Vectors

What is a vector?
- A collection of numbers, called a "tuple" or components.
- Organized
- The # of components denotes the dimension of the vector.

$\underline{v}$ (an underscore denotes a vector)

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ v_1, v_2 are components of $\underline{v}$

Components also denoted by v_i

$\underline{v}$ is a column vector

Another vector $\underline{u} = [u_1 \; u_2]$ is a row vector

The vector is the core of linear algebra.

Linear algebra: manipulation of vectors & groups of vectors.

Vector Operations

Let $\underline{u}$ & $\underline{v}$ be any two vectors of the same dimension.

ex.) $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$ ← A vector of dimension 4

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

① Addition: $\underline{w} = \underline{u} + \underline{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$\underline{w} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ $w_i = u_i + v_i$

② Subtraction: $\underline{w} = \underline{u} - \underline{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \\ u_4 - v_4 \end{bmatrix}$

$w_i = u_i - v_i$

3. Scalar Multiplication

Let c = Scalar number

Now, $\underline{w} = c\,\underline{u} = \begin{bmatrix} c u_1 \\ c u_2 \\ c u_3 \\ c u_4 \end{bmatrix}$ $\qquad w_i = c\,u_i$

4. Special Vectors

- Zero Vector: $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\underline{v} - \underline{v} = \underline{0}$

- One Vector: $\underline{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

5. Order of operation:

$\underline{u} + \underline{v} = \underline{v} + \underline{u}$ $\qquad c\,\underline{u} = \underline{u}\,c$

Multiplication & Division differ from scalar math.

Linear Combination of Vectors

A linear combination of vector is the scaled sum of vectors.

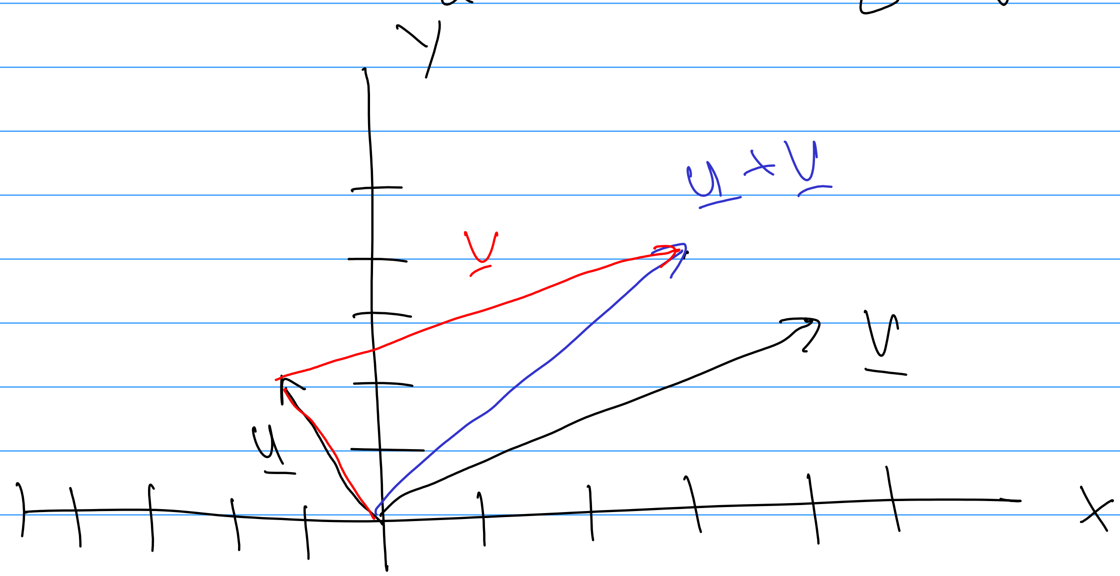
Let $\underline{u}$ & $\underline{v}$ be vectors of the same dimension

Let c & d be scalars.

