C H A P T E R

5

More on Optimum Design Concepts:

Optimality Conditions

5.1

Answer True or False.

1. A convex programming problem always has a unique global minimum point. *False*
2. For a convex programming problem, KKT necessary conditions are also sufficient. *True*
3. The Hessian of the Lagrange function must be positive definite at constrained minimum points. *False*
4. For a constrained problem, if the sufficiency condition of Theorem 5.2 is violated, the candidate point $\mathbf{x}^*$ may still be a minimum point. *True*
5. If the Hessian of the Lagrange function at $\mathbf{x}^*$, $\nabla^2 L(\mathbf{x}^*)$, is positive definite, the optimum design problem is convex. *False*
6. For a constrained problem, the sufficient condition at $\mathbf{x}^*$ is satisfied if there are no feasible directions in a neighborhood of $\mathbf{x}^*$ along which the cost function reduces. *True*

5.2

Formulate the problem of Exercise 4.84. Show that the solution point for the problem is not a regular point. Write KKT conditions for the problem, and study the implication of the irregularity of the solution point. Refer to solution of Exercises 4.84:

A refinery has two crude oils:

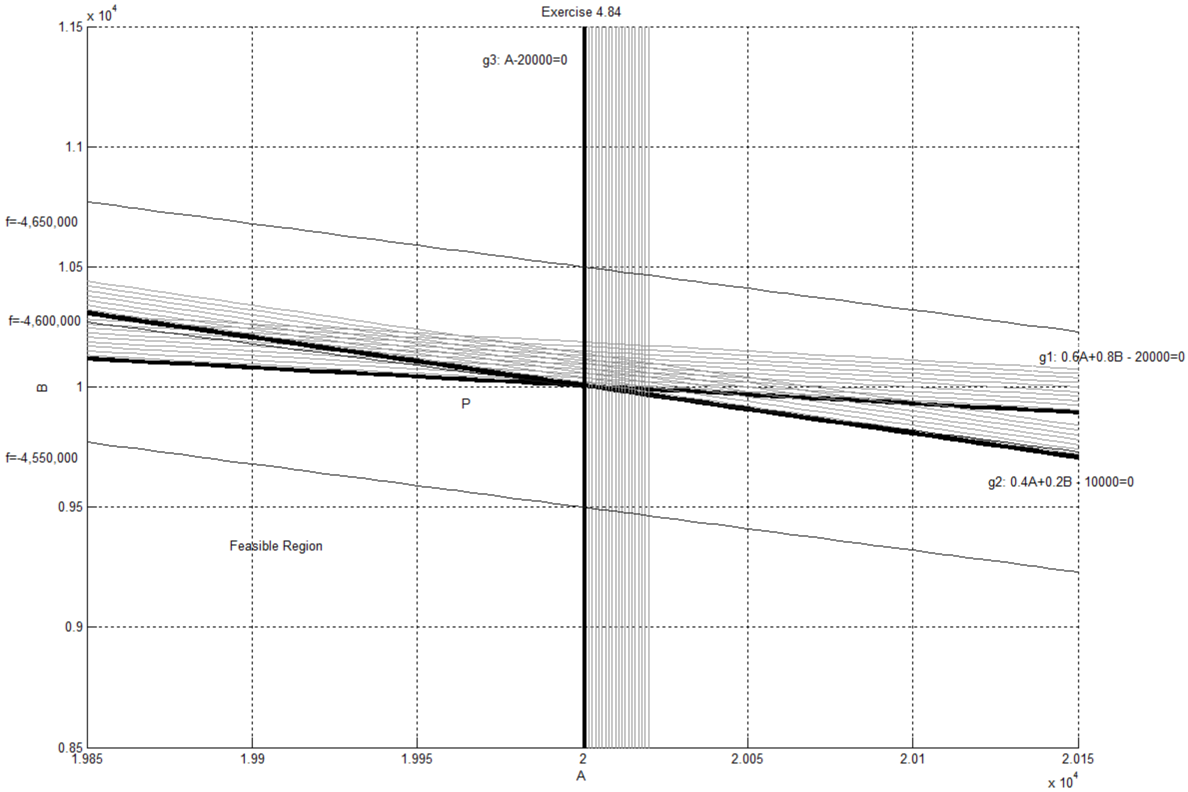
1. Crude *A* costs \$120/barrel (bbl) and 20,000bbl are available.
2. Crude *B* costs \$150/bbl and 30,000 are available.

The company manufactures gasoline and lube oil from the crudes. Yield and sale price barrel of the product and markets are shown in Table E2.2. How much crude oils should the company use to maximize its profit? Formulate the optimum design problem.

Table E2.2 Data for Refinery Operation

Product	Yield/bbl Crude *A*	Yield/bbl Crude *B*	Sale Price per bbl ($)	Market (bbl)
Gasoline	0.6	0.8	200	20,000
Lube oil	0.4	0.2	450	10,000

Solution



Note : $g4 = B - 30000, g5 = -A$ and $g6 = -B$ are not shown on the graph.

According to the graphical solution, the point P (20000, 10000) is the minimum point with *f** = -4,600,000.

Referring to the formulation in Exercise 2.2, we have
Minimize $f = -180A - 100B$
subject to:

$g_1 = 0.6A + 0.8B - 20{,}000 \le 0$, (gasoline market)

$g_2 = 0.4A + 0.2B - 10{,}000 \le 0$ (lube oil market)

$g_3 = A - 20{,}000 \le 0$

$g_4 = B - 30{,}000 \le 0$

$g_5 = -A \le 0; \quad g_6 = -B \le 0$

$L = (-180A - 100B) + u_1(0.6A + 0.8B - 20{,}000) + s_1^2 + u_2(0.4A + 0.2B - 10{,}000 + s_2^2) +$

$u_3\left(A-20{,}000+s_3^2\right)+u_4\left(B-30{,}000+s_4^2\right)+u_5\left(-A+s_5^2\right)+u_6\left(-B+s_6^2\right)$

$\partial L/\partial A=-180+0.6u_1+0.4u_2+u_3-u_5=0$

$\partial L/\partial B=-100+0.8u_1+0.2u_2+u_4-u_6=0$

$g_i+s_i^2=0,\quad u_is_i=0,\quad u_i\ge 0;\quad i=1\text{ to }6$

There are 64 cases because we have six inequality constraints. We shall examine three cases. Two of them yield solutions.

Case 1. $u_2=u_4=u_5=u_6=0, s_1=s_3=0$. The solution is given as A = 20,000, B = 10,000, u_1 = 125, u_3 = 105; s_2 = 0, so g_2 is also active.

Case 2. $u_3=u_4=u_5=u_6=0, s_1=s_2=0$. The solution is given as A = 20,000, B = 10,000, u_1 = 20, u_2 = 420; s_3 = 0, so g_3 is also active.

Case 3. $u_1=u_4=u_5=u_6=0, s_2=s_3=0$. The solution is given as A = 20,000, B = 10,000, u_2 = 500, $u_3=-20$ (violation); so this case does not give a solution.

The candidate minimum point derived in Case 1 and Case 2 can be verified graphically. It is seen that $(20000, 10000)$ is the optimum point. At this optimum point, g_1, g_2 and g_3 (constraints on gasoline and lube oil markets, and limit on crude A) are all active. The optimum cost is $f=$ -4,600,000.. The optimum point is irregular, since there are three active constraints and two design variables. The Lagrange multipliers are not unique.

5.4

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 - 4 = 0$

Solution

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ subject to $h = x_1 + x_2 - 4 = 0$;

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + v(x_1 + x_2 - 4)$;

The necessary conditions give

$\partial L/\partial x_1 = 8x_1 - 5x_2 - 8 + v = 0$; $\partial L/\partial x_2 = 6x_2 - 5x_1 + v = 0$; $h = x_1 + x_2 - 4 = 0$

The solution of these equations is $x_1 = 13/6, x_2 = 11/6, v = -1/6.$

Therefore, (13/6, 11/6) is a KKT point; $f = -25/3$

Check for regularity: $\nabla h = (1, 1)$. Since ∇h is the only vector, regularity of feasible points is satisfied.

Referring to Exercises 4.43/4.97, the point satisfying the KKT necessary conditions is $x_1 = 2.166667$, $x_2 = 1.833333$, $v = -0.166667$, $f = -8.3333$. This point also satisfies second order sufficiency condition for a local minimum. Hessian of the Lagrangian function is calculated as

$$\mathbf{H} = \begin{bmatrix} 8 & -5 \\ -5 & 6 \end{bmatrix}$$

The Hessian is positive definite since $M_1 = 8 > 0$ *and* $M_2 = 8 \times 6 - (-5) \times (-5) = 23 > 0$. Therefore it is positive definite for those $\mathbf{d}$ that satisfy $\nabla h \cdot \mathbf{d} = 0$. Therefore the KKT point satisfies second order necessary and sufficient conditions for a local minimum point.

The problem is solved graphically in Exercise 3.12. The graphical solution confirms this conclusion.

5.5

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 - 4 = 0$

Solution

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ subject to $h = x_1 + x_2 - 4 = 0$;

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8x_1 + v(x_1 + x_2 - 4)$;

The necessary conditions give

$\partial L / \partial x_1 = -8x_1 + 5x_2 + 8 + v = 0$; $\partial L / \partial x_2 = -6x_2 + 5x_1 + v = 0$; $h = x_1 + x_2 - 4 = 0$

The solution of these equations is $x_1 = 13/6, x_2 = 11/6, v = 1/6$.

Therefore, (2.166667, 1.833333) is a KKT point; $F = -25/3$

Check for regularity: Ñh = (1, 1). Since Ñh is the only vector, regularity of feasible points is satisfied.

Referring to Exercises 4.44/4.98, the point satisfying the KKT necessary conditions is x_1 = 2.166667, x_2 = 1.833333, v = 0.166667. However this point violates second order necessary condition. Hessian of the Lagrangian function is calculated as

$$\mathbf{H} = \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix}$$

The Hessian is negative definite since $M_1 = -8 < 0$ *and* $M_2 = -8 \times -6 - (5) \times (5) = 23 > 0$. Therefore it is negative definite for those **d** that satisfy $\nabla h \cdot \mathbf{d} = 0$. Therefore the point violates the second order necessary conditions for a local minimum of $f(\mathbf{x})$, or for the local maximum of $F(\mathbf{x})$. There is no maximum point for $F(\mathbf{x})$.

The problem is solved graphically in Exercise 3.12. The graphical solution confirms this conclusion.

5.6

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 + 1)^2$
subject to $2x_1 + 3x_2 - 4 = 0$

Solution

Minimize $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 + 1)^2$ subject to h= $2x_1 + 3x_2 - 4 = 0$;

$L = (x_1 - 2)^2 + (x_2 + 1)^2 + v(2x_1 + 3x_2 - 4)$;

The KKT necessary conditions give

$\partial L/\partial x_1 = 2(x_1 - 2) + 2v = 0;\ \ \partial L/\partial x_2 = 2(x_2 + 1) + 3v = 0;\ \ \mathrm{h} = 2x_1 + 3x_2 - 4 = 0$

The solution of these equations is $x_1 = 32/13, x_2 = -4/13, v = -6/13.$

Therefore, (32/13, -4/13) is a KKT point; $f = 9/13$

Check for regularity: ∇h = (2, 3). Since ∇h is the only vector, regularity of feasible point is satisfied.

Referring to Exercises 4.45/4.99, the point satisfying the KKT necessary conditions is x_1 = 2.46154, x_2 = -0.307692, v = -0.46154, f = 0.69231. The point satisfies second order sufficiency condition.

5.7

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13$
subject to $x_1 - 3x_2 + 3 = 0$

Solution

Minimize $f(x_1, x_2) = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13$ subject to h $= x_1 - 3x_2 + 3 = 0$;
$L = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13 + v(x_1 - 3x_2 + 3)$
The KKT necessary conditions give

$$\frac{\partial \mathrm{L}}{\partial x_1} = 8x_1 - 4 + v = 0, \qquad \frac{\partial \mathrm{L}}{\partial x_2} = 18x_2 + 6 - 3v = 0, \qquad \frac{\partial \mathrm{L}}{\partial v} = x_1 - 3x_2 + 3 = 0$$

The solution of these equations is $x_1 = -0.4,\ x_2 = 2.6/3,\ v = 7.2$
Therefore, (-0.4, 2.6/3) is a KKT point; $f = 27.2$
Check for regularity: ∇h = (1,-3). Since ∇h is the only vector, regularity of feasible points is satisfied.

Referring to Exercises 4.46/4.100, the point satisfying the KKT necessary conditions is x_1 = -0.4, x_2 = 0.866667, v = 7.2, f = 27.2. The point satisfies second order sufficiency condition.

5.8

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x) = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2$
subject to $2x_1 + 3x_2 - 1 = 0$
$x_1 + x_2 + 2x_3 - 4 = 0$

Solution

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2$ subject to h1= $2x_1 + 3x_2 - 1$; h2= $x_1 + x_2 + 2x_3 - 4$

$L = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2 + v_1(2x_1 + 3x_2 - 1) + v_2(x_1 + x_2 + 2x_3 - 4);$

The KKT necessary conditions give

$\partial L/\partial x_1 = 2(x_1 - 1) + 2v_1 + v_2 = 0;$ $\partial L/\partial x_2 = 2(x_2 + 2) + 3v_1 + v_2 = 0;$ $\partial L/\partial x_3 = 2(x_3 - 2) + 2v_2 = 0;$

$h_1 = 2x_1 + 3x_2 - 1 = 0;$ $h_2 = x_1 + x_2 + 2x_3 - 4 = 0$

The solution of these equations is $x_1 = 1.71698, x_2 = -0.81132, x_3 = 1.547170057$

$v_1 = -0.943396132, v_2 = 0.452829943$

Therefore, (1.71698066, -0.811320724, 1.547170057) is a KKT point; $f = 2.1318$

Check for regularity: Gradients of the constraints are linearly independent; therefore the point is a regular point of the feasible set.

Referring to Exercises 4.47/4.101, the point satisfying the KKT necessary conditions is x_1 = 1.71698, x_2 = -0.81132, x_3 = 1.54714, v_1 =-0.943396, v_2 =0.4528299, f = 2.11318. The point satisfies second order sufficiency condition.

5.9

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 - 16 = 0$

Solution

Minimize $f(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4$ subject to $h = x_1^2 + x_2^2 + 2x_1 - 16 = 0$

$L(x_1, x_2, v) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4 + v(x_1^2 + x_2^2 + 2x_1 - 16)$

$\partial L/\partial x_1 = 18x_1 + 18x_2 + 2vx_1 + 2v = 0$; $\partial L/\partial x_2 = 18x_1 + 26x_2 + 2vx_1 = 0$**;**

$h = x_1^2 + x_2^2 + 2x_1 - 16 = 0$

These equations are nonlinear, which can be solved numerically. Using any nonlinear equation solver, we can find the following KKT points:

1. $x_1 = 1.5088, x_2 = 3.2720, v = -17.151503, f = 244.528$
2. $x_1 = 2.5945, x_2 = -2.0198, v = -1.4390, f = 15.291$
3. $x_1 = -3.630, x_2 = -3.1754, v = -23.2885, f = 453.154$
4. $x_1 = -3.7322, x_2 = 3.0879, v = -2.1222, f = 37.877$

Check for regularity: $\nabla h = (2x_1 + 2,\ 2x_2)$. Since ∇h is the only vector, regularity of feasible points is satisfied for each KKT point.

Refer to Exercises 4.48/4.102:

The points $x_1 = -3.7322, x_2 = 3.0879$; and $x_1 = 2.5945, x_2 = -2.0198$ satisfy second order sufficiency conditions and are isolated local minimum points. The other two KKT points violate the second order necessary conditions.

5.10

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 = 0$

<u>Solution</u>

Minimize $f = (x_1 - 1)^2 + (x_2 - 1)^2$ subject to $h = x_1 + x_2 - 4 = 0$

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 + x_2 - 4)$; the KKT necessary conditions are

$\partial L/\partial x_1 = 2(x_1 - 1) + v = 0;\ \partial L/\partial x_2 = 2(x_2 - 1) + v = 0;\ h = x_1 + x_2 - 4 = 0$

Solution of these equations is $x_1 = 2, x_2 = 2, v = -2$. Therefore, $(2, 2)$ is a KKT point; $f = 2$.

Check for regularity: $\nabla h = (1, 1)$. Since ∇h is the only vector, regularity of feasible points is satisfied.

Refer to Exercises 4.49/4.103

$(2, 2)$ is a KKT point; $f = 2$. The point satisfies second order necessary conditions.

5.11

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 = 4$

Solution

Minimize $f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$ subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 + v\left(x_1 + x_2 - 4\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = 8x_1 - 5x_2 + v = 0;\ \partial L/\partial x_2 = 6x_2 - 5x_1 + v = 0;\ \mathrm{h} = x_1 + x_2 - 4 = 0$

Solution of these equations is $x_1 = 11/6,\ x_2 = 13/6,\ v = -23/6$.

Therefore, $(11/6,\ 13/6)$ is a KKT point; f^* = -1/3.

Check for regularity: $\nabla \mathrm{h} = (1,\ 1)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible point is satisfied.

Referring to Exercises 4.51/4.105, the point satisfying the KKT necessary conditions is x_1 = 1.83333, x_2 = 2.16667, v = -3.83333, f = -0.33333. The point satisfies second order sufficiency condition. The sufficiency check is same as in Exercise 5.5.

5.12

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 = 4$

Solution

Minimize $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8$ subject to $h = x_1 + x_2 - 4 = 0$

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8 + v(x_1 + x_2 - 4)$

The KKT necessary conditions are

$\partial L/\partial x_1 = -8x_1 + 5x_2 + v = 0;\ \partial L/\partial x_2 = -6x_2 + 5x_1 + v = 0;\ h = x_1 + x_2 - 4 = 0$

Solution of these equations is $x_1 = 11/6, x_2 = 13/6, v = 23/6$.

Therefore, $(11/6, 13/6)$ is a KKT point; F = -1/3.

Check for regularity: $\nabla h = (1, 1)$. Since ∇h is the only vector, regularity of feasible point is satisfied.

Referring to Exercises 4.52/4.106, the point satisfying the KKT necessary conditions is x_1 = 1.83333, x_2 = 2.16667, v = 3.83333, f = 0.33333. This point violates second order necessary condition. The check for the second order conditions is the same as in Exercise 5.6. $F(\mathbf{x})$ has no maximum points.

5.13

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 \leq 4$

Solution

Minimize $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8$ subject to $g = x_1 + x_2 - 4 \leq 0$

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8 + u\left(x_1 + x_2 - 4 + s^2\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = -8x_1 + 5x_2 + u = 0;\ \partial L/\partial x_2 = -6x_2 + 5x_1 + u = 0;$

$\partial L/\partial u = x_1 + x_2 - 4 + s^2 = 0; \partial L/\partial s = 2us = 0$

Case 1. $u = 0$; gives a KKT point as (0, 0); $F^* = -8$.
Case 2. $s = 0$ (or g = 0); gives a KKT point as (11/6, 13/6); $u^* = 23/6$, $F^* = -1/3$.
Check for regularity: $\nabla g = (1, 1)$. Since there is only one constraint, regularity is satisfied.

