1

Assignment No. 2

Transforms

Roll. No. 21118

PICT, PUNE

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Q1 Using fourier integral representation, show that

$$\int_0^\infty \left(\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2}\right) d\lambda = \begin{cases} 0, & x<0 \\ \pi/2, & x=0 \\ \pi e^{-x}, & x>0 \end{cases}$$

→

let

$$f(x) = \begin{cases} 0 & x<0 \\ \pi/2 & x=0 \\ \pi e^{-x} & x>0 \end{cases}$$

fourier transform of $f(x)$ can be given as,

$$\boxed{F(\lambda) = \int_{-\infty}^{\infty} f(u) e^{-i\lambda u} du}$$ ... formula

substitute values, we have

$$F(\lambda) = \cancel{\int_{-\infty}^{0} 0 \cdot e^{-i\lambda u} du}^{\,0} + \int_0^\infty \pi e^{-u} e^{-i\lambda u} du$$

$$= \pi \int_0^\infty e^{-(1+i\lambda)u} du$$

$$= -\pi \left[\frac{e^{-(1+i\lambda)u}}{1+i\lambda}\right]_0^\infty$$

$$\boxed{F(\lambda) = \frac{\pi}{1+i\lambda}} \quad \text{———} \quad (1)$$

Inverse fourier transform can be given as,

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} d\lambda}$$ ... formula

here $F(\lambda)$ is fourier transform

Roll No 21118

PICT, PUNE

Substitute the value from eqn ①,
we get

$$f(x) = \int \frac{1}{2\cancel{\pi}} \int_{-\infty}^{\infty} \frac{\cancel{\pi}}{(1+i\lambda)} e^{i\lambda x} d\lambda$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{1-i\lambda}{1+\lambda^2}\right) (\cos \lambda x + i \sin \lambda x)\, d\lambda$$

$\because e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} \left(\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2}\right) d\lambda + i \int_{-\infty}^{\infty} \left(\frac{\sin\lambda x - \lambda \cos\lambda x}{1+\lambda^2}\right) d\lambda\right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left(\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2}\right) d\lambda$$ $\because$ the second integral becomes zero since it's odd f^n on interval $-\infty$ to ∞

$$= \frac{1}{2} \cdot 2 \int_{0}^{\infty} \frac{(\cos\lambda x + \lambda \sin\lambda x)}{1+\lambda^2} d\lambda$$ $\because$ the function is even function

$\therefore \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

$$\boxed{f(x) = \int_{0}^{\infty} \left[\frac{\cos\lambda x + \lambda \sin\lambda x}{1+\lambda^2}\right] d\lambda}$$

Since our $f(x)$ is the RHS of the expression to be proved,

$\therefore$ RHS = RHS

Hence proved

Q2. Find fourier transform of the function
$f(x) = e^{-x^2}$ where $x>0$.

→ here $f(x)$ is an even function.

(2)

Roll No.
21118

PICT, PUNE

∴ The fourier transform of even function is given as

$$\boxed{F_c(\lambda) = \int_0^\infty f(u) \cos \lambda u \, du}$$

substitute value of $f(u)$,
we get

$$F_c(\lambda) = \int_0^\infty e^{-u^2} \cos \lambda u \, du \quad \text{———} \quad (1)$$

According to DUIS Rule,

$$\frac{dF_c}{d\lambda} = -\int_0^\infty u e^{-u^2} \sin \lambda u \, du$$

using integration by parts,
we have,

$$\frac{dF_c}{d\lambda} = \left[\cancel{\frac{+\sin \lambda u \, e^{-u^2}}{2}} \right]_0^\infty + -\int_0^\infty \frac{\cos \lambda u \, \lambda}{2} e^{-u^2} du$$

$$\frac{dF_c}{d\lambda} = \frac{-\lambda}{2} \int_0^\infty \cos \lambda u \, e^{-u^2} du$$

from eq^n (1)

$$\frac{dF_c}{d\lambda} = -\frac{\lambda}{2} F_c .$$

$$\therefore \frac{dF_c}{F_c} = \frac{-\lambda}{2} d\lambda$$

Integrating both sides,
we get

$$\ln F_c = -\frac{\lambda^2}{4} + c \qquad \therefore \; c = \text{const of integration}$$

