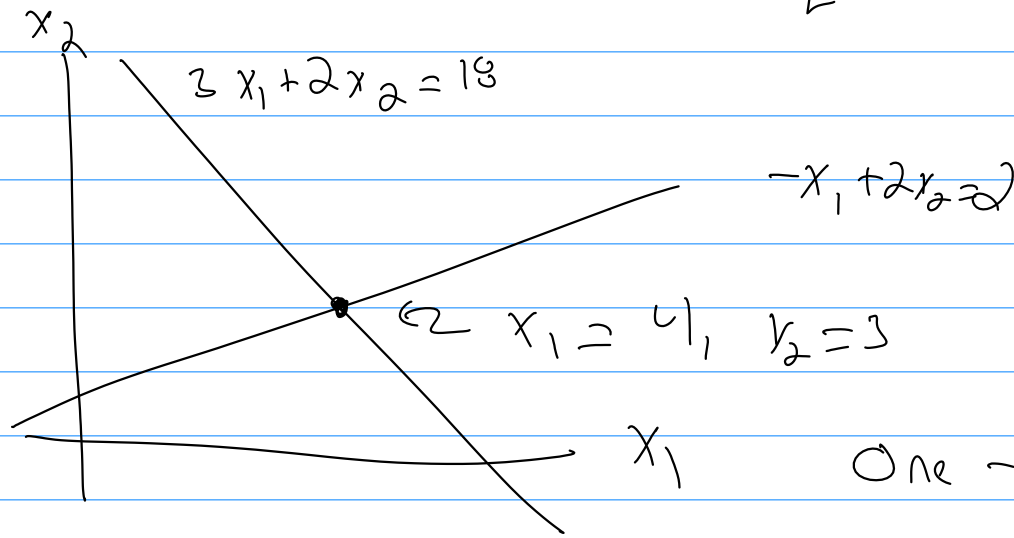
Existence & Uniqueness

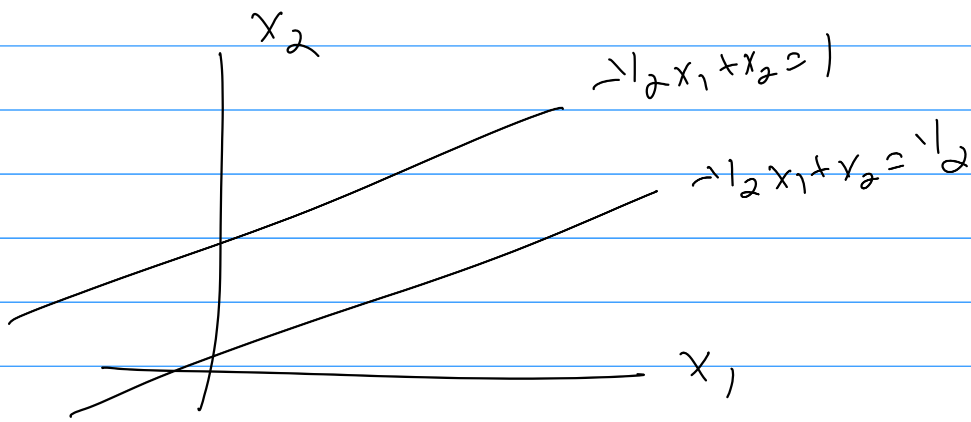
If I need to solve $\underline{A}\underline{x} = \underline{b}$ for $\underline{x}$, when does a solution exist?

Ex.) 2x2 : $\begin{aligned} -x_1 + 2x_2 &= 2 \\ 3x_1 + 2x_2 &= 18 \end{aligned} \Rightarrow \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$



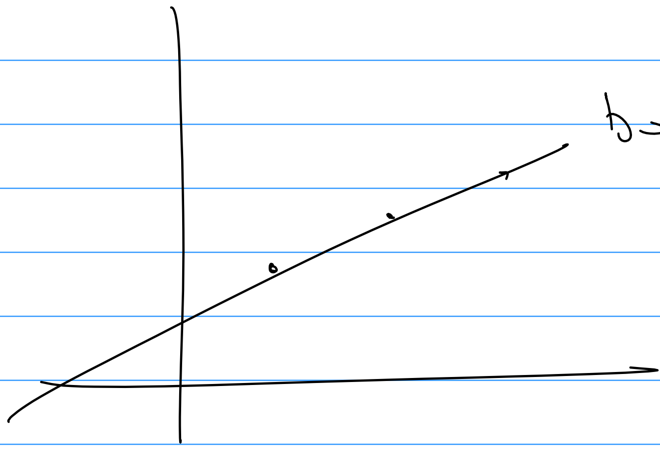
One solution

ex.) $-\frac{1}{2}x_1 + x_2 = 1$ & $-\frac{1}{2}x_1 + x_2 = \frac{1}{2}$



No Solution

ex.) $-\frac{1}{2}x_1 + x_2 = 1$ & $-x_1 + 2x_2 = 2$



both ∞ solutions

3 possibilities: 0, 1, or ∞ solutions

One measure of at least one solution exists is the Determinant of the matrix.

