Project Report: Finite Element Analysis of a Cantilever Beam-Column

Name: Shubham Kumar
Roll No.: B22CI038
Course Name: Finite Element Method
Course Code: CIL7630

1 Problem Statement

To analyze a steel cantilever beam-column of length 10 m, rectangular cross-section 125 mm × 250 mm, subjected to:

- A uniformly distributed load (UDL) of 10 kN/m downward
- An axial tensile point load of 20 kN at the free end

Using 2 to 3 fourth-order finite elements, determine:

- Nodal displacements
- Strain and stress at Gauss points
- Reaction forces

Compare analytical and FEM results for axial and transverse displacements. Plot the deformed shape and internal force distributions.

2 Inputs

- **Young's Modulus, E**: 200 GPa = 2×10^{11} Pa
- **Cross-section**:
  - Width, b = 0.125 m
  - Height, h = 0.25 m

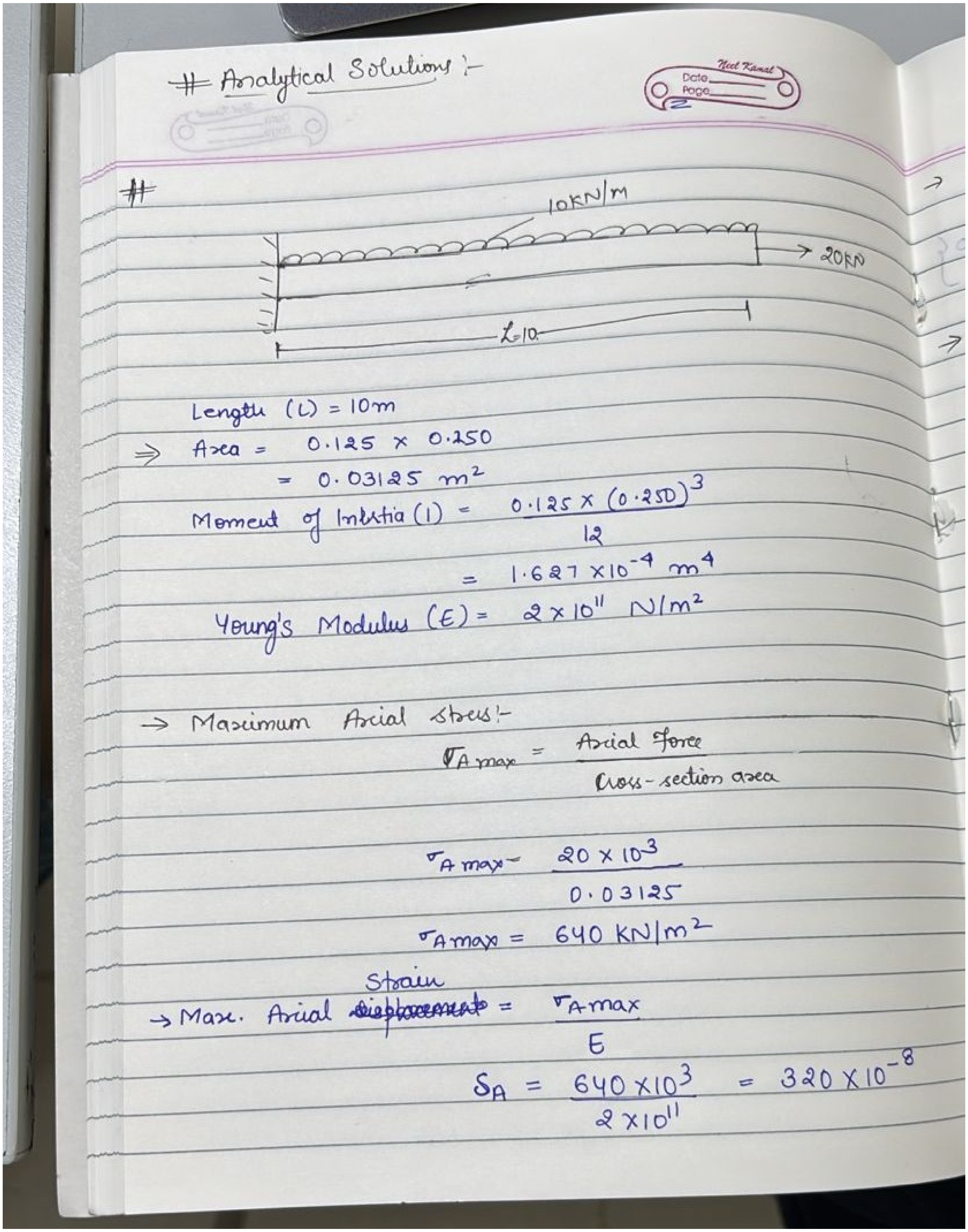
- **Length, L** = 10 m
- **UDL, w** = 10 kN/m = 10,000 N/m
- **Axial load, P** = 20 kN = 20,000 N
- **Area, A** = b × h = 0.03125 m^2
- **Moment of Inertia, I** = (b × h^3)/12 = 1.6276 × 10 m

