Bordered algorithm

consider loop variant:

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) = \left(\begin{array}{c|c} L\backslash U_{TL} & \hat{A}_{TR} \\ \hline \hat{A}_{BL} & \hat{A}_{BR} \end{array}\right)$$

$$\wedge \quad \begin{array}{c|c} L_{TL}U_{TL} = \hat{A}_{TL} & \cancel{L_{TL}U_{TR} = \hat{A}_{TR}} \\ \hline \cancel{L_{BL}U_{TL} = \hat{A}_{BL}} & \cancel{L_{BL}U_{TR} + L_{BR}U_{BR} = \hat{A}_{BR}} \end{array}$$

At the top of the loop, repartioning, A is

$$\begin{array}{c|c|c} L\backslash U_{00} & \hat{a}_{01} & \hat{A}_{02} \\ \hline \tilde{a}_{10}^{T} & \hat{\alpha}_{11} & \hat{a}_{12} \\ \hline \hat{A}_{20} & \hat{a}_{21} & \hat{A}_{22} \end{array}$$

while at the bottom, it must contain

u_{01} instead of $\hat{a}_{01}$
l_{10}^{T} instead of $\tilde{a}_{10}^{T}$
v_{11} instead of $\hat{\alpha}_{11}$.

Now, $LU = \hat{A}$.

$$\begin{array}{c|c|c} L_{00}U_{00} = \hat{A}_{00} & L_{00}u_{01} = \hat{a}_{01} & L_{00}U_{02} = \hat{A}_{02} \\ \hline l_{10}^{T}u_{00} = \hat{a}_{10}^{T} & l_{10}^{T}u_{01} = \hat{\alpha}_{11} & l_{10}^{T}U_{02} + u_{12}^{T} = \hat{a}_{12}^{T} \\ \hline L_{20}u_{00} = \hat{A}_{20} & L_{20}u_{01} + v_{11}l_{21} = \hat{a}_{21} & L_{20}U_{02} + l_{21}u_{12}^{T} + L_{22}U_{22} = \hat{A}_{22} \end{array}$$

→ Solve $L_{00}u_{01} = a_{01}$ for u_{01}, overwriting a_{01}

→ Solve $l_{10}^{T}u_{00} = a_{10}^{T}$ overwriting a_{10}^{T}

→ compute $v_{11} = a_{11} - l_{10}^{T}u_{01}$ overwrite α_{11}.

For crout variant,

- compute $\upsilon_{11} = \alpha_{11} - l_{10}^{T} u_{01}$,
  overwriting α_{11} with this.

- compute $l_{21} = (\alpha_{21} - L_{20} u_{01}) / \upsilon_{11}$

→ compute $u_{12}^{T} = a_{12}^{T} - l_{10}^{T} u_{02}$.

For base case ⇒ $n = 1$,
we have the matrix A as a scalar → α_{11}.
The bordered LU can be equated,
where
$l_{10}^{T} = a_{10}^{T} / \mu_{00}$,
$\upsilon_{11} = \alpha_{11} - a_{10} / \mu_{01}$

Let us define $\hat{L}_{00}$ and $\hat{U}_{00}$ as computed factor of A_{00} by using LU factorization,
then we have

$$A_{00} + \Delta A_{00} = \hat{L}_{00} \hat{U}_{00}, \quad |\Delta A_{00}| \leq \gamma_n |\hat{L}_{00}||\hat{U}_{00}|$$

considering $n = 1$

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^{T} & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^{T} & \delta \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \hat{L}_{00} & 0 \\ \hline \hat{l}_{10}^{T} & 1 \end{array}\right) \left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{\upsilon}_{11} \end{array}\right)$$

For $L \in \mathbb{R}^{n \times n}$ lower triangular matrix,
$y, z \in \mathbb{R}^{n}$, $\check{z}$ be approximate soln,

$$(L + \Delta L)\check{z} = y, \quad |\Delta L| \leq \gamma_n |L|$$

$Lz = y$ → leading to soln of $\check{z}$
$Ux = \check{z}$ → leading to soln of $\check{x}$