MA202 Project Report

Numerical Analysis of Atmospheric Scattering Phenomenon

Rishabh Gupta	Shubh Lavti	Paras Gupta	Tarun Sharma	Nikita
19110025	19110090	19110128	19110140	19110190

Group 44 | 08 May, 2021

Problem Statement

Electromagnetic radiation may be reflected or transmitted by ions or particles in the atmosphere in addition to being absorbed or transmitted. The redirection of electromagnetic radiation by suspended particles (molecules of some substance or on a small particle of matter) in the atmosphere is known as scattering. The nature and amount of scattering that occurs is determined by particle size and energy wavelength. Atmospheric light scattering is the very reason why the atmosphere manifests colour.

The sunlight entering the atmosphere is scattered/ absorbed by air molecules and aerosol, and ozone layers. In this problem statement we will consider two types of scattering phenomena that have a major impact on incoming solar radiation, namely Rayleigh scattering (scattering by small particles such as air molecules) and Mie scattering (scattering by aerosols such as dust). Further, we will be calculating the intensity of light after scattering with respect to various factors such as, height from the ground surface, wavelength of incident light and angle of incidence (w.r.t. Earth), using Simpson's and Trapezoidal model.

This study is useful in space travel simulators and in the simulation of earth surveys (comparisons with observations from weather satellites and weather simulations). This project work may be useful for viewing the earth, and the sea floor, from space, taking into account contaminants (air molecules and aerosols) in the atmosphere as well as water molecules in the sea.

Contents

Physical Model

We consider a parallel beam of sunlight along a direction vector L (see Figure 1) entering the earth's atmosphere at point P_c. The incoming light from sun is considered to have a spectral intensity of $I_1(\lambda)$ These incoming light rays get scattered at different points inside the atmosphere with different intensities along different directions. We want to find the net scattered intensity that reaches the eyes of the observer at the viewpoint (P_o).

The ray of light reaching the viewpoint (P_o) present along the direction vector V is the result of scattering of the incoming light at different points (P) on a line along the direction vector V. The line intersects the atmospheric boundary at points P_a and P_b. The points P_b and P_a are the first and last points of non-zero atmospheric density along V. The line segment P_aP_b is at an angle θ (scattering angle) from the incident beam of light.

