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Assigoment No. 1

Linear Differential Equation

DOP: 12/02/2021

DOS: 19/02/2021

PICT, PUNE

Q1: Solve $(D^3+6D^2+9D)y=0.$

$\rightarrow$

Given D.E is $(D^3+6D^2+9D)y=0.$ ... Homogeneous D.E. with const. coeff.

Auxiliary eqⁿ: $m^3+6m^2+9m=0$

$m(m^2+6m+9)=0$

$m(m+3)^2=0$

roots are $0, 3, 3$

The solution.

$y = c_1e^{0x}+c_2e^{3x}+c_3xe^{3x}$

$\boxed{y = c_1+(c_2+c_3x)e^{3x}}$

Ans

Q2 Solve $(D^2-3D+2)y = \frac{1}{e^{e^{-x}}} + \cos e^{-x}$

$\rightarrow$

Given D.E: $(D^2-3D+2)y = \frac{1}{e^{e^{-x}}} + \cos e^{-x}.$

Complementry Eqⁿ Solution -

The complementry eqⁿ is $(D^2-3D+2)y=0.$

Aux. eqⁿ is $(m^2-3m+2)=0.$

roots of A.E are $2, 1$

$\therefore$ solution of complementry equation

$y_c = c_1e^{x}+c_2e^{2x}$ ——— (A)

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Particular equation solution:

here $y_p = \frac{1}{(D^2-3D+2)}\left[e^{-e^{-x}} + \cos e^{-x}\right]$

$= \frac{1}{(D^2-3D+2)} e^{-e^{-x}} + \frac{\cos e^{-x}}{(D^2-3D+2)}$

$= \frac{e^{-e^{-x}}}{(D-2)(D-1)} + \frac{\cos e^{-x}}{(D-2)(D-1)}$

$= \frac{e^{x}\int e^{-x} e^{-e^{-x}}dx}{(D-2)} + \frac{e^{x}\int e^{-x}\cos e^{-x}dx}{(D-2)}$

$\because \frac{1}{(D-m)} f(x) = e^{mx}\int e^{-mx} f(x)dx$

$= \frac{+e^{x} e^{-e^{-x}}}{(D-2)} + \left(-\frac{e^{x}\sin e^{-x}}{(D-2)}\right)$

$= e^{2x}\int e^{-2x}\cdot e^{x} e^{-e^{-x}} dx$

$+ \left(- e^{2x}\int e^{-2x} e^{x}\sin e^{-x} dx\right)$

$= e^{2x}\int e^{-x} e^{-e^{-x}} dx - e^{2x}\int e^{-x}\sin e^{-x} dx$

$= e^{2x} e^{-e^{-x}} - e^{2x}\cos e^{-x}$

$y_p = e^{2x}\left(e^{-e^{-x}} - \cos e^{-x}\right)$ —— (B)

from eqn (A) & (B)

the solution of given DE is

$y = y_c + y_p$

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$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + e^{+2x}\left(\bar{e}^{\bar{e}^{x}} - \cos \bar{e}^{x}\right)}$$

Ans

Q.3 Solve $(D^2+3D+2)y = x^3+x^2$

$\rightarrow$

Given DE: $(D^2+3D+2)y = x^3+x^2$ .. DE with const. coeff.

Complementry eqn solution:

The comple. DE is $(D^2+3D+2)y=0$

A.E: $m^2+3m+2=0$

solutions of AE are $-2, -1$

Solution of complementry equation

$y_c = c_1 e^{-2x} + c_2 e^{-x}$ — (A)

Particular eqn solution:

here $y_p = \dfrac{x^3+x^2}{(D^2+3D+2)}$

$= \dfrac{x^3+x^2}{2\left(1+\frac{D^2+3D}{2}\right)}$

$= \frac{1}{2}\left(1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2-\left(\frac{D^2+3D}{2}\right)^3\right)(x^3+x^2)$

∵ by series expansion

$\frac{1}{1+x} = 1-x+x^2-x^3+\ldots$

$= \frac{1}{2}\left(x^3+x^2-\left(\frac{6x+\cancel{3}(2x)+3(3x^2)+3(2x)}{2}\right)\right.$