$\det(\underline{A}) = |\underline{A}|$ has a recursive definition:

$\underline{A} = a_{11}$ $\quad |a_{11}| = a_{11}$

2×2: $\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{21}a_{22}$

3×3 : $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$

$$= a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12}\underbrace{\det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right)}_{M_{ij}}$$

$$+ a_{13}\det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right)$$

In general: $\det(\underline{A}) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij}$

Pick an i (usually $i=1$) $\quad = \sum_{j=1}^{n} a_{ij} C_{ij}$

Cofactor

Another method:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ & + a_{13}a_{21}a_{32} - a_{31}a_{32}a_{13} \\ & - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{aligned}$$

Let $\underline{I}_n$ = $n \times n$ identity matrix

$$\det(\underline{A}\,\underline{B}) = \det(\underline{A})\det(\underline{B})$$

$$\det(\alpha \underline{I}_n) = \alpha^n \quad , \alpha \in \mathbb{R}$$

$$\det(\alpha \underline{A}) = \det(\alpha \underline{I}_n \underline{A}) = \det(\alpha \underline{I}_n)\det(\underline{A})$$
$$= \alpha^n \det(\underline{A})$$

$$\Rightarrow \det(\underline{A}^{-1}) = \frac{1}{\det(\underline{A})}$$

$$\det(\underline{A}^T) = \det(\underline{A})$$

Normally, use $\det(\underline{A})$ as a check if $\underline{A}^{-1}$ exists.

If $\det(\underline{A}) \neq 0$, then $\underline{A}^{-1}$ exists.

ex.) Spring - Mass

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$\underline{A} \quad \underline{x} = \underline{f}$

k = spring Constant.

A solution for $\underline{x}$ exists iff $\underline{x} = \underline{A}^{-1} \underline{f}$

$\underline{A}^{-1}$ exists iff $\det(\underline{A}) \neq 0$

$\det(\underline{A}) = 4k^2 - k^2 = 3k^2$

So long as $k \neq 0$, $\det(\underline{A}) \neq 0$

Actual springs have $k > 0$

Let $k = 1$, $f_1 = 1$, $f_2 = 2$

find x_1 & x_2

Write $\underline{A}\underline{x} = \underline{b}$ as an augmented matrix:

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -1 & 2 & 2 \end{array}\right]$$

$\underline{A} \quad \underline{f}$

$2x_1 - x_2 = 1$

$-x_1 + 2x_2 = 2$

Three operations:

1) Row Scaling
2) Row addition
3) Row Swap

Goal is to get : $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow x_1 = a \qquad x_2 = b$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 2 \end{bmatrix} \leftarrow * 2$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -2 & 4 & 4 \end{bmatrix} \downarrow +$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 5 \end{bmatrix} \leftarrow * 1/3$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 5/3 \end{bmatrix} \uparrow +$$

$$\begin{bmatrix} 2 & 0 & 8/3 \\ 0 & 1 & 5/3 \end{bmatrix} \leftarrow * 1/2$$

$$\begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 5/3 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 4/3 \\ x_2 = 5/3 \end{matrix}$$

pivots

Form of $\begin{bmatrix} a & b & c \\ 0 & d & e \end{bmatrix}$ $a, b, c, d, e \in \mathbb{R}$

→ Called row echelon form (ref)

Form of $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \end{bmatrix}$

reduced row echelon form (rref)

Another example!

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 4 \end{array}\right]$$

Row Reduce!

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 + x_3 = 0$$
$$x_2 + x_3 = 0$$
$$0 x_3 = 1 \leftarrow ???$$

This exists because a.) $\det(A) = 0$, A^{-1} does not exist

b.) Non-trivial null space.

Nullspace is all vectors $\underline{v}$ such that $\underline{A}\,\underline{v} = \underline{0}$

In this case, $\underline{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$$\underline{A}\,\underline{v} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 2+1-3 \\ 1+1-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ The columns of $\underline{A}$ are <u>not</u> independent!

Four Sub-spaces of a matrix:

1) Column Space
2) (left) null space
3) Row Space
4) Right null space.

Column Space

Recall that a matrix-vector product is simply a linear combination of the matrix columns:

$$\underline{A}\,\underline{x} = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underline{b}$$

$$\underline{b} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$$

All possible vectors $\underline{b}$ exist in the column space, $\underline{C}(\underline{A})$, of the matrix

The column space is a subspace of $\mathbb{R}^m$.

The column space is very important when solving $\underline{A}\,\underline{x} = \underline{b}$.

