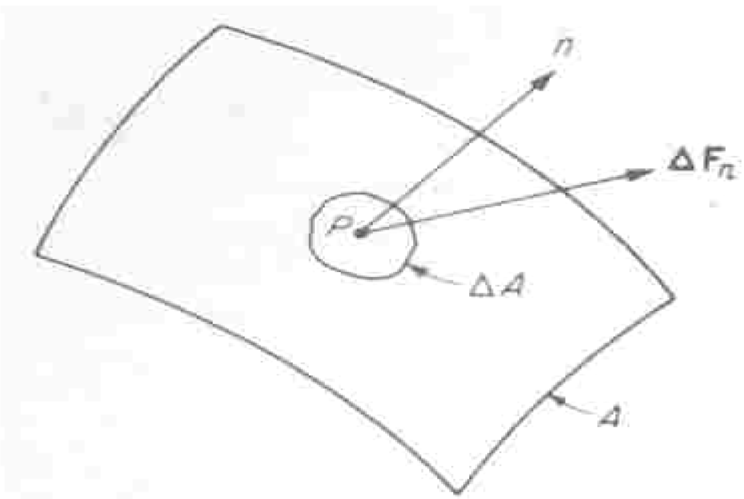
REVIEW OF ELASTICITY

Two basic types of force act on a body to produce stresses:

- Surface Forces
- Body Forces

Surface forces are generally exerted when one body comes in contact with another. Forces of the second type are called body forces since they act on each element of the body e.g. centrifugal, gravitational forces. For many practical applications, however, body forces are so small compared with the surface forces present that they can be neglected without introducing serious error.

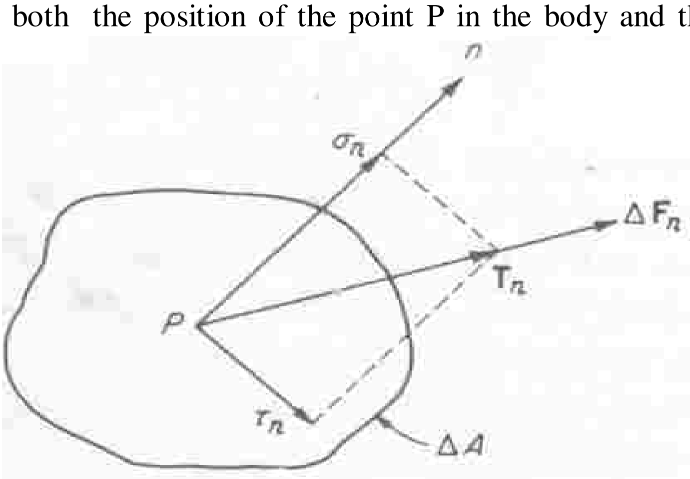
Consider an arbitrary internal or external surface, which may be plane or curvilinear, as shown in Fig. below. Over a small area dA of this surface in the neighborhood of an arbitrary point P, a system of forces acts which has a resultant represented by the vector dF_n in the figure. The line of action of the resultant force vector dF_n does not necessarily coincide with the outer normal n associated with the element of area dA. If the resultant force dF_n is divided by the increment of area dA, the average stress which acts over the area is obtained.



In the limit as dA approaches zero, a quantity defined as the resultant stress $\mathbf{T}_n$ acting at the point p is obtained. This limiting process is illustrated in equation form below.

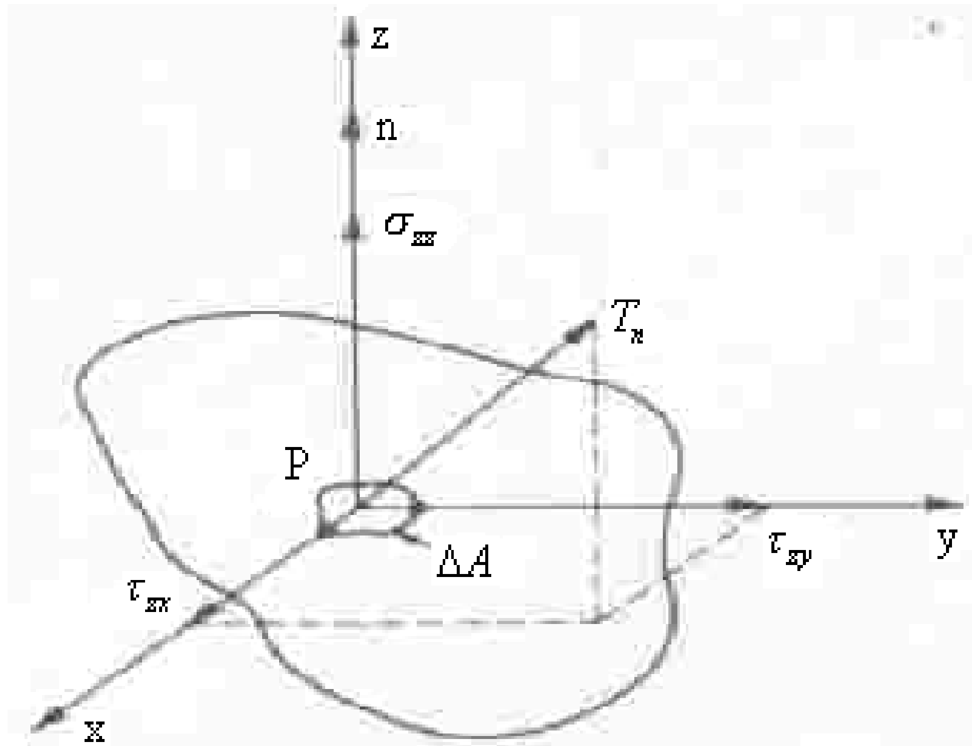
$\mathbf{T}_n$ = limit as dA ->0 $d\mathbf{F}_n/dA$

The line of action of this resultant stress $\mathbf{T}_n$ coincides with the line of action of the resultant force $d\mathbf{F}_n$, as illustrated in Fig. below. The resultant stress $\mathbf{T}_n$ is a function of both the position of the point P in the body and the orientation of the plane which is passed through the point and identified by its outer normal n.



As illustrated in figure above, it is possible to resolve $\mathbf{T}_n$ into two components : one σ_n normal to the surface is known as the resultant normal stress, while the component τ_n is known as the resultant shearing stress.

Cartesian components of stress for any coordinate system can also be obtained from the resultant stress. Consider first a surface whose outer normal is in the positive z direction, as shown in Fig. below. If the resultant stress $\mathbf{T}_n$ associated with this particular surface is resolved into components along the x, y, and z axes, the cartesian stress components τ_{zx}, τ_{zy}, and σ_{zz} are obtained. The components τ_{zx} and τ_{zy} are shearing stresses since they act tangent to the surface under consideration. The component σ_{zz} is a normal stress since it acts normal to the surface.



