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Problems from the book:

Section 4.1: Problem 2

$Y = e^{X}$

CDF of Y: $F_Y(y) = P(Y \le y) = P(e^X \le y) = \begin{cases} P(X \le \ln y), & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$

$$f_Y(y) = \begin{cases} \frac{d}{dx} F_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{y} f_X(\ln y), & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}$$

Ans.

Note: $Y = e^X$ is a strictly monotonic function

$\therefore \quad f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|$. $\quad x = h(y)$, $h(y) = \ln y$ (in this case)

If $X \sim U[0,1]$: $f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore f_X(\ln y) = \begin{cases} 1, & 0 \le \ln y \le 1 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & 1 \le y \le e \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{y}, & 1 \le y \le e \\ 0, & \text{otherwise} \end{cases}$$

Ans.

Problem 5: $X \sim U[0,1]$, $Y \sim U[0,1]$

Let $Z = |X-Y|$

CDF, $F_Z(z) = P(Z \le z) = P(|X-Y| \le z)$

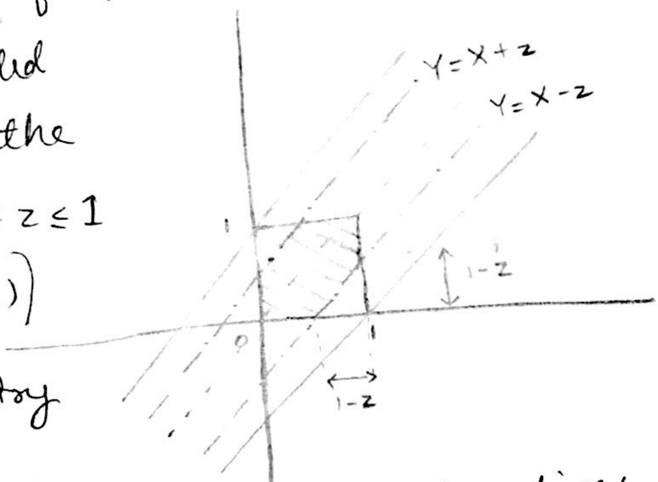
$= P(-z \le X-Y \le z)$

$= P(-\cancel{x}-z \le -Y \le z-x)$

$= P(X-z \le Y \le X+z)$

We can see that Y lies between the values from lines $Y = X-z$ & $Y = X+z$. Since, X & Y $\in [0,1]$ the CDF can be computed by considering the following figure:

For a particular value of z, the area of the shaded portion would give the CDF in the range $0 \le z \le 1$



$$\text{Area} = 1 - \left(2 \times \frac{1}{2} \times (1-z) \times (1-z)\right)$$

∴ By symmetry

$$= 1-(1-z)^2$$

which is basically the area between the lines $Y = X+z$ & $Y = X-z$.

$$\therefore F_Z(z) = \begin{cases} 0, & z \le 0 \\ 1-(1-z)^2, & 0 \le z \le 1 \\ 1 & z > 1 \end{cases}$$

Ans.

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 2(1-z), & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Ans.

Problem 7: Let X and Y be the points which are chosen randomly in the interval [0,1]

Given: $X \sim U[0,1]$, $Y \sim U[0,1]$

Distance between the points is given by $|X-Y|$

From ques. 5 we have CDF of $Z = |X-Y|$

$$\therefore \quad E[Z] = \int_0^1 z f_z(z) dz$$

$$= \int_0^1 z(2(1-z)) dz$$

$$= 2\left[\left[\frac{z^2}{2}\right]_0^1 - \left[\frac{z^3}{3}\right]_0^1\right] = 2\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{3}$$

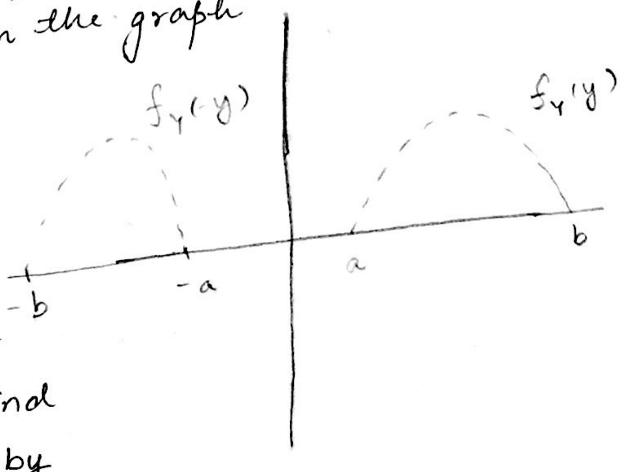
Hence proved.

