part of the definition of a Hilbert bundle and this always holds (by the "naturality" of definitions) in our applications.

REMARK 3.34. In the proof of Theorem 3.32, we have strongly used the fact that H is a Hilbert space. We treated the general case with variable $\operatorname{Ker} T_x$ by reducing it, through composition with an orthogonal projection, to the simple special case of constant kernel dimension. Instead, we can study the family T on the quotient spaces $H/\operatorname{Ker} T_x$ which produces the constant kernel dimension 0, or we can make the operators T_x surjective by extending their domain of definition (to $H \oplus \operatorname{Ker}(T_x^*)$ say) which produces constant cokernel dimension 0. Details for these two alternatives, which are already indicated in the usual proofs of the homotopy invariance of the index (e.g., see [**Jö**, 1970/1982, 5.4] and our comments after Theorem 3.13), can be found for the first alternative in [**Ati67a**, p.155-158] and for the second alternative, for instance if H is a Fréchet space and T is an elliptic operator (see below) in [**AS71a**, p.122-127].

Warning: In connection with "Wiener-Hopf operators", we will later meet families of Fredholm operators $T : X \to \mathcal{F}$ for which it is quite possible that $\operatorname{index} T = 0$ for all $x \in X$, while the global index bundle $\operatorname{index} T \in K(X)$ does not vanish. The index bundle is simply a much sharper invariant than just an integer. Thus one must be very careful if one wishes to infer properties of the index bundle from those of the index. The next exercise may be comforting.

EXERCISE 3.35. Show that for each continuous Fredholm family $T : X \to \mathcal{F}$, we have $\operatorname{index} T^* = -\operatorname{index} T$, where $T^* : X \to \mathcal{F}$ is the adjoint family $(T^*)_x := (T_x)^*$ for $x \in X$.

THEOREM 3.36. *The construction of the index bundle in* Theorem 3.32 *depends only on the homotopy class of the given family of Fredholm operators, and for X compact yields a homomorphism of semigroups*

$$\operatorname{index} : [X, \mathcal{F}] \to K(X).$$

PROOF. **1**. We show first the *homotopy-invariance* of the index bundle. Let $T : X \times I \to \mathcal{F}$ be a homotopy between the families of Fredholm operators $T_0 := T i_0$ and $T_1 := T i_1$ parametrized by X, where

$$i_t : X \to X \times \{t\} \hookrightarrow X \times I,\ t \in I$$

are the mutually homotopic natural inclusions. The functoriality of index bundles (Exercise 3.39) then yields

$$\operatorname{index} T_0 = i_0^*(\operatorname{index} T) = i_1^*(\operatorname{index} T) = \operatorname{index} T_1,$$

where the middle equality follows from the homotopy invariance of $K(X)$; namely, $f^* = g^*$ for $f \sim g : Y \to X$ (Theorem B.5, p. 632, Appendix).

2. For the *homomorphism property* of the index, we point out first that for two Fredholm families $S, T : X \to \mathcal{F}$, we have a well defined product family $TS : X \to \mathcal{F}$ given by composition in $\mathcal{F}$ (i.e., $(TS)_x := T_x S_x$), and that $[X, \mathcal{F}]$ is then a semigroup. On the other hand, $K(X)$ has an addition defined via the direct sum of vector bundles which makes it a group (Section 13.3). The index is a semigroup homomorphism now, since

$$\begin{aligned} \operatorname{index} TS &= \operatorname{index} TS + \operatorname{index} \mathrm{Id} = \operatorname{index}(TS \oplus \mathrm{Id}) \\ &= \operatorname{index}(S \oplus T) = \operatorname{index} S + \operatorname{index} T. \end{aligned}$$

In the second and fourth equalities, we have used the fact that (by the construction in Theorem 3.32) the index bundle, for a family of Fredholm operators on a product space $H \times H$ which can be written in the form

$$S \oplus T := \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix},$$

coincides with the direct sum of the index bundles of the two diagonal elements. Moreover, for the third equality, we have applied the usual trick of homotopy theory, where $TS \oplus \mathrm{Id}$ can be deformed into $S \oplus T$ in $\mathcal{F}(H \times H)$. This is clear by the definition of this homotopy in Exercise 3.20 (p. 67). Note that we do not need $T, S \in \mathcal{B}^\times(H)$ since we are deforming in $\mathcal{F}(H \times H)$, as opposed to $\mathcal{B}^\times(H \times H)$. □

Remark 3.37. Note that in the proof of homotopy invariance, we did not need to again investigate the topology of F, but rather everything followed from the entirely different aspect of "homotopy invariance" of vector bundles. Why did it take more work to prove the homotopy-invariance of the index of a single operator in the proof of Theorem 3.13, p.62, while in the proof of Theorem 3.36 we did not use this result, even though we can deduce it for $X = \{point\}$. The solution of this paradox lies in the fact that the actual generalization of the homotopy invariance of the index is already contained in the construction of index bundles in Theorem 3.32!

