ASSIGNMENT 2 NAME : SHUBHAM SHARMA EAS 595

PROFESSOR : DR. ESFAHANI

PERSON #: 50290293

1. BOOK PROBLEMS

(1) Ch-2 Prob-3

(a) Probability that Fischer wins the match in k^{th} game =

$$(0.4)(0.3)^{k-1}$$ [(k-1) draws and win for Fischer in k^{th} game]

Since after 10 successive draws, match will be declared as drawn : Fischer will have to win in the first 10 games.

$\therefore$ Probability that Fischer wins the game $= \sum_{k=1}^{10} (0.3)^{k-1}(0.4)$

$$= 0.4 \times \frac{(1-.3^{10})}{1-0.3}$$

$$= 0.5714251972 \text{ Ans.}$$

(b) If X is the random variable defining the number of matches(k) to get a win within 10 matches (k<10) or 9 draws and a win (k=10)

Thus, the PMF of the duration of the match is given by,

$P_X(k)$:

Probability that a win happens in a game $= 1-0.3 = 0.7$

Ans.: $$P_X(k) = \begin{cases} 0.7(0.3)^{k-1} & k = 1,2,3,\dots 9 \\ (0.3)^9 & k = 10 \\ 0 & \text{otherwise} \end{cases}$$

Problem 18 let the equal probability with which X takes values between $[2^a, 2^b]$ be p.

$\therefore P_X(x) = p$, $x \in [2^a, 2^b]$ such as x is of the form 2^k $(x \in [a,b)]$ K is an integer

0 otherwise

we know,

$$\sum_x P_X(x) = 1 \Rightarrow p(b-a+1) = 1 \Rightarrow p = \frac{1}{b-a+1}$$

$$\therefore E[X] = \sum_x x P_X(x) = \frac{1}{b-a+1} \sum_{K=a}^{b} 2^k = \frac{1}{b-a+1} \cdot \frac{2^a(2^{b-a+1}-1)}{2-1}$$

$$= \frac{2^{b+1} - 2^a}{b-a+1} \quad \text{Ans.}$$

$$\text{var}(X) = \sum_x (x - E[X])^2 P_X(x)$$

$$= \frac{1}{(b-a+1)} \sum_{K=a}^{b} 2^{2K} + \left(\frac{2^{b+1}-2^a}{(b-a+1)}\right)^2 - 2 \cdot 2^k \cdot \left[\frac{2^{b+1}-2^a}{(b-a+1)}\right]$$

$$= \frac{1}{(b-a+1)} \left[\frac{4^{b+1}-4^a}{3} + \left[\frac{2^{b+1}-2^a}{b-a+1}\right]^2 (b-a+1) - \frac{2(2^{b+1}-2^a)^2}{(b-a+1)} \right]$$

$$= \frac{4^{b+1}-4^a}{3(b-a+1)} - \left(\frac{2^{b+1}-2^a}{b-a+1}\right)^2 \quad \text{Ans.}$$

Problem 21 Since it is a fair coin the probability of head = the probability of tail $= \frac{1}{2}$

If X is the random variable defining the amount gained after k tosses we can write the PMF as

$$P_X(x) = \left(\frac{1}{2}\right)^{K-1} \cdot \frac{1}{2} \quad , \text{ if } x = 2^K \qquad K \in [0, \infty)$$

$$= 0 \qquad \text{otherwise}$$

$$\therefore \quad E[X] = \sum_x x\, P_X(x)$$

$$= \sum_{K=0}^{\infty} 2^K \left(\frac{1}{2}\right)^{K-1} \cdot \frac{1}{2}$$

$$= \sum_{K=0}^{\infty} 1 = \infty$$

Thus, we expect to receive an infinite amount of money from this game. Thus, any amount is suitable to be paid in order to be able to play this game since I expect to gain an infinite amount of money from this game.