Problem 13: Given: $f_X(x) = f_Y(y) \geq 0 \quad \forall x, y \in [a,b]$

$$f_X\left(x + \frac{a+b}{2}\right) = f_X\left(-x + \frac{a+b}{2}\right)$$
$$f_Y\left(y + \frac{a+b}{2}\right) = f_Y\left(-y + \frac{a+b}{2}\right)$$

Since, PDFs are symmetric about the mean, $\frac{a+b}{2}$

Also, $f_Y(y) = f_Y(-y + a+b)$ which is also clear from the graph



$\therefore$ PDF of $X+Y$ is same as $X-Y+(a+b)$

Thus, if we have calculated the PDF of $X+Y$ we can find the PDF of $X-Y$ easily by shifting the PDF of $X+Y$ to the left by a quantity equal to $a+b$.

Section 4.2 : Problem 18

We know,

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{var(X)\, var(Y)}}$$

$$cov(X_1+X_2, Y_1+Y_2) = cov(X_1,Y_1) + cov(X_1,Y_2) + cov(X_2,Y_1) + cov(X_2,Y_2)$$

$$cov(X,X) = var(X)$$

$\rho(R,S)$

$$\therefore \rho(W+X, X+Y) = \frac{cov(W+X, X+Y)}{\sqrt{var(W+X)\, var(X+Y)}}$$

$$= \frac{cov(W,X) + cov(W,Y) + cov(X,X) + cov(X,Y)}{\sqrt{var(X)+var(Y)+2cov(X,Y)}\sqrt{(var(W) + var(X) + 2cov(W,X))}}$$

$$= \frac{var(X)}{\sqrt{1+1+0}\,\sqrt{1+1+0}} = \frac{1}{2} \quad \text{Ans.}$$

... Since W, X, Y, Z are pairwise uncorrelated

thus $cov(W,X) = cov(W,Y) = ..$ etc. $= 0$.

Similarly,

$$\rho(R,T) = \frac{cov(W+X, Y+Z)}{\sqrt{var(W+X)\, var(Y+Z)}} = 0 \quad \text{Ans.}$$

Section 4.3 : Problem 24

(a) Let X be the random variable representing the amount of time that the professor devotes to the task and Y be the random variable that represents the length of the time interval between 9 a.m. and the time of his arrival.

Given: X is a function of y. It is an exponential random variable with $\lambda(y) = \frac{1}{5-y}$

$Y \sim U[0,4]$

$\therefore$ Expected amount of time that the professor devotes to the task = $E[X] = E[E[X|Y]]$ ... by law of iterated expectations

We know, that the expected value of an exponential random variable is $\frac{1}{\lambda}$.

$$\therefore \quad E[X|Y=y] = \frac{1}{1/5-y} = 5-y$$

$$\Rightarrow E[X|Y] = 5-Y$$

$$\therefore E[X] = E[E[X|Y]] = E[5-Y] = 5 - E[Y] = 5-2 = 3$$

$\because Y \sim U[0,4]$

Ans. 3 hours

(b) Time at which the task is completed will be defined by the random variable $X+Y$

$$\therefore \quad E[X+Y] = E[X] + E[Y] = 3+2 = 5 \text{ hours}$$

Expected time at which the task is completed is 2:00 p.m. i.e. 5 hours after 9 a.m.

(c) Let Z be the random variable defining the length of time interval between 9 a.m. and the time of arrival of the Ph.D. student, A be the random variable defining the amount of time the student will spend with the professor, if he meets the professor and B be the random variable defining the amount of time the professor will spend with the student.

Let's assume that the probability of the student meeting the professor be p and the event be C.

$\therefore$ By Total Expectation theorem,

$$E[B] = p\,E[B|C] + (1-p)\,E[B|C^c] \quad \text{.. eq(i)}$$

Clearly,

$$E[B|C] = E[A] = \frac{1}{2} \quad [\text{since } A \sim U[0,1]]$$

A & B are equivalent events when the student meets the professor.

$$E[B|C^c] = 0$$ [since time spent with professor will be 0 if he doesn't find the professor (C^c) & leaves]

∴ From eq(i), we have

$$E[B] = \frac{p}{2}$$

If student needs to meet the professor, he should arrive after the professor has arrived and before the professor leaves.

$$\therefore \quad p = P(Y \le Z \le X+Y)$$

$Z \sim U[0,8]$ $\quad f_Z(z) = \begin{cases} 1/8 & , 0 \le z \le 8 \\ 0 & , \text{otherwise} \end{cases}$