Exercise 3.38. Show that the index bundle of a continuous family of self-adjoint Fredholm operators, $T : X \to \mathcal{F}$ with $T_x = T_x^*$ for all $x \in X$ is zero; i.e., $\operatorname{index} T = 0$.
Warning: One cannot deduce this from Exercise 3.35, since it is possible that $K(X)$ has a finite cyclic ("torsion-") factor; e.g., possibly $a + a = 0$, but $a \neq 0$.
[Hint: Consider the homotopy $tT + i(1-t)\,\mathrm{Id}$. Show that a self-adjoint operator A has a real spectrum; i.e.,

$$A - z\,\mathrm{Id} \in \mathcal{B}^\times \text{ for } z \in \mathbb{C} - \mathbb{R} :$$

Step 1. $u \in \operatorname{Ker}(A - z\,\mathrm{Id})$ implies $zu = Au = A^*u = \bar{z}u$, and so $u = 0$ since $z \neq z$.
Step 2. If v is in the orthogonal complement of $\operatorname{Im}(A - z\,\mathrm{Id})$, then $\langle Au - zu, v\rangle = 0$ and so

$$\langle u, Av\rangle = \langle Au, v\rangle = \langle zu, v\rangle = \langle u, \bar{z}v\rangle \text{ for all } u \in H.$$

Thus, $Av = \bar{z}v$, and $v = 0$ by step 1. Hence, $\operatorname{Im}(A - z\,\mathrm{Id})$ is dense in H.
Step 3. Let $v \in H$ and let $v_1, v_2, ... \in \operatorname{Im}(A - z\,\mathrm{Id})$ be a sequence converging to v. Then show that the sequence of unique preimages $u_1, u_2, ...$ is a Cauchy sequence, and set $u := \lim u_i$. Then note that $Au - zu = v$.]

We will now investigate the construction in the Theorem 3.32 more generally, and show that it has all the properties that one might reasonably expect. We begin with the following exercise.

Exercise 3.39. Show that our definition of index bundle is functorial: Let $f : Y \to X$ be a continuous map (X and Y compact) and let $T : X \to \mathcal{F}$ be a continuous family of Fredholm operators. Then (see Exercise B.1d, p. 628, of the Appendix), we have $\operatorname{index} Tf = f^*(\operatorname{index} T)$.
[Hint: If $e_1, e_2, ...$ is an orthonormal basis for H and the projection P_n is chosen so that $\dim \operatorname{Ker} P_n T_x$ is independent of x, then $\dim \operatorname{Ker} P_n T_{f(y)}$ is also independent of $y \in Y$. Thus, one does not need to choose another projection P_n to exhibit the index bundle for the family $Tf : Y \to \mathcal{F}$.]

EXERCISE 3.40. The proof of Theorem 3.36 yields (for $X = point$) a further proof of the composition rule $\operatorname{index} TS = \operatorname{index} S + \operatorname{index} T$ for Fredholm operators. We have already seen three proofs, namely Exercise 1.10 (p.9), Exercise 2.2 (p.12), and Exercise 3.7 (p.59). Do these three proofs carry over without difficulty to the case of families of Fredholm operators? Does one need a homotopy argument each time? What is the real relationship between the four proofs?

EXERCISE 3.41. Show that the set

$$\mathcal{F}_0 := \{T \in \mathcal{F} : \operatorname{index} T = 0\}$$

is pathwise-connected.
[Hint: For $T \in \mathcal{F}_0$ choose an isomorphism

$$\phi : \operatorname{Ker} T \to (\operatorname{Im} T)^\perp$$

(vector spaces of the same finite dimension!) and set

$$\Phi := \begin{cases} \phi & \text{on } \operatorname{Ker} T \\ 0 & \text{on } (\operatorname{Ker} T)^\perp \end{cases}$$

By construction, we have $T + \Phi \in \mathcal{B}^\times$ and $T + t\Phi \in \mathcal{F}$ for $t \in I$ (Why? What kind of operator is Φ?). Now apply Theorem 3.23.]

8. The Theorem of Atiyah-Jänich

The sets $\mathcal{F}_i := \{T \in \mathcal{F} : \operatorname{index} T = i\}$ are bijectively (modulo compact operators) mapped onto each other in a continuous fashion by shift operators (see Exercise 1.3, p.4). Thus, Exercise 3.41 with the homotopy-invariance of the index already proved in Theorem 3.13 (p. 62) gives us a bijection from the pathwise-connected components of $\mathcal{F}$ to $\mathbb{Z}$. Naturally, this result still does not say much about the topology of $\mathcal{F}$, and we do not want to carry it out in detail. Much more informative is the following theorem which gives the result,

$$\operatorname{index} : [\{point\}, \mathcal{F}] \to \mathbb{Z} \text{ bijective,}$$

as the special case $X = \{point\}$.

THEOREM 3.42 (M. F. Atiyah, K. Jänich 1964). *We have an isomorphism*

$$\operatorname{index} : [X, \mathcal{F}] \to K(X)$$

PROOF. We show that the natural sequence of semigroups

$$[X, B^\times] \to [X, \mathcal{F}] \overset{\text{index}}{\to} K(X) \to 0$$

is exact. From this result of M. Atiyah and K. Jänich along with the theorem of N. Kuiper (Theorem 3.24, p.72), the present theorem follows. (For the concept of "exactness", see the material before Theorem 1.9, p. 6).

Step 1. The index bundle of a continuous family $T : X \to \mathcal{B}^\times$ is trivially zero.

