Problem 6: What is the difference between the sum of the squares and the square of the sums?

First of all, one could use a brute force implementation, because 100 is not a really high limit. This could be done as follows:

$limit = 100$
$sum\_sq = 0$
$sum = 0$
for $i = 1$ to $limit$ **do**
  $sum = sum + i$
  $sum\_sq = sum\_sq + i^2$
end for
print $sum^2 - sum\_sq$

However, such an approach would definitely get in trouble when *limit* becomes very large.
A closer look at the program shows that the *sum* variable, at the end, contains the sum of the integers from 1 to *limit*. As is widely known, this sum can be directly calculated using the formula $sum(n) = n(n+1)/2$. As you might have expected, such a formula also exists for the sum of squares. Let us derive this formula.

Thus, we are looking for a function $f(n)$, that for any n gives the sum of 1^2 up to n^2. Assume it is of the form $f(n) = an^3 + bn^2 + cn + d$, with a, b, c, d constants that we have to determine. This we can do because we can verify that $f(0) = 0, f(1) = 1, f(2) = 5, f(3) = 14$. This yields four equations in four variables, namely

$$\begin{aligned} d &= 0 \\ a + b + c + d &= 1 \\ 8a + 4b + 2c + d &= 5 \\ 27a + 9b + 3c + d &= 14 \end{aligned}$$

Solving this system of equations, we obtain $a = \frac{1}{3}, b = \frac{1}{2}, c = \frac{1}{6}, d = 0$. This gives $f(n) = \frac{1}{6}(2n^3 + 3n^2 + n) = \frac{n}{6}(2n+1)(n+1)$.
What remains is to show this f actually is what we want. This we prove by induction: Assuming f is the correct formula for n, we show it is also for $n + 1$. Then, because we know it is correct for $n = 0, 1, 2, 3$, we know that it's correct for all n. Thus, we have to show $f(n+1) = f(n) + (n+1)^2$. By expanding both sides we get

$$\begin{aligned} f(n+1) &= f(n) + (n+1)^2 \\ \frac{n^3}{3} + \frac{3n^2}{2} + \frac{13n}{6} + 1 &= \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} + n^2 + 2n + 1 \end{aligned}$$

Since both sides are equal, we have proven that f is the correct formula. This means that we can now write a very simple program to calculate the difference between the sum of the squares and the square of the sum:

$limit = 100$
$sum = limit(limit + 1)/2$
$sum\_sq = (2limit + 1)(limit + 1)limit/6$
print $sum^2 - sum\_sq$

This algorithm is limited only by the size of the integer types your programming language (and computer memory) support.

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