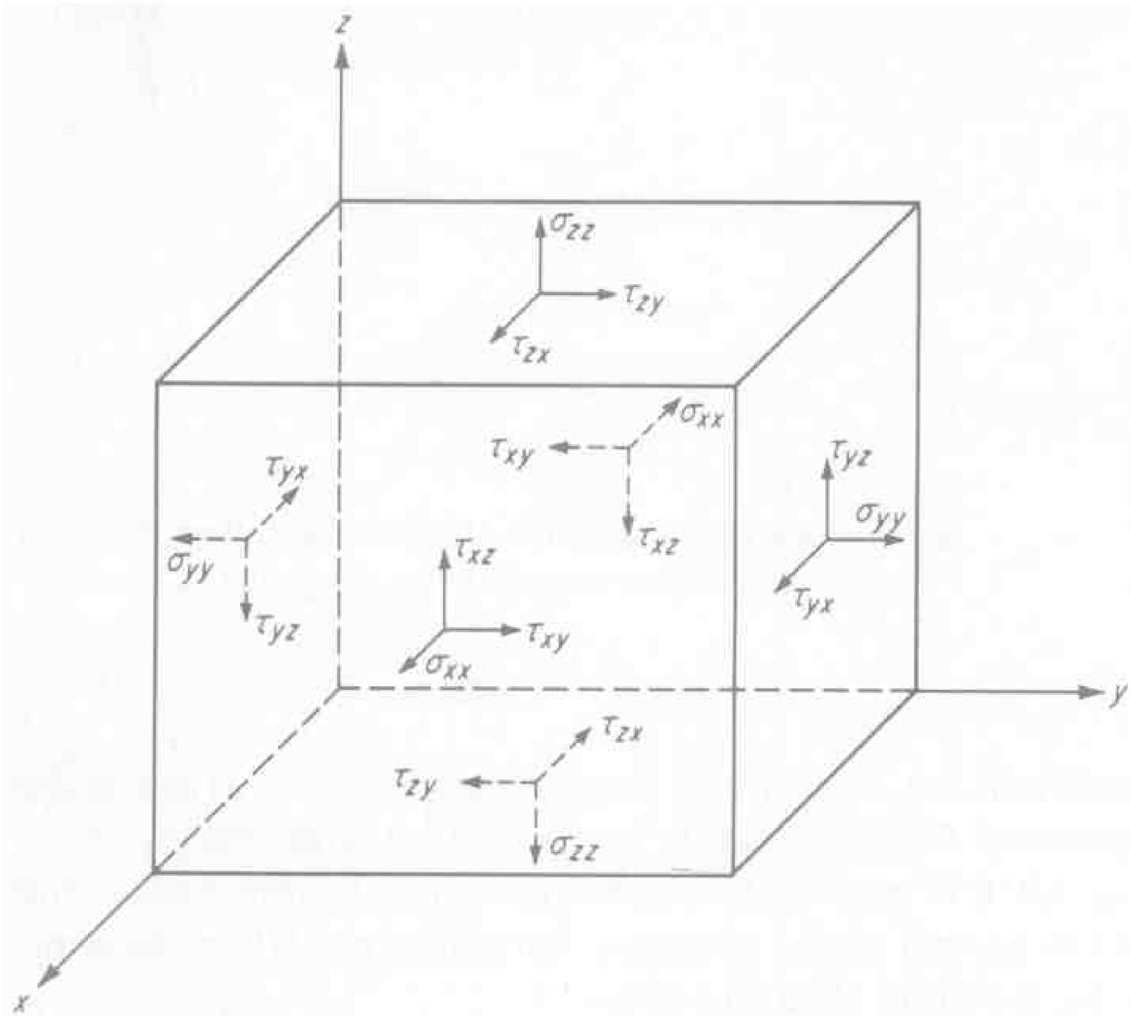
If the same procedure is followed using surfaces whose outer normals are in the. positive x and y directions, two more sets of cartesian components, τ_{xy},τ_{xz} σ_{xx}, and τ_{yx}, τ_{yz}, σ_{yy}, respectively, can be obtained. The three different sets of three cartesian components for the three selections of the outer normal are summarized in the array below:

σ_{xx} τ_{xy} τ_{xz} outer normal parallel to the x axis

τ_{yx} σ_{yy} τ_{yz} outer normal parallel to the y axis

τ_{zx} τ_{zy} σ_{zz} outer normal parallel to the z axis

From this array, it is clear that nine cartesian components of stress exist. These components can be arranged on the faces of a small cubic element, as shown in Fig. below. The sign convention employed in placing the cartesian stress components on the faces of this cube is as follows: if the outer normal defining the cube face is in the direction of increasing x, y, or z, then the associated is in the direction of negative x, y, or z, then the normal and shear stress components are also in the direction of negative x, y, or z. As for one such subscript convention, the first subscript refers to the outer normal and defines the plane upon which the stress component acts, whereas the second subscript gives the direction in which the stress acts. Finally, for normal

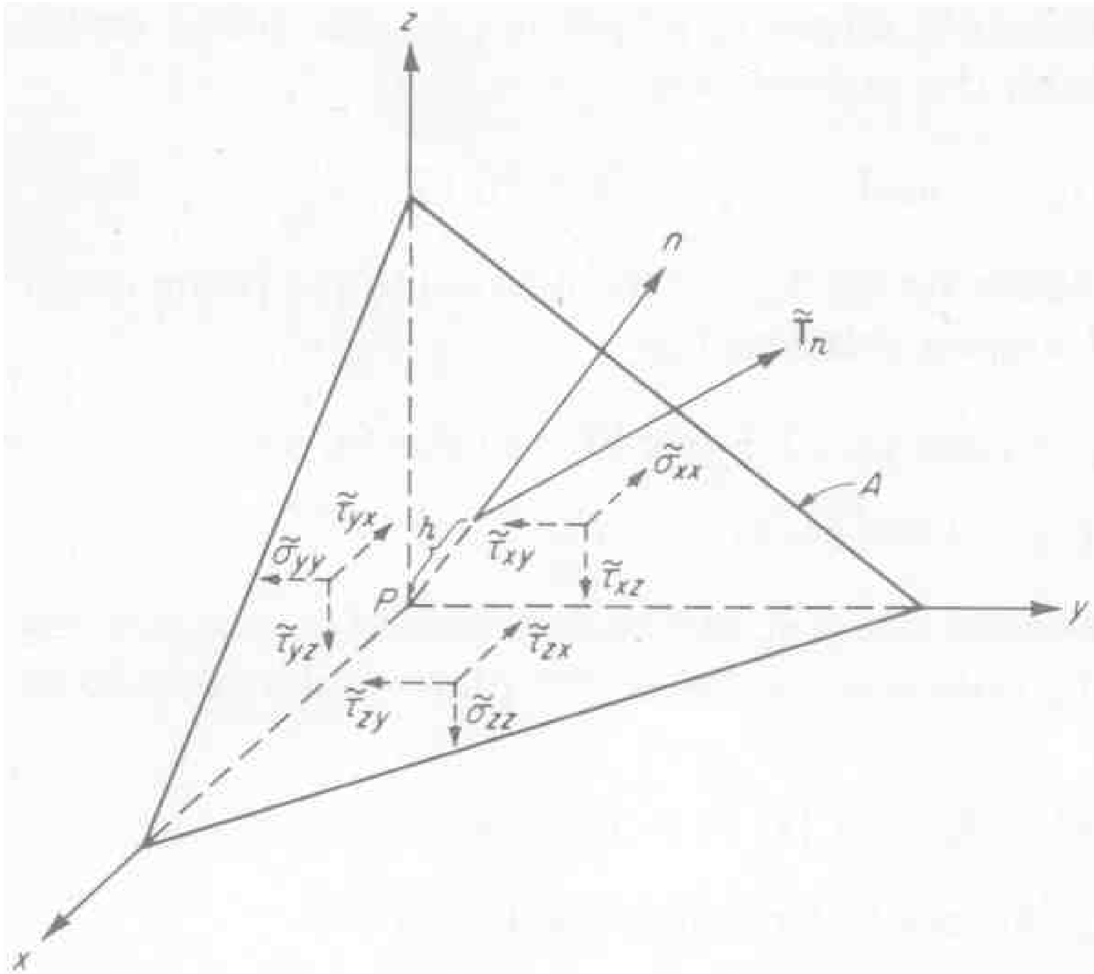


stresses, positive signs indicate tension and negative signs indicate compression.