Problem 36 (a) We know,

$$P_{X,Y}(x,y) = P(X=x, Y=y)$$

$\therefore$ using conditional probability theorem we can write,

$$P_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{P_{X,Y}(x,y)}{P_Y(y)} \quad \ldots \text{eq}^{i)}$$

$$\Rightarrow P_{X,Y}(x,y) = P_{X|Y}(x|y)\, P_Y(y)$$

Similarly,

$$P_{Z|X,Y}(z|x,y) = \frac{P_{X,Y,Z}(x,y,z)}{P_{X,Y}(x,y)}$$

where we consider X, Y as a single unit & apply eq$^{i)}$

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

$$\therefore P_X(x) \cdot P_{Y|X}(y|x) \cdot P_{Z|X,Y}(z|x,y) = P_X(x) \cdot \frac{P_{X,Y}(x,y)}{P_X(x)} \cdot \frac{P_{X,Y,Z}(x,y,z)}{P_{X,Y}(x,y)}$$

$$= P_{X,Y,Z}(x,y,z)$$

Hence Proved.

(b) Multiplication Rule states,

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2) \ldots P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right) \quad \ldots \text{eq(i)}$$

Thus, this formula can be written as:

$$P(X=x, Y=y, Z=z) = P(X=x)\, P(Y=y|X=x)\, P(Z=z|X=x, Y=y)$$

where we use the analogy of random variable X taking value x as corresponding to event A_1 and so forth.

In this way, we can interpret this formula as a special case of our derived formula in part (a)

(c) We can choose random variables, $X_1, X_2, \ldots X_n$

Thus, using eq(i) we can generalise as follows:

$$P_{X_1 \ldots X_n}(x_1, x_2, x_3, \ldots x_n) = P_{X_1}(x_1) \cdot P_{X_2|X_1}(x_2|x_1) \cdot P_{X_3|X_1,X_2}(x_3|x_1,x_2) \ldots P_{X_n|X_1,X_2,X_3\ldots,X_{n-1}}(x_n|x_1,x_2,x_3\ldots,x_{n-1})$$

(2) Ch-3 Prob. 8

(a) We know, for disjoint events $A_1, A_2, \ldots A_n$, Total Probability Theorem states

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n)$$

$$\therefore P(X \leq x) = pP(Y \leq x) + (1-p)P(Z \leq x)$$

$$\Rightarrow F_X(x) = pF_Y(x) + (1-p)F_Z(x) \quad \ldots eq^{n} i)$$

we know,

$$\frac{d}{dx}F_X(x) = f_X(x)$$ [for those points (x) where $f_X(x)$ is continuous]

$\therefore$ Differentiating eq^{n} i) w.r.t. x we get

$$\frac{d}{dx}F_X(x) = \frac{d}{dx}\left[pF_Y(x) + (1-p)F_Z(x)\right]$$

$$\Rightarrow f_X(x) = pf_Y(x) + (1-p)f_Z(x)$$

Hence Proved.

(b) Let us define random variable Y such that

$$f_Y(y) = \begin{cases} \lambda e^{\lambda y} & , \text{if } y < 0 \\ 0 & \text{otherwise} \end{cases}$$

and the random variable Z such that

$$f_Z(z) = \begin{cases} \lambda e^{-\lambda z} & , \text{if } z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, clearly $-Y$ and Z are of the form of exponential random variables

Thus, CDF for Y is given as:

$$F_Y(y) = \begin{cases} e^{\lambda y} & \text{if } y < 0 \\ 1 & \text{if } y \geq 0 \end{cases}$$

CDF for z is given by,

$$F_Z(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1-e^{-\lambda z} & \text{if } z \geq 0 \end{cases}$$

Using eq(i)

$$F_X(x) = \begin{cases} p.e^{\lambda x} + (1-p).0 & \text{if } x < 0 \\ p.1 + (1-p).(1-e^{-\lambda x}), & \text{if } x \geq 0 \end{cases}$$

$$= \begin{cases} pe^{\lambda x} & x < 0 \\ 1-(1-p)e^{-\lambda x}, & x \geq 0 \end{cases}$$ Ans.

Prob 15 (a) We know, for a joint PDF

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$\therefore$ Since $y \geq 0$ & $x^2+y^2 \leq r^2$ $x \in [-r,r]$, $y \in [0,r]$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\therefore f_{X,Y}(x,y) = \begin{cases} c & \text{if } -r \leq x \leq r, \quad 0 \leq y \leq \sqrt{r^2-x^2} \\ 0 & \text{otherwise} \end{cases}$$

This is the case of uniform PDF

$$\therefore f_{X,Y}(x,y) = \begin{cases} \frac{1}{\text{area of } S}, & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi r^2} & \text{for } (x,y) \text{ belonging to semicircle} \\ 0 & \text{otherwise} \end{cases}$$