$X+Y \geq 0$ $\quad f_Y(y) = \begin{cases} 1/4 & , 0 \le y \le 4 \\ 0 & , \text{otherwise} \end{cases}$

$$\therefore \; p = 1 - [P(Z<Y) + P(Z>X+Y)] \quad \ldots \text{eq(ii)}$$

$$P(Z<Y) = \int_0^4 \int_0^y f_{Y,Z}(y,z)\,dz\,dy$$

$$= \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8}\,dz\,dy = \frac{1}{8\times 4}\left[\frac{y^2}{2}\right]_0^4 = \frac{16}{2\times 8\times 4} = \frac{1}{4}$$

$$P(Z>X+Y) = \int_0^4 P(Z>X+Y \mid Y=y)\, f_Y(y)\,dy$$

$$= \int_0^4 P(X < Z-Y \mid Y=y)\, f_Y(y)\,dy$$

$$= \int_0^4 \int_y^8 F_{X|Y}(z-y)\, f_Z(z)\, f_Y(y)\,dz\,dy$$

$$= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{z-y} \frac{1}{5-y} e^{-\frac{x}{5-y}} \, dx\, dz\, dy \qquad \left[\because F_X(x) = \int_0^x f_X(x)\,dx\right]$$

$$= \frac{1}{32} \int_0^4 \int_y^8 \frac{5-y(-1)}{5-y} \left[e^{-\frac{x}{5-y}} \right]_0^{z-y} dz\, dy$$

$$= \frac{1}{32} \int_0^4 \int_y^8 1 - e^{-\frac{z-y}{5-y}} \, dz\, dy$$

$$= \frac{1}{32} \int_0^4 \left[z + (5-y) e^{-\frac{z-y}{5-y}} \right]_y^8 dy$$

$$= \frac{1}{32} \int_0^4 \left((8-y) + (5-y)\left[e^{-\frac{8-y}{5-y}} - 1 \right] \right) dy$$

$$= \frac{12}{32} + \frac{1}{32} \int_0^4 (5-y)\, e^{-\frac{(8-y)}{5-y}} dy$$

Integrating numerically, we get

$$\int_0^4 (5-y)\, e^{-\frac{(8-y)}{5-y}} dy = 1.7584$$

$\therefore$ From eq(ii) we have

$$p = 1 - \frac{1}{4} - \frac{12}{32} - .05495$$

$$= 0.32$$

$\therefore \; E[B] = \frac{p}{2} = 0.16$ hours or 9.6 mins

If Z is the random variable defining the length of the time interval measured from 9 a.m. until he leaves his office.

We can write,

$$\therefore E[Z] = p\, E[Z|C] + (1-p)\, E[Z|C^c]$$

$$= p\, E[X+Y+A] + (1-p)\, E[X+Y]$$

$$= 0.32\left(5+\frac{1}{2}\right) + 0.68 \times 5$$

$$= 5.16 \text{ hours.}$$

Ans. Expected amount of time that the professor will spend with the student is 9.6 mins and the expected time at which he will leave his office is 5.16 hours after 9 a.m.

Section 4.4: Problem 30

$X \sim N(0,1)$

We know, for $X \sim N(\mu, \sigma^2)$

$$M_X(s) = e^{(\sigma^2 s^2/2) + \mu s}$$

$$M_X(s) = E[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x^2/2) + sx}\,dx$$

$$= \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2) + \cancel{s}y - \frac{s^2}{2}}\,dx$$

$$= \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-s)^2/2}\,dx = e^{s^2/2}$$

We know, $\left.\dfrac{d^n}{ds^n} M_X(s)\right|_{s=0} = E[X^n]$

$$\therefore E[X^3] = \frac{d^3}{ds^3} M_X(s) \Big|_{s=0}$$

$$= \frac{d^3}{ds^3} e^{s^2/2} \Big|_{s=0} =$$

$$\frac{d}{ds} e^{s^2/2} = \frac{2s}{2} e^{s^2/2} = s e^{s^2/2}$$

$$\frac{d^2 e^{s^2/2}}{ds^2} = e^{s^2/2} + s \cdot s e^{s^2/2}$$

$$\frac{d^3 e^{s^2/2}}{ds^2} = s e^{s^2/2} + 2s e^{s^2/2} + s^2 \cdot s e^{s^2/2} \quad \text{.. eq(ii)}$$

$$\frac{d^4 e^{s^2/2}}{ds^2} = 3\left(e^{s^2/2} + s^2 e^{s^2/2}\right) + 3s^2 e^{s^2/2} + s^3 \cdot s e^{s^2/2} \quad \text{.. eq(iii)}$$

Ans. $\therefore E[X^3] = 0$ .. from eq(ii)

$E[X^4] = 3$ .. from eq(iii)

Problem 34: X_i is a Bernoulli random variable that takes value 1 if the i^{th} player is successful. $i = 1, 2, 3$

Considering convolutions of PMF of X_1 & X_2. $(Z = X_1 + X_2)$

$$P_Z(z) = \begin{cases} (1-p_1)(1-p_2), & z = 0 \\ (1-p_1)p_2 + p_1(1-p_2), & z = 1 \\ p_1 p_2, & z = 2 \\ 0, & \text{otherwise} \end{cases}$$