$$c\underline{u} + d\underline{v} = \underline{w}$$
$$cu_i + dv_i = w_i$$

Graphical Representation

2D: $\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ "x-comp" "y-comp"



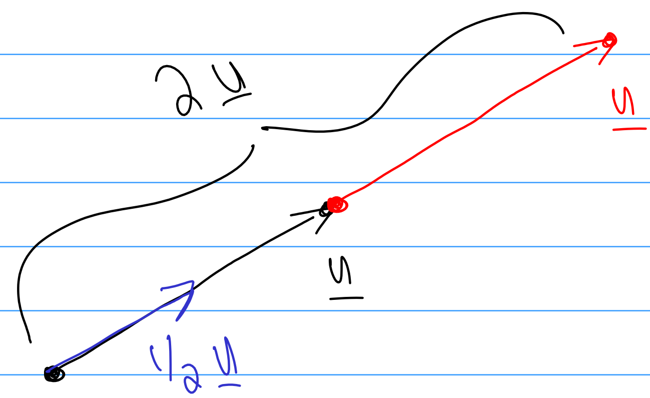
Let $c = 1$ $d = 1$,

$$\underline{w} = c\underline{u} + d\underline{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Graphically: "Head to Tail"

Linear Combination Identities:

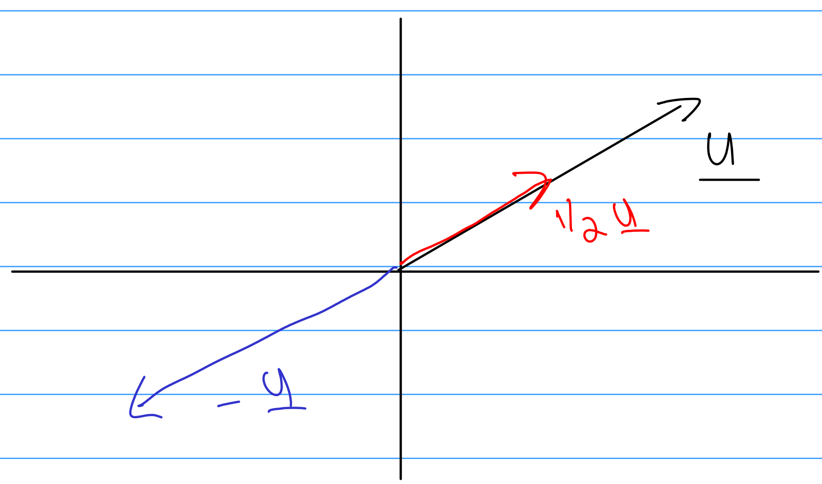
- $0\underline{u} + 0\underline{v} = \underline{0}$
- $c\underline{u} + 0\underline{v} = c\underline{u}$



$2\underline{u} = \underline{u} + \underline{u}$

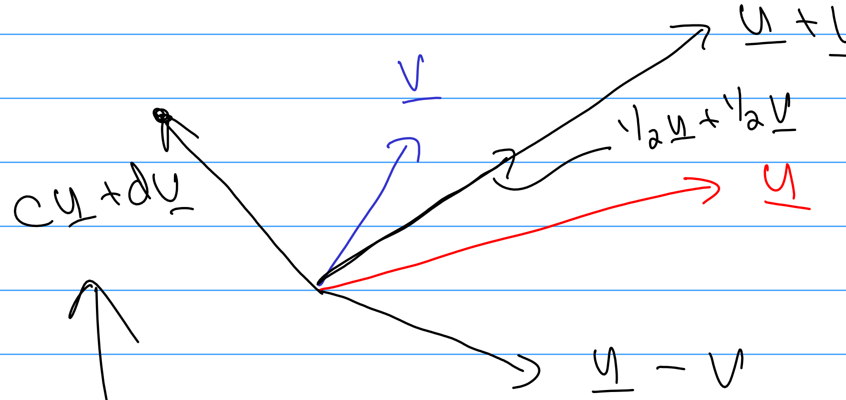
- Geometric Interpretation of linear Combinations

① What are all combinations of $c\underline{u}$



c runs from $[-\infty, \infty]$

② Linear Combination of 2 Vectors, $\underline{u}$ & $\underline{v}$ are $\underline{\underline{\text{not}}}$ aligned.