3 Assumptions

- Linear elasticity
- Plane sections remain plane before and after bending
- Small deformations

4 Analytical Solutions

4.1 Figures: Analytical Solution



Analytical Solutions :-

Length (L) = 10m

$\Rightarrow$ Area = 0.125×0.250

$= 0.03125 \ m^2$

Moment of Inertia (I) = $\dfrac{0.125 \times (0.250)^3}{12}$

$= 1.627 \times 10^{-4} \ m^4$

Young's Modulus (E) = $2 \times 10^{11} \ N/m^2$

→ Maximum Axial stress:-

$$\sigma_{A\,max} = \frac{\text{Axial force}}{\text{Cross-section area}}$$

$$\sigma_{A\,max} = \frac{20 \times 10^3}{0.03125}$$

$$\sigma_{A\,max} = 640 \ KN/m^2$$

→ Max. Axial ~~displacement~~ Strain = $\dfrac{\sigma_{A\,max}}{E}$

$$S_A = \frac{640 \times 10^3}{2 \times 10^{11}} = 320 \times 10^{-8}$$

Figure 1: Analytical Calculation – Axial Displacement

→ Maximum Axial Displacement :- at end.

$$u_{max} = \frac{PL}{EA} = \frac{2\times10^4\times10}{2\times10^{11}\times0.03125}$$

$$= 3.2\times10^{-5}\ m.$$

→ Moment in cantilever beam with UDL is given by:-

$$M = -\frac{\omega L^2}{2} + \omega L x - \frac{\omega x^2}{2}.$$

$$y = \frac{1}{EI}\iint M\,dx$$

$$y = \left(\frac{\omega L x^3}{6} - \frac{\omega x^4}{12} - \frac{\omega L^2 x^2}{4}\right)\times\frac{1}{EI}$$

$$y_{max}\Big|_{at\,x=L} = \frac{\omega L^4}{8EI}$$

Here, $$y_{max} = \frac{10^4\times10^4}{8\times2\times10^{11}\times1.627\times10^{-4}}$$