Referring to Exercises 4.54/4.107, the points satisfying the KKT necessary conditions are: $x_1 = 0$, $x_2 = 0$, u = 0; $x_1 = 1.83333$, $x_2 = 2.16667$, u = 3.83329. Both points violate second order necessary condition. The check for the second order conditions is the same as in Exercise 5.6. $F(\mathbf{x})$ has no maximum points.

5.14

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 \leq 4$

Solution

Minimize $f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$ subject to $g = x_1 + x_2 - 4 \leq 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 + u\left(x_1 + x_2 - 4 + s^2\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = 8x_1 - 5x_2 + u = 0;\ \partial L/\partial x_2 = 6x_2 - 5x_1 + u = 0;$

$\partial L/\partial u = x_1 + x_2 - 4 + s^2 = 0; \partial L/\partial s = 2us = 0$

Case 1. $u = 0$; gives a KKT point as (0, 0); $f\left(\mathbf{x}^*\right) = -8$.

Case 2. $s = 0$; gives no candidate point. (u < 0)

Check for regularity: $\nabla g = (1, 1)$. Since there is only one constraint, regularity is satisfied.

Referring to Exercises 4.55/4.108, the point satisfying the KKT necessary condition are:
$x_1 = 0$, $x_2 = 0$, u = 0. The point satisfies second order sufficiency condition.

5.15

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 \le 4$

Solution

Minimize $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8x_1$ subject to $g = x_1 + x_2 - 4 \le 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + u\left(x_1 + x_2 - 4 + s^2\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = -8x_1 + 5x_2 + 8 + u = 0;\ \partial L/\partial x_2 = -6x_2 + 5x_1 + u = 0;$

$\partial L/\partial u = x_1 + x_2 - 4 + s^2 = 0; \partial L/\partial s = 2us = 0$

Case 1. $u = 0$; gives a KKT point as (48/23, 40/23); $F\left(\mathbf{x}^*\right) = -192/23$.

Case 2. $s = 0$; gives a KKT point as (13/6, 11/6); $u^* = 1/6$, $F\left(\mathbf{x}^*\right) = -8.33333$.

Check for regularity: Ñg $= (1, 1)$. Since there is only one constraint, regularity is satisfied.

Referring to Exercises 4.56/4.109, the points satisfying the KKT necessary conditions are: $x_1 = 2.08696$, $x_2 = 1.73913$, u = 0; $x_1 = 2.16667$, $x_2 = 1.83333$, u = 0.16667. Both points violate second order necessary condition.

5.16

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 \geq 4$
$x_1 - x_2 - 2 = 0$

Solution

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $\mathrm{h} = x_1 - x_2 - 2 = 0$; $\mathrm{g} = -x_1 - x_2 + 4 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 - x_2 - 2) + u(-x_1 - x_2 + 4 + s^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) + v - u = 0$; $\partial L/\partial x_2 = 2(x_2 - 1) - v - u = 0$

$\mathrm{h} = x_1 - x_2 - 2 = 0$; $-x_1 - x_2 + 4 + s^2 = 0$; $us = 0$, $u \geq 0$.

Case 1. $u = 0$; no candidate minimum.

Case 2. $s = 0$; gives $(3, 1)$ as a KKT point with $v = -2$, $u = 2$, $f = 4$.

Since $\nabla \mathrm{h} = (1, -1)$ and $\nabla \mathrm{g} = (-1, -1)$ are linearly independent, regularity is satisfied.

Referring to Exercises 4.57/4.110, the point satisfying the KKT necessary condition is: $x_1 = 3$, $x_2 = 1$, ν = -2, u = 2. The point satisfies second order sufficiency condition.

5.17

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 = 4$
$x_1 - x_2 - 2 \geq 0$

Solution

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$; $\mathrm{g} = -x_1 + x_2 + 2 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 + x_2 - 4) + u(-x_1 + x_2 + 2 + s^2)$

$\partial L / \partial x_1 = 2(x_1 - 1) + v - u = 0$; $\partial L / \partial x_2 = 2(x_2 - 1) + v + u = 0$

$\mathrm{h} = x_1 + x_2 - 4 = 0$; $-x_1 + x_2 + 2 + s^2 = 0$; $us = 0$, $u \geq 0$.

Case 1. $u = 0$; no candidate minimum.

Case 2. $s = 0$; gives $(3, 1)$ as a KKT point with $v = -2$, $u = 2$, $f = 4$.

Since Ñ$\mathrm{h} = (1, 1)$ and Ñ$\mathrm{g} = (-1, 1)$ are linearly independent, regularity is satisfied.

Referring to Exercises 4.58/4.111, the point satisfying the KKT necessary condition is: $x_1 = 3$, $x_2 = 1$, $v = -2$, $u = 2$. The point satisfies second order sufficiency condition.

5.18

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 \geq 4$
$x_1 - x_2 \geq 2$

Solution

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $g_1 = -x_1 - x_2 + 4 \leq 0$; $g_2 = -x_1 + x_2 + 2 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(-x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 + x_2 + 2 + s_2^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) - u_1 - u_2 = 0$; $\partial L/\partial x_2 = 2(x_2 - 1) - u_1 + u_2 = 0$

$-x_1 - x_2 + 4 + s_1^2 = 0$; $-x_1 + x_2 + 2 + s_2^2 = 0$; $u_1 s_1 = 0$, $u_2 s_2 = 0$, $u_1, u_2 \geq 0$.

Case 1. $u_1 = 0$, $u_2 = 0$; no candidate minimum.

Case 2. $u_1 = 0$, $s_2 = 0$; no candidate minimum.
Case 3. $s_1 = 0$, $u_2 = 0$; no candidate minimum.
Case 4. $s_1 = 0$, $s_2 = 0$; gives $(3, 1)$ as a KKT point with $u_1 = 2, u_2 = 2, f = 4$.
Since $\nabla g_1 = (-1, -1)$ and $\nabla g_2 = (-1, 1)$ are linearly independent, regularity is satisfied.

Referring to Exercises 4.59/4.113, the point satisfying the KKT necessary condition is:
$x_1 = 3$, $x_2 = 1$, $u_1 = 2$, $u_2 = 2$. The point satisfies second order sufficiency condition.

5.19

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x, y) = (x-4)^2 + (y-6)^2$
subject to $12 \geq x + y$
$x \geq 6, y \geq 0$

Solution

Minimize $f(x,y) = (x-4)^2 + (y-6)^2$;

subject to $g_1 = x + y - 12 \leq 0$;

$g_2 = -x + 6 \leq 0$;

$g_3 = -y \leq 0$

$$L = (x-4)^2 + (y-6)^2 + u_1(x + y - 12 + s_1^2) + u_2(-x + 6 + s_2^2) + u_3(-y + s_3^2)$$

$\partial L/\partial x = 2(x-4) + u_1 - u_2 = 0$;

$\partial L/\partial y = 2(y-6) + u_1 - u_3 = 0$

$x + y - 12 + s_1^2 = 0$;

$-x + 6 + s_2^2 = 0$;

$-y + s_3^2 = 0$

$u_1 s_1 = 0$, $u_2 s_2 = 0$, $u_3 s_3 = 0$

$u_1, u_2, u_3 \geq 0$.

Case 1. $u_1 = u_2 = u_3 = 0$; no candidate minimum.

Case 2. $u_1 = u_2 = 0, s_3 = 0$; no candidate minimum.

Case 3. $u_1 = u_3 = 0, s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_2 = 4$, $s_1=0$, $s_3=6$, $f = 4$.

Case 4. $u_2 = u_3 = 0, s_1 = 0$; no candidate minimum.

Case 5. $u_1 = 0, s_2 = s_3 = 0$; no candidate minimum.

Case 6. $u_2 = 0, s_1 = s_3 = 0$; no candidate minimum.

Case 7. $u_3 = 0, s_1 = s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_1 = 0, u_2 = 4$, $s_3 = 6, f = 4$.

Case 8. $s_1 = s_2 = s_3 = 0$; no candidate minimum.

Referring to Exercises 4.60/4.108, the point satisfying the KKT necessary condition is:
$x = 6$, $y = 6$, $u_1 = 0$, $u_2 = 4$, $u_3 = 0$. The point satisfies second order sufficiency condition.

5.20

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$
subject to $x_1 + 3x_2 \le 6$
$5x_1 + 2x_2 \le 10$
$x_1, x_2 \ge 0$

Solution

Minimize $f(\mathbf{x}) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2;$

subject to $g_1 = x_1 + 3x_2 - 6 \le 0;$

$g_2 = 5x_1 + 2x_2 - 10 \le 0;\ g_3 = -x_1 \le 0;\ g_4 = -x_2 \le 0;$

$$L = \left(2x_1 + 3x_2 - x_1^3 - 2x_2^2\right) + u_1\left(x_1 + 3x_2 - 6 + s_1^2\right) + u_2\left(5x_1 + 2x_2 - 10 + s_2^2\right)$$
$$+ u_3\left(-x_1 + s_3^2\right) + u_4\left(-x_2 + s_4^2\right)$$

$\partial L/\partial x_1 = 2 - 3x_1^2 + u_1 + 5u_2 - u_3 = 0;$

$\partial L/\partial x_2 = 3 - 4x_2 + 3u_1 + 2u_2 - u_4 = 0;$

$x_1 + 3x_2 - 6 + s_1^2 = 0;\ 5x_1 + 2x_2 - 10 + s_2^2 = 0;\ -x_1 + s_3^2 = 0;\ -x_2 + s_4^2 = 0;$

$u_i s_i = 0;\ u_i \ge 0;$ i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$. There are two possible solution points: $(-0.816, 0.75)$ and $(0.816, 0.75)$. For $(-0.816, 0.75)$, $g_3 = 0.816 > 0$ (violation). For $(0.816, 0.75)$, $g_1 = -2.934 < 0$, $g_2 = -4.42 < 0$, $g_3 = -0.816 < 0$, $g_4 = -0.75 < 0$. All the KKT conditions are satisfied; therefore (0.816,0.75) is a KKT point ($f = 2.214$).

Case 2. $u_1 = u_2 = u_3 = 0, s_4 = 0$. $g_4 = 0 \to x_2 = 0$; $x_1 = \pm 0.816$, $u_4 = 3 > 0$. $x_1 = -0.816 \to g_3 > 0$ (violated). At $x_1 = 0.816$ and $x_2 = 0$, $g_1 = -5.184 < 0$, $g_2 = -5.92 < 0$. All the KKT conditions are satisfied; therefore $(0.816, 0)$ is a KKT point ($f = 1.0887$).

Case 3. $u_1 = u_2 = u_4 = 0, s_3 = 0$. KKT point: $(0, 0.75)$, $u_3 = 2$, $f = 1.125$.

Case 4. $u_1 = u_3 = u_4 = 0, s_2 = 0$. Candidate points: $(-9.8407, 1.2317)$ and $(1.5073, 1.2317)$; first point violates g_3; the second point is a KKT point with $u_2 = 0.9632$; $f = 0.251$.

Case 5. $u_2 = u_3 = u_4 = 0, s_1 = 0.$ Candidate points: $(-1.821, 1.655)$ and $(1.0339, 1.655)$; first point violates g_3; second is a KKT point with $u_1 = 1.2067$; $f = 0.4496$.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0;$ $(0, 0)$ is a KKT point with $u_3 = 2$ and $u_4 = 3$; $f = 0$.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0;$ gives $(2, 0)$ as a KKT point with $u_2 = 2$, $u_4 = 7$; $f = -4$.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0;$ gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0;$ gives no candidate point.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0;$ gives $(0, 2)$ as a KKT point with $u_1 = 5/3, u_3 = 11/3$, $f = -2$.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0;$ gives $(1.386, 1.538)$ as a KKT point with $u_1 = 0.633,\ u_2 = 0.626;\ f = -0.007388$.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0;$ gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0;$ gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

From the above investigation, Cases 1, 2, 3, 4, 5, 6, 7, 10, 11 generate KKT points.

Check for regularity: For cases 1, 2, 3, 4 and 5, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 7, $\nabla g_2 = (5, 2), \nabla g_4 = (0, -1)$. Since ∇g_2 and ∇g_4 are linearly independent, regularity is satisfied. For case 10, $\nabla g_1 = (1, 3)$, $\nabla g_3 = (-1, 0)$. Since ∇g_1 and ∇g_3 are linearly independent, regularity is satisfied. For case 11, $\nabla g_1 = (1, 3)$, $\nabla g_2 = (5, 2)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.

Referring to Exercises 4.61/4.114, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	u_1^*	u_2^*	u_3^*	u_4^*
(1)	0.816	0.75	0	0	0	0
(2)	0.816	0	0	0	0	3
(3)	0	0.75	0	0	2	0
(4)	1.5073	1.2317	0	0.9632	0	0
(5)	1.0339	1.655	1.2067	0	0	0
(6)	0	0	0	0	2	3
(7)	2	0	0	2	0	7
(8)	0	2	1.667	0	3.667	0
(9)	1.386	1.538	0.633	0.626	0	0

For points (6), (7), (8) and (9), the number of active constraints is equal to the number of design variables. There are no feasible directions in the neighborhood of the points that can reduce cost function any further. So, all the points are isolated local minima.

5.21

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 \leq 4$

Solution

Minimize $f(\mathbf{x}) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$; subject to $x_1 + x_2 - 4 \leq 0$.

$L(\mathbf{x}, u) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + u\left(x_1 + x_2 - 4 + s^2\right)$

$\partial L/\partial x_1 = 8x_1 - 5x_2 - 8 + u = 0$; $\partial L/\partial x_2 = 6x_2 - 5x_1 + u = 0$;

Case 1. $u = 0$; gives a KKT point as (48/23, 40/23); $f(\mathbf{x}^*) = -192/23$.

Case 2. $s = 0$ (or g = 0); gives no candidate point ($u = -1/6$).

Check for regularity: $\nabla g = (1, 1)$. Since there is only one constraint, regularity is satisfied.

Referring to Exercises 4.62/4.115, the point satisfying the KKT necessary conditions is:

$x_1^* = 2.0870$, $x_2^* = 1.7391$, $u_1^* = 0$.

The constraint function is linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.22

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$
subject to $x_1 + x_2 \geq 4$

Solution

Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$; subject to $g = -x_1 - x_2 + 4 \leq 0$.

$L = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6 + u(-x_1 - x_2 + 4 + s^2)$

$\partial L/\partial x_1 = 2x_1 - 4 - u = 0$; $\partial L/\partial x_2 = 2x_2 - 2 - u = 0$

$-x_1 - x_2 + 4 + s^2 = 0;\ us = 0,\ u \geq 0$

Case 1. $u = 0$; gives no candidate point $s^2 = -1$.

Case 2. $s = 0$; gives $(2.5, 1.5)$ as a KKT point with $u = 1$ and $f = 1.5$. Since only one constraint is active, regularity is satisfied.

Reference Exercise 4.63/4.116

x* $= (2.5, 1.5)$, u = 1, and $f = 1.5$.

The constraint function is linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.23

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$
subject to $x_1 + 2x_2 \leq 10$
$4x_1 - 3x_2 \leq 20$
$x_i \geq 0; i = 1,2$

Solution

Minimize $f(\mathbf{x}) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$, subject to $g_1 = x_1 + 2x_2 - 10 \leq 0$, $g_2 = 4x_1 - 3x_2 - 20 \leq 0$, $g_3 = -x_1 \leq 0$, $g_4 = -x_2 \leq 0$.

There are 16 cases, but only the case $u_1 = u_3 = u_4 = 0$, $s_2 = 0$ yields a solution: $(6.3, 1.733)$, $u_2 = 0.8$, $f = -56.901$.

Since only one constraint is active, regularity is satisfied.

Reference Exercise 4.64/4.117

$\mathbf{x}^* = (6.3, 1.733)$.

The constraint functions are linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.24

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \leq 0$

Solution

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$, subject to $g_1 = x_1 + x_2 - 4 \leq 0$

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2)$

$\partial L / \partial x_1 = 2(x_1 - 1) + u_1 = 0$;

$\partial L / \partial x_2 = 2(x_2 - 1) + u_1 = 0$;

$x_1 + x_2 - 4 + s_1^2 = 0$; $u_1 s_1 = 0$; $u_1 \geq 0$

Case 1. $u_1 = 0$, gives $(1, 1)$ as a KKT point ($f = 0$). Since no constraint is active, regularity is satisfied.

Case 2. $s_1 = 0$; gives no candidate point.

Reference Exercise 4.65/4.118

x* $= (1, 1)$ and $f = 0$.

The constraint function is linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.25

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \leq 0$
$x_1 - x_2 - 2 \leq 0$

Solution

Minimize $f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $g_1 = x_1 + x_2 - 4 \leq 0$; $g_2 = x_1 - x_2 - 2 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2) + u_2(x_1 - x_2 - 2 + s_2^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) + u_1 + u_2 = 0$; $\partial L/\partial x_2 = 2(x_2 - 1) + u_1 - u_2 = 0$

$x_1 + x_2 - 4 + s_1^2 = 0$; $x_1 - x_2 - 2 + s_2^2 = 0$;

$u_1 s_1 = 0$, $u_2 s_2 = 0$; $u_1 \geq 0$, $u_2 \geq 0$

Case 1. $u_1 = 0$, $u_2 = 0$; gives $(1, 1)$ as a KKT point, $f = 0$

Case 2. $u_1 = 0$, $s_2 = 0$; no candidate minimum.

Case 3. $s_1 = 0$, $u_2 = 0$; no candidate minimum.

Case 4. $s_1 = 0$, $s_2 = 0$; no candidate minimum.

Only the first case gives a solution that satisfies all the KKT necessary conditions. Since no constraint is active, regularity is satisfied.

Reference Exercise 4.66/4.119

$\mathbf{x}^* = (1, 1)$ and $f = 0$.

The constraint function is linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.26

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \le 0$
$2 - x_1 \le 0$

Solution

Minimize $f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to: $g_1 = x_1 + x_2 - 4 \le 0;\ g_2 = 2 - x_1 \le 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2) + u_2(2 - x_1 + s_2^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) + u_1 - u_2 = 0;\ \ \partial L/\partial x_2 = 2(x_2 - 1) + u_1 = 0$

$x_1 + x_2 - 4 + s_1^2 = 0;\ \ 2 - x_1 + s_2^2 = 0$; $u_1 s_1 = 0$, $u_2 s_2 = 0$, $u_1 \ge 0$, $u_2 \ge 0$

$u_1 s_1 = 0;\ u_2 s_2 = 0;\ u_1 \ge 0,\ u_2 \ge 0$

Case 1. $u_1 = 0, u_2 = 0;$ no candidate minimum point $(s_2^2 < 0)$.