STRESS AT A POINT

At a given point of interest within a body, the magnitude and direction of the resultant stress $\mathbf{T}_n$ depend upon the orientation of the plane passed through the point. Thus an infinite number of resultant-stress vectors can be used to represent the resultant stress at each point since an infinite number of planes can be passed through each point. The magnitude and direction of each of these resultant-stress vectors can be specified in terms of the nine cartesian components of stress acting at the point. Consider the equilibrium of the elemental tetrahedron shown in Fig. below.

In this figure the stresses acting over the four faces of the tetrahedron are represented by their average values. The average value is denoted by placing a -sign over the stress symbol. In order for the tetrahedron to be in equilibrium, the following condition must be satisfied. First consider equilibrium in the x direction:



$$T_{nx}A - \sigma_{xx}A \cos(n, x) - \tau_{yx}A \cos(n, y) - \tau_{zx} A \cos(n, z) + F_x hA/3 = 0$$

where :

h = altitude of tetrahedron

A = area of base of tetrahedron

F_x = average body-force intensity in x direction

T_{nx} = component of resultant stress in x direction

and A cos (n, x), A cos (n, y), and A cos (n, z) are the projections of the area A on the yz, xz, and xy planes, respectively.

By letting the altitude h -> 0, after eliminating the common factor A from each term of the expression, it can be seen that the body-force term vanishes, the average stresses become exact stresses at the point P, and the previous expression becomes

$$T_{nx} = \sigma_{xx} \cos(n, x) + \tau_{yx} \cos(n, y) + \tau_{zx} \cos(n, z)$$

Two similar expressions are obtained by considering equilibrium in the y and z directions :

$$T_{ny} = \tau_{xy} \cos(n, x) + \sigma_{yy} \cos(n, y) + \tau_{zy} \cos(n, z)$$

$$T_{nz} = \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_{zz} \cos(n, z)$$

Once the three cartesian components of the resultant stress for a particular plane have been determined by employing Eqs. (1.2), the resultant stress $\mathbf{T}_n$ can be determined by using the expression

$|\mathbf{T}_n|$ = square root($T^2_{nx} + T^2_{ny} + T^2_{nz}$)

The three direction cosines which define the line of action of the resultant stress $\mathbf{T}_n$ are

$\cos(T_n,x) = T_{nx}/|T_n|$

$\cos(T_n,y) = T_{ny}/|T_n|$

$\cos(T_n,z) = T_{nz}/|T_n|$

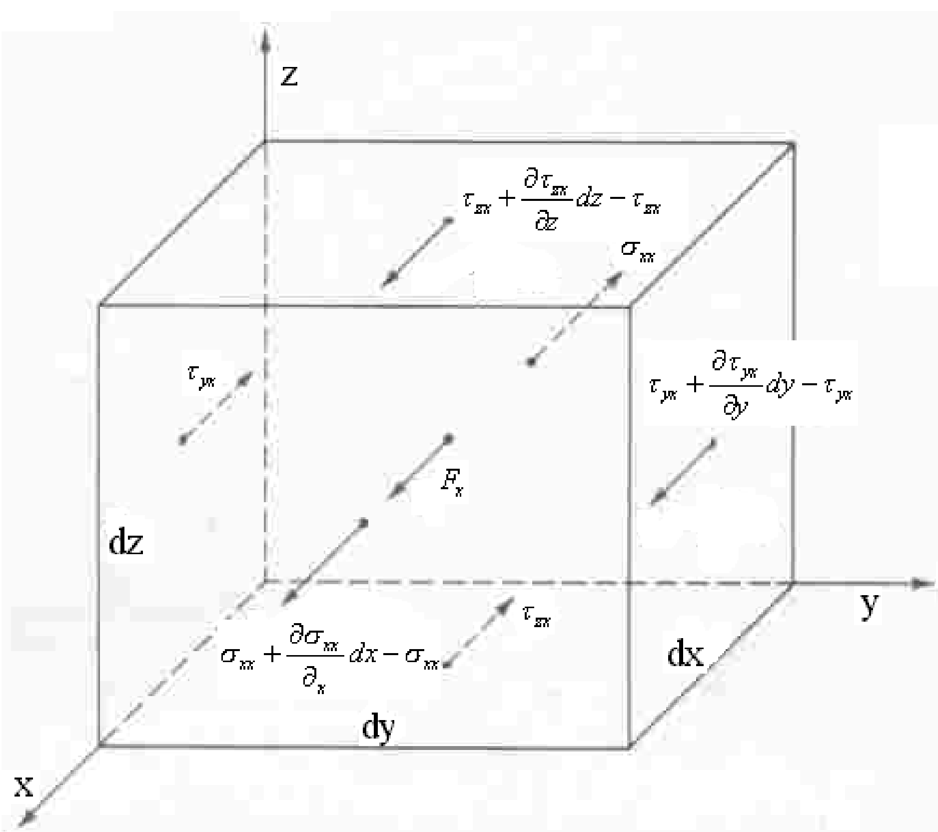
The normal stress σ_n and the shearing stress τ_n which act on the plane under consideration can be obtained from the expressions

$\sigma_n = |\mathbf{T}_n| \cos(T_n,n)$

$\tau_n = |\mathbf{T}_n| \sin(T_n,n)$

STRESS EQUATIONS OF EQUILIBRIUM

In a body subjected to a general system of body and surface forces, stresses of variable magnitude and direction are produced throughout the body. The distribution of these stresses must be such that the overall equilibrium of the body is mai11tained; furthermore, equilibrium of each element in the body must be maintained. This section deals with the equilibrium of the individual ele!11ents of the body. On the element shown in Fig. below, only the stress and body-force components which act in the x direction are shown. Similar components exist and act in the y and z directions. The stress values shown are average stresses over the faces of an element which is assumed to be very small.



A summation of forces in the x direction gives

$$\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial_{x}}dx-\sigma_{xx}\right)dy.dz+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}dy-\tau_{yx}\right)dx.dz$$
$$+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}dz-\tau_{zx}\right)dx.dy+F_{x}dx.dy.dz=0$$

Dividing through by dxdydz gives:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+Fx=0$$

By considering the force and stress components in the y and z directions, it can be established in a similar fashion that

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+Fy=0$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}+Fz=0$$

where F_x, F_y , F_z are body-force intensities (in lb/in^3 or N/m^3) in the x, y, and z directions, respectively.

Above Equations are the well-known stress equations of equilibrium which any theoretically or experimentally obtained stress distribution must satisfy. In obtain- ing these equations, three of the six equilibrium conditions have been employed. The three remaining conditions can be utilized to establish additional relation- ships between the stresses.

