{\Sigma}\,\underline{V}^T \underline{x} = \underline{b}$ Get SVD

$\underline{\Sigma}\,\underline{V}^T \underline{x} = \underline{U}^T \underline{b}$ Due to $\underline{U}^T\underline{U} = \underline{I}$

$\underline{V}^T \underline{x} = \underline{\Sigma}^{-1} \underline{U}^T \underline{b}$ Since $\underline{\Sigma}^{-1}$ is cheap

$\underline{x} = \underline{V}\,\underline{\Sigma}^{-1} \underline{U}^T \underline{b}$ Since $\underline{V}\,\underline{V}^T = \underline{I}$

	Methods
Increasing Stability & Increasing Cost ↓	LU - Gaussian
	QR - Classical G-S
	QR - Modified G-S
	QR - Householder
	SVD - G-K / LHC

Note: Some automatically solve the normal equations

$$\underline{A}^T \underline{A}\,\underline{x} = \underline{A}^T \underline{b}$$

Scalar Differential Equation Basics

A differential equation is simply one which involves one or more derivatives.

Independent variables are the "input"
Dependent variables are the "output"

$\frac{dy}{dt} = t^2$ → $y(t)$ is the unknown fcn
↑ Independent variable (t)
Dependent variable (y)

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ → $u(x,t)$ is the unknown fcn
Independent variables (x, t)
Dependent var. (u)

Focus on scalar equations: the dependent variable (when evaluated), results in a scalar.

Classification:

- Ordinary Differential Eq (ODE): If there is one independent variable
- Partial Differential Eq (PDE): If there are two or more independent variables

- Order: The highest Derivative in the ODE or PDE

- Homogeneous Equation: If the terms which do <u>not</u> include the Dependent variable are zero.

  If not homogeneous, then called non-homogeneous.

- Linear / non-linear: Any differential equation where multiple solutions obey the superposition principle are linear.
  Otherwise non-linear.

  If $y_1(t)$ & $y_2(t)$ are both solutions, then if $c_1 y_1(t) + c_2 y_2(t)$ is also a solution $\rightarrow$ linear.

Notation Note: for simplicity we write

$$\frac{du}{dt} = u_t \quad , \quad \frac{\partial^2 u}{\partial t^2} = u_{tt} \quad , \quad \frac{\partial^3 u}{\partial x \, \partial t^2} = u_{xtt} \quad , \ldots$$

For ODEs, write $\frac{du}{dt} = u_t = \dot{u}$, $\frac{d^2 u}{dt^2} = u_{tt} = \ddot{u}$

ex.)

Type	Eq	Dep Var	Ind Var	Order	Homo	Linear
ODE	$y_{xxx} + y y_{xx} + x = 0$	y	x	3		
ODE	$x_{yyy} + x x_{yy} + x = 0$	x	y	3	✓	
PDE	$u_{xx} + u_{yy} = u_{tt}$	u	x, y, t	2	✓	✓
PDE	$u_{xx} + u_{yy} = 5$	u	x, y	2		✓

Another check on homogeneous:
If you set the dependent variable to zero, does the equation equal zero?

Operators

Let $\phi(x,y,z)$ be a scalar field,

$\underline{u}(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ be a vector field.

(1) Gradient : ∇ = Derivative w.r.t. x-, y-, and z-directions

$$\nabla\phi = \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} \qquad \nabla\underline{u} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

Gradient increases the dimension of the object.