Case 2. $u_1 = 0, s_2 = 0;$ gives $(2, 1)$ as a KKT point with $u_2 = 2,\ f = 1$.

Case 3. $s_1 = 0, u_2 = 0$; no candidate minimum $(u_1 < 0)$.

Case 4. $s_1 = s_2 = 0$; no candidate minimum $(u_1 < 0)$.

Case 2 yields a KKT point. This point is regular since there is only one active constraint.

Reference Exercise 4.67/4.120

$\mathbf{x}^* = (2, 1)$ and $f = 1$.

The constraint function is linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.27

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 \geq 16$

Solution

Minimize $f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$, subject to $g_1 = 16 - (x_1^2 + x_2^2 + 2x_1) \leq 0$.

$L = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4 + u_1(16 - x_1^2 - x_2^2 - 2x_1 + s_1^2)$

$\partial L/\partial x_1 = 18x_1 - 18x_2 - 2u_1x_1 - 2u_1 = 0;\ \partial L/\partial x_2 = -18x_1 + 26x_2 - 2u_1x_2 = 0$

$x_1 + x_2 - 4 + s_1^2 = 0; 2 - x_1 + s_2^2 = 0$; $u_1s_1 = 0$, $u_2s_2 = 0$, $u_1 \geq 0$, $u_2 \geq 0$

$-x_1^2 - x_2^2 - 2x_1 + 16 + s_1^2 = 0; u_1s_1 = 0, u_1 \geq 0$

Case 1. $u_1 = 0$; no candidate minimum $(s_1^2 < 0)$.

Case 2. $s_1 = 0$; Solving the nonlinear system of equations, we get the following KKT points:

$(2.5945,\ 2.0198)$, $u_1 = 1.4390$, $f = 15.291$; $(-3.630, 3.1754)$, $u_1 = 23.2885$, $f = 215.97$;
$(1.5088, -3.2720)$, $u_1 = 17.1503$, $f = 244.53$; $(-3.7322, -3.0879)$, $u_1 = 2.1222$, $f = 37.877$.

Since only one constraint is active, regularity is satisfied.

Referring to Exercise 4.68/4.121, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	u^*
(1)	2.5945	2.0198	1.4390
(2)	-3.630	3.1754	23.2885
(3)	1.5088	-3.2720	17.1503
(4)	-3.7322	-3.0879	2.1222

The Hessian of Lagrangian, and gradient of the constraint are

$$\nabla^2 L = \begin{bmatrix} 18 - 2u & -18 \\ -18 & 26 - 2u \end{bmatrix}; \quad \nabla g_1 = \begin{bmatrix} -2x_1 - 2 \\ -2x_2 \end{bmatrix}$$

1. At point (1), $x_1 = 2.5945$, $x_2 = 2.0198$, $u = 1.4390$, $\nabla^2 L$ is positive definite. It follows from *Theorem 5.3* that the point is an isolated local minimum.

2. At point (2), $x_1 = -3.630$, $x_2 = 3.1754$, $u = 23.2885$, Hessian of Lagrangian is negative definite. Therefore the point violates the second order necessary conditions.

3. At point (3), $x_1 = 1.5088$, $x_2 = -3.2720$, $u = 17.1503$;

$$\nabla^2 L = \begin{bmatrix} -16.3006 & -18 \\ -18 & -8.3006 \end{bmatrix}$$

Let $\mathbf{d} = (d_1, d_2)$. We need to find $\mathbf{d}$ such that $\nabla g_1 \cdot \mathbf{d} = 0$. This gives $\mathbf{d} = c(1, 0.7667)$ where $c \neq 0$ is any constant. $Q = \mathbf{d}^{\mathrm{T}} \left(\nabla^2 L\right) \mathbf{d} = -48.7811c^2 < 0$. So, sufficient condition is not satisfied at this point. It cannot be a local minimum since it violates second order necessary condition.

4. At point (4), $x_1 = -3.7322$, $x_2 = -3.8790$, $u = 2.1222$,

$$\nabla^2 L = \begin{bmatrix} 13.7556 & -18 \\ -18 & 21.7556 \end{bmatrix}$$

$\nabla g_1 \cdot \mathbf{d} = 0$ gives $\mathbf{d} = c(1, -0.8848)$. $Q = \mathbf{d}^T \left(\nabla^2 L\right) \mathbf{d} = 62.6402c^2 > 0$. The sufficient condition is satisfied. Thus, the point is an isolated minimum point.

5.28

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
subject to $x_1 + x_2 \le 4$
$x_1 - 3x_2 = 1$

Solution

Minimize $f(x) = (x_1 - 3)^2 + (x_2 - 3)^2$; subject to $h = x_1 - 3x_2 - 1 = 0$; $g = x_1 + x_2 - 4 \le 0$.

$L = (x_1 - 3)^2 + (x_2 - 3)^2 + v \le (x_1 - 3x_2 - 1) + u(x_1 + x_2 - 4 + s^2)$

$\partial L/\partial x_1 = 2(x_1 - 3) + v + u = 0$; $\partial L/\partial x_2 = 2(x_2 - 3) + 3v + u = 0$

$h = x_1 - 3x_2 - 1 = 0$; $x_1 + x_2 - 4 + s^2 = 0$; $us = 0$, $u \ge 0$.

Case 1. $u = 0$; no candidate minimum $(s^2 < 0)$.

Case 2. $s = 0$; gives $(3.25, 0.75)$ as a KKT point with $v = -1.25, u = 0.75, f = 5.125$.

Since Ñh $= (1, -3)$ and Ñg $= (1, 1)$ are linearly independent, regularity is satisfied.

Referring to Exercise 4.69/4.122, the point satisfying the KKT necessary conditions is $x_1^* = 3.25, x_2^* = -3.8790, u = 0.75, v = -1.25$.

The Hessian of cost function is positive definite and both the constraints are linear. Therefore, this is a convex problem. It follows from *Theorem* 4.11 that the point is an isolated global minimum.

5.29

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$
subject to $x_1 + x_2 \leq 3$

<u>Solution</u>

Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$, subject to $g = x_1 + x_2 - 3 \leq 0$.

$L = x_1^3 - 16x_1 + 2x_2 - 3x_2^2 + u(x_1 + x_2 - 3 + s^2)$

$\partial L/\partial x_1 = 3x_1^2 - 16 + u = 0$; $\partial L/\partial x_2 = 2 - 6x_2 + u = 0$; $x_1 + x_2 - 3 + s^2 = 0$; $u\,g = 0$, $u \geq 0$.

Case 1. $u = 0$; gives $(4/\sqrt{3}, 1/3)$, $f = -24.3$ and $(-4/\sqrt{3}, 1/3)$, $f = 24.967$, as KKT points.

Case 2. $s = 0$; gives $(0, 3)$, $u = 16$, $f = -21$; $(2, 1)$, $u = 4$, $f = -25$, as KKT points.

For both cases, there is only one active constraint; so, regularity is satisfied.

Referring to Exercise 4.70/4.123, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	u^*
(1)	2.3094	0.3333	0
(2)	– 2.3094	0.3333	0
(3)	0	3	16
(4)	2	1	4

The Hessian of the Lagrangian and gradient of the constraint are

$$\tilde{\mathbf{N}}^2 L = \begin{bmatrix} 6x_1 & 0 \\ 0 & -6 \end{bmatrix}; \quad \tilde{\mathbf{N}} g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1. At point (1), $x_1 = 2.3094$, $x_2 = 0.3333$, $\tilde{\mathbf{N}}^2 L$ is indefinite. Since no constraint is active, this is an inflection point for the cost function which does not satisfy the sufficient condition.

2. At point (2), $x_1 = -2.3094$, $x_2 = 0.3333$, Hessian of Lagrangian is negative definite. Since no constraint is active, this is local maximum point (satisfies sufficient condition for local maximum).

3. At point (3), $x_1 = 0$, $x_2 = 3$, $u = 16$, Hessian of Lagrangian is negative semidefinite, so this point does not satisfy second order necessary condition. It cannot be a local minimum.

4. At point (4), $x_1 = 2$, $x_2 = 1$, $u = 4$, $\tilde{\mathbf{N}}^2 L = \begin{bmatrix} 12 & 0 \\ 0 & -6 \end{bmatrix}$.

$\tilde{\mathbf{N}} g \cdot \mathbf{d} = 0$ gives $\mathbf{d} = c(1, -1)$ where $c \neq 0$ is any constant. $Q = \mathbf{d}^T (\nabla^2 L)\mathbf{d} = 6c^2 > 0$, for $c \neq 0$. Since $Q > 0$, sufficient condition is satisfied. Thus the point is an isolated local minimum point.

5.30

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$
subject to $x_1^2 - x_2^2 + 8x_2 \leq 16$

Solution

Minimize $f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$, subject to $g = x_1^2 - x_2^2 + 8x_2 - 16 \leq 0$

$L = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2 + u(x_1^2 - x_2^2 + 8x_2 - 16 + s^2)$

$\partial L/\partial x_1 = 6x_1 - 2x_2 + 2ux_1 = 0$; $\partial L/\partial x_2 = -2x_1 + 10x_2 + 8 - 2ux_2 + 8u = 0.$

$x_1^2 - x_2^2 + 8x_2 - 16 + s^2 \leq 0$; $us = 0$, $u \geq 0$

Case 1. $u = 0$; gives $(-2/7, -6/7)$ as a KKT point $(f = -24/7)$.

Case 2. $s = 0$; no candidate minima $(u < 0)$.

For case 1, since there is no active constraint, the regularity is satisfied.

Referring to Exercise 4.71/4.124, the point satisfying the KKT necessary condition is
$x_1^* = -0.2857$, $x_2^* = -0.8571$, $u^* = 0$.

The Hessian of Lagrangian is positive definite. The sufficient condition is satisfied and the point is an isolated local minimum.

5.31

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x, y) = (x - 4)^2 + (y - 6)^2$
subject to $x + y \leq 12$
$x \leq 6$
$x, y \geq 0$

Solution

Minimize $f(x,y) = (x-4)^2 + (y-6)^2$; subject to $g_1 = x + y - 12 \leq 0$;

$g_2 = x - 6 \leq 0$; $g_3 = -x \leq 0$; $g_4 = -y \leq 0$;

$L = \left((x-4)^2 + (y-6)^2\right) + u_1\left(x + y - 12 + s_1^2\right) + u_2\left(x - 6 + s_2^2\right)$

$+u_3\left(-x + s_3^2\right) + u_4\left(-y + s_4^2\right)$

$\partial L/\partial x = 2(x-4) + u_1 + u_2 - u_3 = 0$; $\partial L/\partial y = 2(y-6) + u_1 - u_4 = 0$;

$x + y - 12 + s_1^2 = 0$; $x - 6 + s_2^2 = 0$; $-x + s_3^2 = 0$; $-y + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \geq 0$; $i = 1$ to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives $(4, 6)$ as a KKT point ; $f = 0$.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives no candidate point.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives no candidate point.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives no candidate point.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; gives no candidate point.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0$; gives no candidate point.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: Only the first case gives a solution that satisfies all the KKT necessary conditions. Since no constraint is active, regularity is satisfied.

Referring to Exercise 4.72/4.125, the point satisfying the KKT necessary condition is $x = 4$**,** $y = 6$**,** $u = 0$.
The constraint functions are linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.32

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Minimize $f(x, y) = (x - 8)^2 + (y - 8)^2$
subject to $x + y \le 12$
$x \le 6$
$x, y \ge 0$

Solution

Minimize $f(x, y) = (x-8)^2 + (y-8)^2$; subject to $g_1 = x + y - 12 \le 0$;

$g_2 = x - 6 \le 0$; $g_3 = -x \le 0$; $g_4 = -y \le 0$;

$$L = \left((x-8)^2 + (y-8)^2\right) + u_1\left(x + y - 12 + s_1^2\right) + u_2\left(x - 6 + s_2^2\right)$$

$$+u_3\left(-x + s_3^2\right) + u_4\left(-y + s_4^2\right)$$

$\partial L/\partial x = 2(x-8) + u_1 + u_2 - u_3 = 0$; $\partial L/\partial y = 2(y-8) + u_1 - u_4 = 0$;

$x + y - 12 + s_1^2 = 0$; $x - 6 + s_2^2 = 0$; $-x + s_3^2 = 0$; $-y + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \ge 0$; $i = 1$ to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives no candidate point.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives no candidate point.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives no candidate point.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives $(6, 6)$ as a KKT point with $u_1 = 4$; $f = 8$.

Case 6. $u_1 = u_2 = 0,\ s_3 = s_4 = 0$; gives no candidate point.

Case 7. $u_1 = u_3 = 0,\ s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0,\ s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0,\ s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0,\ s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0,\ s_1 = s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_1 = 4$; $f = 8$.

Case 12. $u_1 = 0,\ s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0,\ s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0,\ s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For case 5, there is only one active constraint, so regularity is satisfied. For case 11, $\nabla g_1 = (1, 1)$, $\nabla g_2 = (1,0)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.

Referring to Exercise 4.73/4.126, the point satisfying the KKT necessary condition is
$x = 6$, $y = 6$, $u_1 = 4, u_2 = 0, u_3 = 0, u_4 = 0$.
The constraint functions are linear and the Hessian of cost function is positive definite. Therefore this is a convex programming problem and from Theorem 4.11, the point is an isolated global minimum.

5.33

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(x, y) = (x-4)^2 + (y-6)^2$
subject to $x + y \le 12$
$6 \ge x$
$x, y \ge 0$

Solution

Minimize $f(x,y) = -(x-4)^2 - (y-6)^2$; subject to $g_1 = x + y - 12 \le 0$;

$g_2 = x - 6 \le 0$; $g_3 = -x \le 0$; $g_4 = -y \le 0$;

$$L = \left(-(x-4)^2 - (y-6)^2\right) + u_1\left(x + y - 12 + s_1^2\right) + u_2\left(x - 6 + s_2^2\right)$$
$$+ u_3\left(-x + s_3^2\right) + u_4\left(-y + s_4^2\right)$$

$\partial L/\partial x = -2(x-4) + u_1 + u_2 - u_3 = 0$;

$\partial L/\partial y = -2(y-6) + u_1 - u_4 = 0$;

$x + y - 12 + s_1^2 = 0$;

$x - 6 + s_2^2 = 0$;

$-x + s_3^2 = 0$;

$-y + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \ge 0$; $i = 1$ to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives $(4, 6)$ as a KKT point ; $F = 0$.

Case 2. $u_1 = u_2 = u_3 = 0, s_4 = 0$; gives $(4, 0)$ as a KKT point with $u_4 = 12$; $F = 36$.

Case 3. $u_1 = u_2 = u_4 = 0, s_3 = 0$; gives $(0, 6)$ as a KKT point with $u_3 = 8$; $F = 16$.

Case 4. $u_1 = u_3 = u_4 = 0, s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_2 = 4$; $F = 4$.

Case 5. $u_2 = u_3 = u_4 = 0, s_1 = 0$; gives $(5, 7)$ as a KKT point with $u_1 = 2$; $F = 2$.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; gives $(0, 0)$ as a KKT point with $u_3 = 8$ and $u_4 = 12$; $F = 52$.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives $(6, 0)$ as a KKT point with $u_2 = 4$, $u_4 = 12$; $F = 40$.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0$; gives $(0, 12)$ as a KKT point with $u_1 = 12$, $u_3 = 20$; $F = 52$.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0;$ gives $(6, 6)$ as a KKT point with $u_1 = 0$, $u_2 = 4$; $F = 4$.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0;$ gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0;$ gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 1, 2, 3, 4 and 5, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 7, $\nabla g_2 = (1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_2 and ∇g_4 are linearly independent, regularity is satisfied. For case 10, $\nabla g_1 = (1, 1)$, $\nabla g_3 = (-1, 0)$. Since ∇g_1 and ∇g_3 are linearly independent, regularity is satisfied. For case 11, $\nabla g_1 = (1, 1)$, $\nabla g_2 = (1, 0)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.

Refer to Exercise 4.74/4.127. There are seven points that satisfy the KKT necessary conditions. Only the points (0, 0) and (6, 0) are the local maximum points. Others violate second order sufficiency conditions.

5.34

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(r,t) = (r-8)^2 + (t-8)^2$
subject to $10 \geq r + t$
$t \leq 5$
$r, t \geq 0$

Solution

Minimize $f(r,t) = -(r-8)^2 - (t-8)^2$; subject to $g_1 = r + t - 10 \leq 0$;

$g_2 = t - 5 \leq 0$; $g_3 = -r \leq 0$; $g_4 = -t \leq 0$;

$L = \left(-(r-8)^2 - (t-8)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(t - 5 + s_2^2\right)$

$+u_3\left(-r + s_3^2\right) + u_4\left(-t + s_4^2\right)$

$\partial L/\partial r = -2(r-8) + u_1 - u_3 = 0$; $\partial L/\partial t = -2(t-8) + u_1 + u_2 - u_4 = 0$;

$r + t - 10 + s_1^2 = 0$; $t - 5 + s_2^2 = 0$; $-r + s_3^2 = 0$; $-t + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \geq 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives no candidate point.

Case 2. $u_1 = u_2 = u_3 = 0, s_4 = 0$; gives $(8, 0)$ as a KKT point with u_4 = 16; $F = 64$.

Case 3. $u_1 = u_2 = u_4 = 0, s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0, s_2 = 0$; gives no candidate point.

Case 5. $u_2 = u_3 = u_4 = 0, s_1 = 0$; gives no candidate point.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; gives $(0, 0)$ as a KKT point with u_3 = 16, u_4 = 16; F =128.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0$; gives $(10, 0)$ as a KKT point with u_1 = 4, u_4 = 20; F =68.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0$; gives no candidate point.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 2, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 9, $\nabla g_1 = (1, 1)$, $\nabla g_4 = (0, -1)$. Since ∇g_1 and ∇g_4 are linearly independent, regularity is satisfied.

Refer to Exercise 4.75/4.128. There are three points that satisfy the KKT necessary conditions. Only the points (0, 0) and (10, 0) are the local maximum points.

