Final Exam, Part II : EAS 596

Shubham Sharma
Person No.: 50290293
Professor: Dr. David Salac

December 18, 2018

Question 1: (a) The solution of the initial value problem using a fourth order Runge-Kunga method: *ode*45 comes out to be 0.367891 at time step number 57.

(b)

1(b) Given initial value problem:

$$\dot{y} = \begin{cases} y\left(-2t + \frac{1}{t}\right), & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$$\therefore \frac{dy}{dt} = y\left(-2t + \frac{1}{t}\right)$$

$$\Rightarrow \frac{dy}{y} = -2t\,dt + \frac{dt}{t}$$

Above is the integral problem for the given initial value problem.

Integrating both sides,

$$\ln y + c_1 = -t^2 + \ln t + c_2$$

$$\Rightarrow \ln\left(\frac{y}{ct}\right) = -t^2 \quad \text{[Combining all constants as c]}$$

$$\Rightarrow y = c\,t\,e^{-t^2} \quad \ldots \text{eq(i)}$$

To determine the integration constant, c

$$\frac{dy}{dt} = c\left(e^{-t^2} - 2t^2 e^{-t^2}\right) \quad \ldots \text{eq(ii)}$$

we know, $\left.\frac{dy}{dt}\right|_{t=0} = 1$

$\therefore \; c(e^{0} - 0) = 1 \quad \Rightarrow \quad c = 1$

Thus, we get

$$y = te^{-t^2}$$

$$\dot{y} = (1 - 2t^2)\, e^{-t^2}$$

(c) Exact solution, $y(1)$

$$y(1) = 1 \cdot e^{-(1)^2}$$

$$= 0.3678$$

(d) The solution obtained using 2 point Guass Quadrature rule is: 0.367885 for number of guass intervals = 4. Thus, the number of points needed for 2 point Guass Quadrature Rule to get within 1% accuracy of the exact solution is 4 whereas the number of timesteps required using $ode45$ is 57. The function to calculate Guass Quadrature is used from the homework solutions.

Question 2:

(a)

Given Equations:

$$x = L_1 \cos\theta_1 + L_2 \cos\theta_2$$
$$h = L_1 \sin\theta_1 + L_2 \sin\theta_2$$

θ_1, L_1, L_2 and h are given and unknowns are θ_2 and x.

The following equations will be the system of non-linear equations for which we use Root finding methods such as Newton-Raphson:

$$x - L_1 \cos\theta_1 - L_2 \cos\theta_2 = 0 \Rightarrow f_1(x, \theta_2) = 0$$
$$h - L_1 \sin\theta_1 - L_2 \sin\theta_2 = 0 \Rightarrow f_2(x, \theta_2) = 0$$

Thus, our root problem has the above two equations and our system of non-linear equation looks as follows:

$$\underline{f}(\underline{x}) = \underline{0} \qquad \underline{x} = \begin{bmatrix} x \\ \theta_2 \end{bmatrix}$$

$$\underline{f}(\underline{x}) = \begin{bmatrix} x - L_1 \cos\theta_1 - L_2 \cos\theta_2 \\ h - L_1 \sin\theta_1 - L_2 \sin\theta_2 \end{bmatrix}$$

(b) ii: The function newton_sys.m, to solve the root problem for the given nonlinear system of equation using Newton's method is taken from the homework solutions.

(b) iii: The required plot is shown in figures 1.