Consider the moment equilibrium equations, one can prove that

$$\tau_{xy}=\tau_{yx}$$
$$\tau_{yz}=\tau_{zy}$$
$$\tau_{xz}=\tau_{zx}$$

PRINCIPAL STRESSES

The resultant-stress vector $\mathbf{T}_n$ at a given point P depended upon the choice of the plane upon which the stress acted. If a plane is selected such that $\mathbf{T}_n$ coincides with the outer normal n, it is clear that the shear stress tn vanishes and that $\mathbf{T}_n$, σ_n, and n are coincident. If n is selected so that it coincides with $\mathbf{T}_n$ then the plane defined by n is known as a principal plane. The direction given by n is a principal direction, and the normal stress acting on this particular plane is a principal stress. In every state of stress there exist at least three principal planes, which are

mutually perpendicular, and associated with these principal planes there are at most three distinct principal stresses.
We can now write

$$T_{nx} = \sigma_n \cos(n,x) \quad T_{ny} = \sigma_n \cos(n,y) \quad T_{nz} = \sigma_n \cos(n,z) \qquad \text{(a)}$$

If equation (1.2) are substituted into equation (a), the following expressions are obtained :

$$\left\{\begin{array}{l} \sigma_{xx}\cos(n,x) + \tau_{yx}\cos(n,y) + \tau_{zx}\cos(n,z) = \sigma_n \cos(n,x) \\ \tau_{xy}\cos(n,x) + \sigma_{yy}\cos(n,y) + \tau_{zy}\cos(n,z) = \sigma_n \cos(n,y) \\ \tau_{xz}\cos(n,x) + \tau_{yz}\cos(n,y) + \sigma_{zz}\cos(n,z) = \sigma_n \cos(n,z) \end{array}\right\} \qquad \text{(b)}$$

Rearranging equations (b) gives

$$\left\{\begin{array}{l} (\sigma_{xx} - \sigma_n)\cos(n,x) + \tau_{yx}\cos(n,y) + \tau_{zx}\cos(n,z) = 0 \\ \tau_{xy}\cos(n,x) + (\sigma_{yy} - \sigma_n)\cos(n,y) + \tau_{zy}\cos(n,z) = 0 \\ \tau_{xz}\cos(n,x) + \tau_{yz}\cos(n,y) + (\sigma_{zz} - \sigma_n)\cos(n,z) = 0 \end{array}\right\} \qquad \text{(c)}$$

Nontrivial solutions for directions of the principal plane will exist only if the determinant of the coefficients is zero i.e.

$$\begin{vmatrix} \sigma_{xx} - \sigma_n & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} - \sigma_n & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} - \sigma_n \end{vmatrix} = 0$$

Expanding the determinant gives the following cubic equation:

$$\begin{aligned} &\sigma_n^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma_n^2 \\ &+ (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\tau_n \\ &- (\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}) = 0 \end{aligned}$$

The roots of this cubic equation are the three principal stresses. By substituting the six cartesian components of stress into this equation, one can solve for (1n and obtain three real roots. Three possible solutions exist.

1. If σ_1, σ_2, σ_3 are distinct, then n_1, n_2, n_3 are unique and mutually perpendicular.

2. If σ_1 equals σ_2 and is not equal too σ_3, then n_3 is unique and every direction perpendicular to n_3 is a principal direction associated with $\sigma_1 = \sigma_2$.

3. If $\sigma_1 = \sigma_2 = \sigma_3$, then a hydrostatic state of stress exists and every direction is a principal direction.

Once the three principal stresses have been established, they can be substituted individually into Eqs. (c) to give three sets of simultaneous equations which together with the relation

$$\cos^2 (n, x) + \cos^2 (n, y) + \cos^2 (n, z) = 1$$

can be solved to give the three sets of direction cosines defining the principal planes.

Another important concept is that of stress invariants. It can be recalled that the principal stresses σ_1, σ_2, σ_3 are independent of the cartesian coordinate system employed. Thus the coefficients of the cubic equation given above must be independent of or invariant of the co-ordinate system. The three invariants are:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'}$$

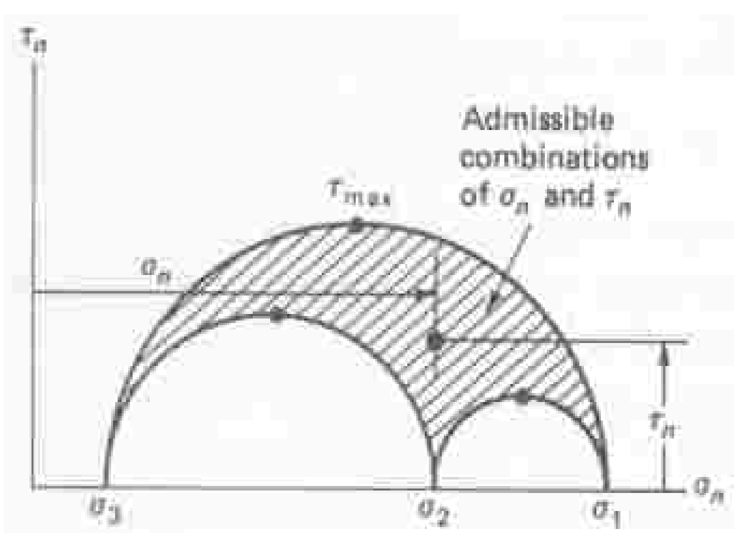
$$I_2 = \sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$
$$= \sigma_{x'x'}\sigma_{y'y'} + \sigma_{y'y'}\sigma_{z'z'} + \sigma_{z'z'}\sigma_{x'x'} - \tau_{x'y'}^2 - \tau_{y'z'}^2 - \tau_{z'x'}^2$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}$$
$$= \sigma_{x'x'}\sigma_{y'y'}\sigma_{z'z'} - \sigma_{x'x'}\tau_{y'z'}^2 - \sigma_{y'y'}\tau_{z'x'}^2 - \sigma_{z'z'}\tau_{x'y'}^2 + 2\tau_{x'y'}\tau_{y'z'}\tau_{z'x'}$$

MAXIMUM SHEAR STRESS

If the principal stresses are such that $\sigma_1 > \sigma_2 > \sigma_3$, then

$$\tau_{max} = (\sigma_{max} - \sigma_{min})/2 = (\sigma_1 - \sigma_3)/2$$



A useful aid for visualizing the complete state of stress at a point is the three-dimenslonal Mohr's circle shown in the figure below.

This representation, which is similar to the familiar two-dimensional Mohr's circle, shows the three principal stresses, the maximum shearing stresses, and the range of values within which the normal and shear-stress components must lie for a given state of stress.

