Test 2 EAS595 Intro to Prob

Name:..
Person number:..

Nov 19, 2018, 7:00pm-9:30 pm

Short Answer Questions – 35%

1. (7 points) We have a sample of a random variable with unknown distribution. What is the probability that this sample is 3 standard deviations away from the mean of the distribution?

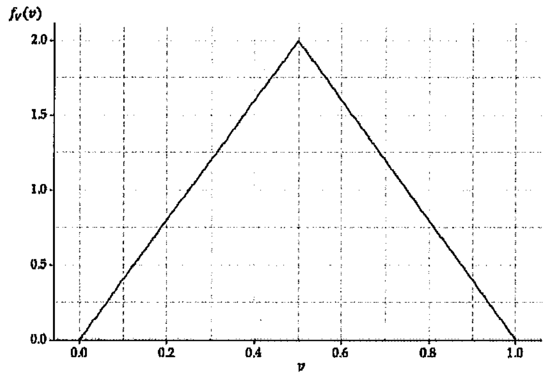
$P(|X-\mu|<c) \leqslant \frac{\sigma^2}{c^2}$ let $c=3\sigma$

$\rightarrow P(|X-\mu|<3\sigma) \leqslant \frac{\sigma^2}{9\sigma^2} = \frac{1}{9}$ The probabality of this sample to be 3σ away from the mean is less then $1/9$

2. (7 points) For continuous random variables X and Y show that $E[E[Y|X]] = E[Y]$ using definitions of conditional expectation.

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \rightarrow E[E[Y|x]] = \underbrace{\int_{-\infty}^{\infty} E[Y|x] f_X(x)\, dx}_{\text{This is the definition of } E[Y] \text{ in}} = E[Y]$$

3. (7 points) We are given a biased coin. Probability(head)=V is a random variable itself with a PDF shown in the following figure. We toss the coin a fixed number of n times and we let X to be the number of heads obtain. What is $E[E[X|V]]$?



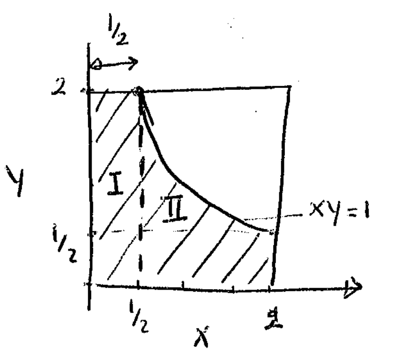
X : Bionomial Distribution

$E[X|V] = nV$

$E[E[X|V]] = E[nV] = nE[V]$

↳ symmetry $E[V] = \frac{1}{2}$ $\Rightarrow$ $E[E[X|V]] = \frac{n}{2}$

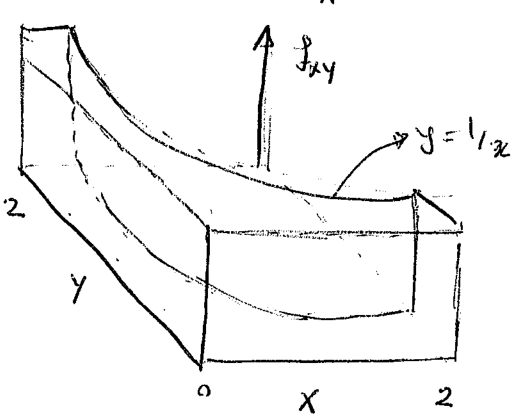
4. (7 points) Let X and Y be two independent Uniform(0,2) random variables. Find $P(XY < 1)$.



The third dimention is the joint PDF which is constant and equal to 1/4.