5.35

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(r,t) = (r-3)^2 + (t-2)^2$
subject to $10 \geq r + t$
$t \leq 5$
$r, t \geq 0$

Solution

Minimize $f(r,t) = -(r-3)^2 - (t-2)^2$; subject to $g_1 = r + t - 10 \leq 0$;

$g_2 = t - 5 \leq 0$; $g_3 = -r \leq 0$; $g_4 = -t \leq 0$;

$L = \left(-(r-3)^2 - (t-2)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(t - 5 + s_2^2\right)$

$+u_3\left(-r + s_3^2\right) + u_4\left(-t + s_4^2\right)$

$\partial L/\partial r = -2(r-3) + u_1 - u_3 = 0$; $\partial L/\partial t = -2(t-2) + u_1 + u_2 - u_4 = 0$;

$r + t - 10 + s_1^2 = 0$; $t - 5 + s_2^2 = 0$; $-r + s_3^2 = 0$; $-t + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \geq 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives $(3, 2)$ as a KKT point ; F = 0.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives $(3, 0)$ as a KKT point with u_4 = 0; $F = 4$.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives $(0, 2)$ as a KKT point with u_3 = 6; $F = 9$.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives $(3, 5)$ as a KKT point with u_2 = 6; $F = 9$.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives $(5.5, 4.5)$ as a KKT point with u_1 = 5; $F = 12.5$.

Case 6. $u_1 = u_2 = 0,\ s_3 = s_4 = 0$; gives $(0, 0)$ as a KKT point with u_3 = 6, u_4 = 4; F =13.

Case 7. $u_1 = u_3 = 0,\ s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0,\ s_2 = s_3 = 0$; gives $(0, 5)$ as a KKT point with u_2 = 6, u_3 = 6; $F = 18$.

Case 9. $u_2 = u_3 = 0,\ s_1 = s_4 = 0$; gives $(10, 0)$ as a KKT point with u_1 = 14, u_4 = 18; F =53.

Case 10. $u_2 = u_4 = 0,\ s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0,\ s_1 = s_2 = 0$; gives $(5, 5)$ as a KKT point with u_1 = 4, u_2 = 2; F =13.

Case 12. $u_1 = 0,\ s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0,\ s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0,\ s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 1, 2, 3, 4 and 5, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 8, $\nabla g_2 = (0, 1)$, $\nabla g_3 = (-1, 0)$. Since ∇g_2 and ∇g_3 are linearly independent, regularity is satisfied. For case 9, $\nabla g_1 = (1, 1)$, $\nabla g_4 = (0, -1)$. Since ∇g_1 and ∇g_4 are linearly independent, regularity is satisfied. For case 11, $\nabla g_1 = (1, 1)$, $\nabla g_2 = (0, 1)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.

Refer to Exercise 4.76/4.129. There are nine points that satisfy the KKT necessary conditions. Only the points (0, 0), (0, 5), (10, 0) and (5, 5) are the local maximum points. Others violate second order sufficiency conditions.

5.36

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(r,t) = (r-8)^2 + (t-8)^2$
subject to $r + t \le 10$
$t \ge 0$
$r \le 0$

Solution

Minimize $f(r,t) = -(r-8)^2 - (t-8)^2$; subject to $g_1 = r + t - 10 \le 0$;

$g_2 = r \le 0$; $g_3 = -t \le 0$;

$L = \left(-(r-8)^2 - (t-8)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(-t + s_2^2\right)$

$+u_3\left(r + s_3^2\right)$

$\partial L/\partial r = -2(r-8) + u_1 + u_3 = 0$; $\partial L/\partial t = -2(t-8) + u_1 + u_2 = 0$;

$r + t - 10 + s_1^2 = 0$; $-t + s_2^2 = 0$; $r + s_3^2 = 0$;

$u_i s_i = 0$; $u_i \ge 0$; i = 1 to 3 (there are 8 cases).

Case 1. $u_1 = u_2 = u_3 = 0$; no candidate minimum.

Case 2. $u_1 = u_2 = 0, s_3 = 0$; no candidate minimum.

Case 3. $u_1 = u_3 = 0, s_2 = 0$; no candidate minimum.

Case 4. $u_2 = u_3 = 0, s_1 = 0$; no candidate minimum.

Case 5. $u_1 = 0, s_2 = s_3 = 0$; no candidate minimum.

Case 6. $u_2 = 0, s_1 = s_3 = 0$; no candidate minimum.

Case 7. $u_3 = 0, s_1 = s_2 = 0$; no candidate minimum.

Case 8. $s_1 = s_2 = s_3 = 0$; no candidate minimum.

Refer to Exercise 4.77/4.130. There are no points that satisfy the KKT necessary conditions.

5.37

Solve the following problem graphically. Check necessary and sufficient conditions for candidate local minimum points and verify them on the graph for the problem.

Maximize $F(r,t) = (r-3)^2 + (t-2)^2$
subject to $10 \geq r + t$
$t \geq 5$
$r, t \geq 0$

Solution

Minimize $f(r,t) = -(r-3)^2 - (t-2)^2$; subject to $g_1 = r + t - 10 \leq 0$;
$g_2 = -t + 5 \leq 0$; $g_3 = -r \leq 0$; $g_4 = -t \leq 0$;
$L = \left(-(r-3)^2 - (t-2)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(-t + 5 + s_2^2\right)$
$+ u_3\left(-r + s_3^2\right) + u_4\left(-t + s_4^2\right)$
$\partial L/\partial r = -2(r-3) + u_1 - u_3 = 0$; $\partial L/\partial t = -2(t-2) + u_1 - u_2 - u_4 = 0$;
$r + t - 10 + s_1^2 = 0$; $-t + 5 + s_2^2 = 0$; $-r + s_3^2 = 0$; $-t + s_4^2 = 0$;
$u_i s_i = 0$; $u_i \geq 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives no candidate point.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives no candidate point.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives $(3, 5)$ as a KKT point with $u_3 = 6$; $F = 9$.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives no candidate point.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; gives no candidate point.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0$; gives $(0, 10)$ as a KKT point with $u_1 = 16$, $u_3 = 22$; $F = 73$.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0$; gives no candidate point.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 4, there is only one active constraint, so regularity is satisfied. For case 10, $\nabla g_1 = (1, 1)$, $\nabla g_3 = (-1, 0)$. Since ∇g_1 and ∇g_3 are linearly independent regularity is satisfied.

Refer to Exercise 4.78/4.131. There are two points that satisfy the KKT necessary conditions; (3, 5) and (0, 10). Only the point (0, 100) is a local maximum point.

5.38

Rewrite the formulation of Exercise 2.23, the problem is written in the standard form as (note that some of the data used here is different from that used in Exercise 2.23):

Minimize $f = 12.331\left(R_o^2 - R_i^2\right)$, subject to $g_1 = \left(6.3662\times10^6\right)R_o \Big/ \left(R_o^4 - R_i^4\right) - 2.5\times10^4 \le 0$

$g_2 = 4244.132\left(R_o^2 + R_oR_i + R_i^2\right) \Big/ \left(R_o^4 - R_i^4\right) - 9000 \le 0,$

$g_3 = R_o - 20 \le 0;\ g_4 = R_i - 20 \le 0;\ g_5 = -R_o \le 0;\ g_6 = -R_i \le 0;$

The optimum solution found by the graphical method is
$R_o^* = 20, R_i^* = 19.84,\ f^* = 79.1$ kg where g_1 and g_3 are active.

1. Check for necessary conditions. Since at the optimum only g_1 and g_3 are active, we can assume $u_2 = u_4 = u_5 = u_6 = 0$ and $s_1 = s_3 = 0$.

The KKT necessary conditions are

$$\frac{\partial L}{\partial R_o} = 24.662R_o + \left(2.5\times10^4\right)u_1\left[254.648\left(R_o^4 - R_i^4\right) - 254.648R_o\left(4R_o^3\right)\right] \Big/ \left(R_o^4 - R_i^4\right)^2 + u_3 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial R_i} = -24.662R_i + \left(2.5\times10^4\right)u_1\left[-254.648R_o\left(-4R_i^3\right)\right] \Big/ \left(R_o^4 - R_i^4\right)^2 = 0 \qquad (2)$$

$g_i + s_i^2 = 0,\ u_is_i = 0,\ u_i \ge 0;\ i = 1$ to 6

Substituting the optimum value into (1) and (2), we get $u_1 = 3.5596\times10^{-3} > 0,\ u_3 = 5.29 > 0$. The other conditions are obviously satisfied. Thus the point $R_o^* = -20, R_i^* = 19.84$ satisfies all of the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints equals the number of design variables, the point is indeed an isolated local minimum.

5.39

Rewriting the formulation of Exercise 2.24, we have

Minimize $f = 1.44086Rt$, subject to $g_1 = 7957.7/Rt - 250 \le 0$; $g_2 = 767.8/R^2 - 3 \le 0$;
$g_3 = 20 - R/t \le 0$; $g_4 = 20 - R \le 0$; $g_5 = R - 200 \le 0$; $g_6 = 1 - t \le 0$; $g_7 = t - 10 \le 0$

The optimum solutions found by the graphical method are the points on g_1 between $(31.83, 1.0)$ and $(25.23, 1.26)$, where $f^* = 45.9$ and g_1 is active.

1. Check for necessary conditions. Since only g_1 is active, we can set $u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 0$ and $s_1 = 0$.

$L = 1.44086Rt + u_1(7957.7/Rt - 250)$.

The KKT necessary conditions are

$\partial L/\partial t = 1.44086R + u_1(-7957.7/Rt^2) = 0$; $\quad \partial L/\partial R = 1.44086t + u_1(-7957.7/R^2 t) = 0$

$g_i + s_i^2 = 0, \ u_i s_i = 0, \ u_i \ge 0; \ i = 1$ to 7

Using the relation, $1.44086Rt = 45.9$ at optimum solutions, and either equation, we can obtain $u_1 = 0.183670$. All the other conditions are also satisfied. Thus, the solutions satisfy all of the necessary conditions.

$\nabla^2 L = \nabla^2 f + u_1 \nabla^2 g_1$

2. Check for sufficient condition.

$$= \begin{bmatrix} 0 & 1.44086 \\ 1.44086 & 0 \end{bmatrix} + (0.1836)\begin{bmatrix} 2(7957.7)/Rt^3 & 7957.7/R^2t^2 \\ 7957.7/R^2t^2 & 2(7957.7)/R^3t \end{bmatrix}$$

Substituting $1.44086Rt = 45.9$ into $\nabla^2 L$ and ∇g_1, we have

$$\nabla^2 L = \begin{bmatrix} 2.88\,R/t & 2.88 \\ 2.88 & 2.88\,t/R \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} -249.8/t \\ -249.8/R \end{bmatrix}$$

$\nabla g_1^T \mathbf{d} = 0$ gives $\mathbf{d} = c(1, -R/t)^T$, $c \ne 0$. Hence, $Q = \mathbf{d}^T \nabla^2 L\, \mathbf{d} = 0$

Thus, the sufficient condition is not satisfied. Hence solution is not an isolated local minimum.

5.40

Rewrite the formulation of Exercise 3.28, we have

Minimize $f = 50.077Rt + 0.1885(R + t/2)^2$, subject to $g_1 = 25 - 25.038(R - t/2)^2 \le 0$;

$g_2 = 3.5R/t - 210 \le 0$; $g_3 = (1.41667 \times 10^{-5})R/t - 0.001 \le 0$; $g_4 = 0.5t - R \le 0$;

$g_5 = -R \le 0$; $g_6 = -t \le 0$

The optimum solution obtained by the graphical method is $R^* = 1.0$, $t^* = 0.0167$, $f^* = 1.0$ where g_1 and g_2 are active.

1. Check for necessary conditions. Since only g_1 and g_2 are active, we can set $u_3 = u_4 = u_5 = 0$ and $s_1 = s_2 = 0$.

$$L = 50.077Rt + 0.1885(R + t/2)^2 + u_1\left[25 - 25.038(R + t/2)^2\right] + u_2(3.5R/t - 210)$$

$$\partial L/\partial R = 50.077t + 0.377(R + t/2) - 2u_1(25.038)(R + t/2) + u_2(3.5/t) = 0 \quad (1)$$

$$\partial L/\partial t = 50.077R + 0.1885(R + t/2) + u_1(25.038)(R + t/2) - u_2(3.5R/t^2) = 0 \quad (2)$$

$$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 5 \quad (3)$$

Substituting the optimum value into (1) and (2);

$1.216434 - 1.9833u_1 + u_2 = 0$, $50.267074 + 0.99165u_1 - 60u_2 = 0$

Solving for u_1 and u_2, we get $u_1 = 0.0417 > 0$ (o.k.), $u_2 = 4.080 \times 10^{-3} > 0$ (o.k.). All the other conditions in (3) are satisfied. Therefore, the point ($R^* = 1.0$m, $t^* = 0.0167$m) satisfies all of the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point ($R^* = 1.0$, $t^* = 0.0167$) is indeed a local minimum.

5.41

Referring to the formulation of Exercise 4.80, we have

Minimize $f = 2.4662\times10^5 Rt$, subject to $g_1 = 795.775/Rt - 2.5\times10^8 \le 0$;

$g_2 = 5\times10^4 - 6.5113\times10^{10} R^3 t \le 0$; $g_3 = R/t - 50 < 0$; $g_4 = -R \le 0$; $g_5 = -t \le 0$

The optimum solution found by the graphical method is $R^* \doteq 0.0787, t^* \doteq 0.00157, f^* \doteq 30.56$ kg; where g_2 and g_3 are active.

1. Check for necessary conditions. Since only g_2 and g_3 are active, we can set $u_1 = u_4 = u_5 = 0$ and $s_2 = s_3 = 0$. The KKT necessary conditions are

$$\partial L/\partial R = 2.4662\times10^5 t + u_2\left[\left(-6.5113\times10^{10}\right)\left(3R^2\right)t\right] + u_3\left(1/t\right) = 0 \qquad (1)$$

$$\partial L/\partial t = 2.4662\times10^5 R + u_2\left(-6.5113\times10^{10} R^3\right) + u_3\left(-R/t^3\right) = 0 \qquad (2)$$

$$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 5 \qquad (3)$$

Substituting the optimum value into (1) and (2), we get

$u_2 = 3.056\times10^{-4} > 0$, $u_3 = 0.3038 > 0$ (o.k.).

All of the other constraints in (3) are also satisfied. Thus, the point $\left(R^* \doteq 0.0787, t^* \doteq 0.00157\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints is equal to the number of design variables, the point is indeed an isolated local minimum.

5.42

Referring to the formulation of Exercise 4.81, we have

Minimize $f = 1.2331\times10^5\left(R_o^2 - R_i^2\right)$, subject to $g_1 = 1.59155\times10^4 \Big/ \left(R_o^2 - R_i^2\right) - 2.5\times10^8 \le 0$;

$g_2 = 5\times10^4 - 1.62783\times10^{10}\left(R_o^4 - R_i^4\right) \le 0$; $g_3 = 0.5\left(R_o + R_i\right) - 50 \le 0$; $g_4 = -R_o \le 0$; $g_5 = -R_i \le 0$

The optimum solution found by the graphical method is $R_o^* \doteq 0.0795$, $R_i^* \doteq 0.0779$, $f^* \doteq 30.56$ kg where g_2 and g_3 are active.

1. Check for necessary conditions. Since only g_2 and g_3 are active, we can set $u_1 = u_4 = u_5 = 0$, and s_2 and $s_3 = 0$.

$$L = 1.2331\times10^5\left(R_o^2 - R_i^2\right) + u_2\left[5\times10^4 - 1.62783\times10^{10}\left(R_o^4 - R_i^4\right)\right] + u_3\left[0.5\left(R_o + R_i\right)\Big/\left(R_o + R_i\right) - 50\right]$$

The KKT necessary conditions are

$$\partial L \Big/ \partial R_o = \left(2.4662\times10^5\right)R_o + u_2\left[-\left(1.62783\times10^{10}\right)\left(4R_o^3\right)\right] + u_3\left[-R_i \Big/ \left(R_o - R_i\right)^2\right] = 0 \quad (1)$$

$$\partial L \Big/ \partial R_i = -\left(2.4662\times10^5\right)R_i + u_2\left[\left(1.62783\times10^{10}\right)\left(4R_i^3\right)\right] + u_3\left[R_o \Big/ \left(R_o - R_i\right)^2\right] = 0 \quad (2)$$

$g_i + s_i^2 = 0$, $u_i s_i = 0$, $u_i \ge 0$; i = 1 to 5 (3)

Substituting the optimum value into (1) and (2), we get

$u_2 = 3.056\times10^{-4} > 0$, $u_3 = 0.3055 > 0$ (o.k.).

All other conditions in (3) are satisfied. Thus, the point $R_o^* \doteq 0.0795$, $R_i^* \doteq 0.0779$ satisfies the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints is equal to the number of design variables, the point is indeed a local minimum. Sufficient condition is deemed satisfied at this point.

5.43

Referring to Exercise 4.79, we have

Minimize $f = \pi DH + \pi D^2/2$, subject to $g_1 = 400 - \pi D^2 H/4 \le 0$; $g_2 = 3.5 - D \le 0$;

$g_3 = D - 8 \le 0$; $g_4 = 8 - H \le 0$; $g_5 = H - 18 \le 0$

The optimum solution found by the graphical method is $H^* = 8, D^* \doteq 7.98$, $f^* \doteq 300.6 \text{ cm}^2$ where g_1 and g_4 are active.

1. Check for necessary conditions. Since only g_1 and g_4 are active, we can set $u_2 = u_3 = u_5 = 0$ and $s_1 = s_4 = 0$.

$L = \pi DH + \pi D^2/2 + u_1\left(400 - \pi D^2 H/4\right) + u_4(8 - H)$

The KKT necessary conditions are

$\partial L/\partial D = \pi H + \pi D + u_1(-\pi DH/2) = 0$; $\partial L/\partial H = \pi D + u_1\left(-\pi D^2/4\right) - u_4 = 0$

$g_i + s_i^2 = 0$, $u_i s_i = 0$, $u_i \ge 0$; i = 1 to 5

Substituting the optimum value into equations, we get $u_1 = 0.5 > 0$, $u_4 = 0.063 > 0$ (o.k.). All the other conditions in (3) are also satisfied. Thus, the point $\left(H^* = 8, D^* = 7.98\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints is equal to the number of design variables, the point is indeed local minimum point.

5.44

Rewriting the formulation of Exercise 4.83, we have

Minimize $f = 0.6h + 0.001A$, subject to $g_1 = 20{,}000 - hA/3.5 \le 0$;

$g_2 = 0.25hA/3.5 + A - 10{,}000 \le 0$; $g_3 = 3.5 - h \le 0$; $g_4 = h - 21 \le 0$; $g_5 = -A \le 0$

The graphical optimum solution is A^* = 5,000, h^* = 14, f^* = 13.4, where g_1 and g_2 are active.

1. Check for necessary conditions.

$L = 0.6h + 0.001A + u_1(20{,}000 - hA/3.5) + u_2(0.25hA/3.5 + A - 10{,}000) + u_3(3.5 - h)$

$+ u_4(h - 21) - u_5 A$

The KKT necessary conditions are

$\partial L/\partial A = 0.01 + u_1(-h/3.5) + u_2(0.25h/3.5 + 1) - u_5 = 0$ (1)

$\partial L/\partial h = 0.6 + u_1(-A/3.5) + u_2(0.25A/3.5) - u_3 + u_4 = 0$ (2)

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; i =1 to 5 (3)

Substituting A = 5,000, h = 14, $u_3 = u_4 = u_5 = 0$ (since only g_1 and g_2 are active) into (1) and (2), we get $u_1 = 5.9 \times 10^{-4} > 0$, $u_2 = 6.8 \times 10^{-4} > 0$ (o.k.). All the other conditions in (3) are satisfied. Therefore, necessary conditions are satisfied.

