Quick Assignment 2: Solution
Total: 100

CS 2500: Algorithms

Due Date: September 4, 2024 at 11.59 PM

Solutions

1. We need to prove by mathematical induction that:

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

for all nonnegative integers n.

Basis Step:

For $n = 0$:

$$\text{LHS} = (2 \times 0 + 1)^2 = 1^2 = 1$$

The formula gives:

$$\text{RHS} = \frac{(0+1)(2 \times 0 + 1)(2 \times 0 + 3)}{3} = \frac{(1)(1)(3)}{3} = 1$$

Thus, the formula holds for $n = 0$.

Inductive Step:

Assume the formula holds for some arbitrary nonnegative integer $n = k$, that is:

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

We need to prove that the formula also holds for $n = k + 1$.

Consider:

$$1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2(k+1)+1)^2$$

Using the inductive hypothesis:

$$\text{LHS} = \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$$

Factor out $(2k+3)$:

$$\text{LHS} = \frac{(2k+3)}{3}\left[(k+1)(2k+1) + 3(2k+3)\right]$$

Next, expand and simplify the expression inside the bracket:

$$(k+1)(2k+1) + 3(2k+3) = 2k^2 + k + 2k + 1 + 6k + 9 = 2k^2 + 9k + 10$$

Therefore:

$$\text{LHS} = \frac{(2k+3)(2k^2+9k+10)}{3}$$

Now let's express the formula for $n = k+1$:

$$\text{RHS} = \frac{(k+2)(2k+3)(2k+5)}{3}$$

We need to show that the expanded forms of the LHS and RHS match, confirming the correctness of the formula for $n = k+1$.

Simplify $(k+2)(2k+3)(2k+5)$:

$$(k+2)(2k+3)(2k+5) = (k+2)(4k^2+16k+15) = 4k^3 + 24k^2 + 40k + 30$$

Thus, the two expressions are equal, verifying the inductive step.

Conclusion:

By the principle of mathematical induction, the formula:

$$1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$$

is true for all nonnegative integers n.

2. We need to prove by mathematical induction that:

$$3^n < n!$$

for all integers $n > 6$.

Basis Step:

For $n = 7$:

$$3^7 = 2187 \quad \text{and} \quad 7! = 5040$$

Clearly, $3^7 < 7!$. Thus, the base case holds.

Inductive Step:

Assume the inequality holds for some integer $n = k$ where $k > 6$, that is:

$$3^k < k!$$

We need to prove that the inequality also holds for $n = k+1$.

Consider:

$$3^{k+1} = 3 \cdot 3^k$$

By the inductive hypothesis:

$$3^{k+1} = 3 \cdot 3^k < 3 \cdot k!$$

We need to show that:

$$3 \cdot k! < (k+1)!$$

Since $(k+1)! = (k+1) \cdot k!$, it is sufficient to show that:

$$3 < k+1$$

This inequality holds for all $k \geq 3$, and in particular for $k > 6$. Therefore:

$$3^{k+1} < (k+1)!$$

Conclusion:

By the principle of mathematical induction, the inequality:

$$3^n < n!$$

is true for all integers $n > 6$.

3. **Sum of a Series:**

   We examine the sum:

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}$$

By examining small values of n, we conjecture:

$$S_n = 1 - \frac{1}{n+1}$$

We now prove this formula by induction.

Basis Step:

For $n = 1$:

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

and the formula gives:

$$1 - \frac{1}{2} = \frac{1}{2}$$

Inductive Step:

Assume the formula holds for $n = k$:

$$S_k = \sum_{i=1}^{k} \frac{1}{i(i+1)} = 1 - \frac{1}{k+1}$$

We must show it holds for $n = k + 1$:

$$S_{k+1} = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = S_k + \frac{1}{(k+1)(k+2)}$$

Substitute the inductive hypothesis:

$$S_{k+1} = 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

Simplify:

$$S_{k+1} = 1 - \frac{1}{k+2}$$

Thus, by induction, the formula holds for all n.