IVP + BVP

Many problems are IVP + BVP

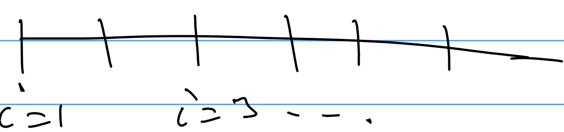
Time-evolving heat equation:

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0 \quad , \quad \alpha > 0$$

how the temperature, $u(\underline{x}, t)$, evolves over time due to conduction.

You combine IVP & BVP concepts.

1D Grid:



$$u(t_n, x_i) = u_i^n \leftarrow \text{time iteration, } \underline{\text{not}} \text{ power}$$

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0 \quad \text{F. Euler}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} = 0 \quad \text{B. Euler}$$

Many other schemes:

ex.) let $\frac{du}{dt} = F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right)$

Denote $F_i^n = F\left(x_i, t_n, u_i^n, \frac{\partial u_i^n}{\partial x}, \frac{\partial^2 u_i^n}{\partial x^2}, \ldots\right)$

F. Euler $\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^n$ $O(\Delta t)$

B. Euler $\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^{n+1}$ $O(\Delta t)$

Crank-Nicholson $\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left(F_i^n + F_i^{n+1}\right)$

$O(\Delta t^2)$

Root Finding & nonlinear equations of one variable.

Recall that $g(x)$ is a linear function if $g(x) + g(y) = g(x+y)$

ex.) $g(x) = ax$, $a \in \mathbb{R}$.

$g(x) + g(y) = ax + ay = a(x+y) = g(x+y)$ ← linear

ex.) $g(x) = ax^2$ $a \in \mathbb{R}$.

$g(x) + g(y) = ax^2 + ay^2 \neq a(x+y)^2 = g(x+y)$ ← nonlinear

Look at methods of solving $g(x) = a$

Write as $f(x) = g(x) - a$,

then find the root of $f(x)$:

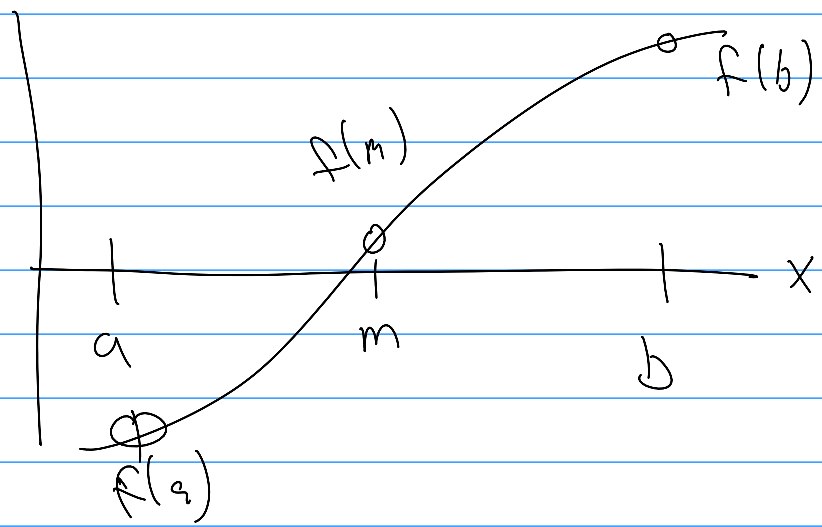
$f(x) = 0$

Methods:
1. Bisection
2. Regula Falsi
3. Newton - Rhapson
4. Secant Method
5. Fixed point.

(1) Bisection method: Requires that a zero exists in the range [a, b]