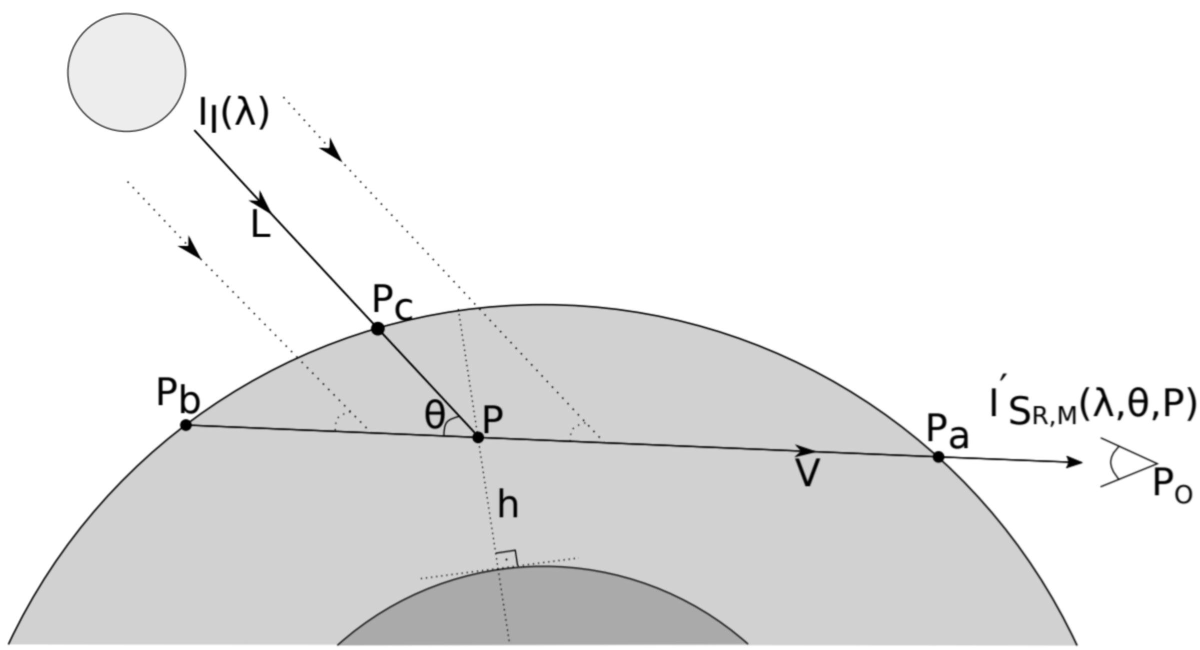

Figure 1: Schematic of single light scattering in Earth's atmosphere

Notation

Symbol / Term	Description
λ	wavelength of light
θ	scattering angle
$F_{R,M}(\theta)$	Rayleigh/Mie phase function
$\beta_{R,M}(\lambda)$	Rayleigh/Mie scattering coefficient
$N_{R,M}$	Rayleigh/Mie molecular number density
$I_I(\lambda)$	incident spectral intensity
$\rho_{R,M}(h)$	Rayleigh/Mie density function
$t_{R,M}(S,\lambda)$	transmittance (optical length)

Assumptions

1. Only single scattering of light between air molecules and particles is taken into consideration. Multiple scattering of light is ignored due to its high computational cost and minimal values.

2. As the distance to the sun can be considered almost infinite, the sunlight can be assumed to be a parallel beam. Thus the scattering angle at every point along P_aP_b can be considered constant.

3. The observer present at the view point (P_o) is considered to be outside the Earth's atmosphere.

4. Only elastic light scattering is considered during a scattering event. Thus, no energy loss occurs

5. Only Rayleigh and Mie scattering is taken into consideration.

6. $F_{R,M}(\theta)$ can be excluded from the integration, because we assume that all light rays coming from the light source are parallel.

7. The densities of air molecules and aerosols are taken to vary exponentially with altitude.

8. It is assumed that light propagates in a straight line even though the actual path is curved (due to the variation of index of refraction with height).

Governing Equations

The equations of a physically dependent mathematical model that was used to calculate single atmospheric scattering are discussed in this section.

3.1 Phase function

The phase function is a 1-Dimensional function that establishes an "angular dependency" between the scattered and the incident light. It represents the directional characteristic of scattering and measures the relative amount of light that gets scattered at a particular angle.

$$F_R(\theta) = \frac{3}{16\pi}(1 + \cos^2\theta) \tag{1}$$

$$F_M(\theta) = \frac{3}{8\pi}\frac{(1 - g^2)(1 + \cos^2\theta)}{(2 + g^2)(1 + g^2 - 2g\cos\theta)^{\frac{3}{2}}}; \quad g \,\epsilon\, (-1, 1) \tag{2}$$

3.2 Density Function

The density function expresses decrease in the atmospheric density as a function of the altitude, with respect to the scaling thickness of the atmosphere $H_{R,M}$ for which the density is uniform.

$$\rho_{R,M}(h) = exp(-\frac{h}{H_{R,M}}) \tag{3}$$

3.3 Scattering Intensity

The amount of incident light that gets scattered at a particular point (P) in a particular direction (at an angle θ from the incident direction) can be measured using the scattering intensity. A generic equation to determine the scattering intensity, for both Rayleigh and Mie Scattering, is given as follows:

$$(I_S(\theta, \lambda, P))_{R,M} = I_1(\lambda)\, \rho_{R,M}(h)\, F_{R,M}(\theta)\, \beta^s_{R,M}(\lambda) \tag{4}$$

3.4 Transmittance

The transmittance or the optical length expresses the attenuation caused by the atmospheric particles for light of a given wavelength (λ) after it has covered a certain distance (s).