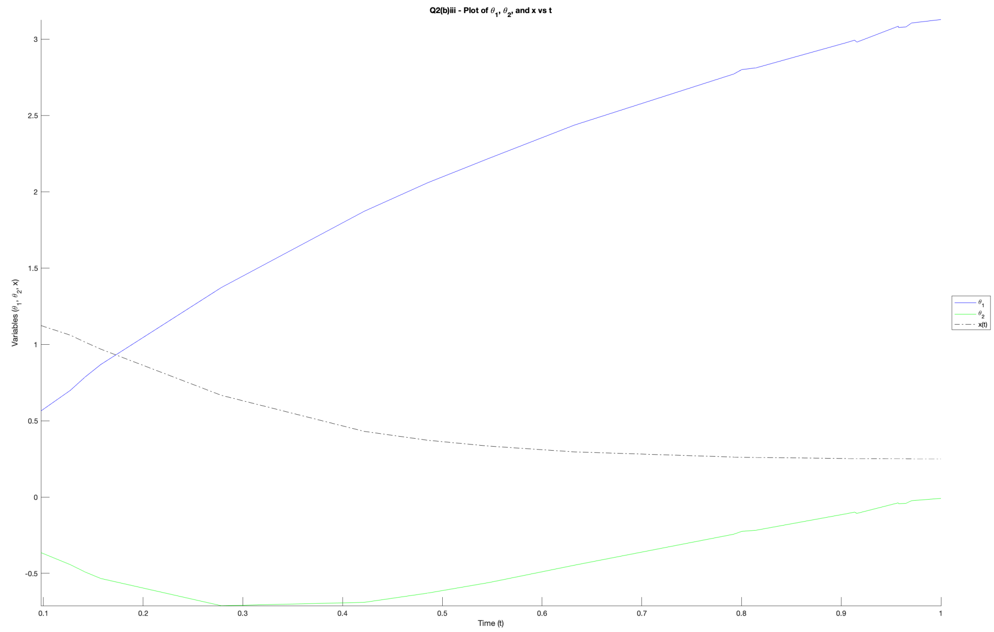


Figure 1: Q2(b) iii: Plot for θ_1, θ_2, x vs. t

(b) vi: The work done by the piston is -29.592988.

(b) v: The plot for computed piston force vs. x is shown in figure 2.

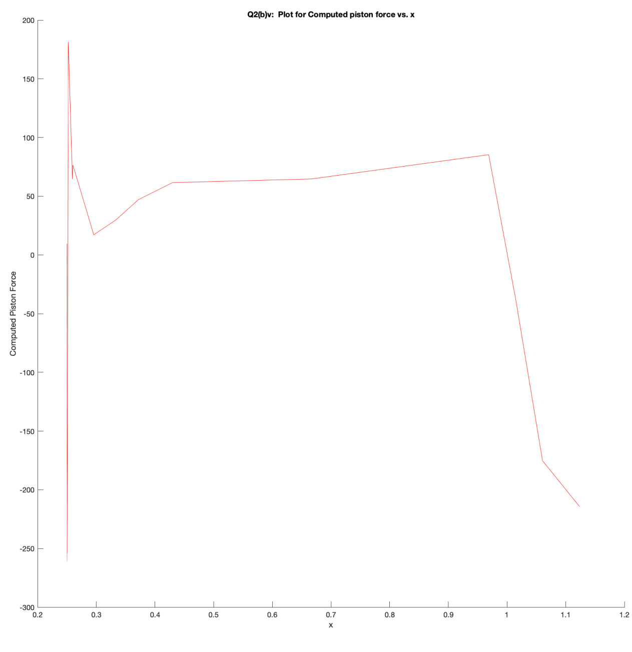


Figure 2: Q2(b) v: Computed piston force vs. x

(c) The coefficients c_1 and c_2 are 6.2185 and 0.9826 respectively. The plot for the data and the best curve fit is shown in figure 3.

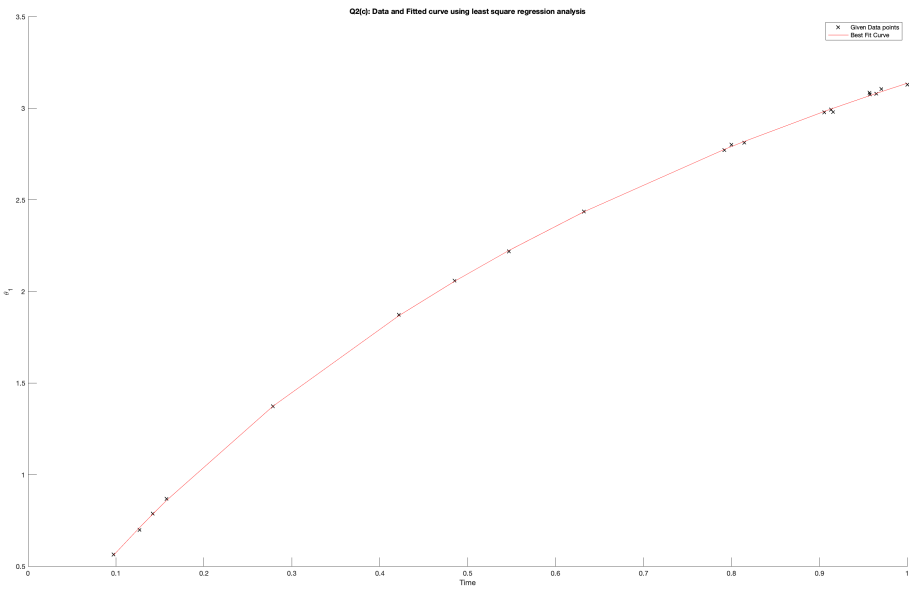


Figure 3: Q2(c): Data and the best curve fit for θ_1 using least square regression analysis

(d) The work done by the piston when using the best curve fit values for θ_1 is -20.102428. The required plots are shown in figures 4 and 5.

