Instructions:

We will practice modeling optimization problems with integer variables in this lab. Integer optimization problems are those where some or all variables can take only integer values in the solution. When all the constraints and the objective function in a problem are linear, we call it an Integer Linear Program or Mixed-Integer Linear Program (MILP). MILPs are used to model many problems, and have found even more applications than linear optimization.

There are only 3 exercises in this lab. Try to solve all problems on your own. If you have difficulties, ask the instructor or TAs. Clarifications will be provided to all questions.

Exercise 1: LP Vs MILP [12 marks]
The simplest application of MILP is when the variables model discrete objects that can only take whole-number values (e.g. number of aeroplanes to manufacture). A commander in Indian Airforce needs to purchase aircraft for transporting goods and people. She wants to have maximum possible carrying capacity. Five different types of aircraft are available. Their capacities are $2, 1, 2, 1, 2$ units per aircraft, respectively. The budget is limited to 13 units (say, crores of rupees). The costs per aircraft are $3, 3, 2, 2, 3$ units, respectively. In addition, parking space is limited to 11 units. The space requirements per aircraft are $2, 1, 3, 1, 2$ units, respectively.

1. [**R**] Write a mathematical model to find the number of aircraft of each type that can be bought in order to maximize the goal. Mention which variables must be integers (We usually replace the notation $x \in \mathbb{R}$ by $x \in \mathbb{Z}$ to denote that variable x is integer).

2. Write your model in AMPL and solve it using GUROBI solver. Remember to specify which variables are integers. In AMPL, the format is:

```
# Variable x1 must have integer values.
var x1 >=0, integer;
```

   or, when using sets,

```
set S;
# All variables ord_q indexed by set S must have integer values.
var ord_q{S} >=0, integer;
```

3. [**R**] Report the optimal value and the solution.

4. [**R**] Let us now compare it to a linear program. Suppose we remove the restrictions that the variables in the above problem are integers. Solve this modified problem and report the solution and the objective value.

5. [**R**] Can the solution of the MILP be obtained by merely rounding the solution of the LP? Why or why not?

6. [**R**] Suppose now we are interested in finding how the solution changes when the right-hand-sides in our constraints change. Suppose the limit on budget is increased to 33 units and the parking space is increased to 21 units, how will the new LP objective value change? How does the new MILP objective value change?

7. [**R**] Now try decreasing limit on budget from 33 units to 31, 30, 29 and 28 units. Correspondingly, increase the parking space to 22, 24, 26 and 28 units. Solve the LP and MILP for each of these four possiblities and comment on the pattern seen in objective values of the LP and MILP.

Exercise 2: Time taken in MILP. [**8 marks**] In this exercise we will compare the time taken to solve LP and MILP. Consider this integer optimization problem

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ \texttt{s.t.} \ & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad i = 1, \dots, m \\ & x_j \in \{0,1\}, \quad j = 1 \dots, n. \end{aligned} \tag{MILP}$$

and its LP relaxation

$$\begin{aligned} \max \ & \sum_{j=1}^{n} c_j x_j \\ \texttt{s.t.} \ & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \dots, m \\ & 0 \le x_j \le 1, \quad j = 1 \dots, n. \end{aligned} \tag{LP}$$

Specific data is available for this problem in Moodle in lp_ip.dat file. It has $n = 500$ variables and $m = 5$ constraints.

1. Modify the data file suitably, and write an AMPL model using sets/parameters than can be used with the data file.
2. [**R**] Solve the MILP using CPLEX solver or Gurobi. Report the optimal objective function value (Do not report the values of variables).
3. [**R**] How much time does it take to solve the MILP? And how many simplex iterations did you observe? To report time, you can type the command

   ```
   option cplex_options ‘timing 1’;
   ```

   or

   ```
   option gurobi_options ‘timing 1’;
   ```

   depending upon your solver. Use this command before the ‘solve’ command and see the time next to the ‘Solve’ header.
4. [**R**] Solve the LP relaxation (LP obtained by removing integer restrictions on all variables) of the same problem using CPLEX or Gurobi. Report only the objective function values of the optimal solution. (You need not report the values of the variables.)
5. [**R**] How much time does it take to solve the problem? And how many simplex iterations did you observe? Compare with the results obtained by solving the MILP.

Exercise 3: Modeling fixed cost using binary variables. [**15 marks**]
A common use of integer variables is to model situations where we make discrete choices. Consider the following scenario. Bharath Steel Works wants to minimize the annual cost of producing steel wires in a rented workshop. There are two costs associated with the production. There is a fixed cost that is incurred if the production facility is rented in a month. If no production is carried out, then this fixed cost is zero; otherwise (independent of the quantity produced) it is ₹500000. The second component of cost associated with production depends on the quantity of wires produced. It changes every month depending upon the market conditions and is given in table below as variable cost. In addition, there is a cost of storing the wire (₹45 per unit length per month). Wire produced in a month can be sold in the same month (then there is no storage cost) or it can be stored and sold in the future. The monthly demands are given in the table below. These must be met every month. Assume that the starting inventory is zero and the inventory after twelve months should also be zero. The production can not exceed 15000 units in a month. Any amount of inventory can be stored.

Month	Variable Cost	Demand		Month	Variable Cost	Demand
1	250	2000		7	200	8000
2	300	1000		8	400	1000
3	700	4000		9	200	2000
4	500	5000		10	250	4000
5	450	500		11	350	3000
6	190	3000		12	150	6000

1. [**R**] First formulate a problem of minimizing the total cost incurred in the year ignoring the fixed cost of using the production facility.

2. [**R**] Solve the problem for the given data and report the objective function value and the production quantity for each month.

3. [**R**] Now model the fixed cost using binary variables. For each month introduce a variable y_i ($i = 1, \dots, 12$) which is 0 if no production is carried out and 1 if there is some production. Notice that when y_i is zero, the variable denoting the production in i^{th} month can only be zero, and when y_i is one, the variable denoting the production in i^{th} month can take any value in $[0, 15000]$. How will you write this constraint using a linear inequality?

4. [**R**] Write the mathematical model in your report.

5. [**R**] Solve the problem and report the solution. How many months have no production?