Rules of a Vector Space

① $\underline{x} + \underline{y} = \underline{y} + \underline{x}$ must be in the V. space

② $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$ must be in V. space

③ There is a unique "zero vector" such that $\underline{0} + \underline{x} = \underline{x} = \underline{x} + \underline{0}$

④ For every $\underline{x}$ there exists $-\underline{x}$ such that

$$\underline{x} + (-\underline{x}) = \underline{0} = (-\underline{x}) + \underline{x}$$

⑤ $a(\underline{x} + \underline{y}) = a\underline{x} + a\underline{y}$ (in V. space)

⑥ $(a+b)\underline{x} = a\underline{x} + b\underline{x}$ " "

⑦ $a(b\underline{x}) = b(a\underline{x})$ " "

⑧ $1\underline{x} = \underline{x}$

If all of these rules are followed, the Vector Space is 'closed.'

Thm: Let V be a vector space. Then for every vector $\underline{v}$ in V & any real number a we have

① $a\underline{0} = \underline{0}$

② $0\underline{v} = \underline{0}$

③ $(-1)\underline{v} = -\underline{v}$

④ If $a\underline{v} = \underline{0}$, then either $a = 0$ or $\underline{v} = \underline{0}$

Proof of Part 1:

	By Property
$a\underline{0} = a\underline{0} + \underline{0}$	3
$= a\underline{0} + (a\underline{0} + (-a\underline{0}))$	4
$= (a\underline{0} + a\underline{0}) + (-[a\underline{0}])$	2
$= a(\underline{0} + \underline{0}) + (-[a\underline{0}])$	5
$= a\underline{0} + (-[a\underline{0}])$	3
$= \underline{0}$	4

Proof of Part 4

Assume that $a\underline{v} = \underline{0}$ and $a \neq 0$, Show that $\underline{v} = \underline{0}$

$$\underline{v} = 1\,\underline{v} \qquad \text{Property } 8$$

$$= \left(\tfrac{1}{a}\cdot a\right)\underline{v} \qquad a \neq 0$$

$$= (1/a)(a\underline{v}) \qquad 7$$

$$= (1/a)\,\underline{0} \qquad \text{because } a\underline{v} = 0$$

$$= \underline{0} \qquad \text{due to part } \underline{1} \text{ above}$$

Subspaces

A portion of a Vector space is called the Subset of that vector space.

Denote this subset of a Vector space V as W.

If W is closed under addition & multiplication, then W is a subspace of V.

Closed means that after addition or multiplication, the result is in W.

Ex1) Let W be all vectors of the form $[a, b, \frac{1}{2}a - 2b]$.

Is this a subspace of $\mathbb{R}^3$?

Check addition:

$$[a, b, \tfrac{1}{2}a - 2b] + [c, d, \tfrac{1}{2}c - 2d]$$

$$= [a+c, b+d, \tfrac{1}{2}a - 2b + \tfrac{1}{2}c - 2d]$$

$$= [a+c, b+d, \tfrac{1}{2}(a+c) - 2(b+d)] \quad \checkmark\checkmark$$

Check Multiplication:

$$k[a, b, \tfrac{1}{2}a - 2b] \qquad k \in \mathbb{R}^1$$

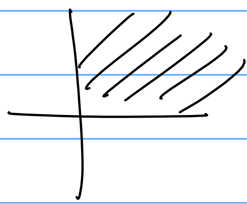
$$= [ka, kb, \tfrac{1}{2}(ka) - 2(kb)] \quad \checkmark\checkmark$$

$\Rightarrow$ This <u>is</u> a subspace of $\mathbb{R}^3$.

Ex of failure

Let W be all vectors of the form

$[a, b]$ $\quad a \geq 0$ & $b \geq 0$



of $\mathbb{R}^2$

This is a subset, but is it a subspace?

Check addition:

$[a, b] + [c, d] \qquad a, b, c, d \geq 0$

$= [a+c, b+d] \qquad a+c \geq 0 \quad b+d \geq 0$ ✓✓

multiplication

$k[a, b] \qquad k \in \mathbb{R}, \; a \geq 0, \; b \geq 0$

$= [ka, kb]$ $\quad$ if $k < 0$, no longer in subset

$\Rightarrow$ Not a subspace, just a subset.

Span

Let S be a non-empty subset of vectors in Vector Space V:

$$S \subseteq V$$

All finite linear combinations of the vectors in S form what is called the span of S: $\text{span}(S)$

ex1) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Then, $\text{span}(S)$ is all of $\mathbb{R}^2$.

Any vector in $\mathbb{R}^2$ can be written as

$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

ex1) Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{Span}(S)$ are all vectors in $\mathbb{R}^4$ of the form

$$[a, 0, 0, b]^T$$

Is this a subspace?

All vectors in this subset are

$$a\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \qquad a, b \in \mathbb{R}$$

Addition:

$$\left(a\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)+\left(c\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+d\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)$$

$$=(a+c)\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+(b+d)\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad \text{in } \mathrm{span}(S)$$

Multiplication:

$$k\left(a\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+b\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=(ka)\begin{bmatrix}1\\0\\0\\1\end{bmatrix}+(kb)\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$\Rightarrow$ Span(S) is a subspace of $\mathbb{R}^4$.