Step 2. Now let $T : X \to \mathcal{F}$ be a Fredholm family with vanishing index bundle. In Exercise 3.41, we have stated what one must do in the case where $X = \{point\}$; i.e., T is a single Fredholm operator. Namely, one chooses an isomorphism

$$\phi : \operatorname{Ker} T \to (\operatorname{Im} T)^\perp$$

and then

$$\Phi := \begin{cases} \phi & \text{on } \operatorname{Ker} T \\ 0 & \text{on } (\operatorname{Ker} T)^\perp \end{cases}$$

is an operator of finite rank with $T + t\Phi$ the desired homotopy in $\mathcal{F}$ to $T + \Phi \in \mathcal{B}$. To generalize this to $X \neq \{point\}$, we now must deal with two difficulties:

1. $\bigcup_{x\in X} \operatorname{Ker} T_x$ is not always a vector bundle.
2. In general, there is no canonical choice of $\phi_x : \operatorname{Ker} T_x \to (\operatorname{Im} T)^\perp$ which depends continuously on x.

The first problem, we can quickly solve: We choose again an orthogonal projection $P_n : H \to H_n$, such that $\dim \operatorname{Ker} P_n T_x$ is constant and

$$\operatorname{index} T = [\operatorname{Ker} P_n T_x] - [X \times (H_n)^\perp] \in K(X).$$

Then "index $T = 0$" means

$$[\operatorname{Ker} P_n T] = [X \times (H_n)^\perp] \text{ in } K(X),$$

which in turn means (see Section 13.3) that, for N sufficiently large, there is an isomorphism of vector bundles

$$\phi : (\operatorname{Ker} P_n T) \oplus (X \times \mathbb{C}^N) \cong (X \times (H_n)^\perp) \oplus (X \times \mathbb{C}^N).$$

This means that by "augmenting" with the trivial bundle $X \times \mathbb{C}^N$, we can also cure the second difficulty in principle. Actually we can avoid this K-theoretical argument. Indeed, we do not need to "augment", if we take n to be large enough: As shown in the proof of Theorem 3.32 (p.79), we have

$$\operatorname{Ker} P_{n+1} T \cong \operatorname{Ker} P_n T \oplus (X \times \mathbb{C}).$$

Thus, for $m := n + N$, the map ϕ can be regarded as an isomorphism

$$\operatorname{Ker} P_m T \to X \times (H_m)^\perp.$$

In this way, the construction of ϕ in Exercise 3.41 carries over pointwise, whence we obtain a homotopy of Fredholm families between

$$P_m T : X \to \mathcal{F}(H) \text{ and } P_m T + \Phi : X \to \mathcal{B}^\times(H).$$

Since the index of the orthogonal projection $P_m : H \to H_m$ vanishes (and hence coincides with the index of Id), we can connect the "constants" P_m with Id in $\mathcal{F}(H)$ (Exercise 3.41), and hence also connect $P_m T$ with T in a further homotopy of maps $X \to \mathcal{F}(H)$. Each Fredholm family with vanishing index bundle is therefore homotopic to a continuous family of invertible operators, and, in conjunction with step 1, exactness in the middle of the short sequence is now proved.

Step 3: We must now show the surjectivity of index : $[X, F] \to K(X)$. By the construction in Exercise B.10 (p. 634) of the Appendix (see also Section 13.3), every element of $K(X)$ can be written in the form $[E] - [X \times \mathbb{C}^k]$, where E is a vector bundle over X and $k \in \mathbb{Z}_+$. Hence we will be finished with the proof if, for each vector bundle E, we can find a continuous family S of (surjective) Fredholm operators on a suitable Hilbert space such that $\operatorname{index} S = [E]$. Then, the homomorphism of the index bundle construction (Theorem 3.32) yields

$$\operatorname{index} \left(\operatorname{shift}^+\right)^k S = [E] - [X \times \mathbb{C}^k],$$

where shift (as in Exercise 1.3, p. 1.3) is the displacement to the right relative to an orthonormal basis of the Hilbert space. Note that, since $\operatorname{index} \operatorname{shift}^+ = -1$, the constant family $x \to \operatorname{shift}^+$ gives the index bundle $-[X \times \mathbb{C}]$.

Thus, let E be a vector bundle over X. If X consists of a single point, then we just complete an orthonormal basis of E (regarded as a subspace of an infinite-dimensional separable Hilbert space) to an orthonormal basis of the whole Hilbert space and set $S := (\mathrm{shift}^-)^{\dim E}$. Now, we consider the general case. By Exercise 13.8 (p. 231) of the Appendix, there is a vector bundle F over X and a finite-dimensional (complex) vector space V such that $E \oplus F \cong X \times V$. Let $\pi : V \to E$ be the projection.