For a Rayleigh/Mie scattering medium, optical length (t) is defined by the equation:

$$t_{R,M}(P_a, P_b, \lambda) = \beta_{R,M} \int_{P_a}^{P_b} e^{(-\frac{h(P)}{H_{R,M}})} dP \tag{5}$$

3.5 Single Scattering

In the considered system, light also gets attenuated and the intensity reduces exponentially with respect to the optical length before arriving at P_V (Viewpoint). Let us consider a variable point P on the line segment P_aP_b to be the point where scattering is observed.

Considering attenuation of light between P_C to P:

$$(I_P(\theta,\lambda,P))_{R,M} = (I_S(\theta,\lambda,P))_{R,M}\, e^{-t(PP_C,\lambda)} \tag{6}$$

Considering attenuation of light from P to the last point of non-zero atmosphere density, P_a, before the light reaches the viewpoint P_v:

$$\begin{aligned}(I_{P_V}(\theta,\lambda,P))_{R,M} &= (I_P(\theta,\lambda,P))_{R,M}\, e^{-t(PP_a,\lambda)} \\ &= (I_S(\theta,\lambda,P))_{R,M}\, e^{-t(PP_C,\lambda)-t(PP_a,\lambda)}\end{aligned} \tag{7}$$

Now, since the point of scattering, P, is variable, we integrate the above equation with respect to P under the limits P_a to P_b.

$$\begin{aligned}(I_V(\theta,\lambda,P))_{R,M} &= \int_{P_a}^{P_b} (I_{P_V}(\lambda))_{R,M} dP \\ &= \int_{P_a}^{P_b} (I_S(\theta,\lambda,P))_{R,M}\, e^{-t(PP_C,\lambda)-t(PP_a,\lambda)} dP \\ &= \int_{P_a}^{P_b} I_1(\lambda)\, \rho_{R,M}(h)\, F_{R,M}(\theta)\, \beta^s_{R,M}(\lambda)\, e^{-t(PP_C,\lambda)-t(PP_a,\lambda)} dP\end{aligned} \tag{8}$$

As scattering angle, $\theta \longrightarrow$ constant, we can take $F_{R,M}(\theta)$ outside integration.

$$(I_V(\theta,\lambda,P))_{R,M} = I_1(\lambda)\, F_{R,M}(\theta)\, \beta^s_{R,M}(\lambda) \int_{P_a}^{P_b} \rho_{R,M}(h)\, e^{-t(PP_C,\lambda)-t(PP_a,\lambda)} dP \tag{9}$$

3.6 Total Single Scattering

The final intensity reaching the viewpoint, P_v is given by the sum of the single-scattering intensities due to Rayleigh (R) and Mie Scattering (M) derived in the previous subsection.

$$I_V(\theta,\lambda,P) = (I_V(\theta,\lambda,P))_R + (I_V(\theta,\lambda,P))_M \tag{10}$$

Parameters

4.1 Scattering coefficients

Rayleigh's scattering equation provides scattering coefficients for a volume for which we know its molecular density. This constant is calculated from measured parameters of the Earth's atmosphere and denotes how well the particle scatters the light.

$$\beta_R^s(\lambda) = \frac{8\pi^3\,(n^2-1)^2}{3N\lambda^4} \tag{11}$$

Mie scattering is analogous to Rayleigh scattering, but it refers to particles of a diameter several times that of the scattered wavelength. These particles (aerosols) can be present in the Earth's atmosphere at low altitudes. As a result, the equation for Mie scattering coefficients must be marginally modified.

$$\beta_M^s = \frac{8\pi^3\,(n^2-1)^2}{3N} \tag{12}$$

Here,
n : Refractive index of air
N : Molecular density at sea level

4.2 Other Parameters

Other parameters used in computation of the problem are given below.