How can I check? Require that

$$f(a)f(b) < 0$$



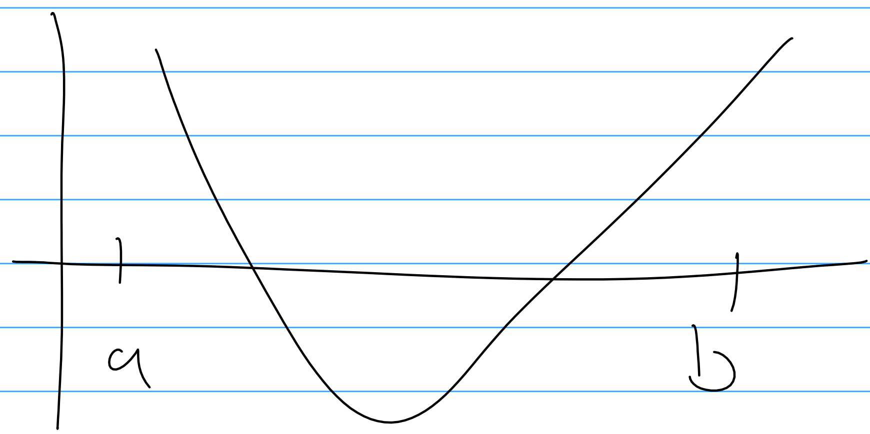
Let $m = \frac{a+b}{2}$, If $f(a)f(m) < 0$, redo with range [a, m], otherwise redo w/ range [m, b]

Keep doing this iteration until $|f(m)| < \varepsilon$

Convergence is linear, as the bound on the root goes down by 1/2 each iteration.

Issue: You <u>must</u> have $f(a)f(b) < 0$ for this to work.

$\Rightarrow$ If $f'(x) = 0$ anywhere in $[a, b]$, you might miss the root:



<u>But</u>, if $f(a)f(b) < 0$ is true, then this will <u>always</u> find the root (if slowly)

If multiple zeros, no idea which one it will choose.

(2) Regula Falsi (rule of false position)

Modification of Bisection method,

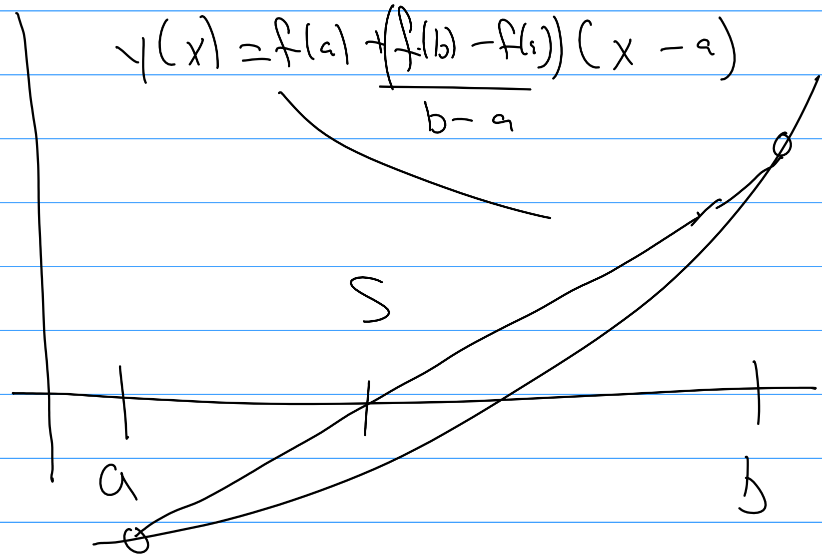
let $f(a)f(b) < 0$ for range $[a, b]$

Instead of checking $[a, \frac{b+a}{2}]$,

check if $f(a)f(s) < 0$ where

$$s = b - \left(\frac{b-a}{f(b)-f(a)}\right) f(b)$$

$$y(x) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$$



$$\Rightarrow y(s) = 0 \Rightarrow s = \frac{b f(a) - a f(b)}{f(a) - f(b)}$$

$$= b - \left(\frac{b-a}{f(b)-f(a)}\right) f(b)$$

If $f(a)f(s) < 0$, use range $[a, s]$ for next iteration.

Otherwise, use $[a, b]$

Still linear convergence

If $f(a)f(b) < 0 \rightarrow$ Guaranteed to converge.

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③ Newton - Rhapson Method

Let x_1 be a point near the root, x^*.

