Matrix Subspaces

let $\underline{A} \in M_{mn}$

# of rows ↑ ↑ # of Cols

① Column Space, $C(\underline{A})$, is the subspace of $\mathbb{R}^m$ that is spanned by the columns of $\underline{A}$.

Also called the Range Space.

$\underline{A}\,\underline{x} = \underline{b}$ $\quad$ $\underline{b}$ is a linear combination of columns of $\underline{A}$.

$\underline{x} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$

② Nullspace, $N(\underline{A})$, is the subspace of $\mathbb{R}^n$ that is spanned by all vectors which are solutions to

$$\underline{A}\,\underline{x} = \underline{0}$$

Also called the Kernel Space

All matrices have a nullspace as $\underline{0}$ is always in $N(\underline{A})$

$$\underline{A}\,\underline{0} = \underline{0}$$

(3) Row Space, $C(\underline{A}^T)$: The subspace of $\mathbb{R}^n$ spanned by the rows of $\underline{A}$.

(4) Left Nullspace, $N(\underline{A}^T)$: All solutions such that

$$\underline{A}^T \underline{x} = \underline{0}$$

Subspace of $\mathbb{R}^m$,

$$\underline{A}^T \underline{x} = \underline{0} \Rightarrow (\underline{A}^T \underline{x})^T = \underline{0}^T$$

$$\underline{x}^T \underline{A} = \underline{0}^T$$

Summary

Let $\underline{A} \in M_{mn}$

Matrix Subspace	Subspace of
$C(\underline{A})$	$\mathbb{R}^m$
$N(\underline{A})$	$\mathbb{R}^n$
$C(\underline{A}^T)$	$\mathbb{R}^n$
$N(\underline{A}^T)$	$\mathbb{R}^m$

Generic Linear Transformation Terms

Let V & W be vector spaces with a linear transformation $L: V \to W$

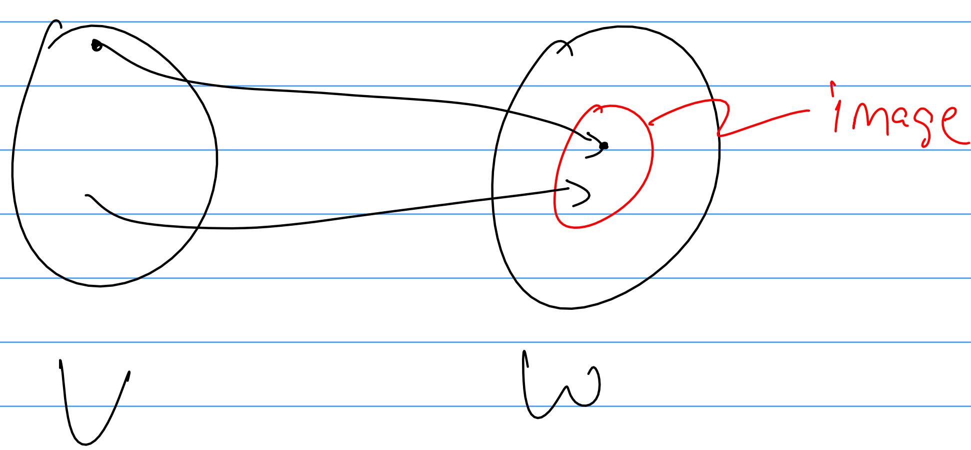
Kernel of L: The $\ker(L)$ is the subspace of V such that

$$\ker(L) = \{ \underline{v} \in V : \underline{L}\,\underline{v} = \underline{0}_W \}$$

The size of $\ker(L)$ is called the nullity of L: $\text{Nullity}(L)$

Rank of L: The rank of a linear operator, $\text{rank}(L)$, is the Dimension of it's image.

Recall that the image of a vector space is the portion of W that maps into it:



Rank-Nullity or Dimension Theorem,

Let V & W be vector spaces w/ a linear transformation $L: V \to W$, then

$$\text{Rank}(L) + \text{Nullity}(L) = |V|$$

dimension/size of V.

Apply to matrices: $\underline{A} \in M_{mn}$

$\underline{A}\,\underline{x} = \underline{b}$ $\quad\quad \underline{x} \in V \in \mathbb{R}^n$

$\underline{b} \in W \in \mathbb{R}^m$

$\underline{b}$ is the image of $\underline{x}$ under the linear transformation of $\underline{A}$.

