Last time: The Dot Product

Given $\underline{u}$ & $\underline{v}$, $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$\underline{u} \cdot \underline{v} = ||\underline{u}|| \, ||\underline{v}|| \cos \theta$

$|\underline{u} \cdot \underline{v}| \leq ||\underline{u}|| \, ||\underline{v}||$ Schwartz Inequality

Now, let's prove the Triangle Inequality

$|a+b| \leq |a| + |b|$

$||\underline{u} + \underline{v}|| \leq ||\underline{u}|| + ||\underline{v}||$ ←

Look at $||\underline{u}+\underline{v}||^2 \leq (||\underline{u}|| + ||\underline{v}||)^2$

① $||\underline{u}+\underline{v}||^2 = (\underline{u}+\underline{v}) \cdot (\underline{u}+\underline{v})$

$= \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{u} + \underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v}$

$= ||\underline{u}||^2 + ||\underline{v}||^2 + 2\, \underline{u} \cdot \underline{v}$

② $(||\underline{u}|| + ||\underline{v}||)^2 = (||\underline{u}|| + ||\underline{v}||)(||\underline{u}|| + ||\underline{v}||)$

$= ||\underline{u}||^2 + ||\underline{v}||^2 + 2||\underline{u}|| \, ||\underline{v}||$

$\cancel{||\underline{u}||^2} + \cancel{||\underline{v}||^2} + \cancel{2}\, \underline{u} \cdot \underline{v} \leq \cancel{||\underline{u}||^2} + \cancel{||\underline{v}||^2} + \cancel{2}\, ||\underline{u}|| \, ||\underline{v}||$

Now, Does $\underline{u} \cdot \underline{v} \leq ||\underline{u}|| \, ||\underline{v}||$ hold?

$\underline{u} \cdot \underline{v} \leq |\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \, \|\underline{v}\|$

Schwartz Inequality

$\Rightarrow \|\underline{u}+\underline{v}\|^2 \leq (\|\underline{u}\| + \|\underline{v}\|)^2$ is true

$\Rightarrow \|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ is true,

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Linear Combination of Vectors.

Let $\underline{u}$, $\underline{v}$ + $\underline{w}$ be vectors of the Same Dimension.

Let a, b, c be any number

A linear combination is

$$a\underline{u} + b\underline{v} + c\underline{w} = \underline{z}$$

Another way to write it: Matrices

Let the matrix $\underline{A}$ have columns of $\underline{u}$, $\underline{v}$, $\underline{w}$:

$$\underline{A} = [\underline{u} \quad \underline{v} \quad \underline{w}] = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ \vdots & \vdots & \vdots \\ u_n & v_n & w_n \end{bmatrix}$$

ex.) If $\underline{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $\underline{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\Rightarrow \underline{A} = [\underline{u}\ \underline{v}\ \underline{w}] = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Now multiply $\underline{A}$ by another vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\underline{A} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [\underline{u}\ \underline{v}\ \underline{w}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\underline{u} + b\underline{v} + c\underline{w}$$

Another motivation? Linear System of equations.

Say that you need to solve

$$x_1 = b_1$$
$$-x_1 + x_2 = b_2$$
$$-x_2 + x_3 = b_3$$

b_1, b_2, b_3 are known
find x_1, x_2, x_3

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

More later.

Matrix Information

Matrices are classified by the number of rows & columns.

In the prior example, $\underline{A}$ is a 3×3 matrix.