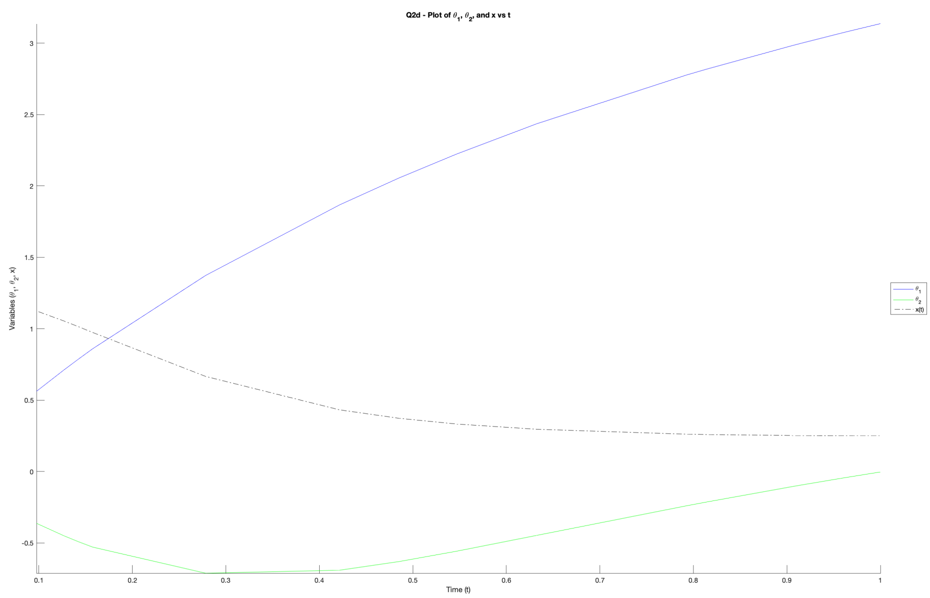


Figure 4: Q2d iii: Plot for θ_1, θ_2, x vs. t in case of best curve fit values

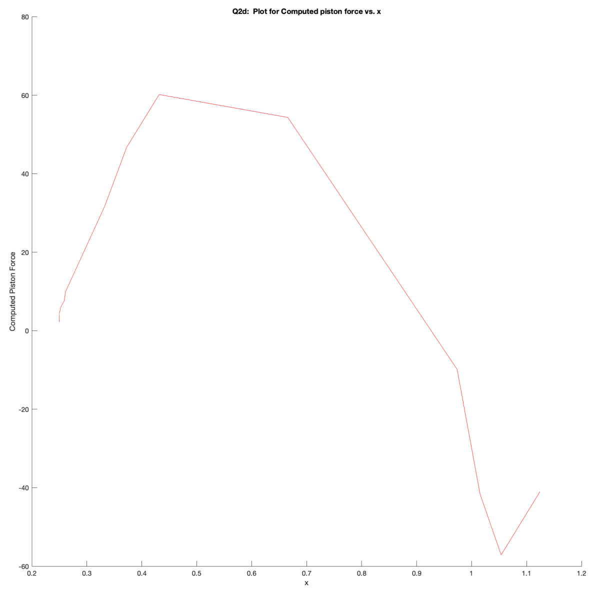


Figure 5: Q2d v: Computed piston force vs. x in case of best curve fit values

Question 3:

3 (a) In the given question points on the interface is $\vec{q}$ where as $\vec{p}$ is any point in the two-dimensional space.

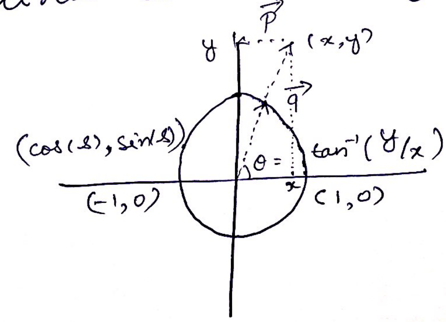
Thus, in two-dimensional space the point which is closest to $\vec{p}$ will have minimum value of the distance from $\vec{p}$.

Thus the function is denoted as $|\vec{q}-\vec{p}|$.

3(b) $x(s) = \cos(s)$

$y(s) = \sin(s) \qquad s \in [0, 2\pi]$

Thus, our parametric interface here is a unit circle centred at origin $(\sin^2 x + \cos^2 x = 1)$



Clearly, $|\vec{q}| = 1$

$|\vec{q}-\vec{p}|$ will be minimum in the case drawn in the figure (i.e. when the points p, q & $(0,0)$

lie on the same line).