Could I find c & d to make this physical vector?

Yes!

$\Rightarrow$ All possible combinations describe a plane.

③ All vectors w/ dimension 3 such that $\underline{z} = c\underline{u} + d\underline{v} + e\underline{w}$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Describes a Volume.

Dot Products: "Multiplying Vectors"

Also called an inner product

This multiplies 2 vectors and results in a scalar.

Dot product of vectors $\underline{u}$ & $\underline{v}$ is $\underline{u} \cdot \underline{v}$

$$\underline{u} \cdot \underline{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2$$

$$= u_i v_i$$

ex.) $\underline{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\underline{u} \cdot \underline{v} = 4(-1) + 2(1) = -4 + 2 = -2$$

Higher Dimensions:

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i$$

n = Dimension of $\underline{u}$ (and $\underline{v}$)

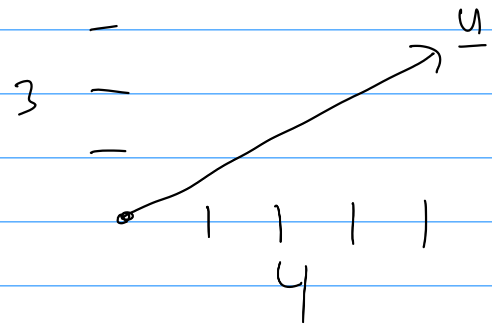
- Dot product of $\underline{u}$ with $\underline{u}$:

$$\underline{u} \cdot \underline{u} = u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 = \|\underline{u}\|^2$$

= "Length" of $\underline{u}$ squared

(Also called the 2-norm)

In 2D: $\underline{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$



$$(4^2+3^2)^{1/2} = \sqrt{\|\underline{u}\|^2}$$

$\Rightarrow \|\underline{u}\| =$ The magnitude ("length") of vector $\underline{u}$.

If $\|\underline{u}\| = 1$, then $\underline{u}$ is called a unit vector

ex.) $\underline{u} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ Is this a unit vector? Check:

$$(1/2)^2 = 1/4$$

$$\underline{u} \cdot \underline{u} = 1/4 + 1/4 + 1/4 + 1/4 = 1 = \|\underline{u}\|^2$$

$$\Rightarrow \|\underline{u}\| = 1$$

- Note: Any vector can be transformed into a unit vector by multiplying by $\dfrac{1}{\|\underline{u}\|}$: $\dfrac{\underline{u}}{\|\underline{u}\|}$ is a unit vector

$$\underline{v} = \frac{\underline{u}}{\|\underline{u}\|} \Rightarrow \underline{v} \cdot \underline{v} = \frac{\underline{u}}{\|\underline{u}\|} \cdot \frac{\underline{u}}{\|\underline{u}\|} = \frac{\underline{u} \cdot \underline{u}}{\|\underline{u}\|^2} = \frac{\|\underline{u}\|^2}{\|\underline{u}\|^2} = 1$$

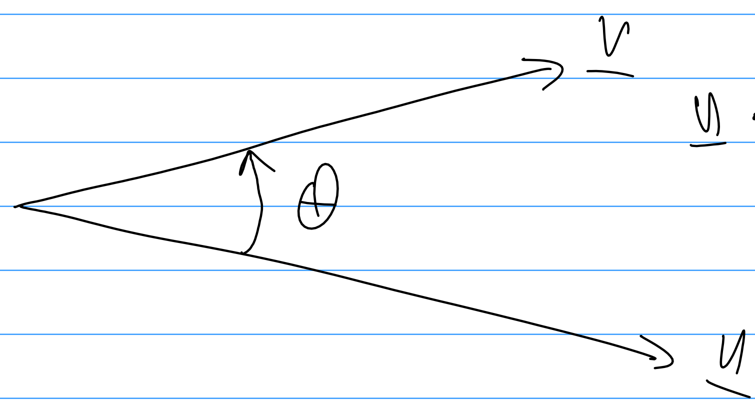
Example Unit Vector: Cartesian Directions

$$\underline{e}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{e}_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{e}_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{e}_j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{e}_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Back to Dot product.