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$$+\left(\frac{9(6x+2)+6(6)}{4}-\frac{27(6)}{8}\right)$$

$$=\frac{1}{2}\left((x^3+x^2)-\left(\frac{9x^2+12x+2}{2}\right)+\left(\frac{54x+54}{4}\right)-\frac{3\times 27}{4}\right)$$

$$=\frac{1}{2}\left(x^3-\frac{7x^2}{2}+\frac{30x}{4}-\frac{27}{4}-1\right)$$

$$=\frac{1}{2}\left(x^3-\frac{7x^2}{2}+\frac{30x}{4}-\frac{31}{4}\right)$$

$$y_p=\frac{1}{8}\left(4x^3-14x^2+30x-31\right) \quad \text{— (B)}$$

The solution of given DE is

$$y=y_c+y_p$$

from eqn (A) & (B)

$$\boxed{y=c_1e^{-2x}+c_2e^{-x}+\frac{1}{8}\left(4x^3-14x^2+30x-31\right)}$$

Ans

Q4 Solve $(D^2+16)y=e^{-3x}+\cos(2x)$

→

Given D.E: $(D^2+16)y=e^{-3x}+\cos(2x)$

∴ 2^{nd} order LDE with const coeff

Complementary eqn

$(D^2+16)y=0$

Auxiliary eqn: $(m^2+16)=0$

$(m+4i)(m-4i)=0$

roots of Auxiliary equation are $+4i, -4i$

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solution of complementary equation

$$y_c = C_1\cos 4x + C_2\sin 4x \quad \text{— (A)}$$

Particular equation:

here $$y_p = \frac{1}{(D^2+16)}(e^{-3x} + \cos 2x)$$

$$= \frac{e^{-3x}}{(D^2+16)} + \frac{\cos 2x}{(D^2+16)}$$

$$= \frac{e^{-3x}}{(9+16)} + \frac{\cos 2x}{(-4+16)}$$

..... (I) $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$

(II) $\frac{\cos ax}{f(D^2)} = \frac{\cos ax}{f(-a^2)}$

$$y_p = \frac{e^{-3x}}{25} + \frac{\cos 2x}{12} \quad \text{— (B)}$$

The solution of given differential eqn is

$$y = y_c + y_p$$

from eqn (A) & (B)

$$\boxed{y = C_1\cos 4x + C_2\sin 4x + \left(\frac{e^{-3x}}{25} + \frac{\cos 2x}{12}\right)}$$

Ans.

Q5 Solve $(D^2+2D+1)y = x\cos x$

→ Given DE is

$(D^2+2D+1)y = x\cos x$ ... 2nd order LDE with const. coefficients

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Complementary eqn:

$(D^2+2D+1)y = 0.$

Auxiliary eqn is $(m^2+2m+1) = 0$

roots of AE are $-1, -1$

$\therefore$ Solution of complementary eqn is

$y_c = c_1 e^{-x} + c_2 x e^{-x}$ —— (A)

Particular eqn:

here $y_p = \dfrac{x\cos x}{(D^2+2D+1)}$

$= \left(x - \dfrac{2D+2}{D^2+2D+1}\right)\dfrac{\cos x}{D^2+2D+1}$

... by formular of linear shift

ie. $\dfrac{(x \cdot V)}{f(D)} = \left(x - \dfrac{f'(D)}{f(D)}\right)\dfrac{V}{f(D)}$

$= \left(x - \dfrac{2(D+1)}{(D+1)^2}\right)\dfrac{\cos x}{(D^2+2D+1)}$

$= \left(x - \dfrac{2}{(D+1)}\right)\dfrac{\cos x}{(D^2+2D+1)}$

$= \left(x - \dfrac{2}{D+1}\right)\dfrac{\cos x}{2D}$ .... $\dfrac{\cos ax}{f(D^2)} = \dfrac{\cos ax}{f(-a^2)}$