1. n = 1.0003
2. N = 2.47×10^{25} molecules per m^3
3. $x\ \epsilon\ [-12,000\,m, 12,000\,m]$, where x represents the horizontal distance with $x = 0$ as the x-coordinate of point P.
4. H_r = 8000 metres
5. H_m = 1200 metres
6. $u\ \epsilon\ [0.7,\ 0.85]$, a constant defined by the atmospheric conditions.
7. $g = \frac{5}{9}u - (\frac{4}{3} - \frac{25}{81}u^2)x^{-1/3} + x^{-1/3}$, where g is an asymmetry factor used in phase functions (Section 3.1) and, $x = \frac{5}{9}u - \frac{125}{729}u^3 + (\frac{64}{27} - \frac{325}{243}u^2 + \frac{1250}{2187}u^4)^{1/2}$

Solution Methodology

To solve the final equation derived in the previous section, we make use of the following solution methodology.

Since the spectral intensity associated with the input wavelength λ is not defined as a function, we generate a Lagrange interpolating polynomial using a considered set of data points. The main integral contained in the final equation consists of another integral associated with optical length inside it, whose limits depend on the integrating variable (P) of the outer integral. The inner integral is first solved independently using numerical integration methods like Simpson's Rule and Trapezoidal Rule, just the way as we would solve a double integral. This is followed by solving the outer integral, again using a numerical integration method, but on the same set of points considered.

Analytical Solution

Optical length (equation 4, Section 3.3) can be written as

$$\begin{aligned} t_{R,M}(P_a, P_b, \lambda) &= \beta_{R,M} \int_{P_a}^{P_b} e^{(-\frac{h(P)}{H_{R,M}})} dP \\ &= H_{R,M}\, \beta_{R,M} (e^{\frac{-h(P_a)}{H_{R,M}}} - e^{\frac{-h(P_b)}{H_{R,M}}}) \end{aligned} \tag{13}$$

Single scattering intensity can be calculated from equations (3, 9, 13) using basic integration and substitution methods,

$$\begin{aligned} (I_V(\theta, \lambda, P))_{R,M} &= I_1(\lambda)\, F_{R,M}(\theta)\, \beta^s_{R,M}(\lambda) \int_{P_a}^{P_b} \rho_{R,M}(h)\, e^{-t(P,P_C,\lambda)-t(P,P_a,\lambda)} dP \\ &= A \int_{P_a}^{P_b} (e^{-\frac{P}{H_{R,M}}}\, e^{-Be^{-\frac{P}{H_{R,M}}}}) dP \\ &= \frac{A\, H_{R,M}}{B} (e^{-Be^{\frac{-P_b}{H_{R,M}}}} - e^{-Be^{\frac{-P_a}{H_{R,M}}}}) \end{aligned} \tag{14}$$

where, A $= I_1(\lambda)\, F_{R,M}(\theta)\, \beta^s_{R,M}(\lambda)\, (e^{H_{R,M} e^{\frac{-P_c}{H_{R,M}}}} - e^{H_{R,M} e^{\frac{-P_a}{H_{R,M}}}})$ and B $= H_{R,M}\, \beta^s_{R,M}$

The above analytical solution can be used to obtain the required intensity for particular values of the parameters θ, P, and λ.

Numerical Methods

7.1 Lagrange's Interpolation Polynomial

The first step is to calculate the spectral intensity, $I(\lambda)$, considering the input wavelength, λ of the incoming light rays. For this, we use n^{th} Order Lagrange Interpolation on a set of data containing the spectral intensity corresponding to a particular wavelength.

$$f_{n-1} = \sum_{i=1}^{n} L_i(x) f(x_i) \tag{15}$$

where,

$$L_i(x) = \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{x - x_i}{x_i - x_j} \tag{16}$$