Transformation Relations (two dimensions)

For two-dimensional stress fields where $\sigma = \tau = \tau = 0$,z' is coincident with z, and θ is the angle between x and x'. The transformation relations are:

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta.\cos\theta$$

$$= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'y'} = \sigma_{yy}\cos^2\theta + \sigma_{xx}\sin^2\theta - 2\tau_{xy}\sin\theta.\cos\theta$$

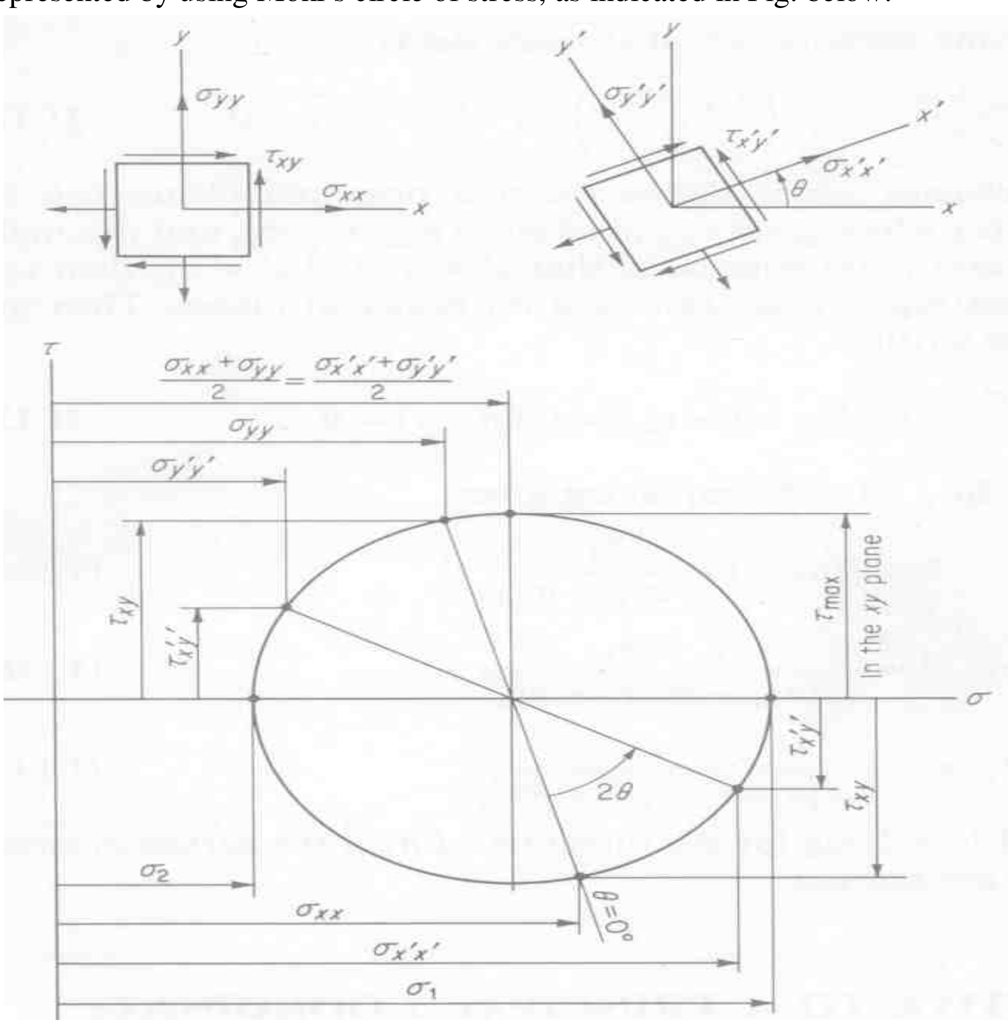
$$= \frac{\sigma_{yy} + \sigma_{xx}}{2} + \frac{\sigma_{yy} - \sigma_{xx}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \sigma_{yy}\cos\theta.\sin\theta - \sigma_{xx}\cos\theta.\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= \frac{\sigma_{yy} - \sigma_{xx}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\sigma_{z'z'} = \tau_{z'x'} = \tau_{y'z'} = 0$$

The relationships between stress components given in above equation can be graphically represented by using Mohr's circle of stress, as indicated in Fig. below:



In this diagram, normal-stress components σ are plotted horizontally, while stress components τ are plotted vertically. Tensile stresses are plotted to the right of the τ axis. Compressive stresses are plotted to the left. Shear-stress components which tend to produce a clockwise rotation of a small element surrounding the point are plotted above the σ axis. Those tending to produce a counterclockwise rotation are plotted below. When plotted in this manner, the stress components associated with each plane through the point are represented by a point on the circle. The diagram thus gives an excellent visual picture of the state of stress at a point

The principal stresses are given by:

$$\sigma_1 \sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_3 = 0$$

The direction of the principal plane w.r.t. x axis is:

$\tan 2\varphi = 2\,\tau_{xy}/(\sigma_{xx}-\sigma_{yy})$

Special Cases

1. A state that of pure shear stress exists if one particular set of axes Oxyz can be found such that σxx = σyy = σzz = 0 . It can be shown that this particular set of axes Oxyz exists if and only if the first invariant of stress I1 =0. The proof of this condition is beyond the scope of this text. Two of the infinite number of arrays which represent a state of pure shearing stress are given below.

$$\begin{Vmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{Vmatrix} \quad \text{or} \quad \begin{Vmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} = -\sigma_{xx} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{Vmatrix}$$

Pure shear

Can be converted to the corm shown on the left by a suitable rotation of the coordinate system

2. A state of stress is said to be hydrostatic if σxx = σyy = σzz = -p and all the shearing stresses vanish.

$$\begin{Vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{Vmatrix}$$

INTRODUCTION TO STRAIN

Previously the state of stress which develops at an arbitrary point within a body as a result of surface or body-force loadings was discussed. The relationships obtained were based on the conditions of equilibrium, and since no assumptions were made regarding body deformations or physical properties of the material of which the body was composed, the results are valid for any material and for any amount of body deformation. Now the subject of body deformation and associated strain will be discussed. Since strain is a pure geometric quantity, no restrictions on body material will be required. However, in order to obtain linear equations relating displacement to strain, restrictions must be placed on the allowable deformations.

DEFINITIONS OF DISPLACEMENT AND STRAIN

If a given body is subjected to a system of forces, individual points of the body will, in general, move. This movement of an arbitrary point is a vector quantity known as a displacement. If the various points in the body undergo different movements, each can be represented by its own unique displacement vector. Each vector can be resolved into components parallel to a set of cartesian coordinate axes such that u, v, and w are the displacement components in the x, y, and z directions, respectively.

Motion of the body may be considered as the sum of two parts:

1. A translation and/or rotation of the body as a whole.
2. The movement of the points of the body relative to each other.