2. Check for sufficient condition. Since the number of constraints active at the candidate point and the number of design variables are the same, the point is indeed a local minimum point.

5.45

Rewriting the formulation of Exercise 3.34, we have

Minimize $f = 3.083\times10^{-3} x_1^2\left(1-x_2^2\right)$, subject to $g_1 = 5.093\times10^7 / x_1^3\left(1-x_2^4\right) - 275 \le 0$;

$g_2 = 6.36619\times10^5 / x_1^4\left(1-x_2^4\right) - 3.49066\times10^{-2} \le 0$; $g_3 = 2.0\times10^7 - \left(4.17246\times10^4\right) x_1^3\left(1-x_2\right)^{2.5} \le 0$;

$g_4 = 20 - x_1 \le 0$; $g_5 = x_1 - 500 \le 0$; $g_6 = 0.6 - x_2 \le 0$; $g_7 = x_2 - 0.999 \le 0$

The optimum solution obtained by the graphical method is $x_1^* = 103, x_2^* = 0.955, f^* = 2.9$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, we can set $u_2 = u_4 = u_5 = u_6 = u_7 = 0$.

$$L = 3.083\times10^{-3} x_1^2\left(1-x_2^2\right) + u_1\left[5.093\times10^7 / x_1^3\left(1-x_2^4\right) - 275\right]$$
$$+u_3\left[2.0\times10^7 - \left(4.17246\times10^4\right) x_1^3\left(1-x_2\right)^{2.5}\right]$$

The KKT necessary conditions are

$$\partial L/\partial x_1 = 6.166\times10^{-3} x_1\left(1-x_2^2\right) + u_1\left[-3\left(5.093\times10^7\right) / x_1^4\left(1-x_2^4\right)\right]$$
$$+u_3\left[-\left(4.17246\times10^4\right)\left(3x_1^2\right)\left(1-x_2\right)^{2.5}\right] = 0 \qquad (1)$$
$$\partial L/\partial x_2 = -6.166\times10^{-3} x_1^2 x_2 + u_1\left[4\left(5.093\times10^7\right) x_2^3 / x_1^3\left(1-x_2^4\right)^2\right]$$
$$+u_3\left[-\left(4.17246\times10^4\right)\left(3x_1^2\right)\left(1-x_2\right)^{2.5}\right] = 0 \qquad (2)$$
$$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7 \qquad (3)$$

Substituting the optimum values into (1) and (2), respectively, we get $u_1 = 4.568\times10^{-3} > 0$, $u_3 = 3.332\times10^{-8} > 0$ (o.k.). All the other conditions in (3) are also satisfied. Therefore, the point $\left(x_1^* = 103, x_2^* = 0.955\right)$ satisfies all of the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(x_1^* = 103, x_2^* = 0.955\right)$ is indeed a local minimum point.

5.46

Rewriting the formulation of Exercise 3.35, we have

Minimize $f = 3.083\times10^{-3}\left(d_o^2 - d_i^2\right)$, subject to $g_1 = \left(5.093\times10^7\right)d_o \Big/ \left(d_o^4 - d_i^4\right) - 275 \le 0$;

$g_2 = \left(6.3662\times10^5\right)\Big/\left(d_o^4 - d_i^4\right) - 3.49066\times10^{-2} \le 0$;

$g_3 = 2.0\times10^7 - \left(4.17246\times10^4\right)d_o^3\left(1 - d_i/d_o\right)^{2.5} \le 0$; $g_4 = 20 - d_o \le 0$;

$g_5 = d_o - 500 \le 0$; $g_6 = 0.6 - d_i/d_o \le 0$; $g_7 = d_i/d_o - 0.999 \le 0$;

The optimum solution obtained by the graphical method is $d_o^* = 103, d_i^* = 98.36, f^* = 2.9$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, we can set $u_2 = u_4 = u_5 = u_6 = u_7$ = 0.

$$L = 3.083\times10^{-3}\left(d_o^2 - d_i^2\right) + u_1\left[\left(5.093\times10^7\right)d_o\Big/\left(d_o^2 - d_i^2\right) - 275\right]$$

$$+u_3\left[2.0\times10^7 - \left(4.17246\times10^4\right)d_o^3\left(1 - d_i/d_o\right)^{2.5}\right]$$

The KKT necessary conditions are

$$\partial L/\partial d_o = 6.166\times10^{-3}d_o + u_1\left(5.093\times10^7\right)\left[1\Big/\left(d_o^4 - d_i^4\right) - 4d_o\Big/\left(d_o^4 - d_i^4\right)^2\right]$$

$$-u_3\left(4.17246\times10^4\right)\left[\left(3d_o^2\right)\left(1 - d_i/d_o\right)^{2.5} + 2.5\left(d_i\, d_o\right)\left(1 - d_i/d_o\right)^{1.5}\right] = 0 \qquad (1)$$

$$\partial L/\partial d_i = -6.166\times10^{-3}d_i + u_1\left[\left(5.093\times10^7\right)d_o\left(4d_i^3\right)\Big/\left(d_o^4 - d_i^4\right)^2\right]$$

$$+u_3\left[\left(4.17246\times10^4\right)\left(2.5\right)d_0^2\left(1 - d_i/d_o\right)^{1.5}\right] = 0 \qquad (2)$$

$$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7 \qquad (3)$$

Substituting the optimum values into (1) and (2) respectively, we get

$u_1 = 4.657\times10^{-3} > 0$, $u_3 = 3.281\times10^{-8} > 0$ (o.k.).

All the other conditions in (3) are also satisfied. Therefore, the point obtained from graphic method satisfies all of the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(d_o^* = 103, d_i^* = 98.36\right)$ is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

5.47

Rewriting the formulation of Exercise 3.36, we have

Minimize $f = 0.02466Rt$, subject to $g_1 = \left(3.1831\times10^6\right)\left(2R+t\right)\big/\left(4R^3t+Rt^3\right)-275 \le 0$;

$g_2 = \left(3.97886\times10^4\right)\big/\left(4R^3t+Rt^3\right)-3.49066\times10^{-2} \le 0$; $g_3 = 2.0\times10^7-\left(3.37972\times10^5\right)\left(R+0.5t\right)^{0.5}t^{2.5} \le 0$;

$g_4 = 50-R \le 0$; $g_5 = R-200 \le 0$; $g_6 = 2-t \le 0$; $g_7 = t-40 \le 0$

The optimum solution obtained by the graphical method is $R^* = 50.3, t^* = 2.34, f^* = 2.9$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, we can set $u_2 = u_4 = u_5 = u_6 = u_7 = 0.$

$$L = 0.02466R_t + u_1\left[\left(3.1831\times10^6\right)\left(2R+t\right)\big/\left(4R^3t+Rt^3\right)\right]$$
$$+u_3\left[2.0\times10^7-\left(3.37972\times10^5\right)\left(R+0.5t\right)^{0.5}t^{2.5}\right]$$

The KKT necessary conditions are

$$\partial L/\partial R = 0.02466t + u_1\left(3.1831\times10^6\right)\left[2\big/\left(4R^3t+Rt^3\right)-\left(2R+t\right)\left(12R^2+t^3\right)\big/\left(4R^3t+Rt^3\right)^2\right]$$
$$-u_3\left(3.37972\times10^5\right)\left[\left(\tfrac{1}{2}\right)t^{2.5}\big/\left(R+0.5t\right)^{0.5}\right] = 0 \qquad (1)$$
$$\partial L/\partial t = 0.02466R + u_1\left(3.1831\times10^6\right)\left[1\big/\left(4R^3t+Rt^3\right)-\left(2R+t\right)\left(4R^3+3Rt^2\right)\big/\left(4R^3t+Rt^3\right)^2\right]$$
$$-u_3\left(3.37972\times10^5\right)\left[\left(\tfrac{1}{4}\right)t^{2.5}\big/\left(R+0.5t\right)+\left(2.5t^{1.5}\right)\left(R+0.5t\right)^{0.5}\right] = 0 \qquad (2)$$
$$g_i \le 0,\ u_ig_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7 \qquad (3)$$

Substituting the optimum value into (1) and (2) respectively, we obtain $u_1 = 4.643\times10^{-3} > 0$, $u_3 = 3.240\times10^{-8} > 0$ (o.k.). All the other conditions in (3) are also satisfied. Therefore, the point obtained from graphical method satisfies all the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(R^* = 50.3, t^* = 2.34\right)$ is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

5.48

Referring to the formulation of Exercise 3.54, we have

Minimize $f = (6.59734 - 10^{-3})D^2(H^2 + 4800)^{1/2}$, subject to

$$g_1 = (2.546475 \times 10^4)(H^2 + 4800)^{\frac{1}{2}} \Big/ (D^2 H) - 1.5 \times 10^4 \le 0;$$

$$g_2 = (2.0 \times 10^4)(H^2 + 4800)^{\frac{1}{2}} \Big/ H - (1.816774 \times 10^6) D^4 \Big/ (H^2 + 4800) \le 0;$$

$$g_3 = H - 500 \le 0;\ \ g_4 = 50 - H \le 0;\ \ g_5 = D - 50 \le 0;\ \ g_6 = 0.5 - D \le 0.$$

The optimum solution obtained by the graphical method is $H^* = 50, D^* \doteq 3.42, f^* \doteq 6.6$, where g_2 and g_4 are active.

1. Check for necessary conditions. Since only g_2 and g_4 are active, we can set $u_1 = u_3 = u_5 = u_6 = 0$ and $s_2 = s_4 = 0$.

$$L = (6.59734 \times 10^{-3}) D^2 (H^2 + 4800)^{\frac{1}{2}} +$$

$$u_2 \left[(2.0 \times 10^4)(H^2 + 4800)^{\frac{1}{2}} \Big/ H - (1.816774 \times 10^6) D^4 \Big/ (H^2 + 4800) \right] + u_4 (50 - H)$$

The KKT necessary conditions are

$$\partial L / \partial D = 0.0131947 D (H^2 + 4800)^{\frac{1}{2}} + u_2 \left[(-7.267096 \times 10^6) D^3 \Big/ (H^2 + 4800) \right] = 0 \quad (1)$$

$$\partial L / \partial H = (6.59734 \times 10^{-3}) D^2 H (H^2 + 4800)^{\frac{1}{2}} + u_2 \left[-(2.0 \times 10^4)(H^2 + 4800)^{\frac{1}{2}} \Big/ H^2 \right.$$

$$\left. + (2.0 \times 10^4)(H^2 + 4800)^{\frac{1}{2}} + (3.633548 \times 10^6) D^4 H \Big/ (H^2 + 4800)^2 \right] - u_4 = 0 \quad (2)$$

$$g_i + s_i^2 = 0,\ \ u_i s_i = 0,\ \ u_i \ge 0;\ \ i = 1 \text{ to } 6 \quad (3)$$

Substituting the optimum value into (1) and (2), we get $u_2 = 9.68 \times 10^{-5} > 0$, $u_4 = 4.68 \times 10^{-2} > 0$ (o.k.)

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $(H^* = 50, D^* = 3.42)$ is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

5.49

Answer True *or False.*

1. Candidate minimum points for a constrained problem that do not satisfy second-order sufficiency conditions can be global minimum designs. *True*
2. Lagrange multipliers may be used to calculate the sensitivity coefficient for the cost function with respect to the right side parameters even if Theorem 4.7 cannot be used. *True*
3. Relative magnitudes of the Lagrange multipliers provide useful information for practical design problems. *True*

5.50

A circular tank that is closed at both ends is to be fabricated to have a volume of 250π m^3. The fabrication cost is found to be proportional to the surface area of the sheet metal needed for fabrication of the tank and is \$400/m^2. The tank is to be housed in a shed with a sloping roof which limits the height of the tank by the relation $H \le 8D$, where H is the height and D is the diameter of the tank. The problem is formulated as minimize $f(D,H)=400(0.5\pi D^2+\pi DH)$ subject to the constraints $\frac{\pi}{4}D^2H = 250\pi$, and $H \le 8D$. Ignore any other constraints.

1. Check for convexity of the problem.
2. Write KKT necessary conditions.
3. Solve KKT necessary conditions for local minimum points. Check sufficient conditions and verify the conditions graphically.
4. What will be the change in cost if the volume requirement is changed to 255π m^3 in place of 250π m^3?

Minimize $f = 400\left(0.5\pi D^2 + \pi DH\right)$, subject to $h_1 = \pi D^2 H/4 - 250\pi = 0$; $g_1 = H - 8D \le 0$

1. Check for convexity of the problem.

$$\nabla f = \begin{bmatrix} \partial f/\partial D \\ \partial f/\partial H \end{bmatrix} = \begin{bmatrix} 400\pi D + 400\pi H \\ 400\pi D \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 400\pi & 400\pi \\ 400\pi & 0 \end{bmatrix}; \quad \begin{array}{l} M_1 = 400\pi > 0 \\ M_2 = -160000\pi^2 < 0 \end{array}$$

Since Hessian of the cost function is not positive definite, this is not a convex programming problem.

2. Write Kuhn-Tucker necessary conditions.

$$L = 400\left(0.5\pi D^2 + \pi DH\right) + v_1\left(\pi D^2 H/4 - 250\pi\right) + u_1\left(H - 8D\right)$$

$$\partial L/\partial D = 400\pi D + 400\pi H + v_1\left(\pi DH/2\right) + u_1(-8) = 0 \qquad (1)$$

$$\partial L/\partial H = 400\pi D + v_1\left(\pi D^2/4\right) + u_1 = 0 \qquad (2)$$

$h_1 = 0$, $g_1 \le 0$, $u_1 g_1 = 0$ and $u_1 \ge 0$

3. Solve the KKT conditions.

Case 1. $g_1 = 0$; $H = 8D$. Solving the equations, we get $D = 5$, $H = 40$, $v_1 = -226.67, u_1 = -1832.5$ < 0. Since $u_1 < 0$, the KKT conditions are violated, and there is no solution for this case.

Case 2. $u_1 = 0$. Solving the equations, we obtain $D = H = 10$, $v_1 = -160$.

Since $g_1 = H - 8D = 10 - 80 = -70 < 0$, all the KKT conditions are satisfied. Thus, (10, 10) is a candidate minimum point. $f^* = 60{,}000\,\pi$

4. Change in cost.

Applying the constraint variation sensitivity theorem, we get the cost increase as

$\Delta f = -u_1 e_1 = 160(225p - 250p) = 800p$

5.51

A symmetric (area of member 1 is the same as area of member 3) three-bar truss problem is described in Section 2.10.

1. Formulate the minimum mass design problem treating A_1 and A_2 as design variables.
2. Check for convexity of the problem.
3. Write KKT necessary conditions for the problem.
4. Solve the optimum design problem using the data: P=50 kN, θ=30°, ρ=7800 kg/m^3, σa=150 MPa. Verify the solution graphically and interpret the necessary conditions on the graph for the problem.
5. What will be the effect on the cost function if σa is increased to 152 MPa?

1. Referring to Section 2.10, the design problem is formulated as

Minimize $f = \rho l\left(2\sqrt{2}A_1 + A_2\right)$, subject to $g_1 = s_1/\sqrt{2}\,A_1 + s_2/\sqrt{2}\left(A_1 + \sqrt{2}A_2\right) - \sigma_a \le 0$

$g_2 = \sqrt{2}\,s_2/\left(A_1 + \sqrt{2}A_2\right) - \sigma_a \le 0,\ g_3 = -A_1 \le 0,\ g_4 = -A_2 \le 0$

2. Check for convexity. Referring to Exercise 4.150, it is shown to be a convex programming problem.

3. Write Kuhn-Tucker necessary conditions.

$$L = \rho l\left(2\sqrt{2}A_1 + A_2\right) + u_1\left[s_1/\sqrt{2}A_1 + s_2/\sqrt{2}\left(A_1 + \sqrt{2}A_2\right) - \sigma_a\right] + u_2\left[\sqrt{2}s_2/\left(A_1 + \sqrt{2}A_2\right) - \sigma_a\right]$$
$$+ u_3\left(-A_1\right) + u_4\left(-A_2\right)$$
$$\partial L/\partial A_1 = 2\sqrt{2}\rho l + u_1\left[-s_1/\sqrt{2}A_1^2 - s_2/\sqrt{2}\left(A_1 + \sqrt{2}A_2\right)^2\right] + u_2\left[-\sqrt{2}s_2/\left(A_1 + \sqrt{2}A_2\right)^2\right] - u_3 = 0$$
$$\partial L/\partial A_2 = \rho l + u_1\left[-s_2/\left(A_1 + \sqrt{2}A_2\right)^2\right] + u_2\left[-2s_2/\left(A_1 + \sqrt{2}A_2\right)^2\right] - u_4 = 0$$
$$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1, 2, 3, 4$$

4. Solve the design problem. Use Newton, meter and kilogram as units for force, length and mass respectively.

P = 50 kN = 5×10^4 N, σ_a = 150 MPa = 1.5×10^8 N/m^2, θ = 30°, ρ = 7800 kg/m^3,

l = 1 m, $s_1 = P\cos\theta = 4.330127\times10^4$ N, $s_2 = P\sin\theta = 2.5\times10^4$ N

Substituting the above values we have:

Minimize $f = 7.8\times10^3\left(2\sqrt{2}A_1 + A_2\right)$

subject to $g_1 = 4.330127\times10^4/\sqrt{2}A_1 + 2.5\times10^4/\sqrt{2}\left(A_1+\sqrt{2}A_2\right) - 1.5\times10^8 \le 0$

$g_2 = \sqrt{2}\left(2.5\times10^4\right)/\left(A_1+\sqrt{2}A_2\right) - 1.5\times10^8 \le 0;\ g_3 = -A_1 \le 0;\ g_4 = -A_2 \le 0$

The KKT conditions become

$$\partial L/\partial A_1 = 2\sqrt{2}\left(7.8\times10^3\right) + u_1\left[-4.330127\times10^4/\sqrt{2}\,A_1^2 - 2.5\times10^4/\sqrt{2}\left(A_1+\sqrt{2}A_2\right)^2\right]$$

$$+u_2\left[-\sqrt{2}\left(2.5\times10^4\right)/\left(A_1+\sqrt{2}A_2\right)^2\right] - u_3 = 0 \quad (1)$$

$$\partial L/\partial A_2 = 7.8\times10^3 + u_1\left[-2.5\times10^4/\left(A_1+\sqrt{2}A_2\right)^2\right]$$

$$+u_2\left[-2\left(2.5\times10^4\right)/\left(A_1+2A_2\right)^2\right] - u_4 = 0 \quad (2)$$

