Systems of 1^{st}-order ODEs

$\dot{\underline{u}} = \underline{A}\,\underline{u} + \underline{f}(t) \rightarrow \underline{u}(t) = \underline{u}_h(t) + \underline{u}_p(t)$

(1) $\underline{u}_h(t) = \underline{x}\, e^{\lambda t} \rightarrow \dot{\underline{u}}_h = \underline{A}\,\underline{u}_h$

$\lambda \underline{x} e^{\lambda t} = \underline{A}\,\underline{x} e^{\lambda t} \Rightarrow \underline{A}\,\underline{x} = \lambda \underline{x}$

Cases for λ:

a.) Real & distinct, $\underline{u}_h = \underline{x}\, e^{\lambda t}$

b.) Real & repeated

If λ is complete (# of independent eigenvectors = # of repeated values of λ), then

$\underline{u}_h(t) = C_1 \underline{x}_1 e^{\lambda t} + C_2 \underline{x}_2 e^{\lambda t} + C_3 \underline{x}_3 e^{\lambda t} + \ldots$

If defective eigenvalues, w/ deficiency d (multiplicity of λ minus # of independent eigenvectors)

Need to do generalized eigen problem.

Let $\underline{x}_1$ be the solution to

$$(\underline{A} - \lambda \underline{I})^{d+1} \underline{x}_1 = 0$$

Then define the sequence

$$(\underline{A} - \lambda \underline{I})\, \underline{x}_1 = \underline{x}_2$$

$$(\underline{A} - \lambda \underline{I})\, \underline{x}_2 = \underline{x}_3$$

$$\vdots$$

$$(\underline{A} - \lambda \underline{I})\, \underline{x}_{d-1} = \underline{x}_d$$

Then $\underline{u}_{h_1}(t) = \underline{x}_1 e^{\lambda t}$

$$\underline{u}_{h_2}(t) = (\underline{x}_1 t + \underline{x}_2) e^{\lambda t}$$

$$\vdots$$

$$\underline{u}_{hd}(t) = \left(\frac{\underline{x}_1 t^{d-1}}{(d-1)!} + \dots + \underline{x}_{d-1} t + \underline{x}_d \right) e^{\lambda t}$$

(3) Complex roots: If $\underline{A}$ is real, then if $\lambda + i\mu$ is an eigen value, then so is $\lambda - i\mu$.

Eigenvectors: $\underline{a} + i\underline{b}$ & $\underline{a} - i\underline{b}$

$$(\underline{a} + i\underline{b})\, e^{(\lambda + i\mu)t} = (\underline{a} + i\underline{b})\, e^{\lambda t}\left[\cos(\mu t) + i \sin(\mu t)\right]$$

$$= \underbrace{e^{\lambda t}\left[\underline{a}\cos(\mu t) - \underline{b}\sin(\mu t)\right]}_{\underline{y}(t)} + i\, \underbrace{e^{\lambda t}\left[\underline{a}\sin(\mu t) + \underline{b}\cos(\mu t)\right]}_{\underline{z}(t)}$$

$$= \underline{y}(t) + i\, \underline{z}(t)$$

It turns out both $\underline{y}(t)$ & $\underline{z}(t)$ solve $\dot{\underline{u}}_h = \underline{A}\, \underline{u}_h$

$$\Rightarrow \underline{u}_h(t) = c_1\, \underline{y}(t) + c_2\, \underline{z}(t)$$

ex1) Solve $\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 2e^{-2t} \\ 3t \end{bmatrix}$

$$\dot{\underline{u}} = \underline{A}\,\underline{u} + \underline{f}(t)$$

(1) $\underline{u}_h(t)$: $\dot{\underline{u}}_h = \underline{A}\,\underline{u}_h$

Eigenpairs of $\underline{A}$ $\Rightarrow$ $2i, \begin{bmatrix} -\frac{1}{2}i \\ 1 \end{bmatrix} \leftarrow$

$-2i, \begin{bmatrix} \frac{1}{2}i \\ 1 \end{bmatrix}$

$\Rightarrow \lambda = 0,\ \mu = 2,\ \underline{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \underline{b} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$

From prior work

$$\underline{y}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\sin(2t)$$

$$\underline{z}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\cos(2t)$$

$$\Rightarrow \underline{u}_h(t) = c_1\, \underline{y}(t) + c_2\, \underline{z}(t)$$

(2) $\underline{u}_p(t)$, $\underline{f}(t) = \begin{bmatrix} 2e^{-2t} \\ 3t \end{bmatrix}$