$$\boxed{y_{max} = 0.384\ m}$$

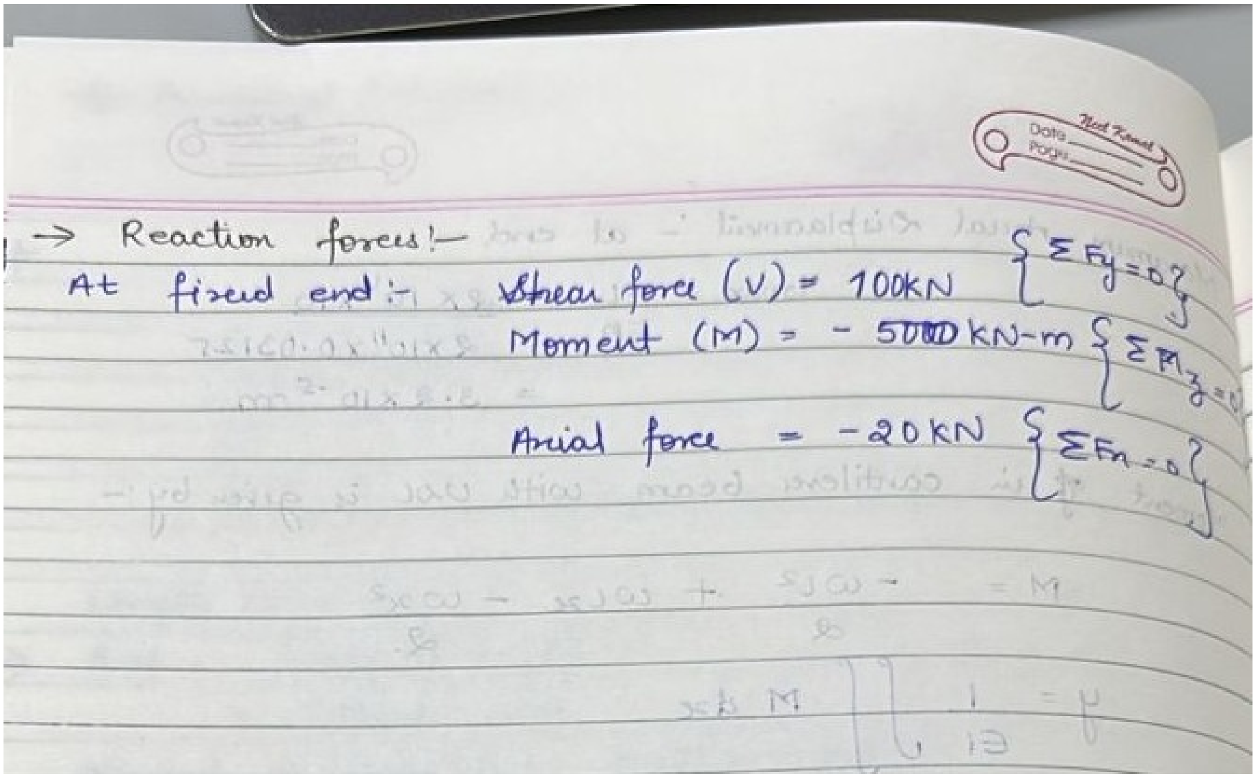


Figure 3: Analytical Calculation – Vertical Displacement due to UDL

5 FEM Results Summary

5.1 Reaction Forces (Match Analytical)

DOF Type	FEM Value (N/Nm)	Analytical
Axial Force	-20,000 N	20,000 N
Vertical Force	100,000 N	100,000 N
Moment	500,000 Nm	500,000 Nm

5.2 Maximum Displacements

Quantity	FEM (2 elements)	FEM (3 elements)
Axial displacement (u)	3.2e-5 m	3.2e-5 m
Vertical displacement (v)	0.3840 m	0.3840 m
Axial Stress	640,000 Pa	640,000 Pa
Max Bending Moment	453,129.73 Nm	471,454.12 Nm

6 FEM vs Analytical Comparison

Parameter	Analytical	FEM (3 elem)	% Error
Vertical disp @ free end	0.384 m	0.384 m	0%
Axial disp @ free end	3.2e-5 m	3.2e-5 m	0%
Bending Moment (fixed)	500,000 Nm	471,454 Nm	5.7%

