PDEs

Separation of Variables : $y(x,t) = X(x)T(t)$

ex.) 1D wave equation $\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$

$\to \frac{1}{X(x)} X_{xx} = \frac{1}{\alpha^2} \frac{1}{T(t)} T(t) = -k^2$

$\to$ 2 ODEs : $X_{xx} + k^2 X = 0$

$T_{tt} + \alpha^2 k^2 T = 0$

ODE 1: $X_{xx} + k^2 X = 0$ w/ $X(0) = 0$, $X(L) = 0$

$X(x) = C_1 \cos(kx) + C_2 \sin(kx) = C_2 \sin(kx)$

$\Rightarrow X(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, \ldots, \infty$

ODE 2: $T_{tt} + k^2 \alpha^2 T = 0$ $T_t(0) = 0$

$\Rightarrow T(t) = D_1 \cos(\alpha k t) + D_2 \sin(\alpha k t)$

$T_t(t) = -\alpha k D_1 \sin(\alpha k t) + \alpha k D_2 \cos(\alpha k t)$

$T_t(0) = D_2 \alpha k = 0 \Rightarrow D_2 = 0$

$\Rightarrow T(t) = D_1 \cos(\alpha k t)$ w/ $k = \frac{n\pi}{L}$ $n = 1, 2, \ldots$

Then, $y(x,t) = X(x)T(t)$

$$= \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L} x\right) D_n \cos\left(\frac{\alpha n\pi}{L} t\right)$$

$$= \sum_{n=1}^{\infty} w_n \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{\alpha n\pi}{L} t\right)$$

I.C. $y(x,0) = f(x)$

$$y(x,0) = \sum_{n=1}^{\infty} w_n \sin\left(\frac{n\pi}{L} x\right) = f(x) = \cos(x)$$

This is just the Fourier Series for an odd function over $[0, L]$

$$\Rightarrow w_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

ex.) let $f(x) = \sin(x)$ w/ $L = \pi$

$$y(x,t) = \sum_{n=1}^{\infty} w_n \sin(nx) \cos(n\alpha t)$$

$$w_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx \Rightarrow w_1 = 1, \quad w_n = 0,\ n \geq 2$$

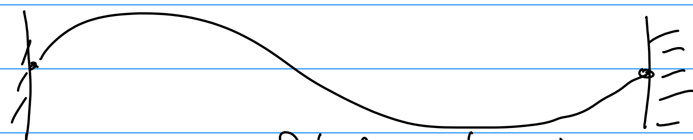
$$\Rightarrow y(x,t) = \sin(x)\cos(\alpha t)$$

Heat Equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u \rightarrow 1D \rightarrow \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

$\alpha =$ thermal conductivity > 0
$u =$ temperature

w/ $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$



$f(x) = u(x,0)$

let $u(x,t) = X(x)\,T(t)$

$u_t = X(x)\,T_t \qquad u_{xx} = X_{xx}\,T(t)$

$\Rightarrow X(x)\,T_t = \alpha X_{xx}\,T(t)$

$$\Rightarrow \frac{1}{\alpha T(t)} T_t = \frac{1}{X(x)} X_{xx} = -k^2$$

ODE 1: $X_{xx} + k^2 X = 0 \qquad X(0) = 0$
$\qquad\qquad\qquad\qquad\qquad X(L) = 0$

$X(x) = C_1 \cos(kx) + C_2 \sin(kx)$

$X(0) = C_1 = 0$

$X(L) = C_2 \sin(kL) = 0 \Rightarrow k = \frac{n\pi}{L} \quad n = 1, 2, \ldots$

$\Rightarrow X_n(x) = C_n \sin\left(\frac{n\pi}{L} x\right) \quad n = 1, 2, \ldots$

ODE2: $\frac{dT}{dt} = -k^2 \alpha T$

$\int \frac{dT}{T} = \int -k^2 \alpha \, dt \Rightarrow T(t) = e^{-k^2 \alpha t} + D_1$