In general matrices are $m \times n$

m rows n columns

The contents of $\underline{A}$ are denoted by a_{ij}

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

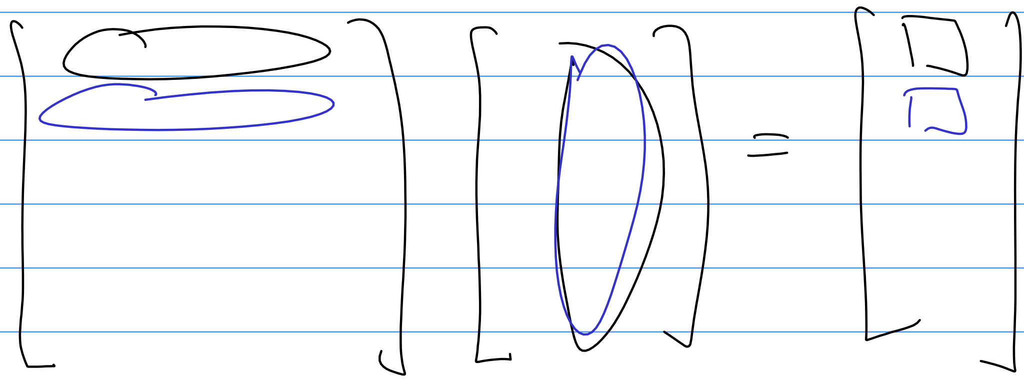
Matrix Operations

① Matrix - Vector Products

→ A matrix-vector product is nothing more than a linear combination of the matrix columns.

It might help to think of this as many dot products.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$



In Compact notation, for $\underline{A}\,\underline{x} = \underline{b}$

$$a_{ij}x_j = b_i$$

In such products, $\underline{A}$ & $\underline{x}$ must be compatible!

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$ ← Doesn't make sense

Let $\underline{A}$ be $m \times n$, $\underline{x}$ must be $\underline{n} \times 1$

$$(m \times n)(n \times 1) = m \times 1$$

2 Matrix - Matrix Products

I want to multiply one matrix $\underline{A}$ by another one $\underline{B}$

$\underline{A}\ \underline{B}$

Let $\underline{B}$ the matrix w/ column vectors:

$$\underline{B} = [\underline{b}_1\ \ \underline{b}_2\ \ \underline{b}_3\ \ \ldots\ \ \underline{b}_q]$$

Column vectors of length p.

$B \in p \times q$

$$\underline{A}\ \underline{B} = \underline{A}\,[\underline{b}_1\ \ \underline{b}_2\ \ \ldots\ \ \underline{b}_q]$$

$$= [\underline{A}\,\underline{b}_1\ \ \underline{A}\,\underline{b}_2\ \ \ldots\ \ \underline{A}\,\underline{b}_q]$$

If $\underline{B}$ is $p \times q$, $\underline{b}_1$ has size $p \times 1$

Thus $\underline{A}$ must be $m \times p$

I need to calculate $\underline{A}\,\underline{b}_1$

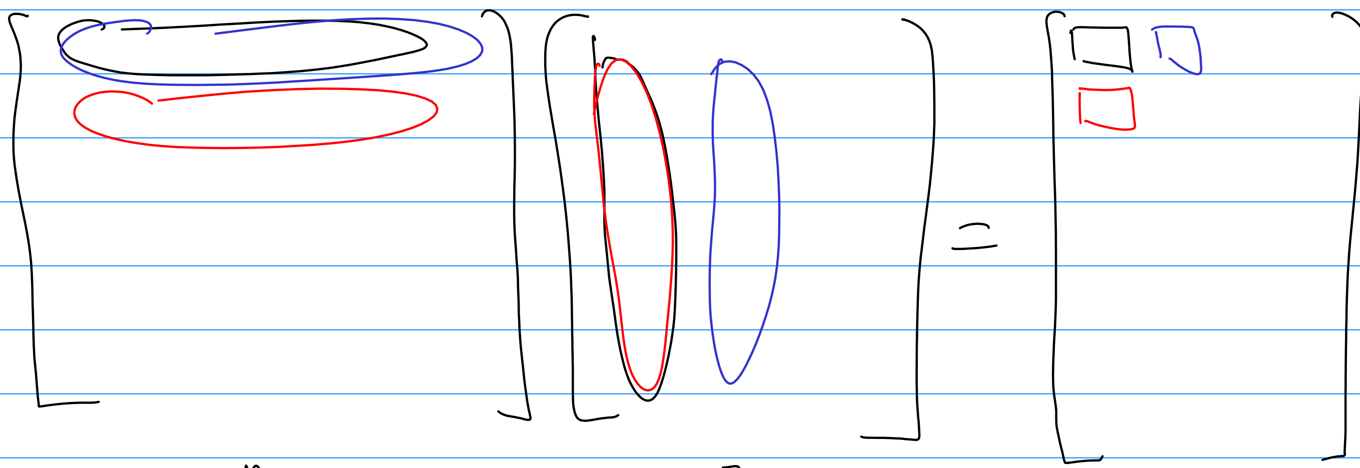
$\Rightarrow (m \times p)(p \times 1)$

To be compatible, $\underline{A}$ must be $m \times p$ &
$\underline{B}$ must be $p \times q$.