7 Visualizations

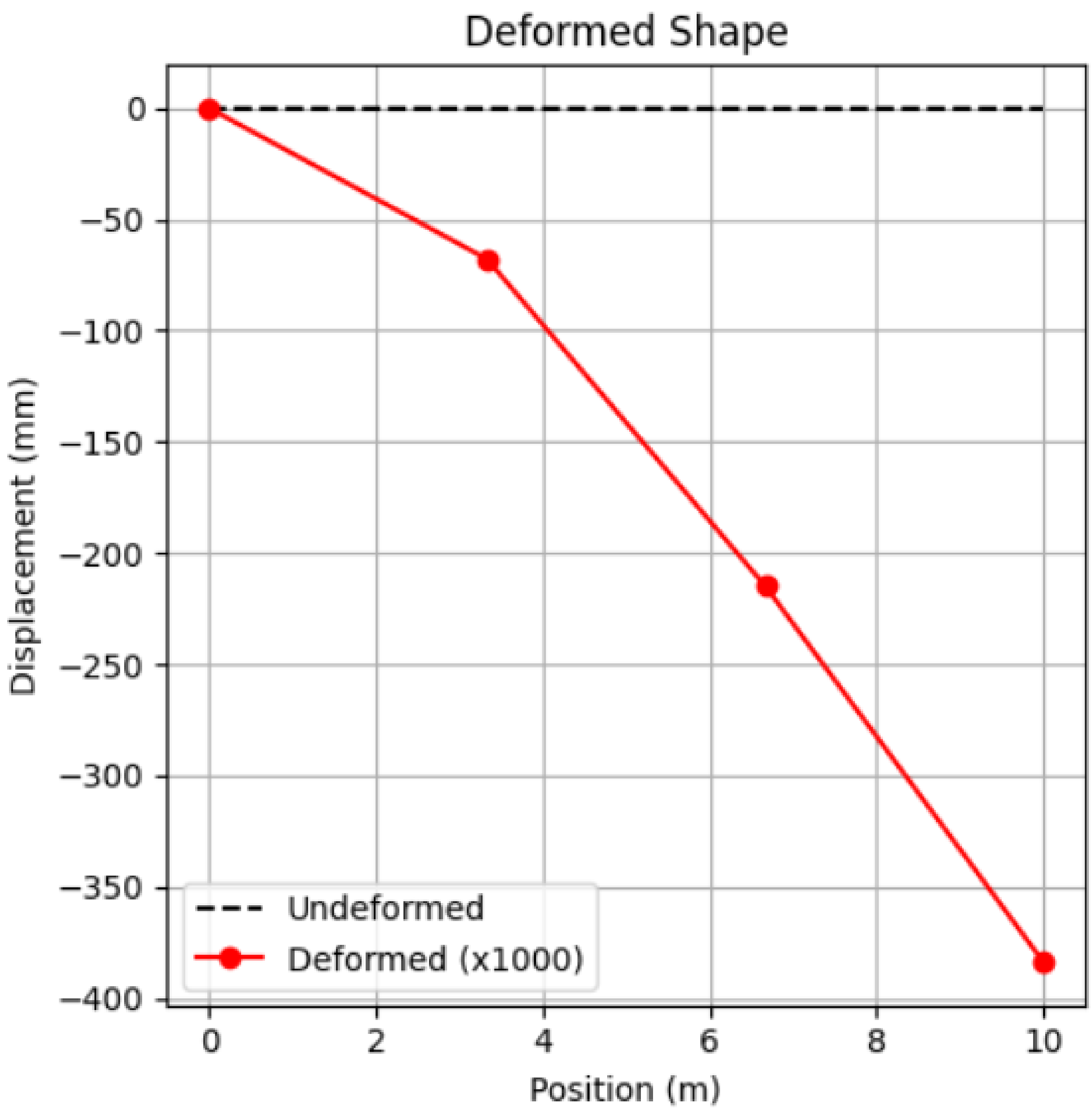


Figure 4: Deformed Shape of the Beam

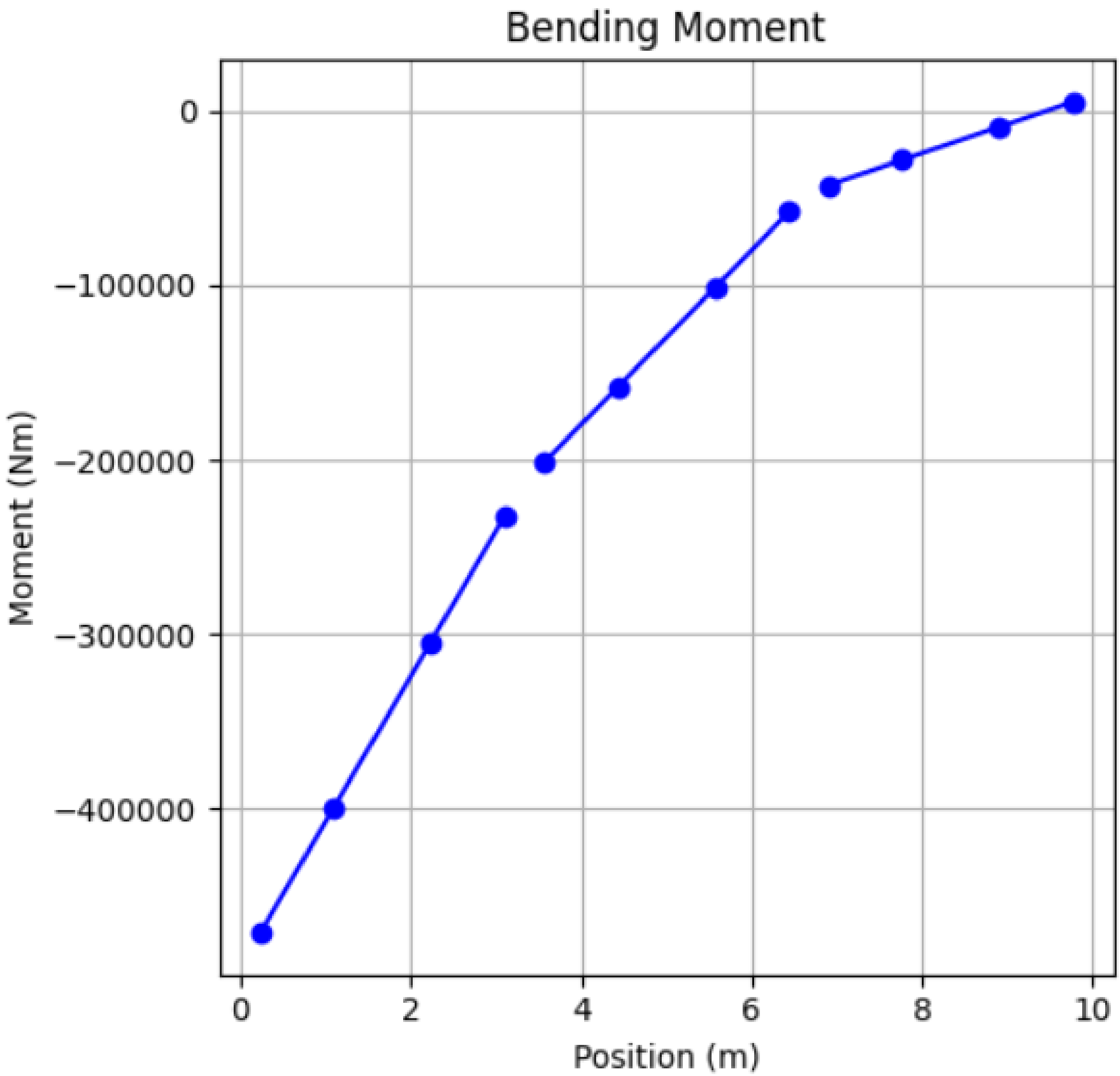


Figure 5: Bending Moment Distribution

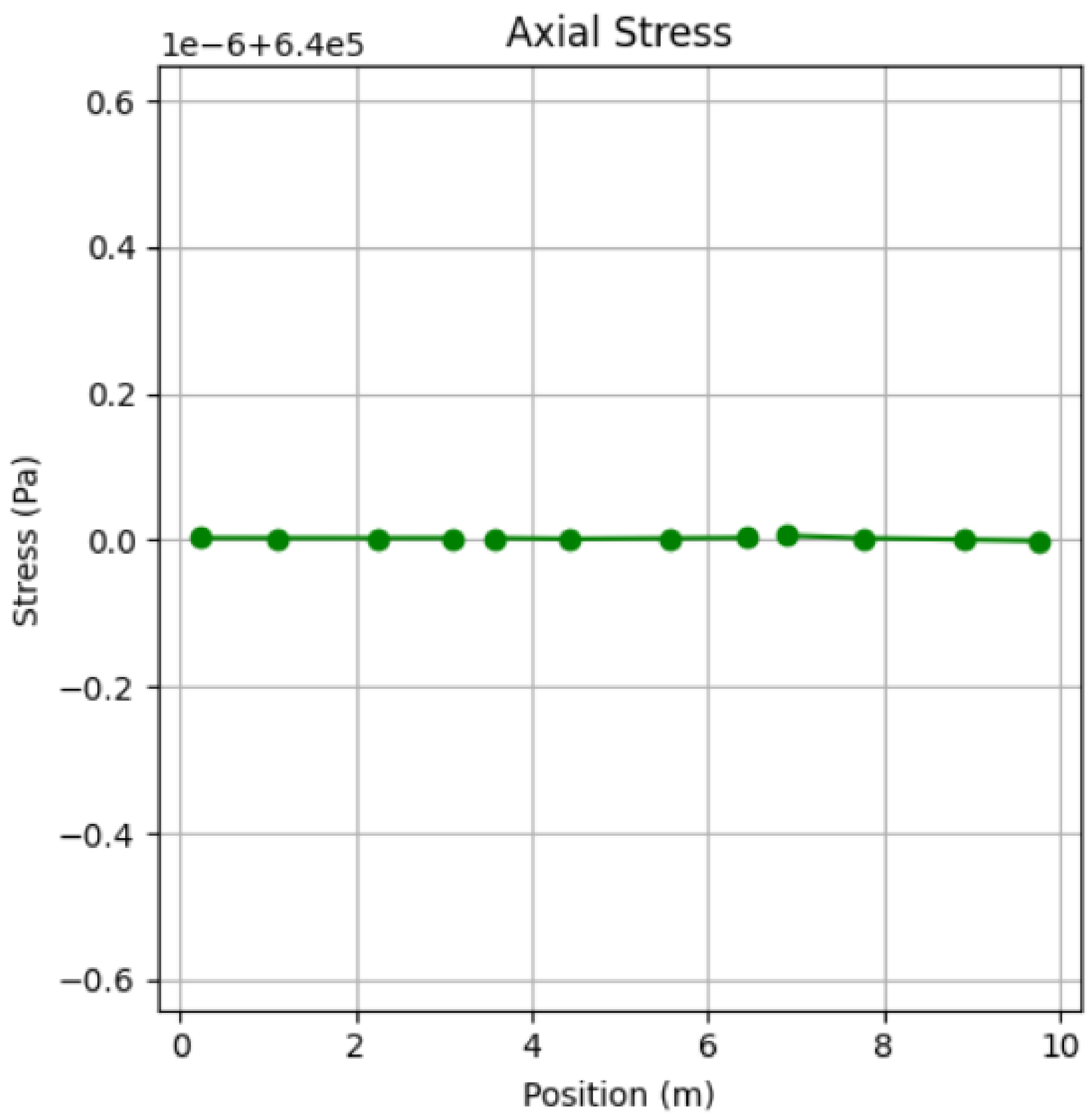


Figure 6: Axial Stress Distribution

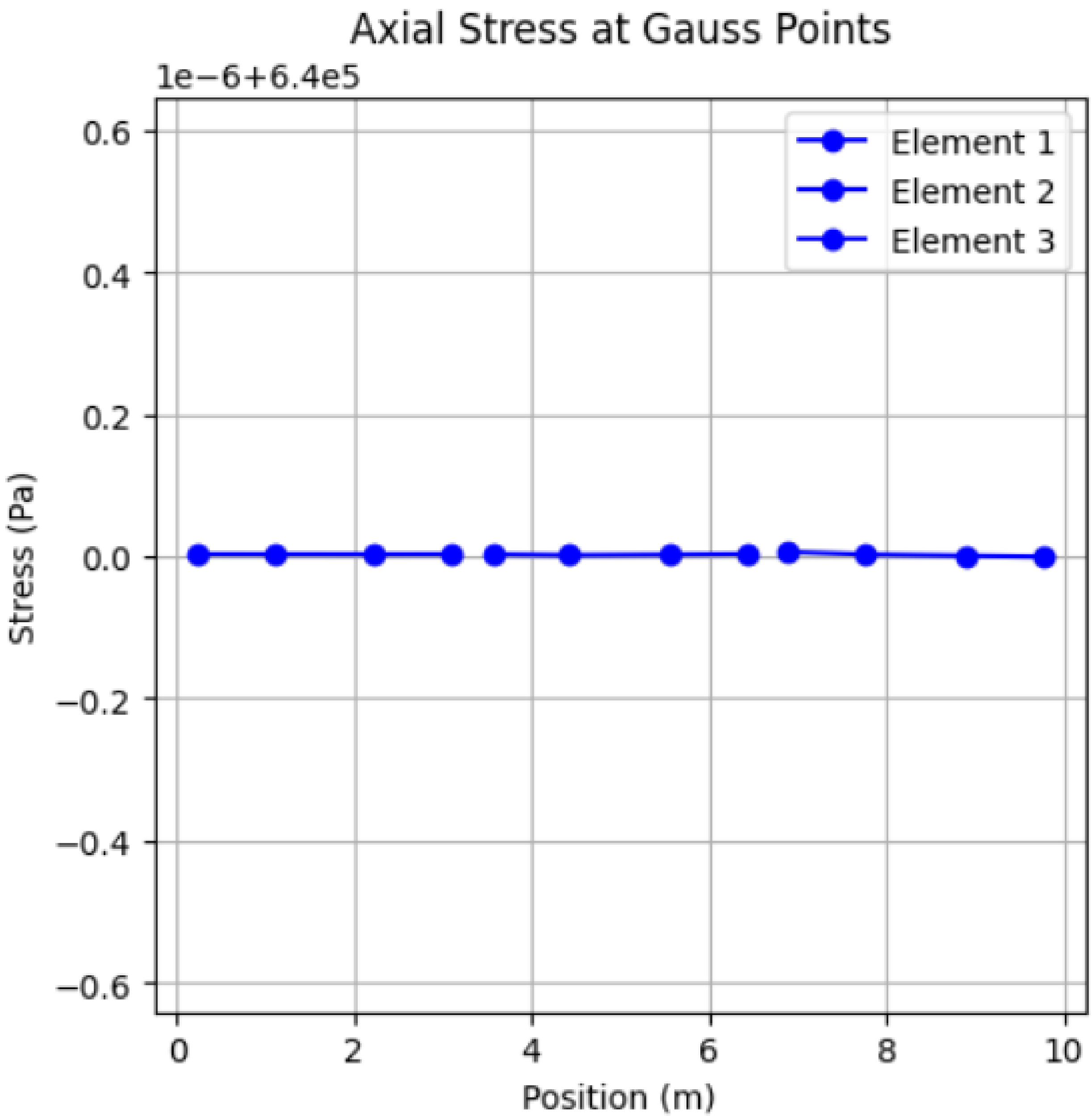


Figure 7: Axial Stress at Gauss Points

