BASICS OF PROGRAMMING ASSIGNMENT - 2

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CHAPTER III MISCELLANEOUS EXAMPLE-VI Q.23

Show that the area of the triangle formed by the lines whose equations a$_s + b_y + c_s = 0, (s = 1, 2, 3) is (\Delta)^2/(2C1C2C3) where \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

SOLUTION

Clearly, we can scale the coefficients of a given linear equation by any (non-zero) constant and the result is unchanged. Therefore, by dividing-through by

$$\sqrt{a_i^2 + b_i^2}$$

, we may assume our equations are in "normal form":

$$x \cos\theta + y \sin\theta - p = 0 \tag{1}$$

$$x \cos\phi + y \sin\phi - q = 0 \tag{2}$$

$$x \cos\psi + y \sin\psi - r = 0 \tag{3}$$

with θ, ϕ, ψ and p, q, r (and A, B, C and a, b, c) as in the figure 2:

Then, C$_1 = \begin{vmatrix} \cos\phi & \sin\phi \\ \cos\psi & \sin\psi \end{vmatrix} =$
$\sin\psi\cos\phi - \cos\psi\sin\phi = \sin(\psi - \phi) =$
$\sin\angle ROQ = \sin A$

Likewise,

C$_2 = \sin B \qquad C_3 = \sin C$

Moreover,

D := $\begin{vmatrix} \cos\theta & \sin\theta & -p \\ \cos\phi & \sin\phi & -q \\ \cos\psi & \sin\psi & -r \end{vmatrix} =$
- $(pC_1 + qC_2 + rC_3) = -(\ p\sin A + q\sin B + r\sin C\ )$

Writing d for the circumdiameter of the triangle, the Law of Sines tells us that,
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = d$

Therefore,

$$D = -\left(\frac{ap}{d} + \frac{bq}{d} + \frac{cr}{d}\right) \tag{4}$$

$$= -\frac{1}{d}\,(\ ap + bq + cr\ ) \tag{5}$$

$$= -\frac{1}{d}\,(\ 2|\triangle COB| + 2|\triangle AOC| + 2|\triangle BOA|\ ) \tag{6}$$

$$= -\frac{2\ |\triangle ABC|}{d} \tag{7}$$

Also,
C$_1C_2C_3 = \sin A \sin B \sin C = \dfrac{a}{d}\dfrac{b}{d}\sin C$
= $\dfrac{2\ |\triangle ABC|}{d^2}$

Finally:

$$\frac{\Delta^2}{2C_1C_2C_3} = \frac{4\ |\triangle ABC|^2/d^2}{4\ |\triangle ABC|/d^2} = |\triangle ABC|$$

Hence Proved.

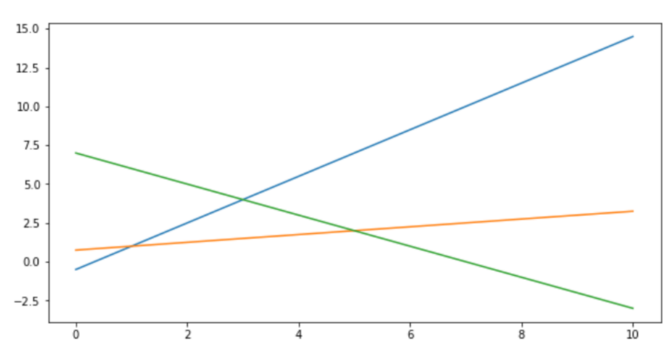

Figure 1: Actual triangle

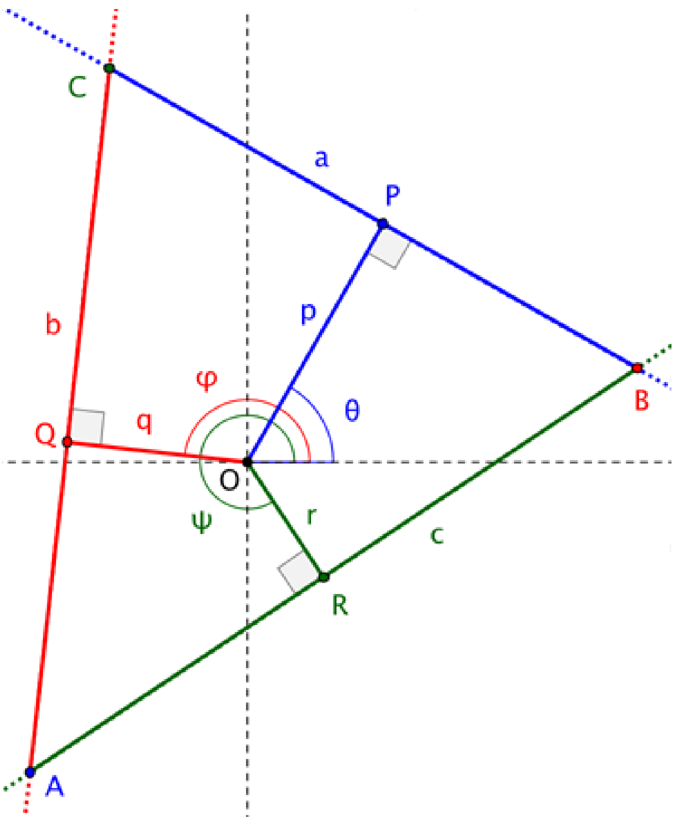

Figure 2: Constructions done on above triangle for proof