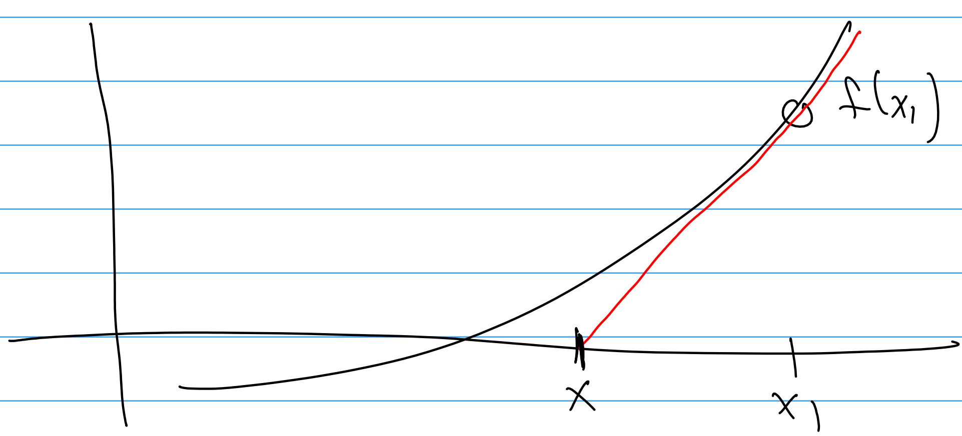
Then $f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{(x-x_1)^2}{2} f''(x_1) + \cdots$

Approximate to

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + O(x - x_1)^2$$

Solve $0 = f(x) = f(x_1) + f'(x_1)(x - x_1)$

$$\Rightarrow \quad x = x_1 - \frac{f(x_1)}{f'(x_1)}$$



$\Rightarrow$ Iteration is then

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Look at convergence:

Let x^* be the true root and x_n, x_{n+1} be estimates at iterations n, $n+1$, such that

$$|x^* - x_n| = \delta << 1$$

Define errors $e_n = x^* - x_n$ & $e_{n+1} = x^* - x_{n+1}$

We know that

$$\rightarrow \quad 0 = f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(\xi)}{2}(x^* - x_n)^2$$

for some $\xi \in (x^*, x_n)$ such that

$$\frac{f''(\xi)}{2}(x^* - x_n)^2 = \frac{f''(x_n)}{2}(x^* - x_n)^2 + \text{H.O.T.},$$

In Newton - Rhapson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow \underline{f(x_n)} = f'(x_n)(x_n - x_{n+1})$$

Then

$$0 = f'(x_n)(x_n - x_{n+1}) + f'(x_n)(x^* - x_n) + \frac{f''(\xi)}{2}(x^* - x_n)^2$$

$$0 = f'(x_n)(x^* - x_{n+1}) + \frac{f''(\xi)}{2}(x^* - x_n)^2$$

$$0 = f'(x_n)\, e_{n+1} + \frac{f''(\xi)}{2}\, e_n^2$$

$$\Rightarrow e_{n+1} = -\frac{f''(\xi)}{2f'(x_n)}\, e_n^2$$

$\Rightarrow e_{n+1} \propto e_n^2 \leftarrow$ 2nd-order convergence,

If $e_n \sim 10^{-3}$

$e_{n+1} \sim 10^{-6}$

$e_{n+2} \sim 10^{-12}$

Issues:

- Your initial guess at the root, x_0, must be "close" to x^*,

This means that you need to check for divergence as well as convergence

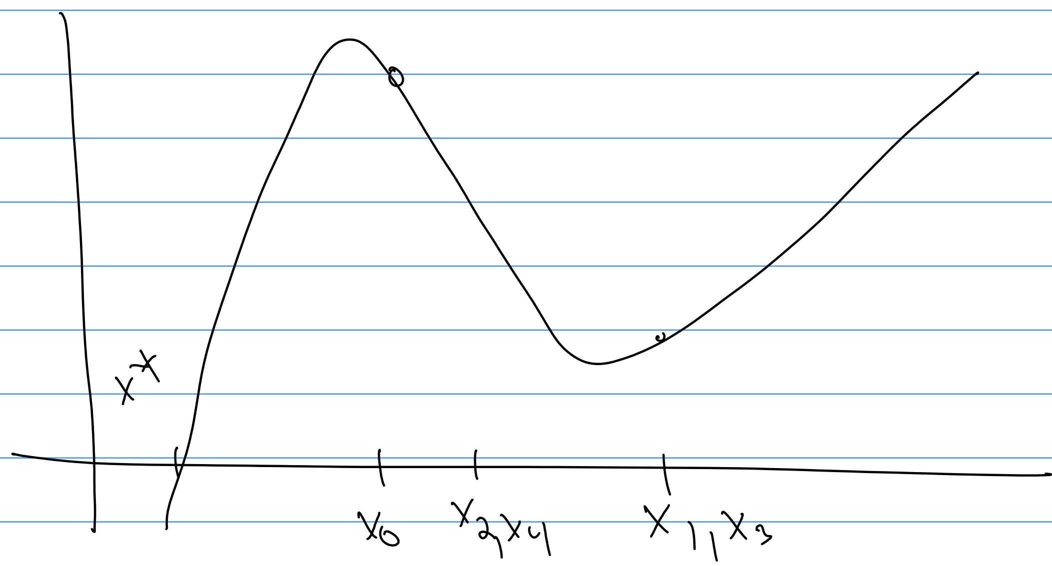
So, converge if $|f(x_i)| < \varepsilon$

