1. a) <u>Bordered algorithm for Cholesky</u>

For a given SPD matrix A, and lower triangular matrix L, we partition A and L as:

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \qquad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

where,

A_{00} is $n1 \times n1$ matrix,

$\alpha_{11} \rightarrow$ scalar

$a_{\phi 0} \rightarrow n1 \times n2$ matrix,

where $n2 = n - n1$

L_{00} is $n1 \times n1$ matrix

l_{10} is an $n2 \times n1$ matrix

λ_{11} is a scalar.

Inspired by bordered LU factorization, Since matrix A is SPD, the cholesky factorization of A is written as:

$$\cancel{A} = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^T$$

$$\Rightarrow \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c|c} L_{00}^T & l_{10}^T \\ \hline 0 & \lambda_{11} \end{array}\right)$$

$\therefore$ Expanding the multiplication and equating to A, we get

$$A_{00} = L_{00} L_{00}^T$$

$$a_{01} = L_{00}^T l_{10}^T$$

$$\alpha_{11} = l_{10} l_{10}^T + \lambda_{11}^2$$

$$\left(\begin{array}{c|c} L_{00} L_{00}^T & 0 \\ \hline L_{00}^T l_{10}^T & l_{10} l_{10}^T + \lambda_{11}^2 \end{array}\right)$$

Thus, for algorithm,

1. Partition $A \rightarrow \left(\begin{array}{c|c} A_{00} & 0 \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$

2. Assume $A_{00} = L_{00} = \text{CHOL}(A_{00})$ has been computed

3. Overwrite $a_{10}^T = l_{10}^T = a_{10}^T L_{00}^{-T}$

4. overwrite $\alpha_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

To derive the algo, ~~we~~. We start by using same approach as bordered LU factorization

→ Let's start by computing L_{00} by taking Cholesky factorization of A_{00}.

→ Then we compute $l_{10} = A_{10} L_{00}^{-T}$

where ~~$A_{01} = a_{001}^T$~~ $A_{10} = a_{10}^T$

→ Compute λ_{11} by applying Cholesky factorization recursively to $\alpha_{11} - l_{10}^T l_{10}$

→ Compute $L_{11} = [\,l_{11}\,]$.

Summarizing bordered Cholesky factorization:

- Input $A \rightarrow$ an $n \times n$ SPD matrix
- Output $L \rightarrow$ an $n \times n$ lower triangular matrix, such that $A = L L^T$

- Partition A an L as described above.
- Compute L_{00}, the cholesky factorization of A_{00}
- compute l_{10}, $l_{10} = A_{\phi 0} L_{00}^{-T}$
  wher $A_{\phi 0} = a_{10}^T$
- Compute λ_{11} reccursively by appling cholesky factorization on $\alpha_{11} - l_{10} l_{10}^T$ $(^{1}/_{2})$
- Set $L_{11} = [\lambda_{11}]$
- Return L.

b) a Proof by induction on n

Base case, $n=1$, A is 1×1 SPD matrix, then it is always true that L exists such that $A = LL^T$, $L_{11} = \sqrt{\alpha_{11}}$.

Inductive step: Assume that Cholesky holds good for all SPD of size $< n$ and then we partion A and L.

$$A = \left(\begin{array}{c|c} A_{00} & 0 \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \qquad L = \left(\begin{array}{c|c} L_{00} L_{00}^T & 0 \\ \hline L_{00} & \end{array}\right)$$

$$L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

$A = L . L^T$,

In case of 1×1 SPD
$L_{11} = \sqrt{\alpha_{11}}$

l_{10}^T is uniquely defined as $a_{10}^T . L_{00}^{-T}$

α_{11} is SPD, hence, cholesky factorization is used recursively.