(2) Divergence : $\nabla\cdot(\ )$ Not applicable to scalar fields

$\nabla\cdot\phi$ Does not exist

$$\nabla\cdot\underline{u} = [\partial_x \ \ \partial_y \ \ \partial_z]\begin{bmatrix} u \\ v \\ w \end{bmatrix} = u_x + v_y + w_z$$

(3) Laplacian : $\nabla \cdot \nabla = \nabla^2 \; (= \Delta)$

$$\nabla \cdot \nabla \phi = (\partial_x, \partial_y, \partial_z) \cdot (\phi_x, \phi_y, \phi_z)$$

$$= \phi_{xx} + \phi_{yy} + \phi_{zz}$$

$$\nabla \cdot \nabla \underline{u} = \begin{bmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{bmatrix}$$

(4) Curl : $\nabla \times (\;)$ Only vectors & above

$$\nabla \times \underline{u} = \det\left(\begin{bmatrix} \hat{e}_i & \hat{e}_j & \hat{e}_k \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{bmatrix} \right) = \varepsilon_{ijk} \nabla \underline{u}$$

$\varepsilon_{ijk} =$ Levi-Civita Operator

$$= \begin{bmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{bmatrix}$$

In 2D: $\nabla \times \underline{u} = (v_x - u_y) \underline{e}_z$

These are used to build (primarily) PDEs.

Operation Output

Operation	Scalar input	Vector Input
Gradient	Vector	Tensor
Divergence	Not defined	Scalar
Laplacian	Scalar	Vector
Curl	Not defined	Vector

ex1) let $\underline{u} = \begin{bmatrix} xy \\ \sin x \end{bmatrix}$, $\phi = zx^2 + 2y + e^{-z}$

$$\nabla \phi = \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} = \begin{bmatrix} 2xz \\ 2 \\ -e^{-z} \end{bmatrix}$$

$$\nabla \cdot \underline{u} = (xy)_x + (\sin x)_y = y$$

$$\nabla \times \underline{u} = \left((\sin x)_x - (xy)_y\right)\underline{e}_z = (\cos x - x)\underline{e}_z$$

$$\nabla^2 \phi = (zx^2)_{xx} + (2y)_{yy} + (e^{-z})_{zz} = 2z + e^{-z}$$

$$\nabla^2 \underline{u} = \begin{bmatrix} (xy)_{xx} + (xy)_{yy} \\ (\sin x)_{xx} + (\sin x)_{yy} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin x \end{bmatrix}$$

Solution of Linear ODEs.

In general, the complete solution to a linear ODE can be written as the sum of a solution to the homogeneous part of the ODE plus the solution to the non-homogeneous part.

Let $g(x(t)) = f(t)$ be the linear ODE

ex.) if $\ddot{x} + 2\dot{x} - 3 = \sin(t)$

$$g(x(t)) = \ddot{x} + 2\dot{x} - 3 \qquad f(t) = \sin(t)$$

Let $x_h(t)$ solve the homogeneous part of $g(x(t)) = f(t)$:

$$\Rightarrow g(x_h(t)) = 0$$

Let $x_p(t)$ solve the non-homogeneous part:

$$\Rightarrow g(x_p(t)) = f(t)$$

Then the complete solution is

$$x(t) = x_h(t) + x_p(t)$$

$x_p(t)$ called the particular solution.

Why? Recall that linear also means

$$f(a+b) = f(a) + f(b)$$

$$g(x(t)) = g(x_h + x_p) = g(x_h) + g(x_p)$$

$$= 0 + f(t)$$

Now, in general there will be multiple homogeneous solutions.

A linear ODE of order n will have n homogeneous solutions.

Let $x_1(t), x_2(t), \ldots, x_n(t)$ be the many homogeneous solutions.

Then $x_h(t) = C_1 x_1(t) + C_2 x_2(t) + \ldots + C_n x_n(t)$

To find $C_1, C_2, \ldots, C_n$ we need n additional conditions.

If the conditions on the dependent variables all occur at the same time → Initial Conditions & the ODE is an initial value problem (IVP)

If the conditions spatially → boundary conditions, the ODE is a boundary value problem (BVP)

ex.) $xy_{xx} + y^2 = 0$ w/ $y = 1$ at $x = 0$
$y_x = 0$ at $x = 0$

An IVP

ex.) $x_{tt} + x = 0$ w/ $x = 1$ at $t = 0$
$x = 1$ at $t = 1$

A BVP

To simplify things, look at 2^{nd}-order ODEs.