$\therefore \; \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \; \vec{q}$ becomes $\cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$$= \frac{x}{\sqrt{x^2+y^2}}\,\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{j}$$

(C) Distance between points p and q,

$$d = \sqrt{(x-\cos(s))^2 + (y-\sin(s))^2}$$

$$\frac{d(d)}{ds} = 0$$

$$\Rightarrow 0 = \frac{1}{2} \times \frac{2((x-\cos(s)) \cdot \sin(s) + 2(y-\sin(s))(-\cos(s))}{\sqrt{(x-\cos(s))^2 + (y-\sin(s))^2}}$$

$$\Rightarrow \quad x\sin(s) - \sin(s)\cos(s) - y\cos(s) + \sin(s)\cos(s) = 0$$

$$\Rightarrow \quad x\sin(s) = y\cos(s)$$

$$\Rightarrow \quad \tan(s) = \frac{y}{x} \quad \text{or } s = \tan^{-1}\left(\frac{y}{x}\right)$$

which is same as the θ obtained in part (b) for point q to be closest to point p

(d) The analytic solution for $\vec{q}$ corresponding to $\vec{p} = (4, 2)$ is (0.8944, 0.447217) which is same as the one obtained using $fminbnd$ which is $(0.894427, 0.447214)$. The distance between $\vec{q}$ and $\vec{p}$ in both the cases is 3.4721.

(e) The closest point $\vec{q}$ to the point $\vec{p}$ using Newtons method(initial guess is taken to be 0.5) is $(0.893410, 0.449242)$ which is quite similar to the previous case.

(f) It is clear from figure 6 that different points are obtained for different initial guesses.

Closest point $\vec{q}$ to the point $\vec{p}$ using Newton's method for Initial Guess s = 0: (0.943991, 0.381088) (red point in the graph). This is the best guess and we get the point on the interface which is closest to the point $\vec{p}$. The initial guess in this case is (1.5,0).

Closest point $\vec{q}$ to the point $\vec{p}$ using Newton's method for Initial Guess s = $\pi/2$: (0.353375, 0.353733) (blue point in the graph). In this case, the solution gets trapped at a local maxima, when we start at an initial guess of (0,1.5).

Closest point $\vec{q}$ to the point $\vec{p}$ using Newton's method for Initial Guess s = $5\pi/4$: (-0.353553, -0.353553) (green point in the graph). In this case, the solution gets trapped at a local minima, when we start at an initial guess of $(-1/2\sqrt(2),-1/2\sqrt(2))$.

The required plot is shown in figures 6.

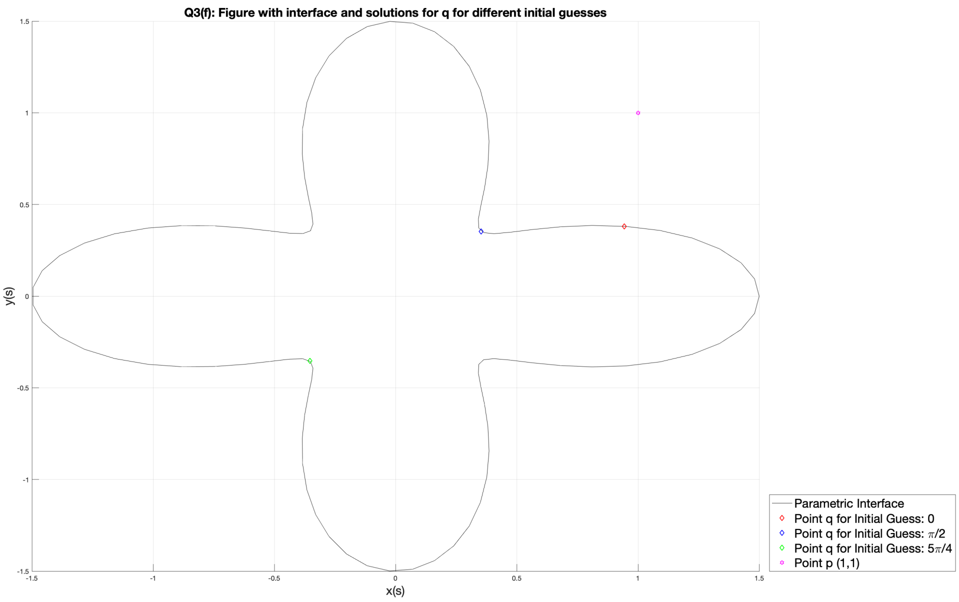


Figure 6: Q3f : Figure with interface and solutions for $\vec{q}$ for different initial guesses