$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1, 2, 3, 4$

There are totally 16 cases, since there are 4 inequality constraints. We shall solve for the case $\left(u_2 = u_3 = u_4 = 0, g_1 = 0\right)$ which yields a solution. The solution is also a global solution because this is a convex programming problem.

$$\partial L/\partial A_2 = 7.8\times10^3 - u_1\left(2.5\times10^4/\sqrt{2}\left(A_1+\sqrt{2}A_2\right)^2\right) = 0 \quad (3)$$

$$\partial L/\partial A_1 = 2\sqrt{2}\left(7.8\times10^3\right) + u_1\left[-4.330127\times10^4/\sqrt{2}\,A_1^2 - 2.5\times10^4/\sqrt{2}\left(A_1+\sqrt{2}A_2\right)^2\right] = 0 \quad (4)$$

$$g_1 = 4.330127\times10^4/\sqrt{2}\,A_1 + 2.5\times10^4/\sqrt{2}\left(A_1+\sqrt{2}A_2\right) - 1.5\times10^8 = 0 \quad (5)$$

Solving the nonlinear system for unknowns, we get

$A_1 = 2.937\times10^{-4}$, $A_2 = 6.556\times10^{-5}$, $u_1 = 4.658\times10^{-8}$, and $f^* = 7.0$

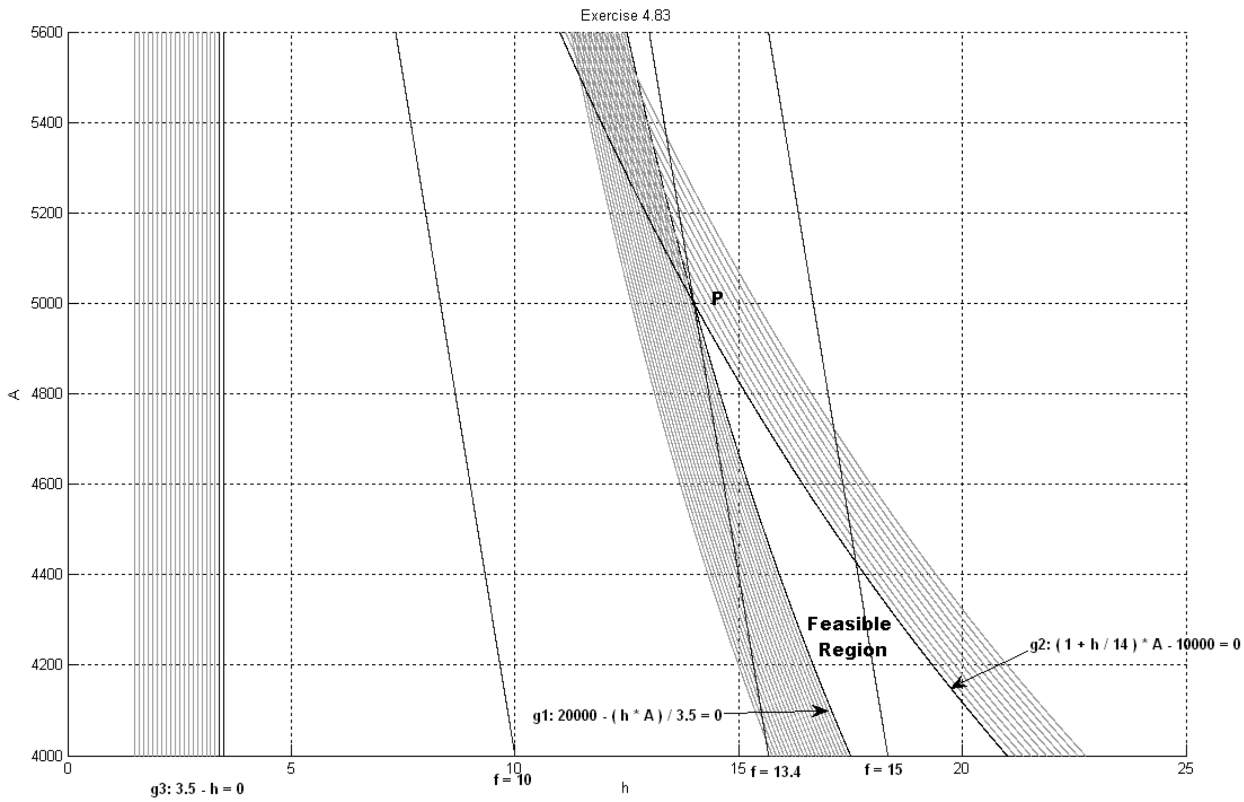
5. If σ_a is increased to 152 MPa, the cost will decrease since $u_1 > 0$. The amount will be

$\Delta f = -u_1\Delta e = -\left(4.658\times10^{-8}\right)\left(2\times10^6\right) = -0.0932$.

5.52

A 100 ×100m lot is available to construct a multistory office building. At least 20,000 m^2 total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21m, and the area for parking outside the building must be at least 25 percent of the floor area. It has been decided to fix the height of each story at 3.5m. The cost of the building in millions of dollars is estimated at 0.6 *h* +0.001 *A*, where *A* is the cross-sectional area of the building per floor and *h* is the height of the building. Formulate the minimum cost design problem.

Solution



Note : $g4 = -A$ is not shown on the graph.

According to the graphical solution, the point P (14, 5000) is minimum point.

Referring to the formulation in Exercise 2.1 we have

Minimize $f = 0.6h + 0.001A$, subject to: $g_1 = 20{,}000 - hA/3.5 \le 0$;

$g_2 = (1 + h/14)A - 10{,}000 \le 0$; $g_3 = 3.5 - h \le 0$; $g_4 = h - 21 \le 0$; $g_5 = -A \le 0$

$$L = (0.6h + 0.001A) + u_1\left(20{,}000 - hA/3.5 + s_1^2\right) + u_2\left[(1 + h/14)A - 10{,}000 + s_2^2\right]$$
$$+u_3\left(3.5 - h + s_3^2\right) + u_4\left(h - 21 + s_4^2\right) + u_5\left(-A + s_5^2\right)$$
$$\partial L/\partial h = 0.6 - u_1(A/3.5) + u_2(A/14) - u_3 + u_4 = 0$$
$$\partial L/\partial A = 0.01 - u_1(h/3.5) + u_2(1 + h/14) - u_5 = 0$$

$g_i + s_i^2 = 0$, $u_i s_i = 0$, $u_i \ge 0$; $i = 1$ to 5; $s_i^2 \ge 0$, Regularity is satisfied.

There are 32 cases because we have five inequality constraints. The case which yields a solution is $u_3 = u_4 = u_5 = 0, s_1 = s_2 = 0.$ The solution is h = 14, A = 5000, $u_1 = 5.9 \times 10^{-4}$, $u_2 = 6.8 \times 10^{-4}$, f = 13.4 mil. dollars. The solution can be verified graphically. It is seen that the point obtained using the KKT conditions is indeed a minimum point.

Referring to Exercises 2.1/4.83, we have already shown the point A = 5000 and h = 14 with $u_1 = 5.9 \times 10^{-4}$, $u_2 = 6.8 \times 10^{-4}$ satisfies the necessary and sufficient conditions. The effect of constraint limit on cost function value is due to the Sensitivity Theorem; $\Delta f = u_1 e_1 + u_2 e_2$ where e_1 and e_2 are the small change of active constraints respectively.

5.53

Referring to the formulation in Exercises 2.3/4.85, we have

Minimize $f = -\pi R^2 H$, subject to $g_1 = 2\pi RH - 900 \le 0$; $g_2 = 5 - R \le 0$;
$g_3 = R - 20 \le 0$; $g_4 = -H \le 0$; $g_5 = H - 20 \le 0$. The optimum solution obtained by the graphical method is $R^* = 20$, $H^* = 7.16$, $f^* = -9000$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, the Lagrange multipliers are all zero except for u_1 and u_3. Therefore,

$L = -\pi R^2 H + u_1(2\pi RH - 900) + u_3(R - 20)$

The KKT necessary conditions are

$\partial L/\partial R = -2\pi RH + 2\pi Hu_1 + u_3 - 0$; $\partial L/\partial H = -\pi R^2 + 2\pi Ru_1 = 0$;

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; i =1 to 5

Substituting the optimum values R^* and H^*, we get $u_1 = 10 > 0$, $u_3 = 449.9 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point obtained from graphical method satisfies all the necessary conditions.

2. Check for sufficient condition (*). Since the number of active constraints is equal to the number of design variables, the point ($R^* = 20$, $H^* = 7.16$) is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

3. Effect of variations in constraint limits on cost function (* *). According to the Constraint Variation Sensitivity *Theorem 4.7*, we have

$\partial f/\partial e_1 = -u_1 = -10$ and $\partial f/\partial e_3 = -u_3 = -499.9$

where e_1 and e_3 are a small variation in the R.H.S. of g_1 and g_3 respectively. Small variations in R.H.S. of the other constraints have no effect on the cost. By using Taylor series expansion, we can show that the change in cost function is $\Delta f = -u_1 e_1 - u_3 e_1 = -10 e_1 - 449.9 e_3$.

(*) The argument used in **2.** is applied to other problems in this section for checking sufficient condition.

(* *) The argument used in **3.** is applied to other problems in this section for studying the effect of variations in constraint limits on cost function.

5.54

Referring to the formulation in Exercises 2.4/4.86, we have:

Minimize $f = -2\pi NR$, subject to $g_1 = 0.5 - R \le 0$; $g_2 = \pi NR^2 - 2000 \le 0$; $g_3 = -N \le 0$. The optimum solution obtained by the graphical method is $R^* = 0.5$, $N^* = 2550$, $f^* = 800$ where g_1 and g_2 are active.

1. Check for necessary conditions. Since only g_1 and g_2 are active, set $u_3 = 0$.

$$L = -2\pi NR + u_1(0.5 - R) + u_2(\pi NR^2 - 2000)$$

The KKT necessary conditions are

$$\partial L/\partial R = -2\pi N - u_1 + 2\pi N - u_1 + 2\pi NRu_2 = 0;$$

$$\partial L/\partial N = -2\pi R + \pi R^2 u_2 = 0;\ g_i \le 0,\ \ u_i g_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 3$$

Substituting the optimum values, we obtain $u_2 = 4 > 0$, and $u_1 = 16022 > 0$ (o.k.). All the other conditions are also satisfied. Therefore, the point obtained from graphical method satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. Effect of variations in constraint limits on cost function. Using the same argument as used in Exercise 5.53, we can conclude that $\Delta f = -u_1 e_1 - u_2 e_2 = -16022 e_1 - 4 e_2$

5.55

Referring to the formulation in Exercises 2.5/4.87, we have:

Minimize $f = 200W + 100D$, subject to $g_1 = W - 100 \le 0$; $g_2 = D - 200 \le 0$; $g_3 = 10{,}000 - WD \le 0$; $g_4 = D - 2W \le 0$; $g_5 = W - 2D \le 0$; $g_6 = -W \le 0$; $g_7 = -D \le 0$. The optimum solution obtained by the graphical method is $W^* = 70.7$, $D^* = 141.4$, $f^* = 28284$ where g_3 and g_4 are active.

1. Check for necessary conditions. Since only g_3 and g_4 are active, set $u_1 = u_2 = u_5 = u_6 = u_7 = 0$.

$L = 200W + 100D + u_3(10{,}000 - WD) + u_4(D - 2W)$

The KKT necessary conditions are:

$\partial L/\partial W = 200 - u_3 D - 2u_4 = 0;$

$\partial L/\partial D = 100 - u_3 W + u_4 = 0;$

$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1$ to 7

Substituting the optimum values, we have $u_3 = 1.414 > 0$, $u_4 = 0 \ge 0$ (o.k.). All the other conditions are also satisfied. Therefore, the point obtained from graphical method satisfies the necessary conditions.

2. Check for sufficient condition. Hessian of the Lagrangian is

$$\mathbf{H} = \begin{bmatrix} 0 & -u_3 \\ -u_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1.414 \\ -1.414 & 0 \end{bmatrix}$$

Since $u_4 = 0$, we only consider the gradient of g_3.

$$\tilde{\mathbf{N}} g_3 = \begin{bmatrix} -D \\ -W \end{bmatrix} = \begin{bmatrix} -141.4 \\ -70.7 \end{bmatrix}; \quad \tilde{\mathbf{N}} g_3^T\, \mathbf{d} = 0 \text{ gives } \mathbf{d} = c(1,\ -2)^T,\ c \ne 0.$$

$Q = \mathbf{d}^T\, \mathbf{H}\,\mathbf{d} = 5.656c^2 > 0$ for $c \ne 0$ (o.k.)

Therefore, the point ($W^* = 70.7$, $D^* = 141.4$) satisfies sufficient condition.

3. Effect of variations in constraint limits on cost function. Using the same argument as in Exercise 5.53, $\Delta f = -u_3 e_3 = -1.414 e_3$. Since the Lagrange multiplier $u_4 = 0$, a small variation in R.H.S. of g_4 has no effect on the cost function.

5.56

Referring to the formulation in Exercises 2.9/4.91, we have

Minimize $f = \pi r^2 + 2\pi rh - 600 = 0$; subject to $h_1 = \pi r^2 h - 600 = 0$; $g_1 = 1 - h/2r \le 0$;
$g_2 = h/2r - 1.5 \le 0$; $g_3 = h - 20 \le 0$; $g_4 = -h \le 0$; $g_5 = -r \le 0$. The optimum solution obtained by the graphical method is $r^* = 4.57$, $h^* = 9.14$, $f^* = 328$ where h_1 and g_1 are active.

1. Check for necessary conditions. Since the active constraints are h_1 and g_1, we can set $u_2 = u_3 = u_4 = u_5 = 0$.

$L = \pi r^2 + 2\pi rh + v_1\left(\pi r^2 h - 600\right) + u_1\left(1 - h/2r\right)$

The KKT necessary conditions are:

$\partial L/\partial r = 2\pi r + 2\pi h + 2\pi rhv_1 + u_1\left(h/2r^2\right) = 0$;

$\partial L/\partial r = 2\pi r + \pi r^2 v_1 - u_1/2r = 0$; $h_1 = 0$, $g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 5

Substituting the optimum values and solving for v_1 and u_1, we get $v_1 = -0.364365$, $u_1 = 43.7562 > 0$ (o.k.). All the other conditions are also satisfied. Therefore, the point ($r^* = 4.57$, $h^* = 9.14$) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. Effect of variations in constraint limits on cost function. Using the same argument as used in Exercise 5.53, $\Delta f = -v_1 b_1 - u_1 e_1 = 0.3647 b_1 - 43.74 e_1$.

5.57

Referring to the formulation in Exercises 2.10/4.92, we have

Minimize $f = 32(b + 2h)/(15bh)$, subject to $g_1 = b - 10 \le 0$; $g_2 = h - 18 \le 0$. The optimum solution obtained by the graphical method is b^* = 10, h^* = 18, f^* = 0.545 where g_1 and g_2 are active.

1. Check for necessary conditions. $L = 32(b + 2h)/(15bh) + u_1(b - 10) + u_2(h - 18)$

The KKT necessary conditions are:

$\partial L/\partial b = (32/15)[bh - (b + 2h)h]/(bh)^2 + u_1 = 0;$

$\partial L/\partial h = (32/15)[2bh - (b + 2h)b]/(bh)^2 + u_2 = 0;$

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1, 2$

Substituting the optimum values, we obtain u_1 = 0.043 > 0, and u_2 = 0.0066 > 0 (o.k.). Thus, the point (b^* = 10, h^* = 18) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. Effect of variations in constraint limits on cost function. Using the same argument as used in Exercise 5.53, $\Delta f = -u_1 e_1 - u_2 e_2 = -0.043 e_1 - 0.0066 e_2$.

5.58

Referring to the formulation in Exercises 2.12/4.94, we have

Minimize $f = 200\pi D^2 + 400\pi DH$, subject to $h_1 = \pi D^2 H/4 - 150 = 0$; $g_1 = H + D/2 - 10 \le 0$; $g_2 = -H \le 0$; $g_3 = -D \le 0$. The optimum solution obtained by the graphical method is $D^* = 5.76$, $H^* = 5.76$, $f^* = 62500$, where h_1 is active.

1. Check for necessary conditions. Since only h_1 is active, set $u_1 = u_2 = u_3 = 0$.

$L = 200\pi D^2 + 400\pi DH + v_1\left(\pi D^2 H/4 - 150\right)$

The KKT necessary conditions are: $\partial L/\partial D = 400\pi D + 400\pi H + \pi DH v_1/2 = 0$;

$\partial L/\partial H = 400\pi D + \pi D^2 v_1/4 = 0$; $h_1 = 0$, $g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1, 3$

Substituting the optimum values we obtain $v_1 = -277.6$. All other conditions are satisfied. Therefore, the point obtained from graphical method satisfies the necessary conditions.

2. Check for sufficient condition. The Hessian of Lagrange function is

$$\nabla^2 L = \begin{bmatrix} 400\pi + \pi H v_1/2 & 400\pi + \pi D v_1/2 \\ 400\pi + \pi D v_1/2 & 0 \end{bmatrix} = \begin{bmatrix} -1255 & -1255 \\ -1255 & 0 \end{bmatrix}$$

$$\nabla h_1 = \begin{bmatrix} \pi DH/2 \\ \pi D^2/4 \end{bmatrix} = \begin{bmatrix} 52.1 \\ 26.05 \end{bmatrix}; \quad \nabla h_1^T \mathbf{d} = 0 \text{ gives } \mathbf{d} = c(1, -2)^T, c \ne 0.$$

$$Q = \mathbf{d}^T\left(\nabla^2 L\right)\mathbf{d} = 3765c^2 > 0 \text{ if } c \ne 0$$

$$L = \left(1 - P_1 + P_1^2\right) + \left(1 + 0.6P_2\right) + u_1\left(60 - P_1 - P_2\right)$$

The KKT necessary conditions are:

$\partial L/\partial P_1 = -1 + 2P_1 - u_1 = 0$

$\partial L/\partial P_2 = 0.6 + 2P_2 - u_1 = 0$; $g_i = 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 3

Substituting the optimum value, we obtain $u_1 = 59.8 > 0$. Therefore, the necessary conditions are satisfied.

2. Check for sufficient condition. The Hessian of Lagrangian is: $\nabla^2 L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Since Hessian of Lagrangian is positive definite, the sufficient condition is satisfied.

3. Effect of variations in constraint limits on cost function. Use the same argument as in Exercise 5.53, $\Delta f = -u_1 e_1 = -59.8 e_1$.

5.59

Referring to the formulation in Exercises 2.14/4.96, we have

Minimize $f = \left(1 - P_1 + P_1^2\right) + \left(1 + 0.6P_2 + P_2^2\right)$; subject to $g_1 = 60 - P_1 - P_2 \le 0$;

$g_2 = -\mathrm{P}_1 \le 0$; $g_3 = -\mathrm{P}_2 \le 0$.