$P(XY<1)$ = Area under the cene $\times \frac{1}{4}$

$$P(XY<1) = \frac{1}{4}\left(2\times\frac{1}{2} + \int_{1/2}^{2} \frac{1}{x}\,dx\right) = \frac{1}{4}\Big[\underbrace{(\ln 2 - \ln 2^{-1})}_{2\ln 2} + 1\Big] = \frac{\ln 2}{2} + \frac{1}{4}$$



5. (7 points) $X_1, X_2, ..., X_n$ are iid with mean μ and variance σ^2. What is the expected value of $Y = \frac{1}{n}\sum_{i=1}^{n} X_i^2$?

$$\left.\begin{array}{l} E[Y] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i^2] = \\ Var[X_i] = E[X_i^2] - E[X_i]^2 \rightarrow E[X_i^2] = \sigma^2 + \mu^2 \end{array}\right\} \Rightarrow E[Y] = \frac{n(\sigma^2+\mu^2)}{n} = \sigma^2 + \mu^2$$

Problems – 65%

6. (25 points) X is a uniform random variable over the interval of (0 2).

- What is the PDF of $Y = X^2 + 2X$.
- What is the correlation of X and Y [Hint: You don't need to solve part 1 to solve this question].

a) $Y = g(x) = \underbrace{X^2+2X+1}_{(X+1)^2} - 1 \longrightarrow X = h(y) = \sqrt{y+1} - 1$

Monotonic function

$\Downarrow$

$f_Y(y) = \left| \frac{dh}{dy}(y) \right| f_X(h(y))$

$f_X(h(y)) = \begin{cases} \frac{1}{2} & 0 \leqslant \sqrt{y+1} - 1 \leqslant 2 \\ 0 & \text{other} \end{cases} \Rightarrow f_X(h(y)) = \begin{cases} \frac{1}{2} & 0 \leqslant y \leqslant 8 \\ 0 & \text{other} \end{cases} \Rightarrow f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y+1}} & 0 \leqslant y \leqslant 8 \\ 0 & \text{othe} \end{cases}$

b) $Var[Y] = E[Y^2] - E[Y]^2 = E[X^4 + 4X^3 + 4X^2] - (E[X]^2 + 2X])^2$

$X \sim U(0,2) \rightarrow E[X] = 1, \quad E[X^2] = \int_0^2 \frac{X^2}{2} dx = \frac{X^3}{6}\Big]_0^2 = \frac{4}{3}, \quad E[X^3] = \int_0^2 \frac{X^3}{2} dx = \frac{x^4}{8}\Big]_0^2 = 2, \quad E[X^4] = \int_0^2 \frac{X^4}{2} dx = \frac{X^5}{10} = 3.2$

$Var[X] = E[X^2] - E[X]^2 = \frac{4}{3} - 1 = \frac{1}{3}$

$Var[Y] = E[X^4] + 4E[X^3] + 4E[X^2] + (E[X^2] + 2E[X])^2 = 3.2 + 8 + 16/3 - (4/3 + 2)^2 = 5.42$

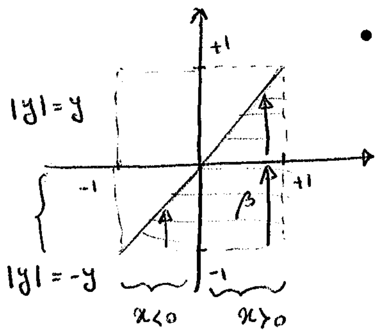
$\rho_{XY} = \frac{COV(X,Y)}{\sqrt{Var[X]Var[Y]}}, \quad Cov(X,Y) = Cov(X, X^2+2X) = \underbrace{COV(X,X^2)}_{E[X^3]-E[X]E[X^2]} + \overbrace{COV(X,2X)}^{2\,Var[X]} = 2 - \frac{4}{3} + \frac{2}{3} = \frac{4}{3}$

$\Rightarrow \rho_{XY} = \frac{4/3}{\sqrt{5.42 \times 1/3}}$

7. (20 points) Consider random variables X,Y with joint PDF.

