Eigensystems

λ is an eigenvalue w/ eigenvector of $\underline{x}$ iff

$\underline{A}\,\underline{x} = \lambda\,\underline{x}$

To determine, we need $(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0}$

To have non-trivial solution, $|\underline{A} - \lambda \underline{I}| = 0$

$\uparrow$

Characteristic Poly.

ex.) let $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 4 = 0$$

$$\lambda^2 - 4\lambda + 1 = 0 \Rightarrow \lambda = 2 \pm \sqrt{3}$$

To find eigenvectors, use each root:

$$\underline{A}\,\underline{x}_1 = \lambda_1 \underline{x}_1 \Rightarrow (\underline{A} - \lambda_1 \underline{I})\underline{x}_1 = 0$$

$$\begin{bmatrix} 1-(2+\sqrt{3}) & 2 \\ 2 & 3-(2+\sqrt{3}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{x}_1 = \begin{bmatrix} 1 \\ (1+\sqrt{3})/2 \end{bmatrix}$$

Similarly, for $\lambda_2 = 2-\sqrt{3}$, $\underline{x}_2 = \begin{bmatrix} 1 \\ (1-\sqrt{3})/2 \end{bmatrix}$

Comments about Eigensystems

(1) The eigenvalues of $\underline{A}^2$ are the square of the eigenvalues of $\underline{A}$, but the eigenvectors are the same.

$\underline{A}\,\underline{x} = \lambda\,\underline{x}$ is known

$$\underline{A}(\underline{A}\,\underline{x}) = \underline{A}(\lambda\underline{x}) = \lambda(\underline{A}\,\underline{x}) = \lambda(\lambda\underline{x}) = \lambda^2\underline{x}$$

$$\Rightarrow \underline{A}^2\underline{x} = \lambda^2\underline{x}$$

Higher powers:

$$\underline{A}^3\underline{x} = \underline{A}(\underline{A}^2\underline{x}) = \underline{A}(\lambda^2\underline{x}) = \lambda^2(\underline{A}\,\underline{x}) = \lambda^2(\lambda\underline{x})$$
$$= \lambda^3\underline{x}$$

$$\Rightarrow \underline{A}^3\underline{x} = \lambda^3\underline{x}$$

eigenvalues of $\underline{A}^n$ are λ^n of $\underline{A}\,\underline{x} = \lambda\,\underline{x}$

② Row reduction does not preserve eigenvalues,

Row reduction is the scaling & addition of the matrix rows.

$$\underline{A} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & 6 \end{bmatrix}$$

$$\begin{vmatrix} 4-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & -3 & 6-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(6-\lambda) = 0$$

$$\lambda = 4, 1, 6$$

flip rows 1 & 3:

$$\begin{vmatrix} 2-\lambda & -3 & 6 \\ 0 & 1-\lambda & 0 \\ 4 & -1 & -\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(-\lambda) - 4(1-\lambda)6 = 0$$

$$\lambda = -4, 1, 6$$

(3) The product of the eigenvalues of $\underline{A}$ equals $\det(\underline{A})$,

and the sum of the eigenvalues equals $tr(\underline{A})$

$tr(\underline{A})$ = trace of $\underline{A}$ = sum of the diagonal

(4) You can have imaginary eigenvalues even if $\underline{A}$ is real.

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 2 \\ 4 & -1 & 0 \end{bmatrix} \rightsquigarrow \begin{vmatrix} 6-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 4 & -1 & -\lambda \end{vmatrix}$$

$$= (6-\lambda)(2-\lambda)(-\lambda) - (-1)(2)(6-\lambda) = 0$$

$$= (6-\lambda)(\lambda^2 - 2\lambda + 2) = 0$$

$$\lambda = 6 \qquad \lambda = 1 \pm i$$

So, $\lambda = 6, 1+i, 1-i$

$\Rightarrow$ Eigenvectors can also be complex

Look at $\lambda = 1+i$ eigenvalue.