I/w can argue that $D_1 = 0$ so that $u(x, \infty) = 0$

$\Rightarrow T(t) = e^{-k^2 \alpha t} \quad$ w/ $\quad k = \frac{n\pi}{L}, \; n = 1, 2, \ldots$

$$\Rightarrow u(x,t) = X(x)\, T(t)$$

$$= \sum_{n=1}^{\infty} w_n e^{-\left(\frac{n^2 \pi^2}{L^2} \alpha t\right)} \sin\left(\frac{n\pi}{L} x\right)$$

$$u(x, 0) = \sum_{n=1}^{\infty} w_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

$w_n =$ Fourier Coefficients

$$w_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

ex,) let $f(x)=100$, $\alpha=1$, $L=\pi$

$$w_n = \frac{2}{\pi}\int_0^{\pi}(100)\sin(nx)\,dx = \frac{200}{n\pi}\left(1-(-1)^n\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\frac{200}{n\pi}\left(1-(-1)^n\right)\sin(nx)\,e^{n^2t}$$

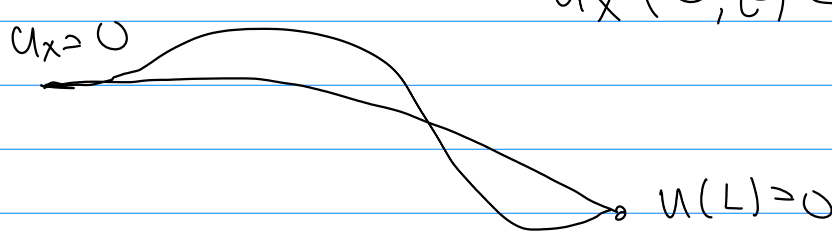
$$= \frac{400}{\pi}\left[\sin(x)e^{-t} + \sin(3x)e^{-9t} + \dots\right]$$

Now look at a different condition:
An insulated end

Find $u(x,t)$ for a rod w/ an insulated end: $u(x,0) = f(x)$

$u(L,t) = 0$

$u_x(0,t) = 0$



$u(x,t) = X(x)\,T(t)$

$u_x(0,t) = X_x(0)\,T(t) = 0 \Rightarrow X_x(0) = 0$

$u(L,t) = 0 \quad X(L)\,T(t) = 0 \Rightarrow X(L) = 0$

$u(x,0) = f(x) = X(x)\,T(0)$

ODE1: $X_{xx} + k^2 X = 0 \qquad X_x(0) = 0$

$X(L) = 0$

$X(x) = C_1 \cos(kx) + C_2 \sin(kx)$

$X_x(x) = -k C_1 \sin(kx) + k C_2 \cos(kx)$

$X_x(0) = k C_2 = 0 \Rightarrow C_2 = 0$

$X(L) = C_1 \cos(kL) = 0 \Rightarrow k = \dfrac{(2n-1)\pi}{2L}$

$n = 1, 2, \ldots,$

ODE2: $T_t = -k^2 \alpha T \Rightarrow T(t) = e^{-k^2 \alpha t}$

$\Rightarrow u(x,t) = X(x)\,T(t)$

$$= \sum_{n=1}^{\infty} \omega_n e^{(-k^2\alpha t)} \cos\left[\frac{(2n-1)\pi}{2L} x\right] \text{ w/ } k = \frac{(2n-1)\pi}{2L}$$

w/ $u(x,0) = \sum_{n=1}^{\infty} w_n \cos\left[\frac{(2n-1)\pi}{2L} x\right] = f(x)$

Determine w_n via cosine series functions.

Procedure: Split solution, Solve individual ODEs & apply all but one condition, Apply final condition on final solution.

Types of Boundary Conditions

1. Dirichlet: The value of the function is supplied at the boundary. If the boundary condition is zero → homogeneous B.C.