Try $\underline{u}_p(t) = \underline{a}t + \underline{b} + \underline{c}e^{-2t}$

$\dot{\underline{u}}_p = \underline{a} - 2\underline{c}e^{-2t}$

$\dot{\underline{u}}_p = \underline{A}\,\underline{u}_p + \underline{f}$

$$\underline{a} - 2\underline{c}e^{-2t} = \underline{A}\,\underline{a}t + \underline{A}\,\underline{b} + \underline{A}\underline{c}e^{-2t} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}e^{-2t} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}t$$

$\Rightarrow$ $\underline{a} = \underline{A}\,\underline{b}$ $\Rightarrow$ $\underline{A}\,\underline{b} = \underline{a}$

$-2\underline{c} = \underline{A}\,\underline{c} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\Rightarrow$ $(\underline{A} - 2\underline{I})\,\underline{c} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\underline{0} = \underline{A}\,\underline{a} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $\Rightarrow$ $\underline{A}\,\underline{a} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\underline{a} = \begin{bmatrix} 3/4 \\ 0 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 0 \\ 3/4 \end{bmatrix}$ $\underline{c} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$$\Rightarrow \underline{u}_p(t) = \begin{bmatrix} 3/4 \\ 0 \end{bmatrix}t + \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}e^{-2t}$$

w/ $\underline{u} = \underline{u}_h(t) + \underline{u}_p(t)$ + 2 boundary conditions

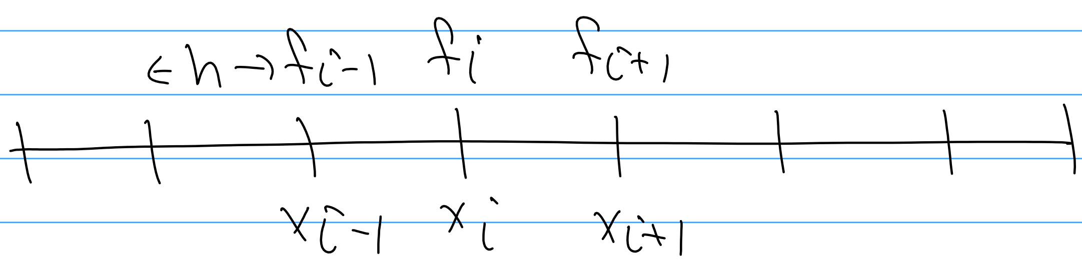
Numeric Differentiation ~ Finite Difference.

Consider the definition of the derivative:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If you evaluate $\frac{f(x+h) - f(x)}{h}$ at finite h ($h > 0$), then this is an approximate derivative.

Consider a 1D Grid:



Try to approximate $\frac{df}{dx}$ at x_i

$$\frac{df}{dx} = a f_i + b f_{i+1}$$ for some constants a & b

Write the Taylor Series of f_i about x_i

$$f_i = f_i$$

$$f_{i+1} = f_i + f_i' h + \tfrac{1}{2} f_i'' h^2 + O(h^3)$$

then

$$a f_i + b f_{i+1} = a f_i + b f_i + b f_i' h + \tfrac{1}{2} b f_i'' h^2 + O(h^3)$$

$$= f_i' = \frac{df}{dx}$$

$\Rightarrow$ $a + b = 0$ (so coefficient of $f_i = 0$)

$bh = 1$ (so " " $f_i' = 1$)

$\Rightarrow b = 1/h$, $a = -1/h$

$\Rightarrow$ Our approximation is thus

$$\frac{df}{dx} = -\frac{1}{h} f_i + \frac{1}{h} f_{i+1} + \frac{1}{h} O(h^2)$$

$$= \frac{f_{i+1} - f_i}{h} + O(h)$$

What about using f_{i-1}, f_i & f_{i+1}?

$$f_{i-1} = f_i - h f_i' + \tfrac{1}{2} h^2 f_i''$$

$$f_i = f_i$$

$$f_{i+1} = f_i + h f_i' + \tfrac{1}{2} h^2 f_i''$$

Use in $a f_{i-1} + b f_i + c f_{i+1}$

$$\Rightarrow (a+b+c) f_i + (-ah + ch) f_i'$$

$$+ \left(\frac{1}{2} ah^2 + \frac{1}{2} ch^2\right) f_i'' + (a+c)\, O(h^3)$$

$$\Rightarrow \quad \begin{array}{l} a+b+c=0 \\ -ah+ch=1 \\ \tfrac{1}{2} ah^2 + \tfrac{1}{2} ch^2 = 0 \end{array} \quad \Rightarrow \quad \begin{array}{l} a = -1/2h \\ b = 0 \\ c = +1/2h \end{array}$$

$$\Rightarrow \frac{\partial f_i}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2h} + O(h^2)$$

Alternative method: fit a polynomial.

ex.) fit a line through (x_i, f_i) & (x_{i+1}, f_{i+1})

$$f(x) = f_i + \frac{(f_{i+1} - f_i)(x - x_i)}{(x_{i+1} - x_i)}$$

$$f'(x) = \frac{f_{i+1} - f_i}{x_{i+1} - x_i} = \frac{f_{i+1} - f_i}{h}$$

Higher-Order Derivatives.