$(\underline{A} - \lambda \underline{I}) \underline{x} = 0$

$$\begin{bmatrix} 6-(1+i) & 0 & 0 \\ 0 & 2-(1+i) & 2 \\ 4 & -1 & -(1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

row echlon form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-i & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{set } x_2 = 1$$

$$\Rightarrow 1 - i + 2x_3 = 0 \Rightarrow x_3 = -1/2 + 1/2\, i$$

$$\Rightarrow \underline{x} = \begin{bmatrix} 0 \\ 1 \\ -1/2(1-i) \end{bmatrix} \quad \text{for } \lambda = 1+i$$

(5) For a real matrix, all complex eigenvectors will come in complex conjugate pairs,

$\Rightarrow$ If $\lambda = a+ib$, & $\underline{A}\,\underline{x} = \lambda \underline{x}$ gives the eigen vector, then

$$(\overline{\underline{A}\,\underline{x}}) = (\overline{\lambda \underline{x}}) \Rightarrow \underline{A}\,\overline{\underline{x}} = \bar{\lambda}\,\overline{\underline{x}}$$

$\Rightarrow$ λ has eigenvector $\underline{x}$

$\bar{\lambda}$ has eigenvector $\underline{\bar{x}}$

(6) You can have repeated eigenvalues

$$\underline{A} = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix} \Rightarrow \quad \lambda = -4, 2, 2$$

We say that $\lambda = 2$ has a multiplicity of 2 $(k=2)$

What does this mean for the eigenvalues?

$$\underset{\lambda=2}{\underline{A} - \lambda \underline{I}} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & -4 \end{bmatrix} \overset{\text{ref}}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ 2 free variables

$\Rightarrow$ $\dim(N(\underline{A} - \lambda \underline{I})) = 2$

$\Rightarrow$ 2 eigenvectors

We state that if the # of eigenvectors equals the multiplicity of an eigenvalue, then that eigenvalue is called complete.

If the # of eigen vectors is $\underline{\text{less}}$ than the multiplicity $\Rightarrow$ that eigenvalue is called defective.

ex.) $\underline{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \rightsquigarrow \lambda = 3, 3$

$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ only has $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \lambda = 3$ is defective.

If $\underline{\text{any}}$ eigenvalue of $\underline{A}$ is defective, we call $\underline{A}$ defective.

Normal Matrix Properties

A Normal matrix is one where

$$\underline{A}^T \underline{A} = \underline{A}\,\underline{A}^T$$

Occurs for symmetric ($\underline{A} = \underline{A}^T$), skew-symmetric ($\underline{A} = -\underline{A}^T$), or orthogonal ($\underline{A}^T \underline{A} =$ Diagonal matrix)

Look at Real Symmetric matrices, but this discussion holds for all normal matrices.

$\Rightarrow$ Since $\underline{A}$ is real & symmetric, $\underline{A}$ is square

① A real normal matrix only has real eigenvalues.

Proof: Let $\underline{A}$ be real & symmetric, and let λ be any (including complex) eigenvalue such that $\underline{A}\,\underline{x} = \lambda \underline{x}$

$$\underline{A}\,\underline{x} = \lambda \underline{x} \qquad \& \qquad \underline{A}\,\bar{\underline{x}} = \bar{\lambda}\,\bar{\underline{x}}$$

Also, $(\underline{A}\,\bar{\underline{x}})^T = (\bar{\lambda}\,\bar{\underline{x}})^T \Rightarrow \bar{\underline{x}}^T \underline{A}^T = \bar{\underline{x}}^T \bar{\lambda}$

but $\underline{A}^T = \underline{A} \Rightarrow \bar{\underline{x}}^T \underline{A} = \bar{\underline{x}}^T \bar{\lambda}$

Take the inner product of $\bar{\underline{x}}$ w/ $\underline{A}\underline{x} = \lambda \underline{x}$ and the inner product of $\underline{x}$ w/ $\bar{\underline{x}}^T \underline{A} = \bar{\underline{x}}^T \bar{\lambda}$

$$\bar{\underline{x}}^T(\underline{A}\underline{x}) = \bar{\underline{x}}^T(\lambda \underline{x}) \quad \& \quad (\bar{\underline{x}}^T \underline{A})\underline{x} = \bar{\underline{x}}^T \bar{\lambda} \underline{x}$$

Thus $\bar{\underline{x}}^T \lambda \underline{x} = \bar{\underline{x}}^T \bar{\lambda} \underline{x}$

$$\lambda \bar{\underline{x}}^T \underline{x} = \bar{\lambda} \bar{\underline{x}}^T \underline{x}$$

Since $\bar{\underline{x}}^T \underline{x} = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 > 0$

Thus $\lambda = \bar{\lambda}$ $\qquad \lambda = a+ib \quad \bar{\lambda} = a-ib$

The only time that $\lambda = \bar{\lambda}$ is true is if the imaginary part of λ is zero.