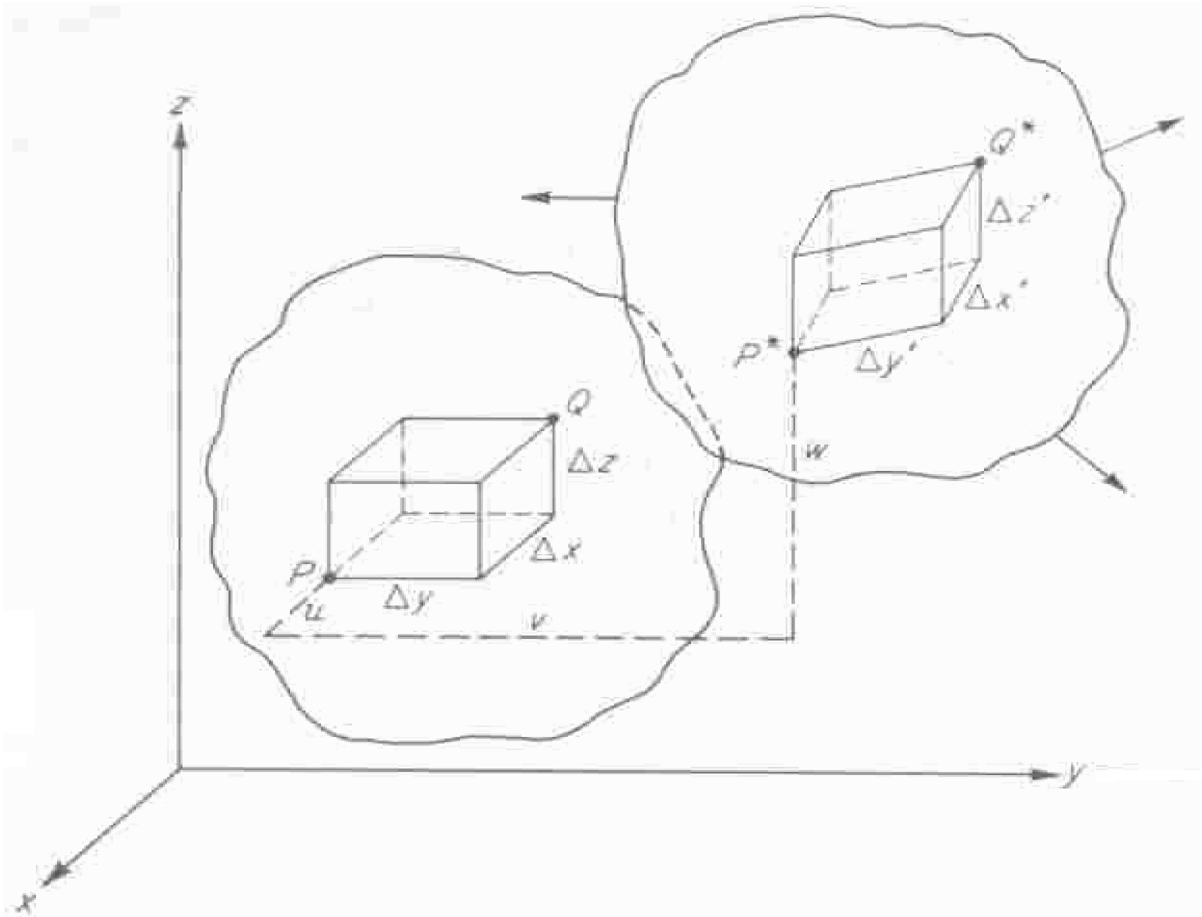
The translation or rotation of the body as a whole is known as rigid-body motion. This type of motion is applicable to either the idealized rigid body or the real deformable body. The movement of the points of the body relative to each other is known as a deformation and is obviously a property of real bodies only. Rigid- body motions can be large or small. Deformations, in general, are small except when rubber like materials or specialized structures such as long, slender beams are involved.

Strain is a geometric quantity which depends on the relative movements of two or three points in the body and therefore is related only to the deformation displacements. Since rigid-body displacements do not produce strains, they will be neglected.

A normal strain is defined as the change in length of a line segment between two points divided by the original length of the line segment. A shearing strain is defined as the angular change between two line segments which were originally perpendicular.

The relationships between strains and displacements can be determined by considering the deformation of an arbitrary cube in a body as a system of loads is applied. This deformation is illustrated in figure below, in which a general point p is moved through a distance u in the ~ direction, v in the y direction, and w in the z direction. The other corners of the cube are also displaced and, in general, they will be displaced by amounts which differ from those at point P. For example the displacements u*, v*, and w* associated with point Q can be expressed in terms of the displacements u, v, and w at point p by means of a Taylor-series expansion.

Figure: The distortion of an arbitrary cube in a body due to the application of a system of forces.



The average normal strain along an arbitrary line segment was previously defined as the change in length of the line segment divided by its original length. This normal strain can be expressed in terms of the displacements experienced by points at the ends of the segment. For example, consider the line PQ originally oriented parallel to the x axis, as shown in figure below:

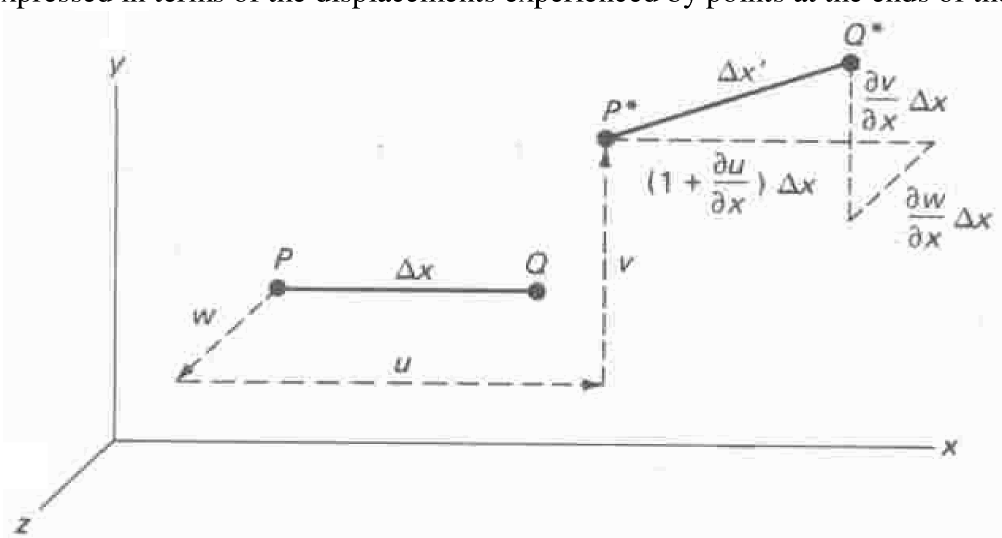
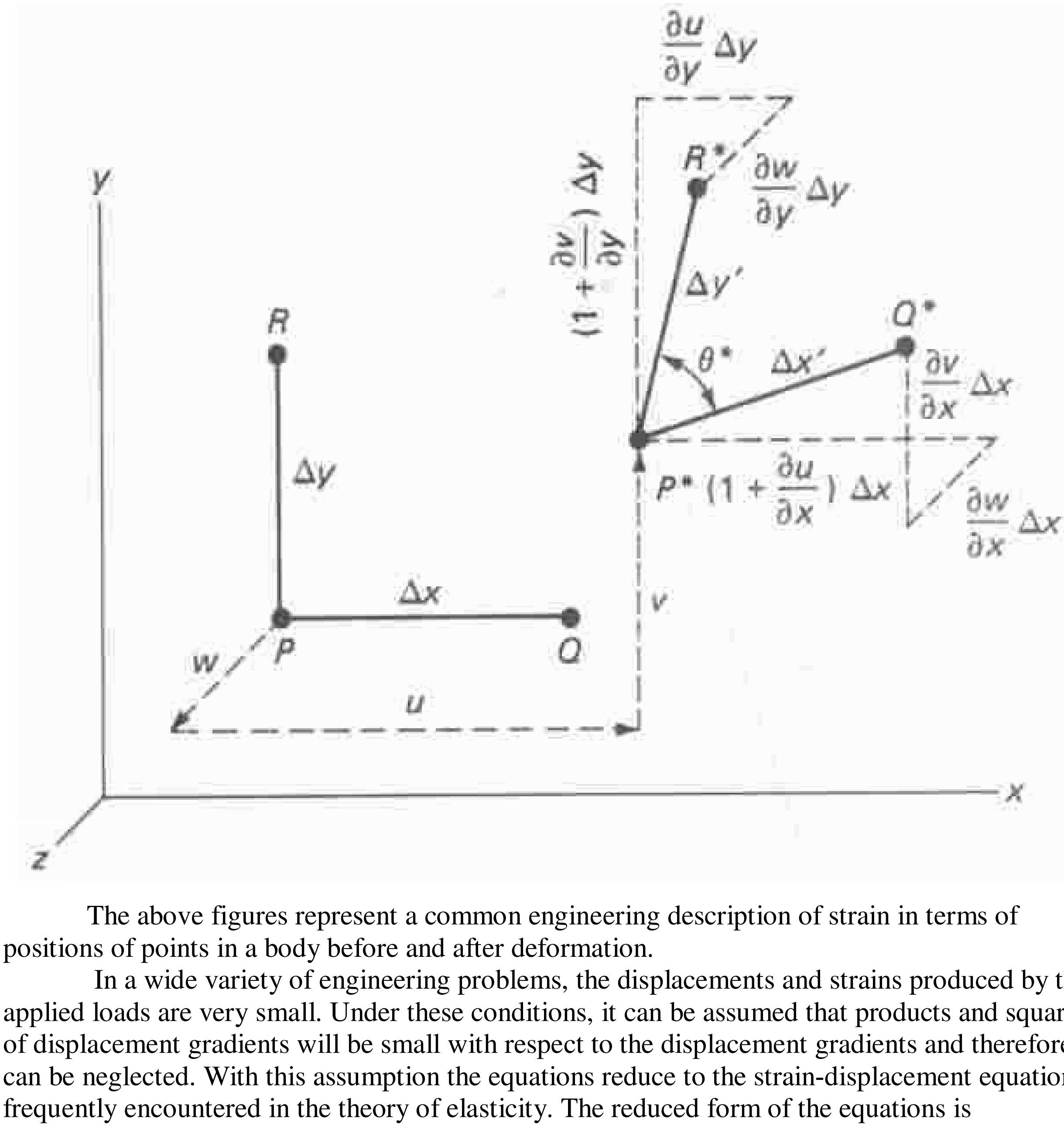