Fit a cubic polynomial through

(x_{i-1}, f_{i-1}), (x_i, f_i), (x_{i+1}, f_{i+1})

Take 2^{nd} derivative at x_i, you get

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$$

Plug in Taylor series for f_{i+1}, f_i & f_{i-1}

$$\frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} = \frac{\partial^2 f}{\partial x_i^2} + \frac{h^2}{12}\frac{\partial^4 f_i}{\partial x_i^4} = \frac{\partial^2 f}{\partial x^2} + O(h^2)$$

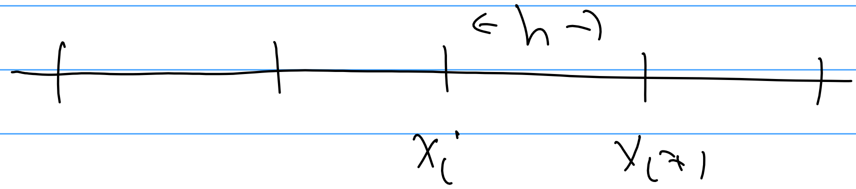
Finite Differences are classified via their direction:

Let the set of points used in a finite difference stencil be f_{i+k},

1. $k \leq 0$: Backward finite difference
2. $k \geq 0$: Forward " "
3. $-d \leq k \leq d$: Center " "

Coefficients in textbooks & online.

ex.) let $f(x) = \sin(x)$ w/ $x_i = \frac{\pi}{2}$, $h = 0.05$



$f_i = \sin(x_i) = 1$

$f_{i+1} = \sin(x_{i+1}) = \sin(x_i + h) = 0.99875\ldots$

$f_{i+2} = \sin(x_{i+2}) = \sin(x_i + 2h) = 0.995\ldots$

$f'(x_i) = 0$

$O(h)$: $f'_i \approx \frac{f_{i+1} - f_i}{h} = \frac{0.99875 - 1}{0.05} = -0.025$

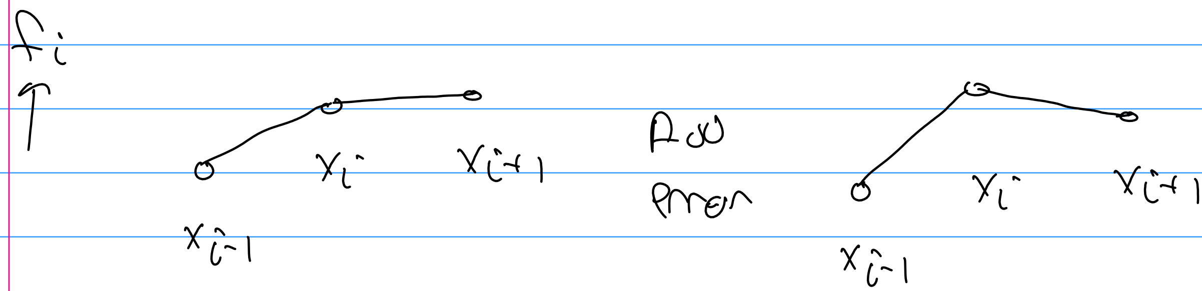
Close to h!

$O(h^2)$: $f'_i \approx \frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2h} = -3.124 \times 10^{-5}$

$\ll h^2$

Issue: Noisy Data

Using finite differences magnifies data errors.



To formally show this, replace f_i by $f_i + \varepsilon$, ε = small error in function value,

$$\frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2(f_i + \varepsilon) + f_{i-1}}{h^2} + O(h^2)$$

$$= \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} - \frac{2\varepsilon}{h^2} + O(h^2)$$

Let $\varepsilon = O(h^2)$ (Still small!)

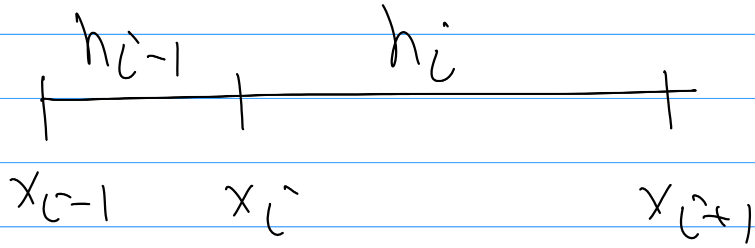
$$\Rightarrow \frac{\partial^2 f}{\partial x^2} - \frac{2\,O(h^2)}{h^2} + O(h^2)$$

$$\frac{\partial^2 f}{\partial x^2} - 2\,O(1) + O(h^2)$$

An order-1 error.