- Linear Combinations of $C(\underline{A})$ give all vectors in the image.

$$\Rightarrow \text{Rank}(\underline{A}) = |C(\underline{A})|$$

Note! Each vector in $C(\underline{A})$ <u>must</u> be an independent vector. E.g. $C(\underline{A})$ contains the minimum # of vectors to span the columns of $\underline{A}$.

ex1) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \underline{A}$

Do <u>not</u> say that

$C(\underline{A}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}} \right\}$

$= \text{Rank}(\underline{A}) = 3$

- All linear combinations of $N(\underline{A})$ give vectors $\underline{b} = \underline{0} = \underline{0}_m$

$\Rightarrow \text{Nullity}(\underline{A}) = |N(\underline{A})|$

Since for $\underline{A}\underline{x} = \underline{b}$, $\underline{x} \in \mathbb{R}^n$

Then $\text{Rank}(\underline{A}) + \text{Nullity}(\underline{A}) = n$

$= \#$ of Columns

Thm: let $\underline{A} \in M_{mn}$, $C(\underline{A})$ be the column space, $N(\underline{A})$ be the null space, $C(\underline{A}^T)$ the row space & $N(\underline{A}^T)$ the left null space,

1) $\text{Rank}(\underline{A}) = |C(\underline{A})| = |C(\underline{A}^T)|$

2) $|N(\underline{A})| = n - \text{rank}(\underline{A})$

3) $|N(\underline{A}^T)| = m - \text{rank}(\underline{A})$

Ex1) let $\underline{A} = \begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix}$

$\text{rref}(\underline{A}) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Independent Columns

free Variable

$\text{rref}(\underline{A}^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Independent Columns

No free Variable

$\Rightarrow \text{Rank}(\underline{A}) = |C(\underline{A})| = 3 = |C(\underline{A}^T)|$

$\Rightarrow |N(\underline{A})| = n - \text{rank}(\underline{A}) = 4 - 3 = 1$

$\Rightarrow |N(\underline{A}^T)| = m - \text{rank}(\underline{A}) = 3 - 3 = 0$

Find all subspaces:

(1) $C(\underline{A}) = \left\{ \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} \right\}$

(2) $N(A)$: $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cancel{x_4}\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{aligned} x_1 + 3 &= 0 \\ x_2 - 1 &= 0 \\ x_3 + 1 &= 0 \end{aligned} \qquad \Rightarrow \begin{aligned} x_1 &= -3 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

$$\Rightarrow N(\underline{A}) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Check: $\begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -24 + 2 - 1 + 23 \\ -12 + 2 - 9 + 19 \\ -30 + 1 - 6 + 35 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

③ $C(\underline{A}^T) = \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 6 \\ 35 \end{bmatrix} \right\}$

④ $N(\underline{A}^T) = \{ \}$

Let $\underline{A} \in M_{mn}$

① The matrix $\underline{A}$ has full column rank if $\operatorname{rank}(\underline{A}) = n$. If $\underline{A}$ has full column rank then the following holds:

1) All columns of $\underline{A}$ are independent.

2) Only vector in $N(\underline{A})$ is $\underline{0}$

3) If $\underline{A}^{-1}$ exists, then the solution to $\underline{A}\,\underline{x} = \underline{b}$ is unique (e.g. only on $\underline{x}$ such that $\underline{A}\,\underline{x} = \underline{b}$).

② The matrix $\underline{A}$ has full row rank if $\operatorname{rank}(\underline{A}) = m$.

1) All rows of $\underline{A}$ are independent

2) $C(\underline{A})$ spans all of $\mathbb{R}^m$ ($\underline{b} \in \mathbb{R}^m$)

3) $\underline{A}\,\underline{x} = \underline{b}$ has at least one solution for any $\underline{b}$, or

$C(\underline{A})$ spans all of $\mathbb{R}^m \Rightarrow$

Any vector in $\mathbb{R}^m$ can be written as a linear combination of the columns of $\underline{A}$,

$\Rightarrow$ Any $\underline{b}$ must be in $\mathbb{R}^m$ $(\underline{A}\,\underline{x} = \underline{b})$

$\Rightarrow$ $\underline{x}$ is that linear combination of columns of $\underline{A}$ which gives $\underline{b}$,

$$\underline{A}\,\underline{x} = [\underline{a}_1 \;\; \underline{a}_2 \;\; \cdots \;\; \underline{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$$

3. Now let $\underline{A} \in M_{nn}$ (square matrix)

We say that A has full rank if rank$(\underline{A}) = n$ (e.g. both full row & column rank)

If $\underline{A}$ is full rank then:

1) $\underline{A}\,\underline{x} = \underline{b}$ has a solution for any $\underline{b}$,

2) $C(\underline{A})$ span all of $\mathbb{R}^n$

3) $N(\underline{A})$ is only the $\underline{0}$

4) $\underline{A}\,\underline{x} = \underline{b}$ only has one solution for any $\underline{b}$,

In other words, if $\underline{A}$ is full rank, $\underline{A}^{-1}$ exists,

Only one solution: $\underline{A}\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$