Divergence if # of iterations is large
or if $|f'(x_i)| < \varepsilon$
or if $|x_{i+1} - x_i| > \delta$

- Why check if $|f'(x_i)| < \varepsilon$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- You may never converge if x_0 is too far from x^* or if you get oscillations



- Last issue: $f'(x)$ might be difficult or expensive to compute.

(4) Secant Method → Tries to alleviate issue if $f'(x)$ is not known or expensive.

Approximate $f'(x_i) \simeq \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$

Then $x_{i+1} = x_i - \left(\dfrac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \right) f(x_i)$

Rate of convergence is $\dfrac{1+\sqrt{5}}{2} \simeq 1.62$

Some of the same issues as Newton Method.

(5) Fixed Point Method

A fixed point is one where

$x = h(x)$ $\qquad y(x) = \sqrt{x} \quad , x = 1$

let $f(x)$ have a linear & nonlinear part:

$f(x) = ax + g(x) \Rightarrow f(x) = 0$ as

$ax + g(x) = 0 \Rightarrow x = -\dfrac{1}{a} g(x)$

Iteration is then $x_{i+1} = -\frac{1}{a} g(x_i)$

Converge if $|x_{i+1} - g(x_{i+1})| < \varepsilon$

or $|f(x_{i+1})| < \varepsilon$

Convergence

Let x^* be the root such that $x^* = g(x^*)$
w/ $x_{n+1} = g(x_n)$

Then $x^* - x_{n+1} = g(x^*) - g(x_n)$

and $\frac{e_{n+1}}{e_n} = \frac{x^* - x_{n+1}}{x^* - x_n} = \frac{g(x^*) - g(x_n)}{x^* - x_n}$

The mean-value theorem of calculus states that if $g(x)$ is continuous over $[x^*, x_n]$, there exists a $\xi \in [x^*, x_n]$ such that

$$g'(\xi) = \frac{g(x^*) - g(x_n)}{x^* - x_n}$$

$\Rightarrow \frac{e_{n+1}}{e_n} = g'(\xi) \Rightarrow e_{n+1} = g'(\xi)\, e_n$

$\Rightarrow$ Convergence if $|g'(\xi)| < 1$

Systems of Nonlinear Equations

Find the roots of!

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}\right\} \quad \underline{f}(\underline{x}) = \underline{0}$$

(1) If possible, write as a fixed point iteration $\underline{x} = g(\underline{x})$

ex.)

$$\begin{aligned} 2x_1 - x_2^2 + x_3^4 &= 1 \\ x_1^2 + x_2 - x_3^3 &= 0 \\ \sin(x_1) + \tan(x_2) + x_3 &= 1 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= \tfrac{1}{2} x_2^2 - \tfrac{1}{2} x_3^4 - 1 \\ x_2 &= -x_1^2 + x_3^3 \\ x_3 &= 1 - \sin(x_1) - \tan(x_2) \end{aligned}$$

$$\underline{x} = g(\underline{x})$$

$$\Rightarrow \underline{x}_{i+1} = \underline{g}(\underline{x}_i)$$

Iterate until $\|\underline{x}_{i+1} - \underline{g}(\underline{x}_{i+1})\|_p < \varepsilon$

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