Scanned by CamScanner

15(b) Marginal PDF of Y,

$$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\, dx = \frac{2}{\pi r^2}\left[\sqrt{r^2-y^2} - \left(-\sqrt{r^2-y^2}\right)\right]$$

$$= \frac{4\sqrt{r^2-y^2}}{\pi r^2} \quad \text{for } y \in [0, r]$$

$$\therefore f_Y(y) = \begin{cases} \dfrac{4\sqrt{r^2-y^2}}{\pi r^2} & \text{if } 0 \le y \le r \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_0^r y \cdot \frac{4\sqrt{r^2-y^2}}{\pi r^2}\, dy$$

$$= \frac{4}{\pi r^2} \int_0^r y\sqrt{r^2-y^2}\, dy$$

Let $z = r^2 - y^2$

$dz = -2y\, dy$

$$\therefore = \frac{4}{\pi r^2} \cdot \frac{-1}{2} \int_{r^2}^{0} \sqrt{z}\, dz$$

$$= \frac{4}{\pi r^2} \cdot \frac{-1}{2} \cdot \frac{2}{3} \left[z^{3/2}\right]_{r^2}^{0}$$

$$= \frac{4r}{3\pi} \quad \text{Ans.}$$

15(c) We know,

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \; f_{X,Y}(x,y)\,dx\,dy$$

Since our area of interest is a semi-circle with radius r we convert the above formula to polar coordinates,

$$E[Y] = \iint_{(x,y)\in D} y\, f_{X,Y}(x,y)\,dx\,dy$$

D denotes the semicircle

$$\therefore \quad = \int_0^{\pi}\int_0^{r} \frac{2}{\pi r^2}\, s(\sin\theta)\, s\, ds\, d\theta$$

$$= \frac{2}{\pi r^2}\int_0^{\pi} \sin\theta \left[\frac{s^3}{3}\right]_0^{r} d\theta$$

$$= \frac{2}{\pi r^2}\cdot\frac{r^3}{3}\left[-\cos\theta\right]_0^{\pi}$$

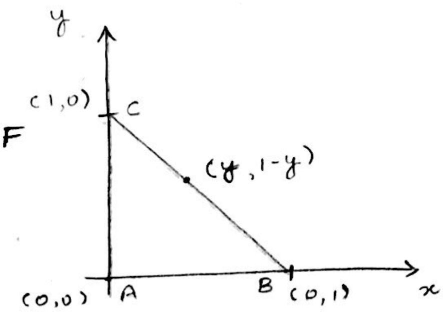
$$= \frac{4r}{3\pi}$$ which is same as Ans. in part (b)

Hence verified.

Prob. 23

(a) Area of the triangle = ½

This is the case of uniform joint PDF



$$\therefore f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of } \Delta}, & \text{for } (x,y)\in \Delta ABC \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2, & \text{for } (x,y)\in \Delta ABC \\ 0, & \text{otherwise} \end{cases}$$

23 (b) Marginal PDF of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 2\,dx$$

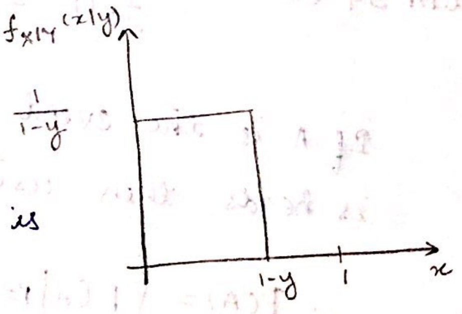
$$= 2(1-y). \quad \text{for} \quad 0 \le y \le 1$$

(c) We know,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y}, \text{ for } x \in [0, 1-y]$$

Basically, the term $\frac{1}{1-y}$ normalises the "slice" of the joint PDF for $Y=y$ so that the conditional PDF integrate to 1.

(d) From the diagram for the conditional PDF, we have



$$E[X|Y=y] = \frac{1-y}{2} \quad 0 \le y < 1$$

(since the conditional PDF, $f_{X|Y}(x|y)$ is a uniform PDF)

For $y=1$, X will be 0 and thus $E[X|Y=1]$ will also be 0.