Figure: Displacement gradients associated with the normal strain.

The shear-strain components can also be related to the displacements by considering the changes in right angle experienced by the edges of the cube during deformation. For example, consider lines PQ and PR, as shown in earlier figure. The angle 0* between P*Q* and P* R * in the deformed state can be expressed in terms of the displacement gradients since the cosine of the angle between any two intersecting lines in space is the sum of the pair wise products of the direction cosines of the lines with respect to the same set of reference axes.

Figure: Displacement gradients associated with the shear strain.



The above figures represent a common engineering description of strain in terms of positions of points in a body before and after deformation.

In a wide variety of engineering problems, the displacements and strains produced by the applied loads are very small. Under these conditions, it can be assumed that products and squares of displacement gradients will be small with respect to the displacement gradients and therefore can be neglected. With this assumption the equations reduce to the strain-displacement equations frequently encountered in the theory of elasticity. The reduced form of the equations is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Strain equations of transformation

A comparison of strain equations of transformation with the stress equations of transformation shows remarkable similarities:

$$\sigma_{xx} \leftrightarrow \varepsilon_{xx} \qquad 2\tau_{xy} \leftrightarrow \gamma_{xy}$$

$$\sigma_{yy} \leftrightarrow \varepsilon_{yy} \qquad 2\tau_{yz} \leftrightarrow \gamma_{yz}$$

$$\sigma_{zz} \leftrightarrow \varepsilon_{zz} \qquad 2\tau_{zx} \leftrightarrow \gamma_{zx}$$

Here the symbol < - > indicates an interchange. This interchange is important since many of the derivations given in the preceding chapter for stresses can be converted directly into strains.

PRINCIPAL STRAINS

From the similarity between the laws of stress and strain transformation it can be concluded that there exist at most three distinct principal strains with their three associated principal directions. By substituting the conversions, the cubic equation whose roots give the principal strains is obtained as follows:

$$\varepsilon_n^3 - (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})\varepsilon_n^2$$

$$+ \left(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx} - \frac{\gamma_{xy}^2}{4} - \frac{\gamma_{yz}^2}{4} - \frac{\gamma_{zx}^2}{4} \right)\varepsilon_n$$

$$-\left(\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}-\varepsilon_{xx}\frac{\gamma_{yz}^2}{4}-\varepsilon_{yy}\frac{\gamma_{zx}^2}{4}-\varepsilon_{zz}\frac{\gamma_{xy}^2}{4}+\frac{\gamma_{xy}\gamma_{yz}\gamma_{zx}}{4}\right)=0$$

Similarly, there are three strain invariants which are analogous to the three stress invariants.The following expressions are obtained for the strain invariants:

$$J_1=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}$$

$$J_2=\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{zz}\varepsilon_{xx}-\frac{\gamma_{xy}^2}{4}-\frac{\gamma_{yz}^2}{4}-\frac{\gamma_{zx}^2}{4}$$

$$J_3=\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}-\frac{\varepsilon_{xx}\gamma_{yz}^2}{4}-\frac{\varepsilon_{yy}\gamma_{zx}^2}{4}-\frac{\varepsilon_{zz}\gamma_{xy}^2}{4}+\frac{\gamma_{xy}\gamma_{yz}\gamma_{zx}}{4}$$

COMPATIBILITY

From a given displacement field, i.e., three equations expressing u, v, and was functions of x, y, and z, a unique strain field can be determined. However, an arbitrary strain field may yield an impossible displacement field, i.e., one in which the body might contain voids after deformation. A valid displacement field can be ensured only if the body under consideration is simply connected and if the strain field satisfies a set of equations known as the compatibility relations. The six equations of compatibility which must be satisfied are

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=\frac{\partial^2\varepsilon_{xx}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy}}{\partial x^2}$$

$$\frac{\partial^2\gamma_{yz}}{\partial y\partial z}=\frac{\partial^2\varepsilon_{yy}}{\partial z^2}+\frac{\partial^2\varepsilon_{zz}}{\partial y^2}$$

$$\frac{\partial^2\gamma_{zx}}{\partial z\partial x}=\frac{\partial^2\varepsilon_{zz}}{\partial x^2}+\frac{\partial^2\varepsilon_{xx}}{\partial z^2}$$

$$2\frac{\partial^2\varepsilon_{xx}}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2\varepsilon_{yy}}{\partial z\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

VOLUME DILATATION

Consider a small, rectangular element in a deformed body, which has its edges oriented along the principal axes. The length of each side of the block may have changed; however, the element will not be distorted since there are no shearing strains acting on the faces. The change in volume of such an element divided by the initial volume is, by definition, the volume dilatation D, that is,

D= (V*- V)/V

Where, V is the initial volume, equal to the product of the three sides of the element, al, a2, a3 , before deformation and V* is the final volume after straining. If the higher-order strain terms are neglected,

$D=(e_1+e_2+e_3)=J_1$

The above equation indicates that the volume dilatation D is equal to the first invariant of strain. Since the first invariant of strain is independent of the coordinate system being used, the volume dilatation of an element is independent of the reference frame forming its sides. Volume dilatation is thus a coordinate- independent concept.