Roll. No 21118

PICT, PUNE

the equation can be written as,

$$f_c = A e^{-\lambda^2/4} \quad \text{—— (2)}$$

From eq^n (1),

$$f_c(0) = \int_0^\infty e^{-u^2} du$$

substitute $u^2 = t$

$$2u\,du = dt$$

$$\therefore du = \frac{dt}{2\sqrt{t}}$$

$$\therefore f_c(0) = \frac{1}{2}\int_0^\infty e^{-t} \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2}\Gamma\tfrac{1}{2} \quad \therefore \text{ property of } \gamma \text{. function.}$$

$$f_c(0) = \frac{\sqrt{\pi}}{2} \quad \therefore \text{ since } \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$\text{—— (3)}$$

from eq^n (2),

$$f_c(\lambda) = A e^{-\lambda^2/4}$$

$$f_c(0) = A e^{-0}$$

$$= A$$

but from eq^n (2), $f_c(0) = \frac{\sqrt{\pi}}{2}$.

$$\therefore A = \frac{\sqrt{\pi}}{2}$$

$\therefore$ we have.

$$\boxed{f_c(\lambda) = A e^{-\lambda^2/4} = \frac{\sqrt{\pi}}{2} e^{-\lambda^2/4}} \quad \text{Ans}$$

Roll No 21118

PICT, PUNE

Q3. Solve the integral equation,

$$\int_0^\infty f(x)\sin\lambda x\,dx = e^{-\lambda},\quad \lambda>0$$

→ here

$$\int_0^\infty f(x)\sin\lambda x\,dx = e^{-\lambda} = f_s(\lambda)$$

= Fourier sine transform of $f(x)$ on $x \in (0,\infty)$

∴ Inverse fourier transform is given as,

$$f(x) = \frac{1}{2\pi}\int_0^\infty f_s(\lambda)\sin\lambda x\,d\lambda$$

putting $f_s(\lambda) = e^{-\lambda}$,
we get,

$$f(x) = \frac{1}{2\pi}\int_0^\infty e^{-\lambda}\sin\lambda x\,d\lambda$$

$$\boxed{f(x) = \frac{1}{2\pi}\left(\frac{x}{1+x^2}\right)}$$ ∵ since $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

Ans

Q4 Using inverse fourier sine transform, find $f(x)$ if,

$$f_s(\lambda) = \frac{1}{\lambda}e^{-a\lambda}$$

→ Inverse fourier sine transform can be given as,

$$f(x) = \frac{2}{\pi}\int_0^\infty f_s(\lambda)\sin\lambda x\,d\lambda$$ ∴ formula

Roll No 21118

PICT, PUNE

Substitute value of $f_s(\lambda)$,
we have,

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \sin \lambda x \, d\lambda \quad \text{— (1)}$$

Using DUIS rule,
we have,

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda} e^{-a\lambda} \lambda \cos \lambda x \, d\lambda$$

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty e^{-a\lambda} \cos \lambda x \, d\lambda$$

$$\frac{df}{dx} = \frac{2}{\pi}\left(\frac{a}{a^2+x^2}\right) \quad \because \text{ since } \int_0^\infty e^{-ax}\cos bx \, dx = \frac{a}{a^2+b^2}$$

$$df = \frac{2}{\pi}\left(\frac{a}{a^2+x^2}\right) dx.$$

integrating both sides,
we get

$$\int df = \int \frac{2}{\pi}\left(\frac{a}{a^2+x^2}\right) dx$$

$$f(x) = \frac{2a}{\pi} \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \quad \therefore C = \text{const of integrat}^n$$

$$f(x) = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right) + C \quad \text{— (2)}$$

From eq^n (1) $f(0) = 0$.
$\therefore$ from eq^n (2) $f(0) = \frac{2}{\pi}\tan^{-1}(0) + C$ $\}$ $C = 0$.

$$\therefore \boxed{f(x) = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right)}$$

Ans

Roll No 21118

PICT, PUNE

Q5 Find z-transform & ROC of $f(k)$

i) $f(k) = \frac{5^k}{k}, \quad k \geq 1$.