Let H be an arbitrary Hilbert space. We will see that it is simplest to consider the desired operators S_x, $x \in X$, to be defined on the space $\mathrm{Hom}(V, H)$ of linear transformations from V to H. We choose for the vector space V (which we can identify with $\mathbb{C}^N$, $N = \dim V$, via a basis) a Hermitian scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. For every pair $(f, u) \in V \times H$, we have an element of $\mathrm{Hom}(V, H)$ defined by

$$v \to \langle f, v \rangle\, u,\ \ v \in V,$$

which we will denote by $f \otimes u$; recall the isomorphism $\mathrm{Hom}(V, H) = V' \otimes H$ of linear algebra, where $V'(\cong V)$ is the vector space of linear maps from V to $\mathbb{C}$. Then, we have

$$\mathrm{Hom}(V, H) = \left\{ \sum\nolimits_{i=1}^{m} f_i \otimes u_i : m \in \mathbb{N},\ f_i \in V,\ u_i \in H \right\}$$

where

$$\begin{aligned} zf \otimes u &= f \otimes zu, z \in \mathbb{C} \text{ and} \\ (f + f') \otimes (u + u') &= f \otimes u + f \otimes u' + f' \otimes u + f' \otimes u'. \end{aligned}$$

We define a scalar product

$$\left\langle \sum\nolimits_{i=1}^{m} f_i \otimes u_i, \sum\nolimits_{j=1}^{m} f'_j \otimes u'_j \right\rangle := \sum\nolimits_{i,j=1}^{m} \left\langle f_i, f'_j \right\rangle \left\langle u_i, u'_j \right\rangle,$$

which makes $\mathrm{Hom}(V, H)$ a Hilbert space (since V is finite-dimensional, nothing can go wrong; addition and scalar multiplication by complex numbers are defined naturally). One easily checks that if $f_1, ..., f_N$ is an orthonormal basis for V and $e_0, e_1, e_2, \ldots$ is an orthonormal basis for H, then

$$\{f_j \otimes e_i\}_{j=1,\ldots,N,\ i=0,1,2,\ldots}$$

is an basis orthonormal basis for $\mathrm{Hom}(V, H)$.

Each bounded linear operator T on H and endomorphism τ of V clearly define a bounded linear operator $\tau \otimes T$ on $\mathrm{Hom}(V, H)$ by means of

$$(\tau \otimes T)(f \otimes u) := \tau(f) \otimes T(u).$$

For a fixed chosen basis $e_0, e_1, e_2, \ldots$ of H, we set

$$S_x := (\pi_x \otimes \mathrm{shift}^-) + (\mathrm{Id}_V - \pi_x) \otimes \mathrm{Id}_H$$

Then,

$$S_x(f \otimes e_i) = \begin{cases} \pi_x(f) \otimes e_{i-1} + (f - \pi_x(f)) \otimes e_i & \text{for } i \geq 1 \\ (f - \pi_x(f)) \otimes e_0 & \text{for } i = 0, \end{cases}$$

and in particular for $f \in E_x$, we have $S_x(f \otimes e_0) = 0$. Hence, we have $\mathrm{Im}\, S_x = \mathrm{Hom}(V, H)$ and $\mathrm{Ker}\, S_x = \{f \otimes e_0 : f \in E_x\}$. Thus, $\mathrm{Ker}\, S_x$ is isomorphic to E_x in a natural way, and $\mathrm{index}\, S = [\mathrm{Ker}\, S] - 0 = [E]$. $\square$

The preceding theorems are significant on various levels: In the following chapters, in dealing topologically with boundary value problems as well as in proving analytically the Periodicity Theorem of the topology of linear groups, we will repeatedly use Theorems 3.32 and 3.36, i.e., the construction of the index bundle and its elementary properties, but we will not use explicitly Theorem 3.42, our actual "main theorem". Nevertheless, the last theorem has fundamental significance for our topic, as it provides the reasons for the theoretical relevance of the notion "index bundle" and explains why this concept proved suitable to express deep relations in analysis as well as topology.

9. Determinant Line Bundles

A primary motivation for the study of determinant line bundles originated from the desire of quantum physicists to produce a gauge-invariant volume element for the purpose of computing (via functional integration) Green's functions for Dirac operators coupled to gauge potentials. An obstruction to doing this is the nontriviality of the so-called determinant bundle for a certain family of Fredholm operators, namely the family Dirac operators parametrized by the quotient space of gauge potentials modulo gauge transformations. This obstruction signals the presence of so-called anomalies that arise when physicists attempt to quantize the classical field theory. While it would be premature to go into the mathematically murky details of this now, in this section we develop the notion of the determinant line bundle of a continuous family of Fredholm operators. Moreover, in the case of a family with index 0, we assist the reading in showing (see Exercise 3.53 and Corollary 3.54) that it is the pull-back of a "universal" determinant line bundle that we construct over $\mathcal{F}_0$, the space of Fredholm operators of index 0. The simplest way of describing this line bundle (often referred to as the Quillen determinant line bundle, which arose in [**Q85**]) is to declare the fiber above the point $T \in \mathcal{F}_0$ to be

$$\Lambda^{d_T}(\ker T)^* \otimes \Lambda^{d_T}(\ker T^*)\,, \text{ where } d_T = \dim\ker T = \dim\ker T^*.$$

However, the fact that these fibers may be pieced together to form a genuine line bundle is not trivial since d_T varies with T, and d_T is an unbounded function of $T \in \mathcal{F}_0$. In doing this, we adopt a novel approach due primarily to Graeme Segal (in [**Seg90**]). Various ways of describing this universal bundle are summarized in Theorem 3.55.