Need higher order to get valid results.

Non-uniform Grids



$f'(x_i) = a\, f_{i-1} + b f_i + c f_{i+1}$ ← Stencil

$f_{i-1} = f_i - h_{i-1} f'_i + \frac{1}{2} h_{i-1}^2 f''_i + O(h^3)$

$f_{i+1} = f_i + h_i f'_i + \frac{1}{2} h_i^2 f''_i + O(h^3)$

Plus into stencil & make coefficients of f_i & f''_i zero, coefficient of f'_i be one,

$\Rightarrow \quad a = \frac{-h_i}{h_{i-1}(h_{i-1}+h_i)} \qquad b = \frac{h_i - h_{i-1}}{h_i h_{i-1}} \qquad c = \frac{h_{i-1}}{h_i(h_{i-1}-h_i)}$

error, $\frac{h_i h_{i-1}}{6} \frac{\partial^2 f}{\partial x^3} + O(h^3)$

If $h_i = h_{i-1} = h$, $\quad a = \frac{-1}{2h}, \quad b = 0, \quad c = \frac{+1}{2h}$

Partial Derivatives

One method: Multi-Dimensional Taylor Series,

Alternative: Stencil Composition

ex 1) $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$$\frac{\partial g}{\partial y} \approx \frac{g_{i,j+1} - g_{i,j-1}}{2h_y} \quad \text{w/} \quad g_{ij} \approx \frac{\partial f}{\partial x} \approx \frac{f_{i+1j} - f_{i-1j}}{2h_x}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} \approx \frac{\frac{f_{i+1j+1} - f_{i-1j+1}}{2h_x} - \frac{f_{i+1j-1} - f_{i-1j-1}}{2h_x}}{2h_y}$$

$$= \frac{f_{i+1j+1} + f_{i-1j-1} - f_{i+1j-1} - f_{i-1j+1}}{4h_x h_y}$$

Plug in Taylor series to see this is

$O(h_x h_y)$ accurate.

Note: The diff command in Matlab is useful:

Let f be an array

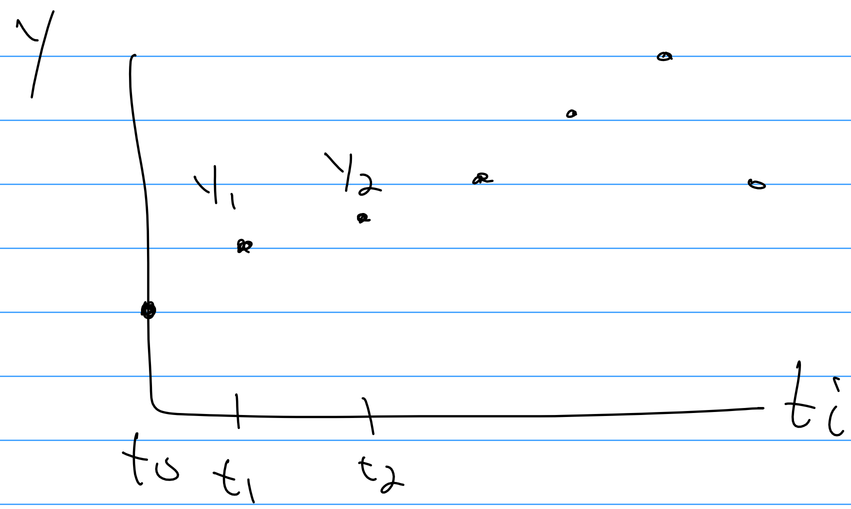
diff(f) returns: for i=1:length(f)-1
$df(i) = f(i+1) - f(i)$

Numerical Solution to IVP

Let's apply these ideas to ODE IVP.

Consider $\frac{dy}{dt} - g(y) = f(t)$ w/ $y(0) = y_0$

Discretize in time & find solution at finite times.



Given y_i & Δt, find y_{i+1}

Look at $O(\Delta t)$ schemes for now.

- Forward / Explicit Euler

$$\frac{y_{i+1} - y_i}{\Delta t} - g(y_i) = f(t_i)$$

y_i is known
y_{i+1} is not

$$\Rightarrow y_{i+1} = y_i + \Delta t\, g(y_i) + \Delta t\, f(t_i)$$ ← fully explicit

- Backward / Implicit Euler

$$\frac{y_{i+1} - y_i}{\Delta t} - g(y_{i+1}) = f(t_{i+1})$$

=> Need to solve

$$\underbrace{y_{i+1} - \Delta t\, g(y_{i+1})}_{\text{Implicit}} = y_i + \Delta t\, f(t_i)$$

Both are $O(\Delta t)$ → errors will be about the same.

So why do the harder (implicit) one?

Stability → Larger Δt.