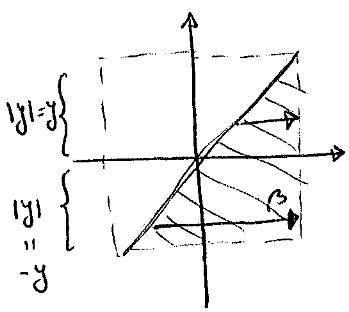
$$f_{X,Y} = \begin{cases} cx|y| & x \in [-1\ 1], \quad -1 \le y \le x \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

- Find $f_X(x), f_Y(y)$
  (Hint: You have to consider the positive and negative values separately)
- For extra credit find c



$$f_X(x) = \int_\beta f_{xy}\, dy \begin{cases} \xrightarrow{x<0} \displaystyle\int_{-1}^{x} -cxy\,dy = -cx\frac{y^2}{2}\Big]_{-1}^{x} = -c\frac{x^3}{2} + \frac{cx}{2} \\ \xrightarrow{x>0} \displaystyle\int_{-1}^{0} -cxy\,dy + \int_{0}^{x} cxy\,dy = -cx\frac{y^2}{2}\Big]_{-1}^{0} + cx\frac{y^2}{2}\Big]_{0}^{x} = \frac{cx^3}{2} + \frac{cx}{2} \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{c}{2}(x - x^3) & x \le 0 \\ \frac{c}{2}(x + x^3) & x > 0 \end{cases}$$



$$f_Y(y) = \int_\beta f_{xy}\, dx \begin{cases} \xrightarrow{y<0} \displaystyle\int_{y}^{1} -cxy\,dx = -cy\frac{x^2}{2}\Big]_{y}^{1} = -\frac{cy}{2} + c\frac{y^3}{2} \\ \xrightarrow{y>0} \displaystyle\int_{y}^{1} cxy\,dx = cy\frac{x^2}{2}\Big]_{y}^{1} = \frac{cy}{2} - c\frac{y^3}{2} \end{cases}$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{c}{2}(y^3 - y) & y \le 0 \\ \frac{c}{2}(y - y^3) & y > 0 \end{cases}$$

8. (20 points) A factory produces X_n gadget on day n, where X_n are iid random variables with mean 5 and variance 9.

- Find an approximation to the probability that the total number of gadgets produced in 100 days is less than 440.
- Find the largest value of n such that $P(X_1 + ... + X_n > 200 + 5n) \leq 0.05$

a) $S_n = \sum_{i=1}^{100} X_i \qquad P(S_n < 440) = ?$

$\bar{X} = \frac{1}{n} S_n \qquad \mu_{\bar{X}} = E[X_i] = 5 \qquad \sigma_{\bar{X}}^2 = \frac{Var[X_i]}{n} \rightarrow \sigma_{\bar{X}} = 0.3$

$P\left(\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} < \varepsilon \right) = \Phi(\varepsilon) \Rightarrow P\left(\frac{\frac{S_n}{100} - 5}{0.3} < \underbrace{\frac{(4.4 - 5)}{0.3}}_{\substack{\varepsilon \\ -2}} \right) \simeq \Phi(\varepsilon)$

$\Phi(-2) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$

b) $P\left(\bar{X} > \frac{200}{n} + 5 \right) = P\left(\frac{\bar{X} - 5}{\frac{3}{\sqrt{n}}} > \underbrace{\frac{\sqrt{n}}{3} \frac{200}{n}}_{\frac{200}{3\sqrt{n}}} \right) = 1 - \Phi\left(\frac{200}{3\sqrt{n}} \right) \leq 0.05$

$\Rightarrow 0.95 \leq \Phi\left(\frac{200}{3\sqrt{n}} \right)$

n is largest when $\Phi(.)$ is closest to 0.95

Table. $\Phi\left(\frac{200}{3\sqrt{n}} \right) = 1.65 \rightarrow n = \left(\frac{200}{3 \times 1.65} \right)^2 \simeq 1632$