$\Rightarrow$ If $\underline{A}$ is real & symmetric, all λ must be real.

(2) All eigenvectors of a real symmetric matrix are orthogonal.

Proof: Let $\underline{A}\underline{x} = \lambda_1 \underline{x}$ & $\underline{A}\underline{y} = \lambda_2 \underline{y}$

for $\lambda_1 \neq \lambda_2$

Since $\underline{A}$ is real & symmetric, λ_1 & λ_2 are real.

$$(\lambda_1 \underline{x}) \cdot \underline{y} = (\lambda_1 \underline{x})^T \underline{y} = (\underline{A}\,\underline{x})^T \underline{y} = \underline{x}^T \underbrace{\underline{A}^T}_{\underline{A}} \underline{y}$$

$$= \underline{x}^T \underline{A}\,\underline{y} = \underline{x}^T (\lambda_2 \underline{y}) \qquad \underline{A} = \underline{A}^T$$

$$\Rightarrow \underline{x}^T \lambda_1 \underline{y} = \underline{x}^T \lambda_2 \underline{y}$$

$$\lambda_1 \underline{x}^T \underline{y} = \lambda_2 \underline{x}^T \underline{y}$$

Since $\lambda_1 \neq \lambda_2 \Rightarrow$ this is only true iff $\underline{x}^T \underline{y} = 0 \Rightarrow$ orthogonal

(3) You can also prove that the eigenvectors for a real & symmetric $\underline{A}$ are orthonormal.

Matrix Diagonalization.

Matrix Diagonalization is the application of a matrix $\underline{P}$ such that

$$\underline{P}^{-1}\,\underline{A}\,\underline{P} = \underline{\Lambda} \quad , \quad \underline{\Lambda} = \text{Diagonal matrix.}$$

Look at the eigen system of $\underline{A}$:

$$\underline{A}\,\underline{x} = \lambda\,\underline{x}$$

Let $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ be the eigen vectors of $\underline{A}$ w/ eigen values of $\lambda_1, \lambda_2, \ldots$

$$\underline{A}\,\underline{x}_1 = \lambda_1 \underline{x}_1$$
$$\underline{A}\,\underline{x}_2 = \lambda_2 \underline{x}_2$$
$$\vdots$$

Look at $\underline{A}[\underline{x}_1 \; \underline{x}_2 \; \cdots \; \underline{x}_n] = \underline{A}\,\underline{S} = [\lambda_1 \underline{x}_1 \; \lambda_2 \underline{x}_2 \; \cdots$

Let $\underline{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & & \\ 0 & \lambda_2 & 0 & & 0 \\ 0 & 0 & \lambda_3 & & \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$

Then I can write $\underline{S}\,\underline{\Lambda} = [\lambda_1 \underline{x}_1 \; \lambda_2 \underline{x}_2 \; \cdots \; \lambda_n \underline{x}_n]$

$\Rightarrow \underline{A}\,\underline{S} = \underline{S}\,\underline{\Lambda}$

Let $\underline{A}$ be non-defective $\Rightarrow$ All eigenvalues of $\underline{A}$ are complete.

$\Rightarrow$ there are n independent eigenvectors

$\Rightarrow \underline{S}^{-1}$ exists

$\Rightarrow \underline{S}^{-1}\underline{A}\,\underline{S} = \underline{\Lambda}$

or $\underline{A} = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1}$ $\leftarrow$ eigen decomposition of $\underline{A}$

If $\underline{A}$ is defective, then $\underline{S}^{-1}$ does not exist.

$\Rightarrow$ If $\underline{A}$ is defective you can not find an eigen decomposition

$\Rightarrow \underline{A}$ is not diagonalizable.