2. Neumann: First derivative of the function is known at the boundary.

3. Robin: A combination of Dirichlet & Neumann.

Systems of ODEs.

These come up in two ways:

① A system of order-1 or higher ODEs,

② A higher-order (≥ 2) ODE that is written as a system of order-1 ODEs,

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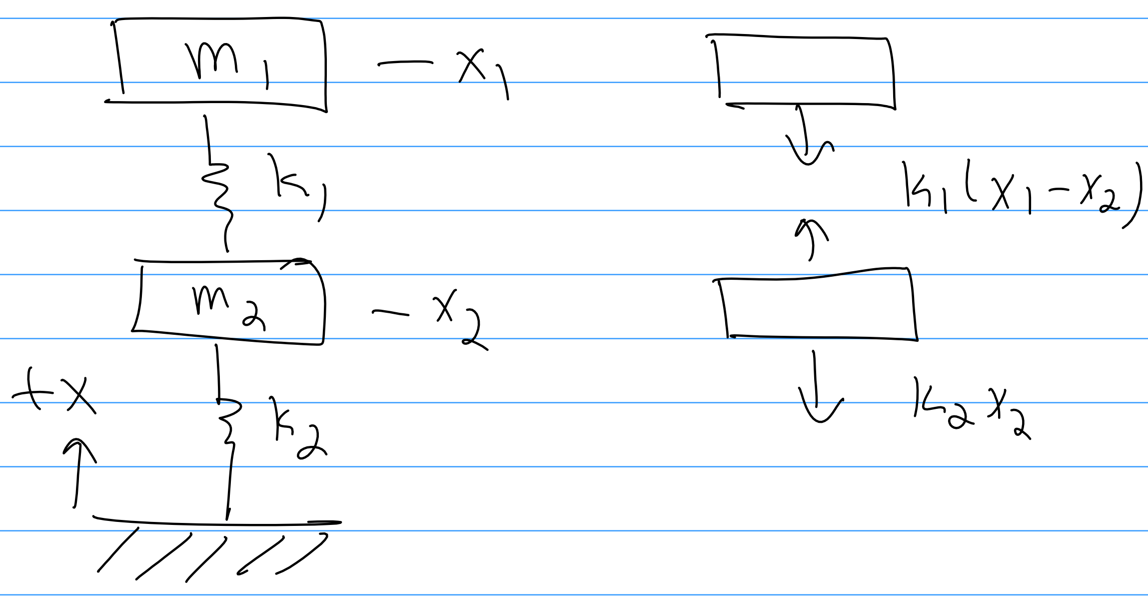
Systems of high-order ODEs.

A 2-mass-2 spring System:



Equations of motion:

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1(x_1 - x_2)$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_1(x_1 - x_2) - k_2 x_2$$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For simplicity, assume

$$x_i(t) = a_i \sin(\omega t) \qquad i = 1, 2$$

$$\Rightarrow \ddot{x}_i(t) = -\omega^2 a_i \sin(\omega t)$$

$$\underbrace{\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}}_{\underline{K}} \underbrace{\begin{bmatrix} a_1 \sin\omega t \\ a_2 \sin\omega t \end{bmatrix}}_{\underline{x}} = \underset{\lambda}{\omega^2} \underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\underline{M}} \underbrace{\begin{bmatrix} a_1 \sin\omega t \\ a_2 \sin\omega t \end{bmatrix}}_{\underline{x}}$$

$$\underline{K}\,\underline{x} = \lambda\, \underline{M}\, \underline{x}$$

$$\Rightarrow \underbrace{\underline{M}^{-1} \underline{K}}\, \underline{x} = \lambda \underline{x}$$

$$\underline{A}\, \underline{x} = \lambda \underline{x} \quad \leftarrow \text{eigen problem}$$

$$\Rightarrow (\underline{A} - \lambda \underline{I})\, \underline{x} = 0 \quad \leftarrow$$

$\underline{A} - \lambda \underline{I}$ must be singular to have non-trivial solutions.