Thm: The system $\underline{A}\,\underline{x} = \underline{b}$ has at least one solution iff $\underline{b}$ is in the column space of $\underline{A}$.

Notice: I said at least one solution.

Case #1: $\underline{A}^{-1}$ exists.

$\Rightarrow$ $\underline{A}\,\underline{x} = \underline{b}$ w/ $\underline{b}$ in $C(\underline{A})$

then $\underline{x} = \underline{A}^{-1}\underline{b}$ $\Rightarrow$ a solution
$\Rightarrow$ any $\underline{b}$ is a valid rhs.

If $\underline{A} \in M_{nn}$ (square matrix)

$\Rightarrow$ If $\underline{A}^{-1}$ exists, then $\underline{A}$ has n independent columns, which means it spans all of $\mathbb{R}^n$.

ex.) $\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Case 2: $\underline{A}^{-1}$ does not exist
$\Rightarrow$ $\underline{b}$ must be in $C(\underline{A})$ to have a solution.

ex.) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Can I write $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x = 1,\ y = 1,\ z = 0$

$$(1)\begin{bmatrix}1\\0\\1\end{bmatrix} + (1)\begin{bmatrix}0\\1\\-1\end{bmatrix} + (0)\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

What about

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\y\end{bmatrix} = \underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\underline{b}}$$

Is $\underline{b}$ in $C(\underline{A})$? $\underline{\underline{\text{No}}}$

$\Rightarrow$ no solution

no x, y, z such that

$$x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}0\\1\\-1\end{bmatrix} + z\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Nullspace

Another important subspace is the nullspace, given by $N(\underline{A})$

The nullspace is all vectors $\underline{v}$ such that

$$\underline{A}\,\underline{v} = \underline{0}$$

Is this a vector space?

Let $\underline{v}$ & $\underline{w}$ be in $N(\underline{A})$

$$\underline{A}\underline{v} = \underline{0} \quad \& \quad \underline{A}\,\underline{w} = \underline{0}$$

① $\underline{A}(\underline{v}+\underline{w}) = \underline{A}\underline{v} + \underline{A}\,\underline{w} = \underline{0} + \underline{0} = \underline{0}$

② $\underline{A}(c\underline{v}) = c(\underline{A}\underline{v}) = c\,\underline{0} = \underline{0}$

The nullspace is always non-empty,

$\underline{0}$ is always in the nullspace,

$$\underline{A}\,\underline{0} = \underline{0}$$

If $\underline{A}^{-1}$ exists, then

$$\underline{A}\,\underline{v} = \underline{0} \Rightarrow \underline{v} = \underline{A}^{-1}\underline{0} = \underline{0}$$

$\Rightarrow$ If $\underline{A}^{-1}$ exists, the only vector in $N(\underline{A})$ is $\underline{0}$

If the nullspace has any vector in addition to $\underline{0}$, $\underline{A}^{-1}$ does not exist.

Ex1) To get $N(\underline{A})$,

$$\underline{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

"Solve" $\underline{A}\,\underline{x} = \underline{0}$

Two options: If I get $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(\underline{A}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\Rightarrow$ Only trivial $N(\underline{A})$ $\Rightarrow$ $\underline{A}^{-1}$ exists

Option #2: Get something like

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} \downarrow -2* \text{ then } + \\ \\ \\ \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} \\ \downarrow + \\ \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{non-trivial } N(A)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

↑ free column / variable

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $x_3 = c$ $(c \in \mathbb{R})$

Make it easy: let $c = 1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 - 1 = 0 \Rightarrow x_1 = 1$$
$$x_2 - 2 = 0 \Rightarrow x_2 = 2$$

$\Rightarrow \underline{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a null-vector

such that $\underline{A}\,\underline{x} = \underline{0}$

$\Rightarrow$ So is $2\underline{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, $3\underline{x} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$, etc.

Nullspaces is not limited to Square matrices:

ex.) $\underline{A} = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix}$

Find $N(\underline{A})$: All vectors which span $N(\underline{A})$

Use rref!

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

← pivots

free columns

$x_1, x_3 =$ "fixed"
$x_2, x_4 =$ "free"

To determine the null space, set one free variable to $\underline{1}$, the others to zero. → find x_1 & x_3

Then set the other free variable to $\underline{1}$, w/ the rest to zero, find x_1 & x_3.

Set $x_2 = 1$ & $x_4 = 0$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 3 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -3 \\ x_3 &= 0 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ is in } N(\underline{A})$$

To get the other: Set $x_2 = 0$ & $x_4 = 1$

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ x_3 + 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -1$, $x_3 = -1$

$$\Rightarrow \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \text{ is in } N(\underline{A})$$

$$\Rightarrow N(\underline{A}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Any linear Combination is in $N(\underline{A})$