Convolution of the PMF of Z and X_3,

Ans.

$$P_X(x) = \begin{cases} (1-p_1)(1-p_2)(1-p_3) & x=0 \\ p_1(1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3 & x=1 \\ (1-p_1)p_2p_3 + p_1(1-p_2)p_3 + p_1p_2(1-p_3) & x=2 \\ p_1p_2p_3 & x=3 \\ 0 & \text{otherwise} \end{cases}$$

$$M_X(s) = (1-p_1+p_1e^s)(1-p_2+p_2e^s)(1-p_3+p_3e^s)$$

In the above equation, the coefficients of the terms e^{ks} gives the probability that x takes the value k.

Ex. $e^{s(0)}$ has the coefficient $(1-p_1)(1-p_2)(1-p_3)$ which is same which we got using convolution.

Hence Verified.

Extra problems:

1. $X \sim U[0,1]$

1.1 $$F_X(x) = \begin{cases} x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$Y = X^2$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \begin{cases} \sqrt{y} - 0 & , \quad 0 \le y \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$\therefore f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \begin{cases} \frac{1}{2\sqrt{y}} & , \quad 0 \le y \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Ans.

1.2 $$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)\, var(X^2)}}$$

$$= \frac{E[X \cdot X^2] - E[X] \cdot E[X^2]}{\sqrt{(E[X^2] - E[X]^2)(E[X^4] - E[X^2]^2)}}$$

$$E[X] = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 \cdot 1\, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$E[X^3] = \int_0^1 x^3 \cdot 1\, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$E[X^4] = \int_0^1 x^4 \cdot 1\, dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$$\therefore \rho(X,Y) = \frac{\frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3}}{\sqrt{\left(\frac{1}{3} - \frac{1}{4}\right)\left(\frac{1}{5} - \frac{1}{9}\right)}}$$

$$= \frac{\frac{1}{12}}{\sqrt{\frac{1}{12} \cdot \frac{4}{45}}} = \frac{\sqrt{15}}{4} \quad \text{Ans.}$$

2. Using convolution,

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x)\, f_Y(z-x)\, dx}_{*}$$

$$y = z - x$$

$$f_Y(y) = \begin{cases} \frac{1-(z-x)}{2}, & x-1 \le z \le 1+x \\ 0, & \text{otherwise} \end{cases}$$

$*$ is non-zero if $-1 \le x \le 1$ & $x-1 \le z \le 1+x$

$$\Rightarrow \underbrace{-2 \le x-1 \le z}_{R_1} \underbrace{\le x+1 \le 2}_{R_2}$$

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<u>Case 1</u> : $x \in R_1$ $\qquad -1 \le x \le z+1$

$$\therefore f_z(z) = \int_{-1}^{z+1} \left(\frac{1+x}{2}\right)\left(\frac{1-z+x}{2}\right) dx \qquad ; z < 0$$

$$-2 \le z \le 0$$

$$= \frac{1}{4}\int_{-1}^{z+1} (1-z+2x-xz+x^2)\,dx$$

$$= \frac{1}{4}\int_{-1}^{1+z} \left[(1-z) + (2-z)x + x^2\right) dx$$

$$= \frac{1}{4}\left[(1-z)(1+z-(-1)) + \left[\left(\frac{2-z}{2}\right)x^2\right]_{-1}^{z+1} \right.$$

$$\left. + \frac{1}{3}(1+z)^3 - (-1)^3 \right]$$

$$= \frac{1}{4}\left[(1-z)(2+z) + \frac{(2-z)}{2}(1+z^2+2z-1) + \frac{1}{3}(1+z^3+3z^2+3z+1\right]$$

$$= \frac{1}{4\times 6}(-z^3+12z+16) \qquad ; -2 \le z < 0$$

<u>Case 2</u> : $x \in R_2$, $\qquad 0 \le z \le 2 \Leftarrow \qquad z-1 \le x \le 1$

$$\therefore f_z(z) = \frac{1}{4}\int_{z-1}^{1} (1-z+x)(1+x)\,dx$$

$$= \frac{1}{4}\left[\left(\frac{x+1}{3}\right)^3 - \frac{z}{2}(x+1)^2\right]_{z-1}^{1}$$

$$= \frac{1}{4}\left[\frac{8}{3} - 2z - \frac{z^3}{3} + \frac{z^3}{2}\right]$$

$$= \frac{z^3}{24} - \frac{z}{2} + \frac{2}{3} = \frac{1}{24}(z^3 - 12z + 16)$$

Ans. $$f_z(z) = \begin{cases} \frac{1}{24}(-z^3 + 12z + 16) & -2 \le z \le 0 \\ \frac{1}{24}(z^3 - 12z + 16) & 0 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}$$