If $\underline{x} = \underline{0}$, trivial solution is possible.

In reality, $x_i(t) = a_i \cos(\omega t) + b_i \sin(\omega t)$
$i = 1, 2$

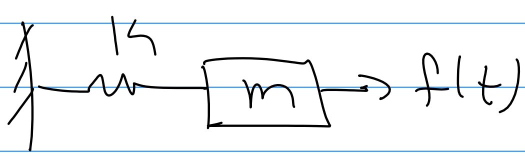
This makes it difficult to apply boundary conditions.

To make the application of boundary & initial conditions easier, move to a system of 1^{st}-order ODEs.

② Systems of Order-1 ODEs

How can we convert order-2 or higher ODE?

Look at a single mass-spring system:



$$m\ddot{x} + kx = f(t)$$

Introduce $v = \dot{x} \Rightarrow \dot{v} = \ddot{x}$

$$\Rightarrow m\dot{v} + kx = f(t)$$
$$\dot{x} = v$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$\underline{\dot{u}} = \underline{A}\,\underline{u} + \underline{f}(t)$$

Note that any higher order ODE can be decomposed like this:

ex1) let $a\dddot{x} + b\ddot{x} + c\dot{x} + dx = f(t)$

$u_1 = x \qquad u_2 = \dot{x} = \dot{u}_1 \qquad u_3 = \ddot{x} = \dot{u}_2$

$$\Rightarrow \begin{aligned} \dot{u}_1 &= u_2 \\ \dot{u}_2 &= u_3 \\ a\dot{u}_3 + bu_3 + cu_2 + du_1 &= f(t) \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -d/a & -c/a & -b/a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f(t)/a \end{bmatrix}$$

$$\underline{\dot{u}} = \underline{A}\,\underline{u} + \underline{f}(t)$$

Focus on $\underline{\dot{u}} = \underline{A}\,\underline{u} + f(t)$

As before, solution will be

$$\underline{u}(t) = \underbrace{\underline{u}_h(t)}_{\text{homogeneous Sol}} + \underbrace{\underline{u}_p(t)}_{\text{Particular Sol}}$$

Homogeneous Part: let $\underline{u}_h(t) = \underline{x} e^{\lambda t}$

$$\Rightarrow \mathring{\underline{u}}_h = \lambda \underline{x} e^{\lambda t}$$

$$\Rightarrow \mathring{\underline{u}}_h = \underline{A} \underline{u}_h$$

$$\lambda \underline{x} e^{\lambda t} = \underline{A} (\underline{x} e^{\lambda t}) \Rightarrow \lambda \underline{x} = \underline{A} \underline{x}$$

$$\Rightarrow \underline{A} \underline{x} = \lambda \underline{x} \quad \leftarrow \text{eigen problem}$$

$$\Rightarrow (\underline{A} - \lambda \underline{I}) \underline{x} = 0 \quad \leftarrow$$ need non-trivial $\underline{x} \Rightarrow \underline{A} - \lambda \underline{I}$ must be singular.

Follow previous procedure for real, repeated & imaginary eigenvalues.

For the particular solution, use method of undetermined coefficients.

ex.) If $\underline{f}(t) = \begin{bmatrix} 2e^{-3t} \\ 3t \end{bmatrix}$

Then try $\underline{x}_p(t) = \underline{a} t + \underline{b} + \underline{c} e^{-3t}$

Instead of unknown coefficients, you have unknown vectors.

plus into the $\dot{\underline{y}} = \underline{A}\,\underline{y} + \underline{f}(t)$ &

Solve for $\underline{a}$, $\underline{b}$, $\underline{c}$

This helps for initial conditions because those are given directly,

ex,) $x(0) = \alpha$ & $\dot{x}(0) = \beta$

$x(0) = \alpha$ & $v(0) = \beta$