STRESS-STRAIN RELATIONS

Thus far stress and strain have been discussed individually, and no assumptions have been required regarding the behavior of the material except that it was a continuous medium. t In this section, stress will be related to strain; therefore, certain restrictive assumptions regarding the body material must be introduced. The first of these assumptions regards linearity of the stress versus strain in the body. With a linear stress-strain relationship it is possible to write the general stress-strain expressions for an isotropic material as follows :

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - v\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - v\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - v\left(\sigma_{yy} + \sigma_{xx}\right)\right]$$

$$\gamma_{xy} + \frac{2(1+v)}{E}\tau_{xy} \qquad \gamma_{yz} + \frac{2(1+v)}{E}\tau_{yz} \qquad \gamma_{zx} + \frac{2(1+v)}{E}\tau_{zx}$$

and for stress in terms of strain and the constants

$$\sigma_{xx} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{xx} + v\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{yy} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{yy} + v\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{zz} = \frac{E}{(1+v)(1-2v)}\left[(1-v)\varepsilon_{zz} + v\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right]$$

$$\tau_{xy} = \frac{E}{2(1+v)}\gamma_{xy} \qquad \tau_{yz} = \frac{E}{2(1+v)}\gamma_{yz} \qquad \tau_{zx} = \frac{E}{2(1+v)}\gamma_{zx}$$

BASIC EQUATIONS AND PLANE ELASTICITY THEORY

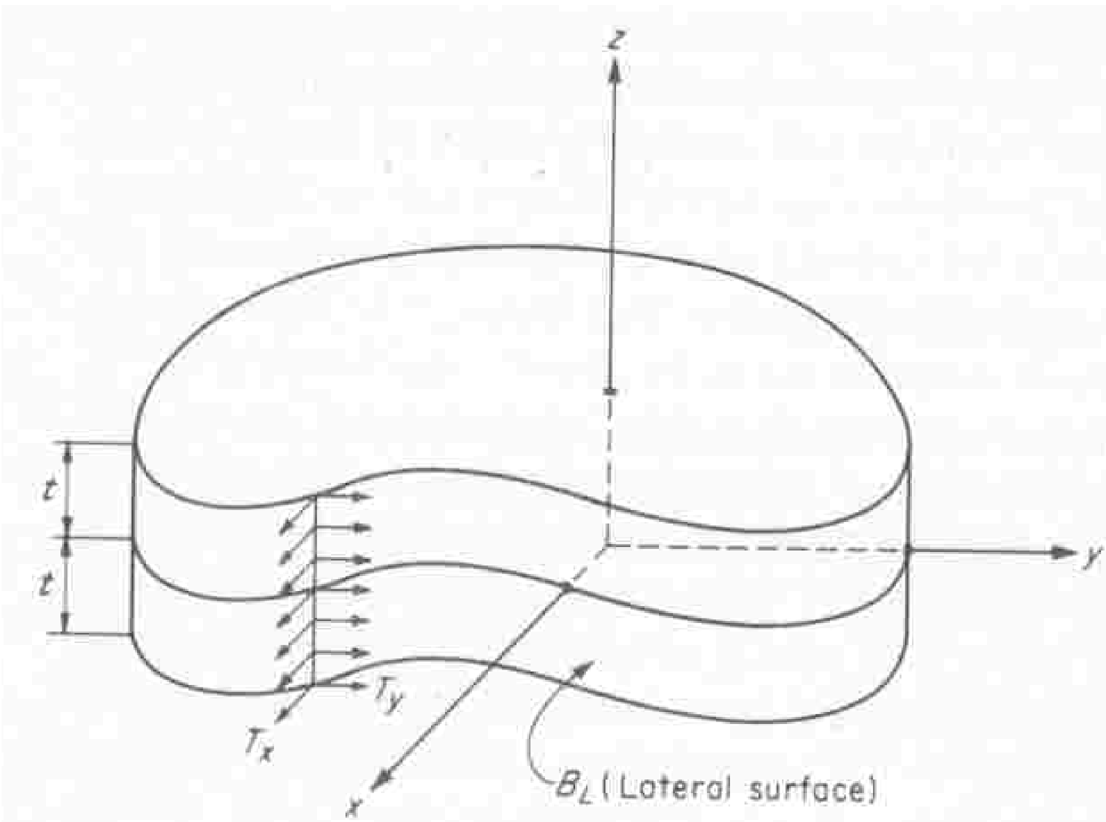
In the general three-dimensional elasticity problem there are 15 unknown quantities which must be determined at every point in the body, namely, the 6 cartesian components of stress, the 6 cartesian components of strain, and the 3 components of displacement. Attempts can be made to obtain a solution to a given problem after the following quantities have been adequately defined :

1. The geometry of the body
2. The boundary conditions
3. The body-force field as a function of position
4. The elastic constants

In order to solve for the above-mentioned 15 unknown quantities, 15 independent equations are required. Three are provided by the stress equations of equilibrium , six are provided by the strain-displacement relations , and the remaining six can be obtained from the stress-strain expressions. A solution to an elasticity problem, in addition to satisfying these 15 equations, must also satisfy the boundary conditions. In other words, the stresses acting over the surface of the body must produce tractions which are equivalent to the loads being applied to the body. Boundary conditions are often classified to define different types of boundary-value problems.

THE PLANE ELASTIC PROBLEM

Figure: A body which may be considered for the plane-elasticity approach is bounded on the top and bottom by two parallel planes and is bounded laterally by any surface which is normal to the top x and bottom planes.



In the theory of elasticity there exists a special class of problems, known as plane problems, which can be solved more readily than the general three-dimensional problem since certain simplifying assumptions can be made in their treatment. The geometry of the body and the nature of the loading on the boundaries which permit a problem to be classified as a plane problem are as follows :

By definition a plane body consists of a region of uniform thickness bounded by two parallel planes and by any closed lateral surface EL, as indicated by Fig. below. Although the thickness of the body must be uniform, it need not be limited. It may be very thick or very thin; in fact, these two extremes represent the most desirable cases for this approach, as will be pointed out later. In addition to the restrictions on the geometry of the body, the following restrictions are imposed on the loads applied to the plane body.

1. Body forces, if they exist, cannot vary through the thickness of the region, that is, F x = F x(x, y) and F y = F y(x, y). Furthermore, the body force in the z direction must equal zero.
2. The surface tractions or loads on the lateral boundary EL must be in the plane of the model and must be uniformly distributed across the thickness, i.e., constant in the z direction. Hence, Tx = Tx(x, y), Ty = Ty(x, y), and Tz = 0.

3. No loads can be applied on the parallel planes bounding the top and bottom surfaces, that is, T" = 0 on z = 1: t.

Once the geometry and loading have been defined, stresses can be determined by using either the plane-strain or the plane-stress approach. Usually the plane- strain approach is used when the body is very thick relative to its lateral dimensions. The plane-stress approach is employed when the body is relatively thin in relation to its lateral dimensions.

THE PLANE-STRAIN APPROACH

If it is assumed that the strains in the body are plane, i.e., the strains in the x and y directions are functions of x and y alone, and also that the strains in the z directions are equal to zero, the strain-displacement relation can be simplified as follows :

$\varepsilon_{zz} = 0$

$\gamma_{xz} = 0$

$\gamma_{yz} = 0$

THE PLANE-STRESS APPROACH

If it is assumed that the body thickness is small relative to its lateral dimensions, it is advantageous to assume that

$\sigma_{zz}=0$

$\tau_{xz}=0$

$\tau_{yz}=0$

throughout the thickness of the plate.