Thus we have,

$$E[X|Y=y] = \frac{1-y}{2}, \quad 0 \le y \le 1$$

undefined, otherwise

To verify, $E[X|Y=y] = \int_0^{1-y} x f_{X|Y}(x|y)\,dx = \int_0^{1-y} \frac{x}{1-y}\,dx$

$$\Rightarrow \frac{1-y}{2} \text{ for } y \in [0,1] \quad \text{Hence verified.}$$

Using total expectation theorem we have,

$$E[X] = \int f_Y(y)\, E[X|Y=y]\, dy$$

$$= \int_0^1 \frac{1-y}{2} f_Y(y)\, dy$$

$$= \frac{1}{2} - \frac{1}{2}\int_0^1 y f_Y(y)\, dy = \frac{1-E[Y]}{2} \quad \text{Ans.}$$

23.(e) Since we have a symmetrical case,

$$E[X] = E[Y]$$

$$\therefore \quad E[X] = \frac{1-E[X]}{2}$$

$$\Rightarrow E[X] = \frac{1}{3} \quad \text{Ans.}$$

Problem 34 (a) $f_P(p) = \begin{cases} pe^p, & p \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

If A is the event that the first coin toss is heads, then using total probability theorem

$$P(A) = \int_0^1 P(A|P=p)\, f_P(p)\, dp$$

$$= \int_0^1 p \cdot pe^p\, dp$$

$$= \int_0^1 p^2 e^p\, dp$$

Let $u = p^2$ $\frac{dv}{dp} = e^p$

$\therefore \frac{du}{dp} = 2p$ $\quad v = \int e^p dp = e^p$

Doing Integration by parts, we get

$$\int p^2 e^p dp = p^2 e^p - \int e^p 2p\, dp = p^2 e^p - 2\int pe^p dp$$

$u = p \quad , \quad \frac{dv}{dp} = e^p$

$\frac{du}{dp} = 1 \qquad v = e^p$

$\therefore \int pe^p dp = pe^p - \int e^p dp$

$= pe^p - e^p$

$\therefore \int_0^1 p^2 e^p = \left[p^2 e^p - 2(pe^p - e^p) \right]_0^1$

$= e - 2 \quad$ Ans.

(b) A is the event that the coin resulted in heads

$\therefore f_{P|A}(p) = \frac{P(A|P=p)\, f_P(p)}{P(A)}$

$= \begin{cases} \frac{p^2 e^p}{e-2} & , \ 0 \le p \le 1 \\ 0 & , \ \text{otherwise} \end{cases}$ Ans.

(c) B is the event that second toss resulted in heads

$P(B|A) = \int_0^1 P(B|P=p, A)\, f_{P|A}(p)\, dp$

$= \int_0^1 P(B|P=p)\, f_{P|A}(p)\, dp$

$= \int_0^1 p \cdot \frac{p^2 e^p}{e-2}$

$= \frac{1}{e-2} \int_0^1 p^3 e^p \, dp$

After Integrating by parts we get,

$$P(B|A) = \frac{1}{e-2}(6-2e)$$

$$= 0.786 \text{ (approx.) Ans.}$$

2. OTHER QUESTIONS

1. Let T be the time at which the police captures the suspect.

A_x is the event that initially the suspect and the police are x unit apart

B_x is the event that after 1 second the suspect and the police are x units apart.

We can find $E[T]$ using total expectation theorem

If $x > 1$:

$$A_x = \underbrace{(A_x \cap B_x)}_{\text{If suspect moves away from police}} \cup \underbrace{(A_x \cap B_{x-1})}_{\text{If suspect does not move}} \cup \underbrace{(A_x \cap B_{x-2})}_{\text{If suspect moves towards the police}}$$

If $x = 1$:

$$A_1 = (A_1 \cap B_1) \cup (A_1 \cap B_0)$$

$\therefore$ Using total expectation theorem,

for $x > 1$

$$E[T|A_x] = P(B_x|A_x) \cdot E[T|A_x \cap B_x] + P(B_{x-1}|A_x) \cdot E[T|A_x \cap B_{x-1}] + P(B_{x-2}|A_x) \cdot E[T|A_x \cap B_{x-2}] \quad \text{... eq(i)}$$

Given: $P(B_1|A_1) = \left(\frac{1-p}{2}\right) \cdot 2$

$P(B_0|A_1) = p$

$E[T|A_1 \cap B_1] = 1 + E[T|A_1]$ ... eq(iii)