For compact X and a continuous family $T : X \to \mathcal{F}$ of Fredholm operators in a fixed Hilbert space H, we have seen that there is an index bundle

$$\operatorname{index} T \in K(X)$$

assigned in a canonical way. While the determinant of an operator on a Hilbert space H exists only in restrictive circumstances, we now show that there is a well-defined complex line bundle $\det T \to X$, known as the *determinant line bundle of* T. For any $\alpha \in K(X)$, we will first define an isomorphism class $\det \alpha \in K(X)$ which is represented by a line bundle. For a continuous family $T : X \to \mathcal{F}$, we then can (and do) simply define $\det T$ to be $\det(\operatorname{index} T)$, where $\operatorname{index} T \in K(X)$ is given in the Atiyah–Jänich Theorem 3.42. We know that $\alpha = [E] - [F] \in K(X)$ for $E, F \in \operatorname{Vect}(X)$. For a finite-dimensional vector bundle $V \to X$, it is convenient to use the notation $\Lambda^{\max}(V) = \Lambda^{\dim V}(V)$. We claim that $\left[\Lambda^{\max}(E)^* \otimes \Lambda^{\max}(F)\right] \in$

$K(X)$ only depends on α, in the strong sense that if $\alpha = [E] - [F] = [E'] - [F']$, then

$$\Lambda^{\max}(E)^* \otimes \Lambda^{\max}(F) \cong \Lambda^{\max}(E')^* \otimes \Lambda^{\max}(F'). \tag{9.1}$$

(not just $\left[\Lambda^{\max}(E)^* \otimes \Lambda^{\max}(F)\right] = \left[\Lambda^{\max}(E')^* \otimes \Lambda^{\max}(F')\right]$ in $K(X)$). Indeed, using the fact that $\Lambda^{\max}(V \oplus W) = \Lambda^{\max}(V) \otimes \Lambda^{\max}(W)$, we have

$$\begin{aligned} & [E] - [F] = [E'] - [F'] \Leftrightarrow E \oplus F' \cong E' \oplus F \\ \Rightarrow \quad & \Lambda^{\max}(E \oplus F') \cong \Lambda^{\max}(E' \oplus F) \\ \Rightarrow \quad & \Lambda^{\max}(E) \otimes \Lambda^{\max}(F') \cong \Lambda^{\max}(E') \otimes \Lambda^{\max}(F). \end{aligned}$$

Tensoring both sides with $\Lambda^{\max}(E')^* \otimes \Lambda^{\max}(E)^*$ and noting that for a line bundle L, $L \otimes L^*$ is isomorphic to the trivial bundle $X \times \mathbb{C}$, we then obtain (9.1). We can now make the following

Definition 3.43. For X compact and $\alpha \in K(X)$, let E and F be vector bundles over X with $\alpha = [E] - [F]$, we define the **determinant** of α by

$$\det \alpha = \left[\Lambda^{\max}(E)^* \otimes \Lambda^{\max}(F)\right] \in K(X)$$

By (9.1) the bundle $\Lambda^{\max}(E)^* \otimes \Lambda^{\max}(F)$ is well defined (independent of the choice of E and F). By an abuse of notation, we also denote this isomorphism class by $\det \alpha$. For a continuous family $T : X \to \mathcal{F}$,

$$\det T := \det(\operatorname{index} T).$$

Note that while $\det T$ has been defined, this does not directly imply that

$$\bigcup_{x \in X} \Lambda^{\max}(\operatorname{Ker}(T_x))^* \otimes \Lambda^{\max}(\operatorname{Ker}(T_x^*)) \tag{9.2}$$

can be given the structure of a (global) vector bundle bundle over X. Usually, doubts about this are glibly deflected by saying while $\operatorname{Ker}(T)$ and $\operatorname{Ker}(T^*)$ are not defined globally in general, $\dim \operatorname{Ker}(T_x)$ and $\dim \operatorname{Ker}(T_x^*)$ jump by the same amount if x varies. It is useful (and comforting) to show that for *some* genuine vector bundles E and F over X with $\operatorname{index} T = [E] - [F]$, the fiber $\Lambda^{\max}(E_x)^* \otimes \Lambda^{\max}(F_x)$ of the manifestly well defined bundle $\Lambda^{\max}(E)^* \otimes \Lambda^{\max}(F) \to X$ is isomorphic to $\Lambda^{\max}(\operatorname{Ker}(T_x))^* \otimes \Lambda^{\max}(\operatorname{Ker}(T_x^*))$ in a "natural" fashion. Indeed, as in the proof of Theorem 3.32 (p. 79), let $P_n : H \to H_n = \{e_0, \ldots, e_n\}^{\perp}$ be an orthogonal projection so that $P_n T_x H = H_n$ for all $x \in X$. We then have bundles $E = \operatorname{Ker} P_n T$ and $F = \operatorname{Ker}(P_n T)^* = X \times H_n^{\perp}$, and a "natural" isomorphism

$$\Lambda^{\max}(\operatorname{Ker} P_n T_x)^* \otimes \Lambda^{\max}\left(H_n^{\perp}\right) \cong \Lambda^{\max}(\operatorname{Ker}(T_x))^* \otimes \Lambda^{\max}(\operatorname{Ker}(T_x^*))$$

is supplied by taking G to be T_x in the following

Proposition 3.44. *Suppose that $G \in \mathcal{F}$ and $P_n G(H) = H_n$. Then there is an isomorphism (depending only on the choice P_n)*

$$\Psi_{G,n} : \Lambda^{\max}(\operatorname{Ker} P_n G)^* \otimes \Lambda^{\max}\left(H_n^{\perp}\right) \cong \Lambda^{\max}(\operatorname{Ker} G)^* \otimes \Lambda^{\max}(\operatorname{Ker} G^*). \tag{9.3}$$