$a x_{tt} + b x_t + c x = a\ddot{x} + b\dot{x} + cx = f(t)$ w/ 2 associated conditions.

1. Find $x_h(t)$: Solution to $a\ddot{x}_h + b\dot{x}_h + c x_h = 0$

Equations of the form $a\dot{x}_h + b x_h = 0$ have solutions of the form
$x_h = Ae^{\alpha t}$

Try $x_h(t) = Ce^{\alpha t} \to \dot{x}_h = \alpha C e^{\alpha t} = \alpha x_h$

$\ddot{x}_h = \alpha^2 C e^{\alpha t} = \alpha^2 x_h$

$a\ddot{x}_h + b\dot{x}_h + cx_h = 0$

$a\alpha^2 x_h + b\alpha x_h + cx_h = 0$

$(a\alpha^2 + b\alpha + c)\, x_h = 0$

We want the non-trivial solution, $x_h \neq 0$

$\Rightarrow a\alpha^2 + b\alpha + c = 0$

$\Rightarrow \alpha = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ form the α values that make $e^{\alpha t}$ a homogeneous sol.

Three Cases:

① $b^2 - 4ac > 0 \rightarrow$ 2 real & distinct roots α_1, α_2.

$\Rightarrow x_h(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}$

② $b^2 - 4ac = 0 \Rightarrow$ One repeated root of $\underline{\underline{\alpha_1 = \dfrac{-b}{2a}}}$

Clearly, $e^{\alpha_1 t}$ is one solution, but what about the other?

Try $x_2 = t x_1 = t e^{\alpha_1 t}$

Check. $\dot{x}_2 = e^{\alpha_1 t} + t\alpha_1 e^{\alpha_1 t}$

$\ddot{x}_2 = \alpha_1 e^{\alpha_1 t} + \alpha_1 e^{\alpha_1 t} + t\alpha_1^2 e^{\alpha_1 t}$

$\Rightarrow a\ddot{x}_2 + b\dot{x}_2 + c x_2$

$= 2\alpha_1 a e^{\alpha_1 t} + a\alpha_1^2 t e^{\alpha_1 t} + b e^{\alpha_1 t} + b\alpha_1 t e^{\alpha_1 t} + c t e^{\alpha_1 t}$

$= (2\alpha_1 a + b) e^{\alpha_1 t} + (a\alpha_1^2 + b\alpha_1 + c) t e^{\alpha_1 t}$

$= \left(2\left(\frac{-b}{2a}\right)a + b\right) e^{\alpha_1 t} + (0) t e^{\alpha_1 t}$

$= (0) e^{\alpha_1 t} + (0) t e^{\alpha_1 t} = 0$

$\Rightarrow x_h(t) = C_1 e^{\alpha_1 t} + C_2 t e^{\alpha_1 t}$

$= e^{\alpha_1 t}(C_1 + C_2 t)$

Note!! This only works because $b^2 - 4ac = 0$

(3) $b^2 - 4ac < 0$: Two imaginary roots:

$$\alpha_1 = p + iq \qquad \alpha_2 = p - iq$$

$$\Rightarrow x_h(t) = \hat{C}_1 e^{(p+iq)t} + \hat{C}_2 e^{(p-iq)t}$$

Use the Euler Formula: $e^{i\phi} = \cos\phi + i\sin\phi$

$$\Rightarrow x_h(t) = \hat{C}_1 e^{pt} e^{iqt} + \hat{C}_2 e^{pt} e^{-iqt}$$

$$= e^{pt}\left[\hat{C}_1 (\cos qt + i\sin qt) + \hat{C}_2 (\cos qt - i\sin qt)\right]$$

$$= e^{pt}\left[(\hat{C}_1 + \hat{C}_2)\cos qt + i(\hat{C}_1 - \hat{C}_2)\sin qt\right]$$

$$= e^{pt}\left(C_1 \cos(qt) + C_2 \sin(qt)\right) = x_h(t)$$