**Instructions:**

We will practice some nonlinear optimization in this lab. In many real applications, the functions associated with an optimization problem can be nonlinear. Nonlinear optimization is usually more difficult and error-prone as compared to linear optimization.

Generally speaking, there are no practical methods to verify whether a given point is an optimal solution. Most solvers iteratively converge to a point that satisfies 'Karush-Kuhn-Tucker' (KKT) conditions. SNOPT is one such solver. If the problem is convex or if certain other conditions are satisfied, then the KKT point is also an optimal point. In this session, we will encounter convex functions in the first two exercises and nonconvex in the remaining two.

In this lab, we will model some problems that require nonlinear objective function or constraints. Just like we used Gurobi or CPLEX to solve linear problems, we can use the option

```
option solver snopt;
```

to ensure that AMPL calls SNOPT to solve the problem. One can write nonlinear functions of constraints and objective in the same way as linear functions. The nonlinear expressions in AMPL are written just like you would write them on paper. $x*y$ denotes product of two variables. Similarly $x/y, \ln(x), x^\wedge y, \cos(x), \sin(x)$ have the usual meanings. `sqrt(x)` is used to denote the square-root of x.

There are 4 exercise problems in this lab. Solve all the problems. One of them is a practice problem; you must include the results for practice problem as well in the report though they will not be considered for evaluation. Clarifications will be provided for all questions by the instructor/TAs.

**Exercise 1: Unconstrained Nonlinear Optimization.** [**5 marks**]
You are assigned the task of placing a reactor at an optimal position in an alloy manufacturing plant. The reactor receives through pipes eight different raw materials from feeders distributed spatially in the plant. The coordinates of the eight feeders along the three spatial axis are

| Feeder | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Coordinates | (0.5,0,0) | (10,10,0) | (0,0,5) | (2,10,0) | (3,0,15) | (8,0,8) | (0,12,0) | (15,1,0) |
| Quantity | 6 | 9 | 3 | 12 | 5 | 8 | 8 | 5 |

The last row denotes the quantity of material that needs to be moved from each feeder to the reactor for each production-run. Your goal is to find the location such that the total distance travelled by all the material is minimized.

1. [**R**] Formulate this problem as an unconstrained nonlinear optimization problem.
2. Model this problem in AMPL and solve using SNOPT.
3. [**R**] Report the optimal solution and the number of iterations the solver took to solve it.
4. [**R**] What is the minimum quantity that should flow from feeder-B so that the optimal location of the main reactor is within 4 units of feeder-B?

**Exercise 2: Pipeline Layout.** [**8 marks**]
Sonu owns a rather irregular piece of farm. It is a polyhedron with four corners, with his house at one of the corners. The second corner is 100 meters to the west and 100 meters to the south of his house. The third one is 200 meters south of his house. The fourth one is 150 meters east and 50 meters south of his house. There is a circular lake of radius 30 meters whose center is 200 meters north and 200 meters east of his house. He wants to lay a pipeline from the lake to his farm.

1. [**R**] Write a mathematical program to find the two end points of the shortest possible pipeline. Let $(xf, yf)$ and $(x\ell, y\ell)$ be the end points at the farm side and the lake side respectively. Remember that $(xf, yf)$ must lie inside the farm and $(x\ell, y\ell)$ inside the lake.

2. Model the problem in AMPL and solve using SNOPT solver. **Hint:** If you see an error message of type: `can't evaluate sqrt'(0).`, then it means AMPL can not evaluate derivatives at some iteration. You should try to solve by specifying a different starting point to the solver by using commands like

   ```
   let xf := 1;
   let yf := 0.5;
   ```

   and similarly for $(x\ell, y\ell)$. Remember that the default starting point value for a variable is 0, and sqrt() function is not differentiable when its value is 0.

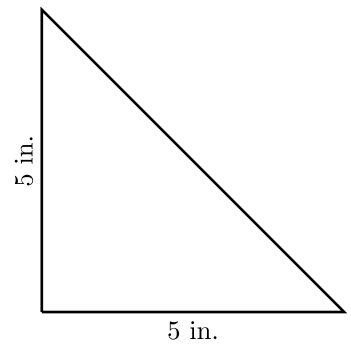
3. [**R**] Report the locations of the two ends.

4. [**R**] Sonu also wants to have a small pipe coming into his store-house which is 15m east and 20m north of his house. He wants to lay a 'Y' shaped pipe one end of which is in the lake and the others are in the store-house and the farm. The pipe from the junction to the store-house is thinner and its unit-cost is only 20% of that of the pipe from lake to junction and that from the junction to the farm. Write a mathematical program to find the locations of the ends in the farm and the lake, and also of the junction.

5. Model the problem in AMPL and solve using SNOPT solver.

6. [**R**] Report the locations of the two ends.

**Exercise 3: Packing carrom coins.** [**8 marks**]
Guru Sports manufactures the popular indoor game of Carrom. Each carrom coin is a circle of $\frac{2}{3}$ inch radius (we can assume for now that they have negligible height). To make their packing attractive they have designed a new box for their coins. The box looks like the right-angled triangle below, with all lengths in inches.




We want to find whether we can pack 2 coins into this pack. Let $(x[i], y[i])$ be the coordinates of center of the $i-$th coin.

1. [**R**] First, write linear constraints to force each coin to lie inside the triangular box.

2. [**R**] Next, write nonlinear constraints that forbid the coins from overlapping each other.

3. Write your model in AMPL. Use $x[i], y[i]$ as coordinates of the center of the $i-$th coin. Keep $R$, radius and $n$, the number of coins as parameters. Solve your problem of fitting two coins using SNOPT to test whether you find a feasible solution.

4. If SNOPT give a message 'Nonlinear infeasibilities minimized', it means that it could not find a feasible solution. It is a sign that the feasible region of the problem is not convex.

5. [**R**] Explain why the feasible region of your problem is not convex.

6. [**R**] We can use statements like

   ```
   let x[1] := 1;
   let x[2] := 3.5;
   ```

   to give initial values to our variables. Put these statements before the solve command. When the problem is not convex, a solver may converge to different points depending upon the starting point. Try different starting values of x, y in your model to obtain a feasible solution. Report the feasible solution.

7. [**R**] Now solve the problem for 4, 6, 8, 10 and 12 coins. In each case report if you can find a solution.

**Exercise 4** [**Practice Problem; not for evaluation. Include results in report.**]
Consider the optimization problem in two variables.

$$\begin{aligned}
&\min \frac{x_1}{3.0+\sin(x_1)} + \frac{x_2}{3.0+\cos(x_2)} \\
&\text{s.t. } \big((x_1-20)^2 + (x_2-20)^2 - 2\big) \times \big((x_1+20)^2 + (x_2+20)^2 - 5\big) \le 0;
\end{aligned}$$

1. [**R**] Write the problem in AMPL and solve it using SNOPT solver. Report the solution value and the solution.
2. [**R**] Solve the problem using eight different starting points. All the six points should be selected from the intervals $-40 \le x_1 \le 40$ and $-40 \le x_2 \le 40$. They should be distributed roughly randomly in these intervals. Report the solution values.
3. [**R**]Explain why we are observing different 'optimal solution values' for the same problem.