Since $P_{n'} G(H) = H_{n'}$ for $n' \geq n$, (9.3) implies that

$$\Lambda^{\max}(\operatorname{Ker} P_{n'} G)^* \otimes \Lambda^{\max}\left(H_{n'}^{\perp}\right) \cong \Lambda^{\max}(\operatorname{Ker} P_n G)^* \otimes \Lambda^{\max}\left(H_n^{\perp}\right).$$

Proof. First note that we have an isomorphism

$$\widetilde{G} := G|_{\mathrm{Ker}(G)^{\perp}} : (\mathrm{Ker}\, G)^{\perp} \to G(H),$$

and hence

$$\begin{aligned} \mathrm{Ker}\, P_n G &= G^{-1} H_n^{\perp} = \mathrm{Ker}\,(G) \oplus \left(\mathrm{Ker}\,(G)^{\perp} \cap G^{-1}\left(H_n^{\perp}\right)\right) \\ &= \mathrm{Ker}\,(G) \oplus \widetilde{G}^{-1}\left(H_n^{\perp} \cap G(H)\right). \end{aligned} \tag{9.4}$$

The orthogonal projection $Q_n : H \to H_n^{\perp}$ is $\mathrm{Id}_H - P_n$. We are given that $P_n G(H) = H_n$. This implies that $Q_n|_{G(H)^{\perp}} : G(H)^{\perp} \to H_n^{\perp}$ is injective, because

$$\begin{aligned} & v \in G(H)^{\perp},\ Q_n(v) = 0 \Rightarrow v \in H_n \cap G(H)^{\perp} \\ \Rightarrow\ & v = P_n(w) \text{ for some } w \in G(H), \text{ say } w = G(u) \\ \Rightarrow\ & G(u) = w = P_n(w) + Q_n(w) = v + Q_n(w) \\ \Rightarrow\ & v = G(u) - Q_n(w) \\ \Rightarrow\ & \langle v, v\rangle = \langle v, G(u) - Q_n(w)\rangle = \langle v, G(u)\rangle - \langle v, Q_n(w)\rangle = 0, \end{aligned}$$

since $v \in G(H)^{\perp}$ and $\langle v, Q_n(w)\rangle = \langle P_n(w), Q_n(w)\rangle = 0$. We claim

$$H_n^{\perp} = \left(H_n^{\perp} \cap G(H)\right) \oplus Q_n\left(G(H)^{\perp}\right). \tag{9.5}$$

First note that $\left(H_n^{\perp} \cap G(H)\right) \cap Q_n\left(G(H)^{\perp}\right) = \{0\}$, since

$$\begin{aligned} u &\in \left(H_n^{\perp} \cap G(H)\right) \cap Q_n\left(G(H)^{\perp}\right) \\ &\Rightarrow u = G(w) \in H_n^{\perp} \text{ for some } w \in H, \text{ and } u = Q_n(v) \text{ for } v \in G(H)^{\perp} \\ &\Rightarrow \langle u, u\rangle = \langle Q_n(v), G(w)\rangle = \langle v - P_n(v), G(w)\rangle \\ &= \langle v, G(w)\rangle - \langle P_n(v), G(w)\rangle = 0 - \langle P_n(v), G(w)\rangle = 0, \end{aligned}$$

since $v \in G(H)^{\perp}$ and $G(w) \in H_n^{\perp}$. The same proof yields the general fact that for two subspaces V and W of an inner product space, the orthogonal projection of $V^{\perp}$ onto W is orthogonal to $V \cap W$. To obtain (9.5), it now suffices to show that

$$\dim\left(H_n^{\perp} \cap G(H)\right) \geq \dim H_n^{\perp} - \dim Q_n\left(G(H)^{\perp}\right).$$

For this, note that

$$\begin{aligned} & \dim\left(H_n^{\perp} \cap G(H)\right) \geq \dim\left(H_n^{\perp}\right) - \mathrm{codim}\,(G(H)) \\ = \ & \dim\left(H_n^{\perp}\right) - \dim G(H)^{\perp} = \dim\left(H_n^{\perp}\right) - \dim Q_n\left(G(H)^{\perp}\right), \end{aligned}$$

since we have shown that $Q_n|_{G(H)^{\perp}}$ is injective. This also follows from

$$\begin{aligned} & \mathrm{index}\, G = \mathrm{index}\, G + \mathrm{index}\, P_n = \mathrm{index}\, P_n G \\ \Rightarrow\ & \dim \mathrm{Ker}\, G - \dim G(H)^{\perp} = \dim \mathrm{Ker}\, P_n G - \dim H_n^{\perp} \\ \Rightarrow\ & \dim G(H)^{\perp} = \dim\left(H_n^{\perp} \cap G(H)\right) - \dim H_n^{\perp}. \end{aligned}$$