The optimum solution obtained by the graphical method is $P_1^* = 30.4$, $P_2^* = 29.6$, $f^* = 1790$ where g_1 is active.

1. Check for necessary conditions. Since only g_1 is active, set $u_2 = u_3 = 0$.

$L = \left(1 - P_1 + P_1^2\right) + \left(1 + 0.6P_2 + P_2^2\right) + \mathrm{u}_1\left(60 - P_1 - P_2\right)$

The KKT necessary conditions are:

$\partial L/\partial P_1 = -1 + 2P_1 - u_1 = 0$; $\partial L/\partial P_2 = 0.6 + 2P_2 - u_1 = 0$; $g_i = 0$; $u_i g_i = 0$; $u_i \ge 0$; $i = 1\ to\ 3$

Substituting the optimum value, we obtain $u_1 = 59.8 > 0$. Therefore, the necessary conditions are satisfied.

2. Check for sufficient condition. The Hessian of Lagrangian is: $\tilde{\mathbf{N}}^2 L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Since Hessian of Lagrangian is positive definite, the sufficient condition is satisfied.

3. The effect of variations in constraint limits on cost function. Use the same argument as in Exercise 5.53, $\Delta f = -u_1 e_1 = -59.8 e_1$.

5.60

Referring to the formulation in Exercises 2.23/5.38, we have

Minimize $f = 12.331\left(R_o^2 - R_i^2\right)$; subject to $g_1 = \left(6.3662 \times 10^6\right)R_o \big/ \left(R_o^4 - R_i^4\right) - 2.5 \times 10^4 \le 0$;

$g_2 = 4244.132\left(R_o^2 + R_o R_i + R_i^2\right) \big/ \left(R_o^4 - R_i^4\right) - 9000 \le 0$; $g_3 = R_o - 20 \le 0$; $g_4 = R_i - 20 \le 0$;

$g_5 = -R_o \le 0$; $g_6 = -R_i \le 0$. The optimum solution obtained by the graphical method is $R_o^* = 20$, $R_i^* = 19.84$, $f^* = 79.1$ where g_1 and g_3 are active.

1. Check for necessary conditions. This point satisfies the necessary conditions which have been checked in Exercise 5.38.

2. Check for sufficient condition. Use the same argument in Exercise 5.53.

3. The effect of variations in constraint limits on cost function. Using the same argument used in Exercise 5.53, $\Delta f = -u_1 e_1 - u_3 e_3 = -3.5596 \times 10^{-3} e_1 - 5.29 e_3$.

5.61

Formulation is given in Exercise 2.24. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.39.

Rewriting the formulation of Exercise 2.24, we have

Minimize $f = 1.44086Rt$, subject to $g_1 = 7957.7/Rt - 250 \le 0$; $g_2 = 767.8/R^2 - 3 \le 0$;

$g_3 = 20 - R/t \le 0$; $g_4 = 20 - R \le 0$; $g_5 = R - 200 \le 0$; $g_6 = 1 - t \le 0$; $g_7 = t - 10 \le 0$

The optimum solutions found by the graphical method are the points on g_1 between $(31.83, 1.0)$ and $(25.23, 1.26)$, where $f^* = 45.9$ and g_1 is active.

1. Check for necessary conditions. Since only g_1 is active, we can set $u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 0$ and $s_1 = 0$.

$L = 1.44086Rt + u_1\left(7957.7/Rt - 250\right)$.

The KKT necessary conditions are

$\partial L/\partial t = 1.44086R + u_1\left(-7957.7/Rt^2\right) = 0$; $\quad \partial L/\partial R = 1.44086t + u_1\left(-7957.7/R^2 t\right) = 0$

$g_i + s_i^2 = 0$, $u_i s_i = 0$, $u_i \ge 0$; $i = 1$ to 7

Using the relation, $1.44086Rt = 45.9$ at optimum solutions, and either equation, we can obtain $u_1 = 0.183670$. All the other conditions are also satisfied. Thus, the solutions satisfy all of the necessary conditions.

$\nabla^2 L = \nabla^2 f + u_1 \nabla^2 g_1$

2. Check for sufficient condition.

$$= \begin{bmatrix} 0 & 1.44086 \\ 1.44086 & 0 \end{bmatrix} + (0.1836) \begin{bmatrix} 2(7957.7)/Rt^3 & 7957.7/R^2t^2 \\ 7957.7/R^2t^2 & 2(7957.7)/R^3 t \end{bmatrix}$$

Substituting $1.44086Rt = 45.9$ into $\nabla^2 L$ and ∇g_1, we have

$$\nabla^2 L = \begin{bmatrix} 2.88\,R/t & 2.88 \\ 2.88 & 2.88\,t/R \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} -249.8/t \\ -249.8/R \end{bmatrix}$$

$\nabla g_1^T \mathbf{d} = 0$ gives $\mathbf{d} = c\left(1, -R/t\right)^T$, $c \ne 0$. Hence, $Q = \mathbf{d}^T \nabla^2 L\, \mathbf{d} = 0$

Thus, the sufficient condition is not satisfied. Hence solution is not an isolated local minimum.

The Lagrange multiplier of active constraint is $u_1 = 0.1836$. Therefore,

$\Delta f = -u_1 e_1 = -0.1836 e_1$.

5.62

Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.41. Referring to the formulation of Exercise 4.80, we have

Minimize $f = 2.4662\times10^5 Rt$, subject to $g_1 = 795.775/Rt - 2.5\times10^8 \le 0$;

$g_2 = 5\times10^4 - 6.5113\times10^{10} R^3 t \le 0$; $g_3 = R/t - 50 < 0$; $g_4 = -R \le 0$; $g_5 = -t \le 0$

The optimum solution found by the graphical method is $R^* \doteq 0.0787, t^* \doteq 0.00157, f^* \doteq 30.56$ kg; where g_2 and g_3 are active.

1. Check for necessary conditions. Since only g_2 and g_3 are active, we can set $u_1 = u_4 = u_5 = 0$ and $s_2 = s_3 = 0$. The KKT necessary conditions are

$$\partial L/\partial R = 2.4662\times10^5 t + u_2\left[\left(-6.5113\times10^{10}\right)\left(3R^2\right)t\right] + u_3\left(1/t\right) = 0 \qquad (1)$$

$$\partial L/\partial t = 2.4662\times10^5 R + u_2\left(-6.5113\times10^{10} R^3\right) + u_3\left(-R/t^3\right) = 0 \qquad (2)$$

$$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 5 \qquad (3)$$

Substituting the optimum value into (1) and (2), we get

$u_2 = 3.056\times10^{-4} > 0$, $u_3 = 0.3038 > 0$ (o.k.).

All of the other constraints in (3) are also satisfied. Thus, the point $\left(R^* \doteq 0.0787, t^* \doteq 0.00157\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints is equal to the number of design variables, the point is indeed an isolated local minimum.

The Lagrange multipliers of the active constraints are $u_2 = 3.056\times10^{-4}$, $u_3 = 0.3038$.

Therefore, $\Delta f = -u_2 e_2 - u_6 e_6 = -3.056\times10^{-4} e_2 - 0.3038 e_6$.

5.63

The formulation is given in Exercise 3.24. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.42.

Referring to the formulation of Exercise 4.81, we have

Minimize $f = 1.2331 \times 10^5 \left(R_o^2 - R_i^2\right)$, subject to $g_1 = 1.59155 \times 10^4 \Big/ \left(R_o^2 - R_i^2\right) - 2.5 \times 10^8 \le 0$;

$g_2 = 5 \times 10^4 - 1.62783 \times 10^{10} \left(R_o^4 - R_i^4\right) \le 0$; $g_3 = 0.5\left(R_o + R_i\right) - 50 \le 0$; $g_4 = -R_o \le 0$; $g_5 = -R_i \le 0$

The optimum solution found by the graphical method is $R_o^* \doteq 0.0795$, $R_i^* \doteq 0.0779$, $f^* \doteq 30.56$ kg where g_2 and g_3 are active.

1. Check for necessary conditions. Since only g_2 and g_3 are active, we can set $u_1 = u_4 = u_5 = 0$, and s_2 and $s_3 = 0$.

$$L = 1.2331 \times 10^5 \left(R_o^2 - R_i^2\right) + u_2\left[5 \times 10^4 - 1.62783 \times 10^{10}\left(R_o^4 - R_i^4\right)\right] + u_3\left[0.5\left(R_o + R_i\right) \big/ \left(R_o + R_i\right) - 50\right]$$

The KKT necessary conditions are

$$\partial L \big/ \partial R_o = \left(2.4662 \times 10^5\right) R_o + u_2\left[-\left(1.62783 \times 10^{10}\right)\left(4R_o^3\right)\right] + u_3\left[-R_i \big/ \left(R_o - R_i\right)^2\right] = 0 \quad (1)$$

$$\partial L \big/ \partial R_i = -\left(2.4662 \times 10^5\right) R_i + u_2\left[\left(1.62783 \times 10^{10}\right)\left(4R_i^3\right)\right] + u_3\left[R_o \big/ \left(R_o - R_i\right)^2\right] = 0 \quad (2)$$

$$g_i + s_i^2 = 0,\; u_i s_i = 0,\; u_i \ge 0;\; i = 1 \text{ to } 5 \quad (3)$$

Substituting the optimum value into (1) and (2), we get

$u_2 = 3.056 \times 10^{-4} > 0$, $u_3 = 0.3055 > 0$ (o.k.).

All other conditions in (3) are satisfied. Thus, the point $R_o^* \doteq 0.0795$, $R_i^* \doteq 0.0779$ satisfies the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints is equal to the number of design variables, the point is indeed a local minimum. Sufficient condition is deemed satisfied at this point.

The Lagrange multipliers of active constraints are $u_2 = 3.056 \times 10^{-4}$, $u_3 = 0.3055$. Therefore,

$\Delta f = -u_2 e_2 - u_4 e_4 = -3.056 \times 10^{-4} e_2 - 0.3038 e_3$.

5.64

The formulation is given in Exercise 4.79. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.43.

Referring to Exercise 4.79, we have

Minimize $f = \pi DH + \pi D^2/2$, subject to $g_1 = 400 - \pi D^2 H/4 \le 0$; $g_2 = 3.5 - D \le 0$;

$g_3 = D - 8 \le 0$; $g_4 = 8 - H \le 0$; $g_5 = H - 18 \le 0$

The optimum solution found by the graphical method is $H^* = 8, D^* \doteq 7.98, f^* \doteq 300.6 \text{ cm}^2$ where g_1 and g_4 are active.

1. Check for necessary conditions. Since only g_1 and g_4 are active, we can set $u_2 = u_3 = u_5 = 0$ and $s_1 = s_4 = 0$.

$L = \pi DH + \pi D^2/2 + u_1\left(400 - \pi D^2 H/4\right) + u_4\left(8 - H\right)$

The KKT necessary conditions are

$\partial L/\partial D = \pi H + \pi D + u_1\left(-\pi DH/2\right) = 0$; $\partial L/\partial H = \pi D + u_1\left(-\pi D^2/4\right) - u_4 = 0$

$g_i + s_i^2 = 0$, $u_i s_i = 0$, $u_i \ge 0$; $i = 1$ to 5

Substituting the optimum value into equations, we get $u_1 = 0.5 > 0$, $u_4 = 0.063 > 0$ (o.k.). All the other conditions in (3) are also satisfied. Thus, the point $\left(H^* = 8, D^* = 7.98\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Since this is the case that the number of active constraints is equal to the number of design variables, the point is indeed local minimum point.

The Lagrange multipliers of active constraints are $u_1 = 0.5$, $u_4 = 0.063$. Therefore,

$\Delta f = -u_1 e_1 - u_4 e_4 = -0.5 e_1 - 0.063 e_4$.

5.65

The formulation is given in Exercise 3.28. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.40.

Rewrite the formulation of Exercise 3.28, we have

Minimize $f = 50.077Rt + 0.1885(R + t/2)^2$, subject to $g_1 = 25 - 25.038(R - t/2)^2 \le 0$;

$g_2 = 3.5R/t - 210 \le 0$; $g_3 = (1.41667\times10^{-5})R/t - 0.001 \le 0$; $g_4 = 0.5t - R \le 0$;

$g_5 = -R \le 0$; $g_6 = -t \le 0$

The optimum solution obtained by the graphical method is $R^* = 1.0$, $t^* = 0.0167$, $f^* = 1.0$ where g_1 and g_2 are active.

1. Check for necessary conditions. Since only g_1 and g_2 are active, we can set $u_3 = u_4 = u_5 = 0$ and $s_1 = s_2 = 0$.

$$L = 50.077Rt + 0.1885(R + t/2)^2 + u_1\left[25 - 25.038(R + t/2)^2\right] + u_2(3.5R/t - 210)$$

$$\partial L/\partial R = 50.077t + 0.377(R + t/2) - 2u_1(25.038)(R + t/2) + u_2(3.5/t) = 0 \qquad (1)$$

$$\partial L/\partial t = 50.077R + 0.1885(R + t/2) + u_1(25.038)(R + t/2) - u_2(3.5R/t^2) = 0 \qquad (2)$$

$$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 5 \qquad (3)$$

Substituting the optimum value into (1) and (2);

$1.216434 - 1.9833u_1 + u_2 = 0$, $50.267074 + 0.99165u_1 - 60u_2 = 0$

Solving for u_1 and u_2, we get $u_1 = 0.0417 > 0$ (o.k.), $u_2 = 4.080\times10^{-3} > 0$ (o.k.). All the other conditions in (3) are satisfied. Therefore, the point ($R^* = 1.0$m, $t^* = 0.0167$m) satisfies all of the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point ($R^* = 1.0$, $t^* = 0.0167$) is indeed a local minimum.

The Lagrange multipliers of active constraints are

$u_1 = 0.0417$, $u_2 = 4.080\times10^{-3}$. Therefore, $\Delta f = -u_1 e_1 - u_2 e_2 = -0.0417e_1 - 4.080\times10^{-3} e_2$.

5.66

The formulation is given in Exercise 3.34*. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.45.

Rewriting the formulation of Exercise 3.34, we have

Minimize $f = 3.083\times10^{-3} x_1^2\left(1-x_2^2\right)$, subject to $g_1 = 5.093\times10^7 / x_1^3\left(1-x_2^4\right) - 275 \le 0$;

$g_2 = 6.36619\times10^5 / x_1^4\left(1-x_2^4\right) - 3.49066\times10^{-2} \le 0$; $g_3 = 2.0\times10^7 - \left(4.17246\times10^4\right)x_1^3\left(1-x_2\right)^{2.5} \le 0$;

$g_4 = 20 - x_1 \le 0$; $g_5 = x_1 - 500 \le 0$; $g_6 = 0.6 - x_2 \le 0$; $g_7 = x_2 - 0.999 \le 0$

The optimum solution obtained by the graphical method is $x_1^* = 103, x_2^* = 0.955, f^* = 2.9$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, we can set $u_2 = u_4 = u_5 = u_6 = u_7 = 0$.

$$L = 3.083\times10^{-3} x_1^2\left(1-x_2^2\right) + u_1\left[5.093\times10^7 / x_1^3\left(1-x_2^4\right) - 275\right]$$
$$+u_3\left[2.0\times10^7 - \left(4.17246\times10^4\right)x_1^3\left(1-x_2\right)^{2.5}\right]$$

The KKT necessary conditions are

$$\partial L/\partial x_1 = 6.166\times10^{-3} x_1\left(1-x_2^2\right) + u_1\left[-3\left(5.093\times10^7\right) / x_1^4\left(1-x_2^4\right)\right]$$
$$+u_3\left[-\left(4.17246\times10^4\right)\left(3x_1^2\right)\left(1-x_2\right)^{2.5}\right] = 0 \qquad (1)$$

$$\partial L/\partial x_2 = -6.166\times10^{-3} x_1^2 x_2 + u_1\left[4\left(5.093\times10^7\right)x_2^3 / x_1^3\left(1-x_2^4\right)^2\right]$$
$$+u_3\left[-\left(4.17246\times10^4\right)\left(3x_1^2\right)\left(1-x_2\right)^{2.5}\right] = 0 \qquad (2)$$

$$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7 \qquad (3)$$

Substituting the optimum values into (1) and (2), respectively, we get $u_1 = 4.568\times10^{-3} > 0$, $u_3 = 3.332\times10^{-8} > 0$ (o.k.). All the other conditions in (3) are also satisfied. Therefore, the point $\left(x_1^* = 103, x_2^* = 0.955\right)$ satisfies all of the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(x_1^* = 103, x_2^* = 0.955\right)$ is indeed a local minimum point.

The Lagrange multipliers of active constraints are $u_1 = 4.568\times10^{-3}$, $u_3 = 3.332\times10^{-8}$. Therefore, $\Delta f = -u_1 e_1 - u_3 e_3 = -4.568\times10^{-3} e_1 - 3.332\times10^{-8} e_3$.

5.67

The formulation is given in Exercise 3.35*. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.46.

Rewriting the formulation of Exercise 3.35, we have

Minimize $f = 3.083\times10^{-3}\left(d_o^2 - d_i^2\right)$, subject to $g_1 = \left(5.093\times10^7\right)d_o\big/\left(d_o^4 - d_i^4\right) - 275 \le 0$;

$g_2 = \left(6.3662\times10^5\right)\big/\left(d_o^4 - d_i^4\right) - 3.49066\times10^{-2} \le 0$;

$g_3 = 2.0\times10^7 - \left(4.17246\times10^4\right)d_o^3\left(1 - d_i/d_o\right)^{2.5} \le 0$; $g_4 = 20 - d_o \le 0$;

$g_5 = d_o - 500 \le 0$; $g_6 = 0.6 - d_i/d_o \le 0$; $g_7 = d_i/d_o - 0.999 \le 0$;

The optimum solution obtained by the graphical method is $d_o^* = 103, d_i^* = 98.36, f^* = 2.9$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, we can set $u_2 = u_4 = u_5 = u_6 = u_7$ = 0.

$$L = 3.083\times10^{-3}\left(d_o^2 - d_i^2\right) + u_1\left[\left(5.093\times10^7\right)d_o\big/\left(d_o^2 - d_i^2\right) - 275\right]$$
$$+u_3\left[2.0\times10^7 - \left(4.17246\times10^4\right)d_o^3\left(1 - d_i/d_o\right)^{2.5}\right]$$

The KKT necessary conditions are

$$\partial L/\partial d_o = 6.166\times10^{-3}d_o + u_1\left(5.093\times10^7\right)\left[1\big/\left(d_o^4 - d_i^4\right) - 4d_o\big/\left(d_o^4 - d_i^4\right)^2\right]$$
$$-u_3\left(4.17246\times10^4\right)\left[\left(3d_o^2\right)\left(1 - d_i/d_o\right)^{2.5} + 2.5\left(d_i\, d_o\right)\left(1 - d_i/d_o\right)^{1.5}\right] = 0 \qquad (1)$$

$$\partial L/\partial d_i = -6.166\times10^{-3}d_i + u_1\left[\left(5.093\times10^7\right)d_o\left(4d_i^3\right)\big/\left(d_o^4 - d_i^4\right)^2\right]$$
$$+u_3\left[\left(4.17246\times10^4\right)\left(2.5\right)d_0^2\left(1 - d_i/d_o\right)^{1.5}\right] = 0 \qquad (2)$$

$$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7 \qquad (3)$$

Substituting the optimum values into (1) and (2) respectively, we get

$u_1 = 4.657\times10^{-3} > 0$, $u_3 = 3.281\times10^{-8} > 0$ (o.k.).