How does this help? One location! Markov Chains

$\underline{A}^n p_0$

Let $\underline{A}$ be complete, and look at

$\underline{A}^2$

$$\underline{A}^2 = \underline{A}\,\underline{A} = (\underline{S}\,\underline{\Lambda}\,\underline{S}^{-1})(\underline{S}\,\underline{\Lambda}\,\underline{S}^{-1})$$
$$= \underline{S}\,\underline{\Lambda}^2\,\underline{S}^{-1}$$

$$\underline{\Lambda}^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 & \\ 0 & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_n^2 \end{bmatrix}$$

$$\Rightarrow \underline{A}^n p_0 = \underline{S}\,\underline{\Lambda}^n\,\underline{S}^{-1} p_0$$

Now, what if all $|\lambda_i| < 1$

$$\Rightarrow \underline{A}^\infty p_0 \to \underline{0}$$

Now let $\underline{A}$ be normal $(\underline{A}\,\underline{A}^T = \underline{A}^T\underline{A})$

$\Rightarrow$ All eigen vectors are orthonormal

$\Rightarrow$ $\underline{S}$ = Unitary Matrix such that

$$\underline{S}^T\underline{S} = \underline{S}\,\underline{S}^T = \underline{I} \quad \Rightarrow \underline{S}^{-1} = \underline{S}^T$$

We denote this by $\underline{Q}$

$\Rightarrow$ If $\underline{A}$ is normal then $\underline{A} = \underline{Q}\,\underline{\Lambda}\,\underline{Q}^T$

$\underline{A} = \underline{Q}\,\underline{\Lambda}\,\underline{Q}^T$ = Unitary Decomposition

Any Unitary decomposition is the summation of rank-1 matrices

Rank-1 matrix has the form $\underline{u}\,\underline{v}^T$

$$\underline{A} = \underline{Q}\,\underline{\Lambda}\,\underline{Q}^T = [\underline{x}_1 \ \underline{x}_2 \ \cdots \ \underline{x}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \vdots \\ \underline{x}_n^T \end{bmatrix}$$

$$= \lambda_1 \underline{x}_1 \underline{x}_1^T + \lambda_2 \underline{x}_2 \underline{x}_2^T + \cdots + \lambda_n \underline{x}_n \underline{x}_n^T$$

Or, another way to look at it:

$\underline{x}_i \underline{x}_i^T$ is the projection onto the eigenspace w/ a basis given by $\{\underline{x}_i\}$

What if $\underline{A}$ is non-Diagonalizable (e.g. $\underline{A}$ is defective?)

Use Schur's Theorem which states that every square matrix $\underline{A}$ can be written as

$$\underline{A} = \underline{Q}\,\underline{T}\,\underline{Q}^T \quad \text{where } \underline{T}$$

is an upper-triangular matrix and $\underline{Q}$ is unitary.

Summary:

- All square matrices: $\underline{A} = \underline{Q}\,\underline{T}\,\underline{Q}^T$
- If $\underline{A}$ is complete: $\underline{A} = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1}$
- If $\underline{A}$ is normal: $\underline{A} = \underline{Q}\,\underline{\Lambda}\,\underline{Q}^T$

Now, look at a real, symmetric $\underline{A}$ w/ only positive eigenvalues.

$\Rightarrow$ This is called positive-Definite.

One way to show positive-definite is to compute all the eigenvalues

$\Rightarrow$ this is really expensive $O(n^3)$

$\Rightarrow$ Another method:

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

$$\underline{x}^T \underline{A}\,\underline{x} = \lambda \underline{x}^T \underline{x} > 0$$

$$\underline{x}^T \underline{x} = |x_1|^2 + \dots + |x_n|^2 > 0$$

we want the condition that shows $\lambda > 0$ for any λ,

$\Rightarrow$ if $\underline{x}^T \underline{A}\,\underline{x} > 0$, then $\lambda > 0$

In fact, it can be shown that if $\underline{x}^T \underline{A}\,\underline{x} > 0$ for $\underline{\text{any}}$ $\underline{x}$, then all λ's will be positive.

$\underline{x}^T \underline{A}\,\underline{x}$ is the "energy" definition of positive-definite.