Now, All of the following are equivalent statements

1. $\underline{A}$ is invertable
2. The Columns of $\underline{A}$ are independent
3. the rows of $\underline{A}$ are independent
4. $\mathrm{Det}(\underline{A}) \neq 0$
5. $\underline{A}\underline{x} = \underline{0}$ only has $\underline{x} = \underline{0}$ as a solution
6. $\underline{A}$ has n pivots ($\underline{A} \in M_{nn}$)
7. $\underline{A}$ is full rank: $\mathrm{rank}(\underline{A}) = n$
8. $\mathrm{rref}(\underline{A}) = \underline{I}$
9. $C(\underline{A})$ spans all of $\mathbb{R}^n$
10. $C(\underline{A}^T)$ spans all of $\mathbb{R}^n$

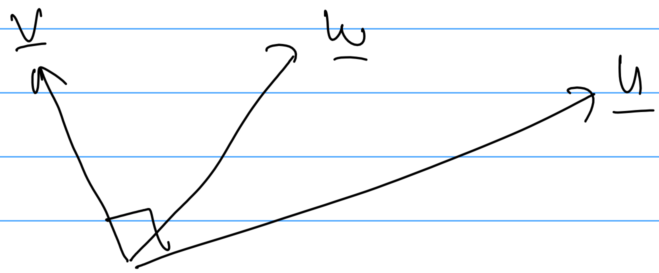
If any <u>one</u> of these are true, then <u>all</u> are true,

<u>Must be square matrix</u>

Orthogonality

We state that two vectors are orthogonal (perpendicular) to each other iff

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = 0$$



$\underline{u}$ & $\underline{v}$ are orthogonal : $\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta = 0$

$\underline{u}$, $\underline{w}$ → not orthogonal.

We also say that two subspaces are othogonal to each if any vector in one subspace is orthogonal to all vectors in the other subspace.

If $\underline{u}$ is in subspace S, & $\underline{v}$ is in subspace T, then if for any $\underline{u} \in S$ & $\underline{v} \in T$ we have

$\underline{u} \cdot \underline{v} = 0$, then S & T are orthogonal.

For a matrix $\underline{A} \in M_{mn}$

(1) The row space, $C(\underline{A}^T)$, is an orthogonal subspace in $\mathbb{R}^n$ of the null space, $N(\underline{A})$

To show this, look at

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$

Let $\underline{x} \in N(\underline{A}) \Rightarrow \underline{A}\,\underline{x} = \underline{0}$

$$\underline{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \underline{row}_1 \cdot \underline{x} \\ \underline{row}_2 \cdot \underline{x} \\ \vdots \\ \underline{row}_m \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The row space is the linear combination of the rows of $\underline{A}$,

Since any $\underline{x}$ in $N(\underline{A})$ gives $\underline{A}\underline{x} = \underline{0}$

$\therefore$ Since $\underline{A}\,\underline{x}$ = a dot product between rows of $\underline{A}$ & $\underline{x}$ $\Rightarrow$ $C(\underline{A}^T)$ & $N(\underline{A})$ are orthogonal to each other.

A Cleaner way:

Let $\underline{y}$ be any vector compatible w/ $\underline{A}^T$:

$\underline{A}^T\underline{y}$ $\Rightarrow$ A linear combination of the rows of $\underline{A}$.

Let $\underline{x}$ be in $N(\underline{A})$

$$\Rightarrow \underline{x} \cdot (\underline{A}^T\underline{y}) = \underline{x}^T\underline{A}^T\underline{y} = (\underline{A}\,\underline{x})^T\underline{y}$$

$$= \underline{0}^T\underline{y} = 0$$

(2) The Column space, $C(\underline{A})$, is an orthogonal subspace in $\mathbb{R}^m$ of the left-nullspace, $N(\underline{A}^T)$

$\underline{A}\,\underline{y}$ = any vector in $C(\underline{A})$

Let $\underline{x} \in N(\underline{A}^T)$: $\underline{x}^T\underline{A} = \underline{0} = \underline{A}^T\underline{x}$

$$\underline{x} \cdot (\underline{A}\,\underline{y}) = \underline{x}^T\underline{A}\,\underline{y} = (\underline{x}^T\underline{A})\,\underline{y} = \underline{0}^T\underline{y} = 0$$

One step further:

1. $N(\underline{A})$ is the orthogonal complement of $C(\underline{A}^T)$ in $\mathbb{R}^n$,

2. $N(\underline{A}^T)$ is the orthogonal complement of $C(\underline{A})$ in $\mathbb{R}^m$

The orthogonal complement to a subspace contains <u>every</u> possible vector that is perpendicular to that subspace,