7.2 Composite Simpson's Rule

The Single Scattering Equation (equation (9), Section 3.5) consists of two integrals. We van use the Simpson's 1/3 Rule as the numerical integration method to solve the both the integrals. The choice of the numerical method is not rigid and thus we have also used other methods like the Trapezoidal Method, as described in the next subsection. Given two points x_0 and x_2, we can determine the area of the unique parabola connecting the two points and its midpoint x_1 using the result given below,

$$I = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \tag{17}$$

It is to be noted that the $\frac{1}{3}$ Rule can used only when number of segments is even and each segment has the same width. In case of odd number of segments, we can use another rule, Simpson's $\frac{3}{8}$ Rule, along with the $\frac{1}{3}$ Rule to obtain the result of integration. Given two points x_0 and x_2, we can determine the area under the third order Lagrange Polynomial connecting the two points and their points of trisection x_1 and x_2 using the result given below,

$$I = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] \tag{18}$$

We can use the Composite Simpson's $\frac{1}{3}$ rule and Composite Simpson's $\frac{3}{8}$ to calculate the integrals given in equation (9) by considering a number of integrating segments rather than two or three respectively.

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-2}}^{x_n} f(x)dx \tag{19}$$

For each integration interval (segment), we use the Simpson's Rule independently and add those independent results to obtain the final result of integration,

Composite Simpson's $\frac{1}{3}$ Rule:

$$I = h\frac{f(x_0)+4f(x_1)+f(x_2)}{3} + ... + h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{3} \tag{20}$$

Composite Simpson's $\frac{3}{8}$ Rule:

$$I = 3h\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8} + ... + 3h\frac{f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n)}{8} \tag{21}$$

7.3 Composite Trapezoidal Rule

Given two points x_0 and x_2, we can determine the area under the straight line joining the two points and the result given below,

$$I = \frac{h}{2}[f(x_0)+f(x_1)] \tag{22}$$

We use the Composite Trapezoidal rule to calculate the integrals given in equation (9).

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-2}}^{x_n} f(x)dx \tag{23}$$

For each integration interval (segment), we use the Simpson's Rule independently and add them

$$I = h\frac{f(x_0)+f(x_1)}{2} + ... + h\frac{f(x_{n-1})+f(x_n)}{2} \tag{24}$$

Algorithm Implemented

A Python Code was written to solve the problem statement. The algorithm followed can be stated in the following steps:

1. All the parameters and the different functions used in the final scattering equation (9) are defined.
2. A linear space on x axis is considered with a step size of 100.
3. Angle α between the vector direction V and the horizontal x direction, and the considered variable point P is used to form the equation of the line P_aP_b
4. Angle $(\theta - \alpha)$ is used to determine the point P_c
5. Using the line equation, the y coordinates for different x are found which are further used to calculate the height (h) of the corresponding point P (x, y) from the ground.
6. Simpson's Rule is used to calculate the the inner integrals (from P_a to P_b and from P_a to P_c) associated with optical length and the corresponding values obtained are stored in a lookup tables (or arrays) to be used later.
7. The outer integral is also computed using the Simpson's Rule and the required values of optical length corresponding to each point P is obtained from the lookup table.
8. Trapezoidal Rule is also used to compute the required integral. We also compared our results with Trapezoidal method.
9. The above procedure is repeated for Mie Scattering with the corresponding functions and parameters.
10. The scattering intensities associated with both types of scatterings are added to find the net intensity.
11. The analytical solution for the final equation is obtained using Python functions to determine the accuracy of the applied numerical methods.
12. Graphs to study the parametric effects of height (point P), wavelength (λ), and angle($\angle$) on scattering intensity are plotted using the MATPLOTLIB framework in Python.