$E[T|A_1 \cap B_0] = 1$ ... eq(iv)

Substituting values of eq(iii) and eq(iv) in the following equation we get:

for $x = 1$

$$E[T|A_1] = P(B_1|A_1)\, E[T|A_1 \cap B_1] + P(B_0|A_1)\, E[T|A_1 \cap B_0]$$

$$= (1-p)(1 + E[T|A_1]) + p$$

$$= 1 - p + (1-p)\, E[T|A_1] + p$$

$$= 1 + (1-p)\, E[T|A_1]$$

$$\therefore E[T|A_1] = \frac{1}{p}$$

Using eq(i) for $x = 2$, we get

$$E[T|A_2] = P(B_2|A_2)\, E[T|A_2 \cap B_2] + P(B_1|A_2) \cdot E[T|A_2 \cap B_1] + P(B_0|A_2) \cdot E[T|A_2 \cap B_0]$$

$$= \left(\frac{1-p}{2}\right) E[T|A_2 \cap B_2] + p\, E[T|A_2 \cap B_1] + \left(\frac{1-p}{2}\right) E[T|A_2 \cap B_0]$$

... eq(v)

Clearly,

$E[T|A_2 \cap B_0] = 1$

$E[T|A_2 \cap B_1] = 1 + E[T|A_1]$

$E[T|A_2 \cap B_2] = 1 + E[T|A_2]$

Using the above values in eq(v) we get

$$E[T|A_2] = \left(\frac{1-p}{2}\right)\left(1+E[T|A_2]\right) + p\left(1+E[T|A_1]\right) + \left(\frac{1-p}{2}\right)\times 1$$

$$\Rightarrow E[T|A_2] = \left(\frac{1-p}{2}\right)\left(1+E[T|A_2]+1\right) + p\left(1+\frac{1}{p}\right)$$

$$\Rightarrow E[T|A_2] = (1-p) + \left(\frac{1-p}{2}\right)E[T|A_2] + p + 1$$

$$\Rightarrow E[T|A_2] = 2 + \frac{E[T|A_2]}{2} - \frac{pE[T|A_2]}{2}$$

$$\Rightarrow \frac{E[T|A_2]}{2} + \frac{p}{2}E[T|A_2] = 2$$

$$\Rightarrow \frac{E[T|A_2]}{2}(1+p) = 2$$

$$\Rightarrow E[T|A_2] = \frac{4}{1+p}$$

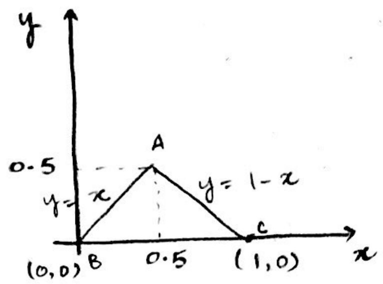
Similarly for $x>2$, we can write

$$E[T|A_x] = \left(\frac{1-p}{2}\right)\left(1+E[T|A_x]\right) + p\left(1+E[T|A_{x-1}]\right) + \left(\frac{1-p}{2}\right)\left(1+E[T|A_{x-2}]\right)$$

From total expectation theorem, the expected value of T for initial distance x and the given PMF $P_X(x)$ can be written as follows:

$$E[T] = \sum_x P_X(x)\, E[T|A_x] \qquad \text{Ans.}$$

(2)



The joint PDF in this case is uniform

$$\therefore \quad f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{area of } \Delta ABC} & \text{for } y \in [0, 0.5] \\ & \quad\; x \in [y, 1-y] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 4, & y \in [0, 0.5], x \in [y, 1-y] \\ 0, & \text{otherwise} \end{cases}$$

$$\left(\because \text{ Area of } \Delta ABC = \tfrac{1}{2} \times 1 \times 0.5 = \tfrac{1}{4} \right)$$

we know,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \ldots \text{eq(i)}$$

Marginal PDF $f_Y(y) = \int_x f_{X,Y}(x,y)\,dx$

$$= \int_y^{1-y} 4\,dx$$

$$= 4\,[x]_y^{1-y}$$

$$= 4(1-y-y)$$

$$= 4(1-2y)$$

Using eq(i) we get

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{4}{4(1-2y)} = \frac{1}{1-2y} \quad \text{Ans.}$$