→

consider $g(k) = 5^k$

z-transform of $g(k)$ is

$$G(z) = \sum_{k=1}^{\infty} g(k)\, z^{-k}$$

$$= \sum_{k=1}^{\infty} 5^k z^{-k}$$

$$= \sum_{k=1}^{\infty} \left(\frac{5}{z}\right)^k$$

$$= \frac{5/z}{1-5/z} \quad \text{where } \left|\frac{5}{z}\right| < 1 \text{ i.e. } 5 < |z|$$

$$G(z) = \frac{5}{z-5} = Z\{5^k\} \quad \text{--- (1)}$$

We know the property of z-transform as

$$\boxed{\begin{array}{l} \text{If } Z\{f(k)\} = F(z) \\ \text{then } Z\left\{\frac{f(k)}{k}\right\} = \int_z^{\infty} \frac{F(z)}{z}\, dz \end{array}}$$

using the above property,

$$Z\left\{\frac{5^k}{k}\right\} = \int_z^{\infty} \frac{Z\{5^k\}}{z}\, dz \qquad |z| > 5$$

$$= \int_z^{\infty} \frac{5}{z(z-5)}\, dz \quad \text{... substituting value from eq}^n \text{ (1)}$$

$$= \int_z^{\infty} \left(\frac{1}{z-5} - \frac{1}{z}\right) dz$$

Roll No 21118

PICT, PUNE

$$= \left[\ln\left(\frac{z-5}{z}\right) \right]_z^{\infty}$$

$$= \left[\ln\left(1-\frac{5}{z}\right) \right]_z^{\infty}$$

$$= -\ln\left(1-\frac{5}{z}\right)$$

$$= \ln\left(\frac{z}{z-5}\right)$$

$$\boxed{\therefore F(z) = \ln\left(\frac{z}{z-5}\right)} \quad \& \text{ ROC is } |z|>5$$

Ans

ii) $f(k) = e^k \cos\left(\frac{k\pi}{2}\right), \; k \geq 0$

→ We know that,

if $f(k) = \cos \alpha k, \; k \geq 0$

then $Z\{\cos \alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ — ① $|z|>1$

consider $Z\{\cos(k\pi/2)\}$.

from eqn ①.

$$Z\{\cos(k\pi/2)\} = \frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1} \quad \ldots \alpha = \pi/2$$

$$= \frac{z^2}{z^2+1} \quad \text{— ①} \quad |z|>1$$

property of Z transform, (change of scale)

$$\boxed{\begin{array}{l} \text{if } Z\{f(k)\} = F(z) \text{ then} \\ Z\{a^k f(k)\} = F(z/a) \end{array}}$$

Roll No
21118

PICT, PUNE

using the above property,

$$Z\{e^k \cos(\pi/2\, k)\} = F(z/e) \quad \text{where } F(z) \text{ is a z-transform of } \cos(\pi/2\, k)$$

substituting the value from eqn ①, we get

$$Z\{e^k \cos(\pi/2\, k)\} = \frac{(z/e)^2}{(z/e)^2+1} \quad \because \text{ ROC is } \left|\frac{z}{e}\right|>1$$

$$\therefore |z|>e$$

$$= \frac{z^2}{z^2+e^2}$$

$$= f(z)$$

$$\boxed{\therefore f(z) = \frac{z^2}{z^2+e^2}} \quad \text{where ROC is } |z|>e.$$

Ans

Q6 find inverse z-transform of $f(z)$

i) $f(z) = \frac{z^2}{z^2+1}$, $|z|>1$ by any method.