$\Rightarrow$ $\underline{A}\,\underline{B}$ is going to be

$$(m \times p)(p \times q) = m \times q$$

Mechanically: A bunch of dot products.



$\underline{A}$ $\underline{B}$

③ Matrix Addition

$\underline{A}$ & $\underline{B}$ $\boxed{\text{must}}$ be the same size.

$\underline{A} + \underline{B}$ $\rightarrow$ Add each individual component

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$a_{ij} + b_{ij} = c_{ij}$$

Matrix Rules

(1) Commutative Rule of Addition:

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}$$

(2) Scalar Distribution Rule:

$$c(\underline{A} + \underline{B}) = c\underline{A} + c\underline{B}$$

(3) Associative of Multiplication:

$$\underline{A}(\underline{B}\,\underline{C}) = (\underline{A}\,\underline{B})\underline{C}$$

(4) Matrix-matrix is <u>not</u> commutative,

In general, $\underline{A}\,\underline{B} \neq \underline{B}\,\underline{A}$

Ex.) Let $\underline{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \underline{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\underline{A}\,\underline{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{B}\,\underline{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

5. Matrix Distributive Properties:

Left: $\underline{C}(\underline{A}+\underline{B}) = \underline{C}\,\underline{A} + \underline{C}\,\underline{B}$

Right: $(\underline{A}+\underline{B})\underline{C} = \underline{A}\,\underline{C} + \underline{B}\,\underline{C}$

Vector: $(\underline{A}+\underline{B})\underline{x} = \underline{A}\,\underline{x} + \underline{B}\,\underline{x}$

More Matrix Stuff

1. Transpose: The transpose of a vector or matrix is given by the symbol ^{T} & is obtained by flipping row & columns.

Let $\underline{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\quad \underline{A} \in 2 \times 3$

$\underline{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\quad \underline{A}^T \in 3 \times 2$

$\underline{A}^T = (a_{ij})^T = a_{ji}$

For vectors: Let $\underline{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$\underline{x}^T = [1 \quad 2 \quad 3 \quad 4]$

You can write dot products using transpose,

$\underline{u} \circ \underline{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

$$\underline{u}^T \underline{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n$$

$((1 \times n)(n \times 1)) = 1 \times 1$

② Matrix Powers: $\underline{A}^p = \underbrace{\underline{A}\,\underline{A}\,\underline{A} \cdots \underline{A}}_{p \text{ times}}$

$\underline{A}^p \underline{A}^q = \underline{A}^{p+q} \qquad (A^p)^q = A^{pq}$

In scalar algebra, $c^0 = 1$

In linear algebra:

$\underline{A}^0 = \underline{I}$ ← Identity matrix

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrix:

$\underline{A}\,\underline{I} = \underline{A}$ $\quad\quad$ $\underline{I}\,\underline{A} = \underline{A}$ $\quad\quad$ $\underline{I}\,\underline{x} = \underline{x}$

(3) Matrix Inverse:

Let $\underline{A}$ be a square matrix $(n \times n)$

The inverse of $\underline{A}$ is denoted by $\underline{A}^{-1}$ such that

$$\underline{A}^{-1}\underline{A} = \underline{A}\,\underline{A}^{-1} = \underline{I} \qquad \left(\frac{1}{a}a = a\frac{1}{a} = 1\right)$$

You can write solutions $\underline{x}$ to $\underline{A}\,\underline{x} = \underline{b}$

$$\underline{A}^{-1}\underline{A}\,\underline{x} = \underline{A}^{-1}\underline{b}$$

$$\underline{I}\,\underline{x} = \underline{A}^{-1}\underline{b}$$

$$\underline{x} = \underline{A}^{-1}\underline{b} \quad \leftarrow \text{ more later}$$

(Note: Never actually compute $\underline{A}^{-1}$)

(4) Block Matrices

A regular matrix can be written as a bunch of column vectors,

Now can also write a matrix as a bunch of sub-matrices.