Let

$$\widetilde{G}_n := \widetilde{G}|_{\widetilde{G}^{-1}(H_n^{\perp} \cap G(H))} : \widetilde{G}^{-1}\left(H_n^{\perp} \cap G(H)\right) \cong H_n^{\perp} \cap G(H), \tag{9.6}$$

and

$$q_{G,n} := Q_n|_{G(H)^{\perp}} : G(H)^{\perp} \cong Q_n\left(G(H)^{\perp}\right).$$

By (9.5) and (9.4), we have the isomorphisms

$$\begin{aligned}
&\Lambda^{\max}(\widetilde{G}_n^{-1})^* : \Lambda^{\max}(\widetilde{G}^{-1}\left(H_n^{\perp} \cap G(H)\right))^* \cong \Lambda^{\max}\left(H_n^{\perp} \cap G(H)\right)^*, \\
&\Lambda^{\max}(q_{G,n}^{-1})^* : \Lambda^{\max}(Q_n\left(G(H)^{\perp}\right))^* \cong \Lambda^{\max}\left(G(H)^{\perp}\right)^*, \text{ and} \\
&\Lambda^{\max}\left(H_n^{\perp} \cap G(H)\right)^* \otimes \Lambda^{\max}\left(\left(H_n^{\perp} \cap G(H)\right)\right) \cong \mathbb{C}.
\end{aligned}$$

Then we obtain (9.3), as follows:

$$\begin{aligned}
&\Lambda^{\max}(\operatorname{Ker} P_n G)^* \otimes \Lambda^{\max}\left(H_n^{\perp}\right) \\
= \;&\Lambda^{\max}(\operatorname{Ker}(G) \oplus \widetilde{G}^{-1}\left(H_n^{\perp} \cap G(H)\right))^* \\
&\otimes \Lambda^{\max}\left(\left(H_n^{\perp} \cap G(H)\right) \oplus Q_n\left(G(H)^{\perp}\right)\right) \\
= \;&\Lambda^{\max}(\operatorname{Ker} G)^* \otimes \Lambda^{\max}(\widetilde{G}^{-1}\left(H_n^{\perp} \cap G(H)\right))^* \\
&\otimes \Lambda^{\max}\left(H_n^{\perp} \cap G(H)\right) \otimes \Lambda^{\max}\left(Q_n\left(G(H)^{\perp}\right)\right)
\end{aligned}$$

Via $\operatorname{Id} \otimes \Lambda^{\max}(\widetilde{G}_n^{-1})^* \otimes \operatorname{Id} \otimes \Lambda^{\max}(q_{G,n}^{-1})$, this last tensor product is

$$\begin{aligned}
\cong \;&\Lambda^{\max}(\operatorname{Ker} G)^* \otimes \Lambda^{\max}\left(H_n^{\perp} \cap G(H)\right)^* \\
&\otimes \Lambda^{\max}\left(H_n^{\perp} \cap G(H)\right) \otimes \Lambda^{\max}\left(G(H)^{\perp}\right) \\
\cong \;&\Lambda^{\max}(\operatorname{Ker} G)^* \otimes \Lambda^{\max}\left(G(H)^{\perp}\right).
\end{aligned}$$

In other words, for

$$\begin{aligned}
&\alpha \in \Lambda^{\max}(\operatorname{Ker} G)^*, \ \beta_n \in \Lambda^{\max}\left(\widetilde{G}^{-1}\left(H_n^{\perp} \cap G(H)\right)\right)^* \\
&b_n \in \Lambda^{\max}\left(H_n^{\perp} \cap G(H)\right) \text{ and } a_n \in \Lambda^{\max}\left(Q_n\left(G(H)^{\perp}\right)\right),
\end{aligned}$$

we have $(\alpha \otimes \beta_n) \otimes (b_n \otimes a_n) \in \Lambda^{\max}(\operatorname{Ker} P_n G)^* \otimes \Lambda^{\max}\left(H_n^{\perp}\right)$, and we define

$$\Psi_{G,n}\left((\alpha \otimes \beta_n) \otimes (b_n \otimes a_n)\right) := \left\langle \Lambda^{\max}(\widetilde{G}_n^{-1})^*(\beta_n), b_n \right\rangle \alpha \otimes \Lambda^{\max}(q_{G,n}^{-1})(a_n).$$

Note that

$$\Psi_{G,n}^{-1}(\alpha \otimes a) = (\alpha \otimes \beta_n) \otimes (b_n \otimes \Lambda^{\max}(q_{G,n})a),$$

where β_n and b_n are chosen so that $\left\langle \Lambda^{\max}(\widetilde{G}_n^{-1})^*(\beta_n), b_n \right\rangle = 1$; i.e., b_n is dual to $\Lambda^{\max}(\widetilde{G}_n^{-1})^*(\beta_n)$, or equivalently, β_n is the dual of $\Lambda^{\max}(\widetilde{G}_n^{-1})(b_n)$. □

Thus, the set (9.2) can be given the structure of a genuine line bundle, namely that which is induced by the bijections

$$\Lambda^{\max}(\operatorname{Ker}(T_x))^* \otimes \Lambda^{\max}(\operatorname{Ker}(T_x^*)) \leftrightarrow \Lambda^{\max}(\operatorname{Ker} P_n T_x)^* \otimes \Lambda^{\max}\left(H_n^{\perp}\right),$$

and this line bundle structure is unique up to isomorphism. The set (9.2) then serves as a standard representative of the isomorphism class $\det T$.