Results and Discussion

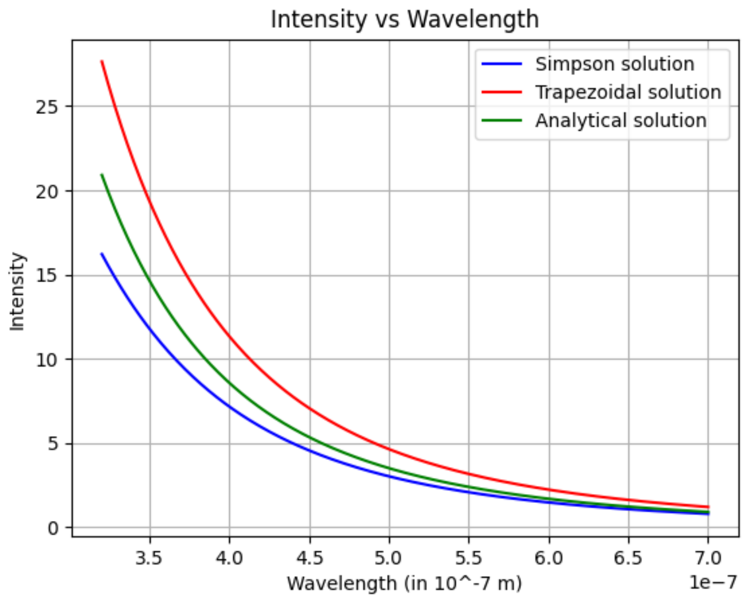

Figure 2: Intensity Vs Wavelength

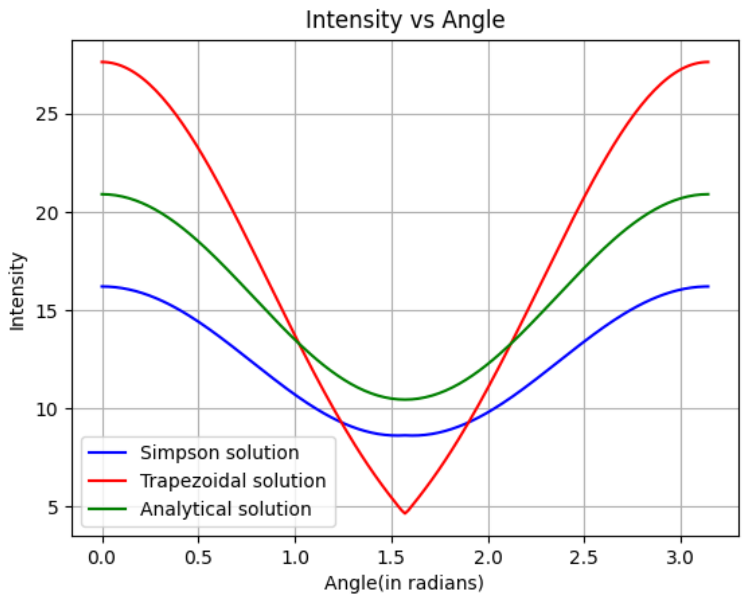

Figure 3: Intensity Vs Angle

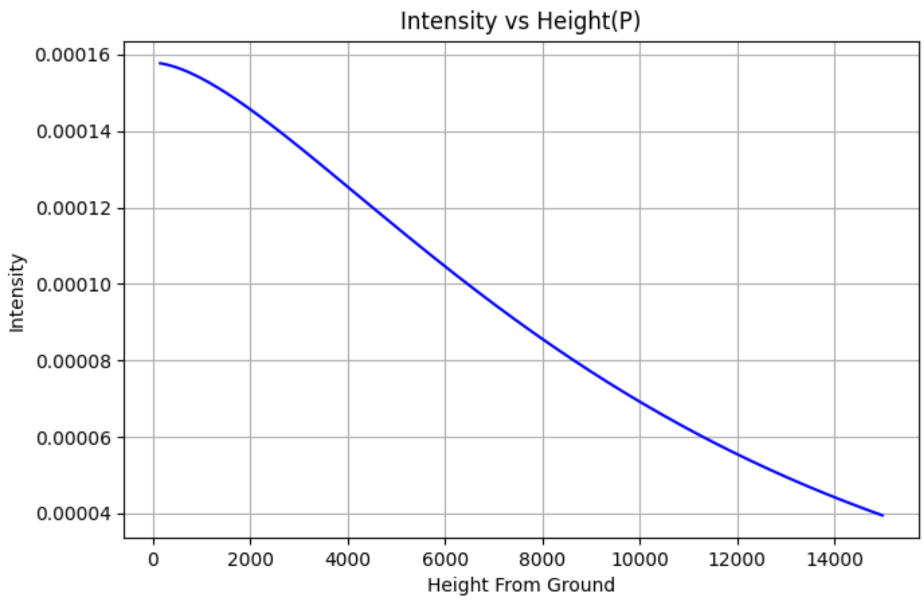

Figure 4: Intensity Vs Height

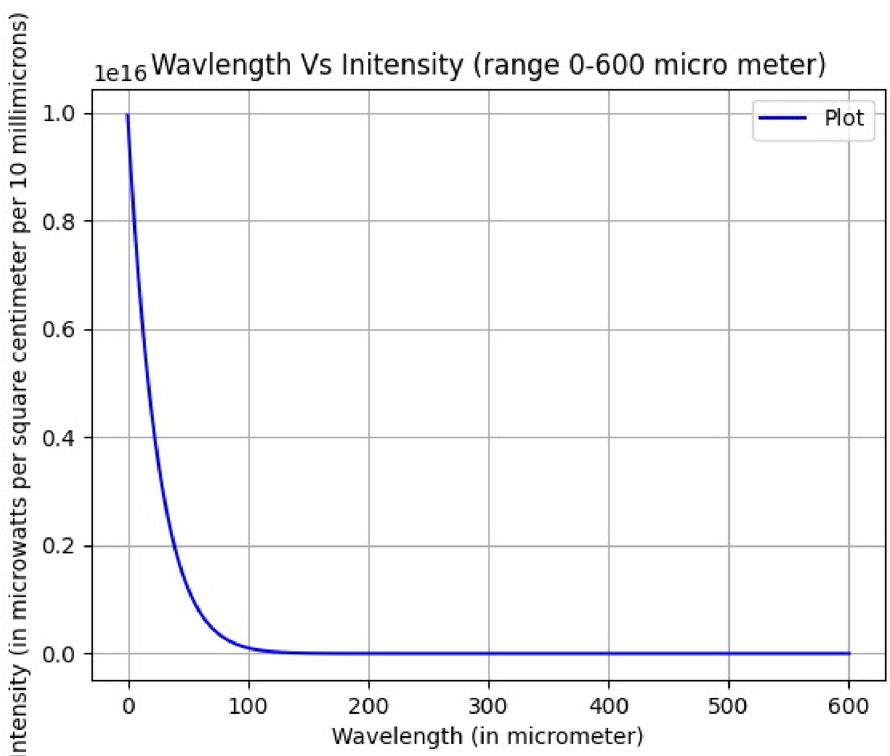

Figure 5: Intensity of incident light Vs Wavelength

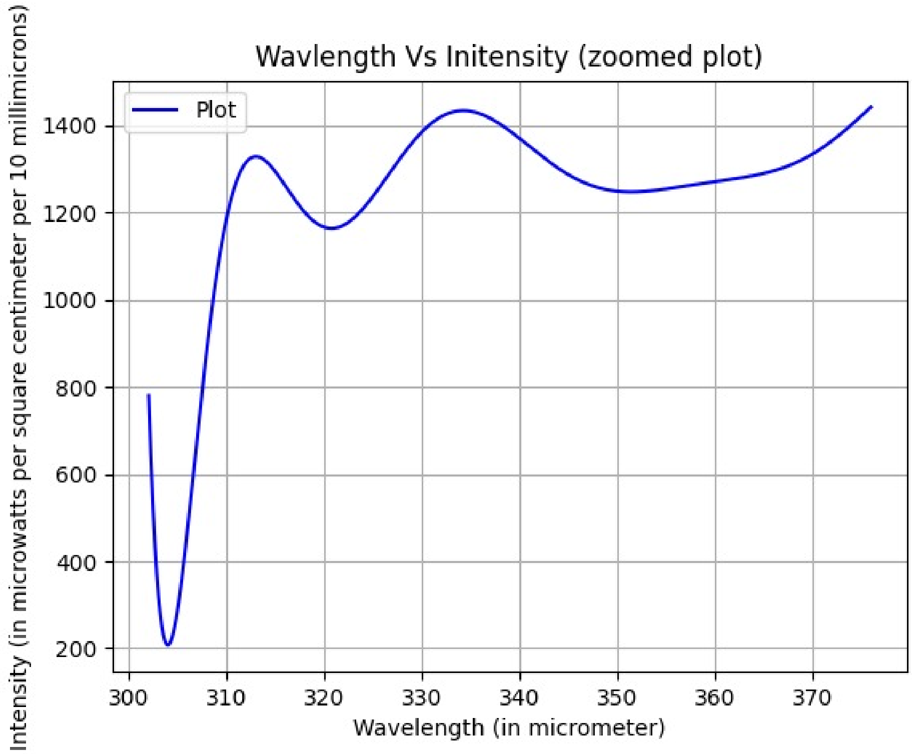

Figure 6: Magnified Image of Fig. 5

- Here as we can see that there are some variations in our answers when using 2 different methods (Simpson's and Trapezoidal)
- Simpson Method is more preferred over trapezoidal method because the difference in consecutive data points is same over the entire range of P *i.e.*, $\forall\ (X_i - X_{i-1} = constant)$
- The density of the particles decreases exponentially as we increase the altitude (equation 3, from Section 3.2). Therefore, with scarce density of particles, light scattering is minimal (tending to 0). Thus, to reduce the computation, we have considered a boundary condition as $x\ \epsilon\ [-12000,\ 12000]$
- The number of segments chosen also determines the accuracy of their methods and thus their closeness to the actual solution, given by the analytical solution.