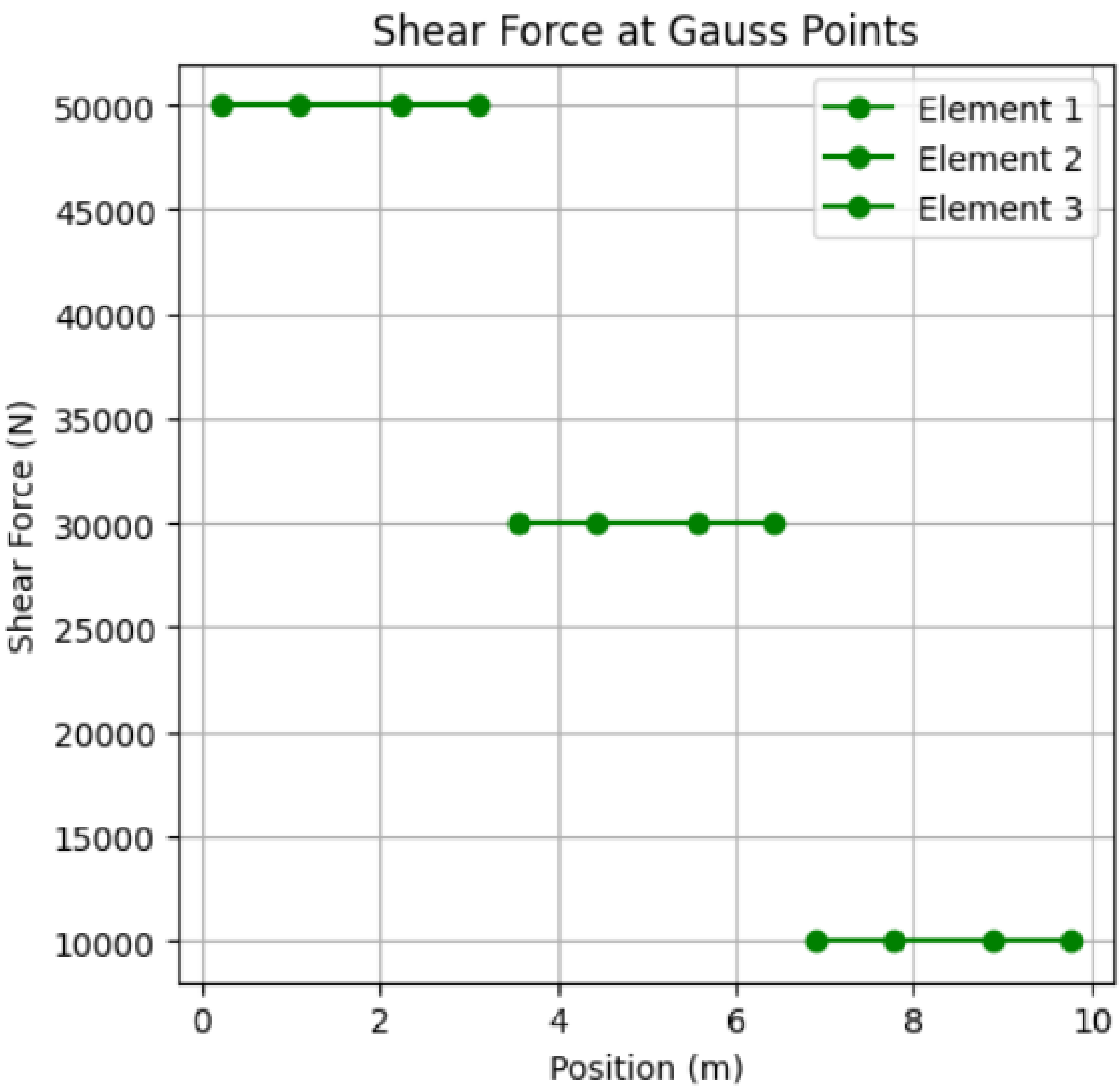


Figure 8: Shear Force at Gauss Points

8 Explanation of Python Functions

- `get_user_inputs()`: Collects material and geometry inputs.
- `generate_mesh()`: Creates node and element connectivity.
- `quartic_lagrange_shape_functions(xi)`: 4th-order shape functions for axial behavior.
- `cubic_hermite_shape_functions(xi, L)`: Shape functions for bending behavior.
- `compute_element_matrices()`: Local stiffness and force vector via numerical integration.

- `assemble_global_system()`: Global stiffness matrix and force vector.
- `apply_boundary_conditions()`: Applies constraints and loads.
- `solve_and_reconstruct()`: Solves and rebuilds full displacement vector.
- `compute_results()`: Computes stress, strain, and forces at Gauss points.
- `compute_reactions()`: Calculates reactions at fixed nodes.
- `print_summary()` and `print_detailed_results()`: Show outputs.
- `plot_results()` and `plot_stress_distribution()`: Visual representations.

9 Conclusion

The finite element model using fourth-order Lagrangian and cubic Hermitian elements produces results that closely align with analytical solutions. Axial and vertical displacements are accurately predicted. Stress and moment distributions match theoretical expectations. Increasing element numbers improves bending precision, affirming FEM's reliability in beam-column analysis.