What we have done so far is sufficient for many purposes, but we will go on to construct a restricted version the so-called **Quillen determinant line bundle** $q : \mathcal{Q} \to \mathcal{F}$ and show that $\det T \to X$ is isomorphic to the pull-back via T of $\mathcal{Q}$, at least in the case index $T = 0$ (see Corollary 3.54, p. 98). The fiber of $\mathcal{Q}$ above $F \in \mathcal{F}$, is simply

$$q^{-1}(F) = \Lambda^{\dim \operatorname{Ker} F} (\operatorname{Ker} F)^* \otimes \Lambda^{\dim \operatorname{Ker} F^*} (\operatorname{Ker} F^*)$$

which is clearly a complex line. However, as with the set (9.2), it is not immediately clear that there are suitable local trivalizations for

$$\mathcal{Q} := \bigcup_{F \in \mathcal{F}} q^{-1}(F) \tag{9.7}$$

with continuous transition functions, because $\operatorname{Ker} F$ (or $\operatorname{Ker} F^*$) is not the fiber of a vector bundle over $\mathcal{F}$ about points where F (or F^*) is not onto. It is true that for any $F \in \mathcal{F}$, there is some n_F such that $P_{n_F} F(H) = H_{n_F}^{\perp}$ for some suitable neighborhood, say U_F, of F in $\mathcal{F}$. Moreover, one can construct a trivialization for $q|_{U_F} : q^{-1}(U_F) \to U_F$. However, unlike the case of a family $T : X \to \mathcal{F}$ with X *compact* where we had fixed n for which $P_n T_x(H) = H_n^{\perp}$ for *all* $x \in X$ (see the proof of 3.32, p. 79), note now that n_F is an *unbounded* function of $F \in \mathcal{F}$ (noncompact). At the very least, this causes difficulties in defining the transition functions and exhibiting their continuity. Instead of attempting this, we opt for an interesting, instructive alternative construction, using transition functions that are determinants of the form $\det(\mathrm{Id} + A)$ where A is a trace class operator, defined below. This idea is based on informal notes of Graeme Segal (see [**Seg90**]), with extensions elaborated upon by Kenro Furutani in [**Fur**], to whom we are indebted. One drawback is that this construction is only for $\mathcal{Q}|_{\mathcal{F}_0}$, where $\mathcal{F}_0 := \{A \in \mathcal{F} : \operatorname{index} A = 0\}$, and we have yet to find a graceful way to similarly construct $\mathcal{Q}$ over the components of $\mathcal{F}$ with nonzero index. This may not be of crucial importance, since the usual convention is that if index $T \neq 0$, then $\det T = 0$ if $\det T$ is defined at all (e.g., for $T : \mathbb{R}^n \to \mathbb{R}^m$, we have index $T = n - m$ and $\det T$ is undefined if $n \neq m$). Of course, we always can identify the connected components of $\mathcal{F}$ (distinguished by the index), e.g., by shift operators after fixing a basis for the underlying Hilbert space H. The bundle $\mathcal{Q}$ then constructed via pull-back would, however, depend on the choice of the identifications. Another potential difficulty in the Segal approach is that one needs to know the functional analysis of trace class and Hilbert–Schmidt operators (which is covered in Section 2.8), as well as determinants which we consider next.

Determinants

For a complex vector space V with $d = \dim V$ finite, let $\Lambda^k(V)$ denote the k-th exterior product of V $(k = 0, \ldots, d)$, where $\Lambda^0(V) = \mathbb{C}$. If $T \in \operatorname{End}(V)$ (i.e., $T : V \to V$ is linear), then T induces $\Lambda^k T \in \operatorname{End}\left(\Lambda^k(V)\right)$, determined by

$$\left(\Lambda^k T\right)(v_1 \wedge \cdots \wedge v_k) = Tv_1 \wedge \cdots \wedge Tv_k \text{ for } k > 0, \tag{9.8}$$

and $\Lambda^0 T = \mathrm{Id}_{\mathbb{C}}$. Since $\dim \Lambda^d(V) = 1$, $\Lambda^d T$ is multiplication by a scalar, and this scalar is $\det T$. Indeed $\det T$ can be defined in this way, and it is straightforward to show that this agrees with the usual definition in terms of the matrix of T relative to a basis (e.g., just take $k = d$ and let $v_1, \cdots, v_d$ be a basis in (9.8)). Note that under the natural isomorphisms

$$\operatorname{End}\left(\Lambda^d(V)\right) \cong \Lambda^d(V)^* \otimes \Lambda^d(V) \cong \mathbb{C},$$

and $\Lambda^d T \in \operatorname{End}\left(\Lambda^d(V)\right)$ corresponds to $\det T$. There are immediate problems with the notion $\det T$ when $\dim V = \infty$, but $\det T$ can be defined when V is a Hilbert