- We observed that scattering intensity in case of Mie scattering is much less than that of Rayleigh scattering. This is due to the fact that density of dust, fog, etc. particles are negligible compared to the air molecules in the atmosphere. The main distinction between these two forms of scattering is that Rayleigh scattering's amplitude is dependent on the wavelength of scattered radiation, while Mie scattering is not. This means that the blue sky colour is caused by Rayleigh scattering in combination. Mie scattering, on the other hand, causes the grey shades of halos around the sun, clouds, and fog.

- Simpson's 1/3 rule considers the data set to follow a polynomial of order 3, while Simpson's 3/8 rule considers 4_{th} order polynomial.
- Trapezoidal method considers the data set to follow linear relation.
- The Global Error in Trapezoidal Method $\propto$ h_s^2 ; where h_s is the distance step that we have considered while obtaining our solution.
- The Global Error in both Simpson's Method $\propto$ h_s^4 ; where h_s is the distance step that we have considered while obtaining our solution.
- The Truncation Error or Local Error in Trapezoidal Method $\propto$ h_s^3; where h_s is the distance step that we have considered while obtaining our solution.
- The Truncation Error or Local Error in both Simpson's Method $\propto$ h_s^5; where h_s is the distance step that we have considered while obtaining our solution.
- Since the number of segments in the interval of x is odd, therefore, our algorithm will use Simpson's 3/8 rule till the number of segments remaining becomes less than 2.The integral of remaining segments is computed using either Simpson's 1/3 rule or Trapezoidal method depending upon the number of segments left (if the number of segments remaining are 2, then Simpson's 1/3 rule is used, else Trapezoidal method will be used).
- If the number of segments were even then we would only use Simpson's 1/3 rule.
- Atmospheric molecular density and refractive index vary constantly pertaining to the weather conditions. This may account for slight amount of error in computation of the above results.
- By using look up tables the optical depth and the illuminance by the sky light are effectively measured using the information that the earth is spherical and that the sun rays is parallel.

References

[1] Gałaj, Tomasz and Wojciechowski, Adam *A study on numerical integration methods for rendering atmospheric scattering phenomenon*. Open Physics, vol. 17, no. 1, 2019, pp. 241-249.

[2] Ralph Stair, Russell G. Johnston, and Thomas C . Bagg *Spectral Distribution of Energy From the Sun* Journal of Research of the National Bureau of Standards, Vol. 53, No.2, August 1954

[3] Nishita, Tomoyuki and Sirai, Takao and Tadamura, Katsumi and Nakamae, Eihachiro *Display of the Earth Taking into Account Atmospheric Scattering*. SIGGRAPH '93: Proceedings of the 20^{th} annual conference on Computer graphics and interactive techniques, September 1993