ex.) Let $\underline{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{A} = \begin{bmatrix} \underline{I} & \underline{I} & \underline{I} \\ \underline{I} & \underline{I} & \underline{I} \end{bmatrix}$ ← block matrix

$$\underline{A} = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

It simplifies calculations later on,

Regular Rules apply:

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{bmatrix} \begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix} = \begin{bmatrix} \underline{A}_{11}\underline{B}_{11} + \underline{A}_{12}\underline{B}_{21} & \cdots \\ \underline{A}_{21}\underline{B}_{11} + \underline{A}_{22}\underline{B}_{21} & \cdots \end{bmatrix}$$

Vector Independence

Let $\underline{a}_1, \underline{a}_2, \underline{a}_3$ be 3 vectors of the same dimension,

If the <u>only</u> linear combination of $\underline{a}_1, \underline{a}_2$ & $\underline{a}_3$ that results in the zero vector is:

$$0\,\underline{a}_1 + 0\,\underline{a}_2 + 0\,\underline{a}_3 = \underline{0},$$

then $\underline{a}_1, \underline{a}_2$ & $\underline{a}_3$ are independent.

One the other hand, if some non-trivial combination exists $\Rightarrow$ dependent.

ex.) Is $\underline{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\underline{a}_2 = \begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix}$ $\underline{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

independent? No!

$$2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \underline{1}\begin{bmatrix} -1 \\ -4 \\ -5 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{a}_3 = 2\,\underline{a}_1 + \underline{a}_2$$

This becomes important when the columns of a matrix $\underline{A}$ are independent or dependent.

If Columns are independent $\Rightarrow$ $\underline{A}^{-1}$ exists

" " " Dependent $\Rightarrow$ $\underline{A}^{-1}$ does not exist & we say that $\underline{A}$ is singular.

Vector Spaces & Subspaces

A vector space is the collection of vectors w/ the same dimension that follows a set of rules.

- The vector space of vectors of real numbers is $\mathbb{R}^n$.

ex.) $\underline{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is in $\mathbb{R}^3$

- Scalars live in $\mathbb{R}^1$ or simply $\mathbb{R}$.

  4π is in $\mathbb{R}$

- Complex vectors live in $\mathbb{C}^n$

  $\underline{u} = \begin{bmatrix} -i \\ 1+i \end{bmatrix}$ is in $\mathbb{C}^2$

- Real matrices of size $m \times n$ live in the vector space $\mathbb{R}^{m \times n}$. Typically denoted by $\underline{M}$

- Real functions live in some space F.

Rules of a Vector Space

Vector spaces are not only defined by what they hold, but how operations in that vector space occur,

"inside remains inside"

Definition of a Vector Space.

Let $\underline{x}, \underline{y}, \underline{z}$ be in a particular vector space, and let a & b be in $\mathbb{R}$ (any real number)

All vector spaces must obey:

① $\underline{x} + \underline{y} = \underline{y} + \underline{x}$ must be in the V. space

② $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$ must be in V. space

③ There is a unique "zero vector" such that $\underline{0} + \underline{x} = \underline{x} = \underline{x} + \underline{0}$

④ For every $\underline{x}$ there exists $-\underline{x}$ such that

$$\underline{x} + (-\underline{x}) = \underline{0} = (-\underline{x}) + \underline{x}$$

⑤ $a(\underline{x} + \underline{y}) = a\underline{x} + a\underline{y}$ (in V. space)

(6) $(a+b)\underline{x} = a\underline{x} + b\underline{x}$ " "

(7) $a(b\underline{x}) = b(a\underline{x})$ " "

(8) $1\underline{x} = \underline{x}$

If all of these rules are followed, the vector space is closed.

Example of non-closed space:

Let the space of continuous functions be $[0, 1]$ such that $f(1/2) = 1$

Check $(f+g)(1/2) = f(1/2) + g(1/2) = 1+1 = 2$

Another: Is all vectors in $\mathbb{R}^2$ such that $\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\begin{matrix} a \geq 0 \\ b \geq 0 \end{matrix}$

Is this a vector space? let $\underline{u} = \underline{0}$

$a(\underline{u}+\underline{v}) = (-1)(\underline{u}+\underline{v}) = -1\underline{0} + (-\underline{v})$
$= -\underline{v} = \begin{bmatrix} -a \\ -b \end{bmatrix}$