Span, subspace etc, also applies to matrix & function spaces.

ex1) Let U_2 be the set of 2×2 upper triangular matrices and L_2 be the set of 2×2 lower triangular matrices.

$$U_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$$L_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Let $S = U_2 \cup L_2$ (Union)

Then the span(S) contains all 2×2 matrices, called M_{22}

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Thm

Let S be a non-empty subset of Vector Space V. Then:

① $S \subseteq \text{Span}(S)$

② $\text{Span}(S)$ is a subspace of V

③ If W is a subspace of V with $S \subseteq W$, then $\text{span}(S) \subseteq W$

④ $\text{Span}(S)$ is the smallest subset of V containing S.

---

① : Any vector in S: $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$

can be written as a linear combination of the subset S.

$$\underline{v}_1 = 1\,\underline{v}_1 + 0\,\underline{v}_2 + \dots + 0\,\underline{v}_n$$

② : $\text{Span}(S)$ is a subspace of V.

$$\text{Span}(S): a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n$$

$$(a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n) + (b_1\underline{v}_1 + b_2\underline{v}_2 + \dots + b_n\underline{v}_n)$$

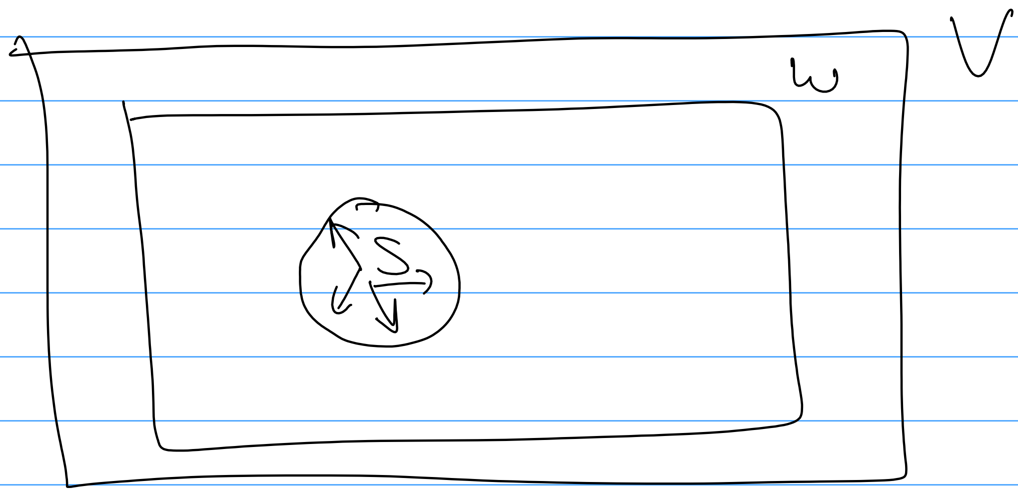
$$= (a_1+b_1)\underline{v}_1 + (a_2+b_2)\underline{v}_2 + \dots + (a_n+b_n)\underline{v}_n$$

$k(a_1\underline{v}_1 + a_2\underline{v}_2 + \cdots + a_n\underline{v}_n)$

$= (ka_1)\underline{v}_1 + (ka_2)\underline{v}_2 + \cdots + (ka_n)\underline{v}_n$

$\rightarrow$ Span(S) is a subspace,

3



4 Summary of 1 $\rightarrow$ 3

Linear Independence & Dependence.

Let S be a Subset of vector space V.

S is linearly dependent if some non-zero linear combination of S results in the zero vector.

$S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$

Some $a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_n \underline{v}_n = \underline{0}$

for some $a_1 \neq 0$ or $a_2 \neq 0, \dots$ or $a_n \neq 0$

& the span(S) is linearly dependent.

ex.) $S \in \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

$$(1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If not linearly dependent $\Rightarrow$ linearly independent.

Basis

We say that B is a basis for vector space V iff

1. B spans all of V
2. B is linearly independent.

ex.) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

Ex.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \right\}$

also form a basis for $\mathbb{R}^3$.

Ex.) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

is a basis for all of M_{22}

ex.) $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} = B$

S is a subspace of $\mathbb{R}^4$.

It is also a basis for that subspace.

Dimension

We say that the Dimension of a vector space U is given by the minimum # of vectors needed in a basis B of U.

If the # of vectors in B is finite, then $\dim(U)$ → Dimension of U, is finite.

Otherwise U has infinite Dimension.

ex.) $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is

a basis for $\mathbb{R}^3$. # of elements in the set

$\Rightarrow \dim(\mathbb{R}^3) = |B| = 3$

$\dim(\mathbb{R}^n) = n$

ex,) $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$V = \text{span}(B) \Rightarrow \dim(V) = 3$

ex,) Vector Space P_3 : All polynomials of order 3 & below,

$P_3 : \{1, x, x^2, x^3\} \leftarrow$ the basis for P_3

$\Rightarrow a(1) + b(x) + c(x^2) + d(x^3)$

$\dim(P_3) = 4$

$\dim(P_n) = n+1$

ex,) $\dim(M_{22}) = 4$

$\dim(M_{mn}) = mn$

ex.) Infinite Dimensional Space:

Taylor Series

$B = \{ (x-a)^0, (x-a)^1, (x-a)^2, \ldots, (x-a)^n, \ldots \}$

$F = \alpha (x-a)^0 + \beta (x-a)^1 + \ldots$

dim(Taylor Series) $= \infty$

Comes up alot in the 4 subspaces of Matrices;

- Column Space : $C(\underline{A})$
- Nullspace : All vectors $\underline{x} \neq \underline{0}$ such that $\underline{A}\underline{x} = \underline{0}$
- Row Space : $C(\underline{A}^T)$
- Right Nullspace : $\underline{x}^T \underline{A} = \underline{0}$ $\underline{x} \neq 0$