→ here $f(z) = \frac{z^2}{z^2+1}$ where $|z|>1$

The poles of $f(z)$ are at $+i$ and $-i$.
both the poles are simple poles.

$$\operatorname*{Res}_{z=i} f(z) z^{k-1} = \lim_{z\to i}\left[(z-i)\, z^{k-1} f(z)\right]$$

$$= \lim_{z\to i}\left[(z-i)\, z^{k-1} \frac{z^2}{(z-i)(z+i)}\right]$$

Roll No
21118

PICT, PUNE

$$= \lim_{z \to i} \frac{z^{k+1}}{z+1}$$

$$= \frac{i^{k+1}}{2i}$$

$$= \frac{i^k}{2} \quad \text{--- (1)}$$

$$\underset{z=-i}{\mathrm{Res}}\, f(z) z^{k-1} = \lim_{z \to -i} \left[\cancel{(z+i)} \frac{z^{k-1}\, z^2}{(z-i)\cancel{(z+i)}} \right]$$

$$= \lim_{z \to -i} \frac{z^{k+1}}{z-i}$$

$$= \frac{(-i)^{k+1}}{-2i}$$

$$= \frac{(-i)^k}{2} \quad \text{--- (2)}$$

As we know,

$$\boxed{f(k) = \Sigma \left(\text{Residues of } f(z) z^{k-1} \text{ at the poles of } f(z) \right)}$$

substitute values from eqn (1) & (2),
we get

$$f(k) = \frac{i^k}{2} + \frac{(-i)^k}{2}$$

$$= \frac{(e^{i\pi/2})^k}{2} + \frac{(e^{-i\pi/2})^k}{2}$$

$$= \frac{e^{ik\pi/2} + e^{-ik\pi/2}}{2}$$

$$= \frac{2\cos(k\pi/2)}{2}$$

(5)

Roll No 21118

PICT, PUNE

$$\boxed{f(k) = \cos(k\pi/2)} \quad k \geq 0.$$

Ans

1) $F(z) = \dfrac{2z^2+3z}{z^2+z+1}$, using inverse integral method.

→ Given that,

$$F(z) = \frac{2z^2+3z}{z^2+z+1}$$

poles of $F(z)$:

$$z^2+z+1 = 0$$

$$z = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}.$$

∴ poles are $\dfrac{-1+\sqrt{3}i}{2}$ & $\dfrac{-1-\sqrt{3}i}{2}$ (let say α & β) — (1)

both poles are simple poles.

we know that,

$$\boxed{f(k) = \Sigma\left[\text{Residues of } F(z)\, z^{k-1} \text{ at the } \cancel{\text{simple}} \text{ poles of } F(z)\right]}$$

$= R_1 + R_2$ ∵ since we have two poles α & β. — (2)

$$R_1 = \underset{z=\alpha}{\text{Res}} \left[F(z)\, z^{k-1}\right]$$

$$= \lim_{z\to\alpha} \left[(z-\alpha)\, z^{k-1} F(z)\right]$$

. here $F(z) = \dfrac{2z^2+3z}{(z-\alpha)(z-\beta)}$

$$= \lim_{z\to\alpha} \left[\frac{\cancel{(z-\alpha)}\, z^{k-1}\, (2z^2+3z)}{\cancel{(z-\alpha)}\, (z-\beta)}\right]$$