$= \left(x - \dfrac{2}{D+1}\right)\dfrac{\sin x}{2}$

$= \dfrac{x\sin x}{2} - \dfrac{\sin x}{D+1}$

$= \dfrac{x\sin x}{2} - \dfrac{(D-1)\sin x}{D^2-1}$ ... rationalization

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$$= \frac{x\sin x}{2} - \frac{(\cos x - \sin x)}{D^2 - 1}$$

$$= \frac{x\sin x}{2} + \frac{(\cos x - \sin x)}{2} \quad \ldots \frac{\cos ax}{f(D^2)} = \frac{\cos ax}{f(-a^2)}$$

$$\& \ \frac{\sin(ax)}{f(D^2)} = \frac{\sin(ax)}{f(-a^2)}$$

$$= \frac{x\sin x}{2} - \frac{\sin x}{2} + \frac{\cos x}{2}$$

$$y_p = \frac{(x-1)\sin x + \cos x}{2} \quad \text{—— (B)}$$

The solution of given D.E is,

$y = y_c + y_p$

From eqn (A) & (B),

$$\boxed{y = C_1 e^{-x} + C_2 x e^{-x} + \frac{(x-1)\sin x + \cos x}{2}}$$

Ans

Q6. Solve (by method of variation of parameters)
$(D^2 + 2D + 1)y = e^{-x}\log x$

$\rightarrow$

Given DE is $(D^2 + 2D + 1)y = e^{-x}\log x$
where $e^{-x}\log x = f(x)$ —— (1)

complementary eqn:
$(D^2 + 2D + 1)y = 0.$

Auxiliary equation is $(m^2 + 2m + 1) = 0$
$(m+1)^2 = 0$

solution of auxiliary eqn $-1, -1$

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hence solution of complementry eq^n is

$$y_c = c_1 e^{-x} + c_2 x e^{-x} \quad \text{--- (A)}$$

comparing y_c with $y_c = c_1 y_1 + c_2 y_2$

we have $y_1 = e^{-x}$ & $y_2 = x e^{-x}$. — (2)

particular solution can be given as,

$$y_p = u y_1 + v y_2 \quad \text{--- (3)}$$

where $u = \int \frac{-y_2 f(x)}{w(x)} dx$ & $v = \int \frac{y_1 f(x)}{w(x)} dx$

here $w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ = wronskian value.

calculating u & v,

taking values of f(x), u, v from eq^n (1) & (2)

we have

$$w(x) = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x}(1-x) + x e^{-2x} = e^{-2x}$$

now $u = \int - \frac{x e^{-x} \, e^{-x} \log x}{e^{-2x}} dx$

$= -\int x \log x \, dx$

$= -\frac{x^2}{2} \log x + \int \frac{x^2}{2x} dx$ ... using by-parts method

$= -\frac{x^2}{2} \log x + \frac{x^2}{4}$

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$\& \; v = \int \frac{e^{x} \, e^{x} \log x}{e^{2x}} \, dx$

$= \int \log x \, dx$

$= \log x \cdot x - \int \frac{x}{x} \, dx$ .. by using by-parts

$= x \log x - x$.

put values of u & v in eqn (3)
we get

$y_p = \left(-\frac{2x^2 \log x + x^2}{4} \right) e^{-x} + (x \log x - x)\, x e^{-x}$

$y_p = \frac{(2\log x + 1)}{4} x^2 e^{-x} + (\log x - 1)\, x^2 e^{-x}$ — (B)

from eqn (A) & (B)
solution of differential eqn can be given as

$y = y_c + y_p$

$= c_1 e^{-x} + c_2 x e^{-x} + \frac{(2\log x + 1)}{4} x^2 e^{-x} + (\log x - 1) x^2 e^{-x}$.

$= c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x} \left(\frac{2\log x + 4\log x + 1 - 4}{4} \right)$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + \frac{x^2 e^{-x}}{4} (2\log x - 3)}$$

Ans

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Q 7. Solve $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

→ let $x = e^t$ ie $t = \log x$ — (1)

hence,

$x\frac{dy}{dx} = Dy$ & $x^2\frac{dy}{dx} = D(D-1)y$ where $D = \frac{d}{dt}$

substitute values in the given eq^n.

i.e.

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 4y = x^2$$

substituting values we get,

$$D(D-1)y - 2Dy - 4y = e^{2t}$$

$$(D^2 - 3D - 4)y = e^{2t} \quad \text{— (2)}$$

eq^n (2) is a 2^nd order D.E with const coefficients.