The Dot product also gives you the inner angle between vectors in the plane defined by those two vectors.



Show that
$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta$

To show that, use the Law of Cosine

$$\|\underline{u}-\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\|\underline{v}\|\cos\theta$$

$$\hookrightarrow \|\underline{u}-\underline{v}\|^2 = (\underline{u}-\underline{v}) \cdot (\underline{u}-\underline{v})$$

$$= \underline{u}\cdot\underline{u} - \underline{u}\cdot\underline{v} - \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}$$

$$= \underline{u}\cdot\underline{u} - 2\,\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

$$= \underbrace{\|\underline{u}\|^2 - 2\,\underline{u}\cdot\underline{v} + \|\underline{v}\|^2}$$

$$\|\underline{u}-\underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\,\underline{u}\cdot\underline{v} = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\,\|\underline{v}\|\cos\theta$$

$$\Rightarrow \underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta$$

- What if $\underline{u}\cdot\underline{v} = 0$?

If $\underline{u}\cdot\underline{v} = 0$ then $\underline{u}$ & $\underline{v}$ are perpendicular.

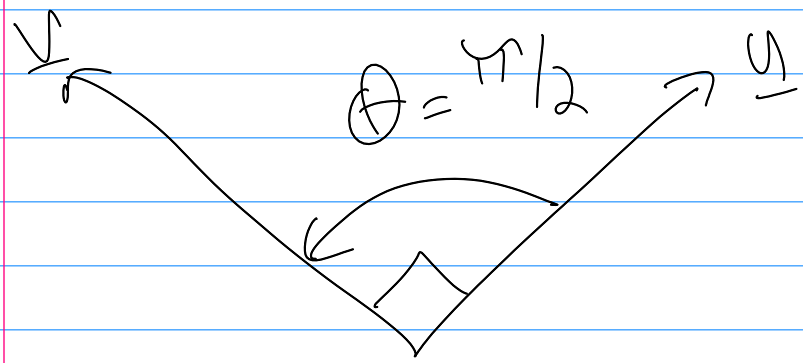
Show that $\underline{u}\cdot\underline{v} = 0$ iff they are orthogonal.

To Do this show:

① If $\underline{u}$ & $\underline{v}$ are $\perp$, then $\underline{u}\cdot\underline{v} = 0$

② If $\underline{u}\cdot\underline{v} = 0$, then $\underline{u}$ & $\underline{v}$ are $\perp$.

① If $\underline{u}$ & $\underline{v}$ are $\perp$



$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta = \|\underline{u}\|\,\|\underline{v}\|\cos\frac{\pi}{2} = 0$$

② If $\underline{u}\cdot\underline{v} = 0$, are they $\perp$?

$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta = 0$$

In general, $\|\underline{u}\| \neq 0$ & $\|\underline{v}\| \neq 0$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2$$

Look again at $\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta$

$$\Rightarrow |\underline{u}\cdot\underline{v}| = \Big|\,\|\underline{u}\|\,\|\underline{v}\|\cos\theta\,\Big| \le \Big|\,\|\underline{u}\|\,\|\underline{v}\|\,\Big| = \|\underline{u}\|\,\|\underline{v}\|$$

$$\Rightarrow |\underline{u}\cdot\underline{u}| \le \|\underline{u}\|\,\|\underline{v}\|$$ ← Schwartz Inequality