Roll No 21118

PICT, PUNE

$$= \frac{2\alpha^{k+1} + 3\alpha^k}{\alpha - \beta} \quad \text{--- (3)}$$

$$R_2 = \underset{z=\beta}{\text{Res}} \left[f(z)\, z^{k-1} \right]$$

$$= \lim_{z \to \beta} \cancel{(z-\beta)}\, z^{k-1} \frac{(2z^2+3z)}{(z-\alpha)\cancel{(z-\beta)}}$$

$$= \frac{2\beta^{k+1} + 3\beta^k}{(\beta - \alpha)} \quad \text{--- (4)}$$

substitute values from eqⁿ (3) & (4) into eqⁿ (2),
we get

$$f(k) = \frac{2\alpha^{k+1} + 3\alpha^k}{\alpha - \beta} + \frac{2\beta^{k+1} + 3\beta^k}{(\beta - \alpha)}$$

$$= \frac{\alpha^k}{\alpha - \beta}(2\alpha + 3) + \frac{\beta^k}{\beta - \alpha}(2\beta + 3)$$

substitute values of α & β from eqⁿ (1)
we get

$$f(k) = \left(\frac{-1+\sqrt{3}i}{2}\right)^k \frac{1}{\sqrt{3}i}(2+\sqrt{3}i) + \left(\frac{-1-\sqrt{3}i}{2}\right)^k \frac{1}{(-\sqrt{3}i)}(2-\sqrt{3}i)$$

$$\boxed{f(k) = \frac{1}{\sqrt{3}i}\left((2+\sqrt{3}i)\left(\frac{-1+\sqrt{3}i}{2}\right)^k - (2-\sqrt{3}i)\left(\frac{-1-\sqrt{3}i}{2}\right)^k\right)}$$

Ans

⑦

Roll No
21118

PICT, PUNE

Q7. Solve difference equation using z-transform

$f(k+1) + \frac{1}{4} f(k) = \left(-\frac{1}{4}\right)^k, \quad k \geq 0, f(0)=0$

→ ~~Ute~~

Given equation,

$f(k+1) + \frac{1}{4} f(k) = \left(-\frac{1}{4}\right)^k, \quad k \geq 0, f(0)=0$ —①

we know that,

$Z\{f(k+1)\} = zF(z) - zf(0)$ —(i)

$Z\{f(k)\} = F(z)$ —(ii)

take z-transform of both sides in eqn ①, we get

$Z\{f(k+1)\} + \frac{1}{4} Z\{f(k)\} = Z\left\{\left(-\frac{1}{4}\right)^k\right\} \quad k \geq 0$

from eqn (i) & (ii)

$zF(z) - zf(0) + \frac{1}{4} F(z) = \frac{z}{z+\frac{1}{4}} \quad |z| > \frac{1}{4}$

... since $Z\{a^k\} = \frac{z}{z-a}, \ |z| > |a|$

given that $f(0)=0$

$\therefore zF(z) + \frac{1}{4} F(z) = \frac{z}{(z+\frac{1}{4})}$

$F(z) = \frac{z}{(z+\frac{1}{4})(z+\frac{1}{4})}$

$F(z) = \frac{z}{(z+\frac{1}{4})^2}$ —②

Roll No
21118

PICT, PUNE

Inverse z-transform of $F(z)$ is,

$$f(k) = \Sigma \left[\text{Residues of } F(z)\, z^{k-1} \text{ at the poles of } F(z) \right] \quad \text{——— (1)}$$

$z = -1/4$ is only pole of order 2 of $F(z)$

∴ residue of $f(z) z^{k-1}$ for a pole of order r at $z = a$ is

$$= \frac{1}{\lfloor r-1} \lim_{z\to a} \frac{d^{r-1}}{dz^{r-1}} \left[(z-a)^r z^{k-1} f(z) \right]$$

$$\underset{z=-1/4}{\text{Res}} f(z) z^{k-1} = \frac{1}{\lfloor 2-1} \lim_{z\to -1/4} \frac{d}{dz} \left[(z+1/4)^2 z^{k-1} \frac{z}{(z+1/4)^2} \right]$$

$\therefore r = 2$ & $a = -1/4$

$$= \lim_{z\to -1/4} k (z^{k-1})$$

$$= k \left(-1/4\right)^{k-1}$$

substitute this in eqn (1),

we get

$$\boxed{f(k) = k\left(-1/4\right)^{k-1}} \quad k \geq 0.$$

Ans