Complementry eq^n:

$$(D^2 - 3D - 4)y = 0.$$

Auxiliary eq^n is $m^2 - 3m - 4 = 0$

solutions of auxiliary eq^n $4, -1$

∴ sol^n of complementry DE can be given as

$$y_c = c_1 e^{4t} + c_2 e^{-t} \quad \text{— (A)}$$

Particular eq^n:

$$y_p = \frac{e^{4t}}{(D^2 - 3D - 4)}$$

$$= \frac{e^{2t}}{4 - 3(2) - 4} \quad \cdots \frac{e^{at}}{f(D)} = \frac{e^{at}}{f(a)}$$

$$= \frac{-e^{2t}}{6} \quad \text{— (B)}$$

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$= \dfrac{\cancel{2}te^{4t}}{2D-3}$ ...... case of failure

$= \dfrac{te^{4t}}{2(4)-3}$ ...... $\dfrac{e^{at}}{f(D)} = \dfrac{e^{at}}{f(a)}$

$= \dfrac{te^{4t}}{5}$ —— (B)

Solution of given DE can be written as

$y = y_c + y_p$

substituting values from eqn (A) & (B)

$y = c_1 e^{4t} + c_2 e^{-t} + \dfrac{te^{4t}}{5}$

here $t = \log x$ (G)

$\therefore x = e^t$

$y = c_1 x^4 + c_2 x^{-1} + \dfrac{\log x}{5} \cdot x^4$

$$\boxed{y = c_1 x^4 + \dfrac{c_2}{x} + \dfrac{x^4 \log x}{5}}$$

Ans

Q8 Solve: $\dfrac{x\,dx}{z^2 - 2yz - y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

→

Given eqn $\dfrac{x\,dx}{z^2-2yz-y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

consider second inequality ie.

$\dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

$\dfrac{dy}{dz} = \dfrac{y+z}{y-z}$

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this is homogeneous D.E.

put $y = vz$

$\therefore \frac{dy}{dz} = v + z\frac{dv}{dz}$

$\therefore$ substitute values,

$$v + z\frac{dv}{dz} = \frac{vz+z}{vz-z}$$

$$v + z\frac{dv}{dz} = \frac{v+1}{v-1}$$

$$z\frac{dv}{dz} = \frac{v+1}{v-1} - v$$

$$z\frac{dv}{dz} = \frac{v+1 - v^2 + v}{v-1}$$

$$z\frac{dv}{dz} = \frac{1+2v-v^2}{v-1}$$

$$\frac{(v-1)}{1+2v-v^2}\,dv = \frac{dz}{z}$$

integrating both sides,
we get.

$$\int \frac{(v-1)}{(1+2v-v^2)}\,dv = \int \frac{dz}{z}$$

$$-\frac{1}{2}\int \frac{(2-2v)\,dv}{(1+2v-v^2)} = \ln z + c$$

$\therefore$ c = const of integrat?

$$-\frac{1}{2}\ln(1+2v-v^2) = \ln z + c_1$$

put $v = y/z$

we get $\therefore -\frac{1}{2}\ln\left(1 + \frac{2y}{z} - \frac{y^2}{z^2}\right) = \ln z + c_1$

rearranging $\ln\left(\frac{z^2+2yz-y^2}{z^2}\right) = \ln\left(\frac{1}{z^2}\right) + c_1$

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$$\boxed{z^2+2yz-y^2=c_1} \quad \text{—} ①$$

Consider multipliers $1, y, z$.

$= l(z^2-2yz-y^2)+m(y+z)+n(y-z)$

$= 1(z^2-2yz-y^2)+y(y+z)+z(y-z)$

$= z^2-2yz-y^2+y^2+yz+zy-z^2$

$= 0$

$\therefore \quad x\,dx+y\,dy+z\,dz=0$

On integrating we get

$$\int x\,dx+\int y\,dy+\int z\,dz = C$$

$$\boxed{x^2+y^2+z^2=2C=C_2} \quad \text{—} ②$$

From eqn ① & ② we get solution of given simultaneous equations.