All the other conditions in (3) are also satisfied. Therefore, the point obtained from graphic method satisfies all of the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(d_o^* = 103, d_i^* = 98.36\right)$ is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

The Lagrange multipliers of active constraints are

$u_1 = 4.657\times10^{-3}$, $u_3 = 3.281\times10^{-8}$.

Therefore, $\Delta f = -u_1 e_1 - u_3 e_3 = -4.657\times10^{-3}e_1 - 3.281\times10^{-8}e_3$.

5.68

The formulation is given in Exercise 3.36*. Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.47.

Rewriting the formulation of Exercise 3.36, we have

Minimize $f = 0.02466Rt$, subject to $g_1 = \left(3.1831\times10^6\right)\left(2R+t\right)/\left(4R^3t+Rt^3\right) - 275 \le 0$;

$g_2 = \left(3.97886\times10^4\right)/\left(4R^3t+Rt^3\right) - 3.49066\times10^{-2} \le 0$; $g_3 = 2.0\times10^7 - \left(3.37972\times10^5\right)\left(R+0.5t\right)^{0.5}t^{2.5} \le 0$;

$g_4 = 50 - R \le 0$; $g_5 = R - 200 \le 0$; $g_6 = 2 - t \le 0$; $g_7 = t - 40 \le 0$

The optimum solution obtained by the graphical method is $R^* = 50.3, t^* = 2.34, f^* = 2.9$ where g_1 and g_3 are active.

1. Check for necessary conditions. Since only g_1 and g_3 are active, we can set

$u_2 = u_4 = u_5 = u_6 = u_7 = 0.$

$$L = 0.02466R_t + u_1\left[\left(3.1831\times10^6\right)\left(2R+t\right)/\left(4R^3t+Rt^3\right)\right]$$
$$+u_3\left[2.0\times10^7 - \left(3.37972\times10^5\right)\left(R+0.5t\right)^{0.5}t^{2.5}\right]$$

The KKT necessary conditions are

$$\partial L/\partial R = 0.02466t + u_1\left(3.1831\times10^6\right)\left[2/\left(4R^3t+Rt^3\right) - \left(2R+t\right)\left(12R^2+t^3\right)/\left(4R^3t+Rt^3\right)^2\right]$$
$$-u_3\left(3.37972\times10^5\right)\left[\left(\tfrac{1}{2}\right)t^{2.5}/\left(R+0.5t\right)^{0.5}\right] = 0 \qquad (1)$$
$$\partial L/\partial t = 0.02466R + u_1\left(3.1831\times10^6\right)\left[1/\left(4R^3t+Rt^3\right) - \left(2R+t\right)\left(4R^3+3Rt^2\right)/\left(4R^3t+Rt^3\right)^2\right]$$
$$-u_3\left(3.37972\times10^5\right)\left[\left(\tfrac{1}{4}\right)t^{2.5}/\left(R+0.5t\right) + \left(2.5t^{1.5}\right)\left(R+0.5t\right)^{0.5}\right] = 0 \qquad (2)$$
$$g_i \le 0,\ u_ig_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7 \qquad (3)$$

Substituting the optimum value into (1) and (2) respectively, we obtain $u_1 = 4.643\times10^{-3} > 0$, $u_3 = 3.240\times10^{-8} > 0$ (o.k.). All the other conditions in (3) are also satisfied. Therefore, the point obtained from graphical method satisfies all the necessary conditions.

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(R^* = 50.3, t^* = 2.34\right)$ is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

The Lagrange multipliers of active constraints are

$u_1 = 4.643\times10^{-3}$, $u_3 = 3.240\times10^{-8}$.

Therefore, $\Delta f = -u_1e_1 - u_3e_3 = -4.643\times10^{-3}e_1 - 3.240\times10^{-8}e_3$.

5.69

(a) Referring to the formulation in Exercise 3.39 (3.23), we have

Minimize $f = 0.2466Rt$, subject to $g_1 = 7957.5/(Rt) - 250 \le 0$; $g_2 = 5\times10^4 - 0.26045R^3t \le 0$;
$g_3 = R/t - 50 \le 0$; $g_4 = 10 - R \le 0$; $g_5 = R - 1000 \le 0$; $g_6 = 5 - t \le 0$; $g_7 = t - 200 \le 0$

The optimum solution is obtained by the graphical method is R^* = 33.7, t^* = 5.0, f^* = 41.6 where g_2 and g_6 are active.

1. Check for necessary conditions.

Since only g_2 and g_6 are active, we can set $u_1 = u_3 = u_4 = u_5 = u_7 = 0$.

$$L = 0.2466Rt + u_2\left[5\times10^4 - 0.26045R^3t\right] + u_6(5-t)$$

The KKT necessary conditions are:

$$\partial L/\partial R = 0.266t - u_2(0.78135)\left(R^2t\right) = 0;$$

$$\partial L/\partial t = 0.2466R - u_2\left(0.26045R^3\right) - u_6 = 0;$$

$$g_i \le 0,\ u_ig_i = 0,\ u_i \ge 0;\quad i = 1 \text{ to } 7$$

Substituting the optimum values, we obtain $u_2 = 2.779\times10^{-4} > 0$, $u_6 = 5.540 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point (R^* = 33.7, t^* = 5.0) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$$\Delta f = -u_2e_2 - u_6e_6 = -2.779\times10^{-4}e_2 - 5.54e_6$$

(b) Referring to the formulation in Exercise 3.42 (3.24), we have

Minimize $f = 0.1233\left(R_o^2 - R_i^2\right)$; subject to $g_1 = 15915.5/\left(R_o^2 - R_i^2\right) - 250 \le 0$;

$g_2 = 5\times10^5 - 0.06511\left(R_o^4 - R_i^4\right) \le 0$; $g_3 = (R_o - R_i)/\left[2(R_o - R_i)\right] - 50 \le 0$; $g_4 = R_i - R_o + 5 \le 0$;

$g_5 = R_o - R_i - 200 \le 0$; $g_6 = -0.5(R_o + R_i) + 10 \le 0$; $g_7 = 0.5(R_o + R_i) - 1000 \le 0$

The optimum solution obtained by the graphical method is R_o^* = 36.0, R_i^* = 31.0, f^* = 41.3 where g_2 and g_4 are active.

1. Check for necessary conditions.

Since only g_2 and g_4 are active, we can set $u_1 = u_3 = u_5 = u_6 = u_7 = 0$.

$$L = 0.1233\left(R_o^2 - R_i^2\right) + u_2\left[5\times10^5 - 0.06511\left(R_o^4 - R_i^4\right)\right] + u_4(R_i - R_o + 5)$$

The KKT necessary conditions are:

$$\partial L/\partial R_o = 0.2466R_o - u_2\left(0.026044R_o^3\right) - u_4 = 0;\ \partial L/\partial R_i = -0.2466R_i + u_2\left(0.26044R_i^3\right) + u_4 = 0;$$

$$g_i \le 0,\ u_ig_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 8$$

Substituting the optimum values, we get u_2 = 2.807 $2.807\times10^{-4} > 0$, $u_4 = 5.467 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point $\left(R_o^* = 36.0,\ R_i^* = 31.0\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$$\Delta f = -u_2e_2 - u_4e_4 = -2.807\times10^{-4}e_2 - 5.467e_4$$

5.70

(a) Referring to the formulation in Exercise 3.40 (3.23), we have

Minimize $f = 0.2466Rt$, subject to $g_1 = 7957.7/(Rt) - 250 \le 0$; $g_2 = 5\times10^4 - 1.04181R^3t \le 0$;

$g_3 = -R/t - 50 \le 0$; $g_4 = 10 - R \le 0$; $g_5 = R - 1000 \le 0$; $g_6 = 5 - t \le 0$; $g_7 = t - 200 \le 0$

The optimum solution obtained by the graphical method is $R^* = 21.3$, $t^* = 5.0$, $f^* = 26.0$ where g_2 and g_6 are active.

1. Check for necessary condition.

Since only g_2 and g_6 are active, we can set $u_1 = u_3 = u_4 = u_5 = u_7 = 0$.

$L = 0.2466Rt + u_2(5\times10^4 - 1.04181R^3t) + u_6(5 - t)$

The KKT necessary conditions are:

$\partial L/\partial R = 0.2466t - u_2(3.12543R^2t) = 0$;

$\partial L/\partial t = 0.2466R - u_2(1.04181R^3) - u_6 = 0$;

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 7

Substituting the optimum values, we obtain $u_2 = 1.739\times10^{-4} > 0$, $u_6 = 3.491 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point ($R^* = 21.3$, $t^* = 5.0$) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$\Delta f = -u_2 e_2 - u_6 e_6 = -1.739\times10^{-4} e_2 - 3.491 e_6$

(b) Referring to the formulation in Exercise 3.43(3.24), we have

Minimize $f = 0.1233(R_o^2 - R_i^2)$, subject to $g_1 = 15915.5/(R_o^2 - R_i^2) - 250 \le 0$;

$g_2 = 5\times10^4 - 0.26045(R_o^4 - R_i^4) \le 0$; $g_3 = (R_o + R_i)/[2(R_o - R_i)] - 50 \le 0$;

$g_4 = R_i - R_o + 5 \le 0$; $g_5 = R_o - R_i - 200 \le 0$; $g_6 = -0.5(R_o + R_i) + 10 \le 0$;

$g_7 = 0.5(R_o + R_i) - 1000 \le 0$;

The optimum solution obtained by the graphical method is $R_o^* = 24.0$, $R_i^* = 19.0$, $f^* = 26.0$ where g_2 and g_4 are active.

1. Check for necessary conditions.

Since only g_2 and g_4 are active, we can set $u_1 = u_3 = u_5 = u_6 = u_7 = 0$.

$L = 0.1233(R_o^2 - R_i^2) + u_2[5\times10^4 - 0.26045(R_o^4 - R_i^4)] + u_4(R_i - R_o + 5)$

The KKT necessary conditions are:

$\partial L/\partial R_o = 0.2466R_o - u_2(1.0418R_o^3) - u_4 = 0$;

$\partial L/\partial R_i = 0.2466R_i + u_2(1.0418R_i^3) + u_4 = 0$; $g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 8

Substituting the optimum values, we get $u_2 = 1.699\times10^{-4} > 0$, $u_4 = 3.471 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point ($R_o^* = 24.0$, $R_i^* = 19.0$) satisfies the necessary conditions.

2. Check for sufficient conditions. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$\Delta f = -u_2 e_2 - u_4 e_4 = -1.699\times10^{-4} e_2 - 3.471 e_4$

5.71

(a) Referring to the formulation in Exercise 3.41 (3.23), we have

Minimize $f = 0.2466Rt$, subject to $g_1 = 7957.7/(Rt) - 250 \le 0$;

$g_2 = 5\times10^4 - 0.52091\mathrm{R}^3\mathrm{t} \le 0$; $\mathrm{g}_3 = \mathrm{R}/\mathrm{t} - 50 \le 0$; $\mathrm{g}_4 = 10 - \mathrm{R} \le 0$;

$g_5 = \mathrm{R} - 1000 \le 0$; $\mathrm{g}_6 = 5 - \mathrm{t} \le 0$; $\mathrm{g}_7 = \mathrm{t} - 200 \le 0$;

The optimum solution obtained by the graphical method is $R^* = 27.0$, $t^* = 5.0$, $f^* = 33.0$ where g_2 and g_6 are active.

1. Check for necessary conditions.

Since only g_2 and g_6 are active, we can set $u_1 = u_3 = u_4 = u_5 = u_7 = 0$.

$L = 0.2466Rt + \mathrm{u}_2 (5\times10^4 - 0.52091R^3t) + u_6 (5 - t)$

The KKT necessary conditions are:

$\partial L/\partial R = 0.2466t - u_2 (1.56273\mathrm{R}^3\mathrm{t}) = 0$; $\partial L/\partial t = 0.2466R - u_2\left(0.52091R^3\right) - u_6 = 0$;

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 7

Substituting the optimum values, we obtain $u_2 = 2.165\times10^{-4} > 0$, $u_6 = 4.439 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point ($\mathrm{R}^* = 27.0$, $\mathrm{t}^* = 5.0$) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$\Delta f = -u_2 e_2 - u_6 e_6 = -2.065\times10^{-4} e_2 - 4.439 e_6$.

(b) Referring to the formulation in Exercise 3.43 (3.24), we have

Minimize $f = 0.1233\left(R_o^2 - R_i^2\right)$, subject to $g_1 = 15915.5 / \left(R_o^2 - R_i^2\right) - 250 \le 0$;

$g_2 = 5\times10^4 - 0.13022\left(R_o^4 - R_i^4\right) \le 0$; $g_3 = \left(R_o + R_i\right)/\left[2\left(R_o - R_i\right)\right] - 50 \le 0$;

$g_4 = \left(R_i - R_o\right) + 5 \le 0$; $g_5 = R_o - R_i - 200 \le 0$; $g_6 = -0.5\left(R_o - R_i\right) + 10 \le 0$; $g_7 = 0.5\left(R_o + R_i\right)$

The optimum solution obtained by the graphical method is $R_o^* = 29.5$, $R_i^* = 24.5$, $f^* = 33.0$ where g_2 and g_4 are active.

1. Check for necessary conditions.

Since only g_2 and g_4 are active, we can set $\mathrm{u}_1 = \mathrm{u}_3 = \mathrm{u}_5 = \mathrm{u}_6 = \mathrm{u}_7 = 0$.

$L = 0.1233\left(R_o^2 - R_i^2\right) + u_2\left[5\times10^4 - 0.13022\right] + u_4\left(R_i - R_o + 5\right)$

The KKT necessary conditions are:

$\partial L/\partial R_o = 0.2466R_o = 0.2466R_o - u_2\left(0.52088R_o^3\right) - u_4 = 0$

$\partial L/\partial R_i = -0.2466R_i + u_2\left(0.52088R_i^3\right) + u_4 = 0$;

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 8

Substituting the optimum values, we get $u_2 = 2.1586\times10^{-4} > 0$, $\mathrm{u}_4 = 4.388 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point ($R_o^* = 29.5$, $R_i^* = 24.5$) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$\Delta f = -u_2 e_2 - u_4 e_4 = -2.1586\times10^{-4} e_2 - 4.388 e_4$.

5.72

Referring to the formulation in Exercise 3.46, we have

Minimize $f = 9.8125\times10^{-3}(A_1 + A_2)$,

subject to $g_1 = (202.97A_1+512.95A_2)/(A_1A_2) - 50 \le 0$; $g_2 = 103420/A_1 - 250 \le 0$;

$g_3 = A_1 - 5000 \le 0$; $g_4 = 40925/A_2 - 250 \le 0$; $g_5 = A_2 - 5000 \le 0$;

The optimum solution obtained by the graphical method is $A_1^* = 413.68$, $A_2^* = 163.7$, $f^* = 5.7$ where g_2 and g_4 are active.

1. Check for necessary conditions. Since only g_2 and g_4 are active, we can set $u_1 = u_3 = u_5 = 0$.

$L = 9.8125\times10^{-3}\left(A_1 + A_2\right) + u_2\left(103420/A_1 - 250\right) + u_4\left(40925/A_2 - 250\right)$

The KKT necessary conditions are:

$\partial L/\partial A_1 = 9.8125\times10^{-3} - u_2\left(103420/A_1^2\right) = 0;$

$\partial L/\partial A_1 = 9.8125\times10^{-3} - u_2\left(103420/A_1^2\right) = 0;$

$g_i \le 0, u_i g_i \le 0, u_i \ge 0$; i =1 to 5

From the equations, we get $u_2 = 1.6237\times10^{-2}$, $u^4 = 6.425\times10^{-3} > 0$ (o.k.). All the conditions are satisfied. Thus, the point ($A_1^* = 413.68$, $A_2^* = 163.7$) satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

5.73

Referring to the formulation in Exercise 3.47, we have

Minimize $f = 0.1233Rt$, subject to $g_1 = 16459.5/(Rt) - 250 \le 0$;

$g_2 = 1.034188 \times 10^5 - 4.16726R^3t \le 0;\ \ g_3 = R/t - 50 \le 0;$

$g_4 = 20 - R \le 0;\ \ g_5 = R - 400 \le 0;\ g_6 = 2 - t \le 0;\ g_7 = t - 40 \le 0$

The optimum solutions obtained by the graphical method are the points on the line segment between "a" and "b". We shall check the point "a" where $R^* = 20.$, $t^* = 3.3$ together with g_1 and g_4 active.

1. Check for necessary conditions.

Since only g_1 and g_4 are active, we can set $u_2 = u_3 = u_5 = u_6 = u_7 = 0$.

$L = 0.1233Rt + \underline{u}_1[16459.5/(Rt) - 250] + u_4(20 - R)$

The KKT necessary conditions are:

$\partial L/\partial R = 0.1233t - u_1\left[16459.5/R^2t\right] - u_4 = 0;$

$\partial L/\partial t = 0.123R - u_1\left[16459.5/Rt^2\right] = 0;$

$g_i \le 0,\ \ u_i\, g_i = 0;\ \ u_i \ge 0;\ \ i = 1 \text{ to } 7$

Substituting the optimum values, we get $u_1 = 0.03263 > 0$, $u_4 = 0 \ge 0$ (o.k.). All the other conditions are also satisfied. Thus, the point ($R^* = 20$, $t^* = 3.3$) satisfies the necessary conditions.

2. Check for sufficient condition. We shall consider the general case where only g_1 is active. The Hessian of Lagrangian is

$$\tilde{\mathbf{N}}^2 L = \begin{bmatrix} \dfrac{u_1(16459.5)(2)}{R^3t} & 0.1233 + \dfrac{u_1(16459.5)}{R^2t^2} \\ 0.1233 + \dfrac{u_1(16459.5)}{R^2t^2} & \dfrac{u_1(16459.5)(2)}{Rt^3} \end{bmatrix}$$

$\tilde{\mathbf{N}}g_1 = \begin{bmatrix} 16459.5/R^2t \\ -16359.5/Rt^2 \end{bmatrix}$; $\tilde{\mathbf{N}}g_1^T\, \mathbf{d} = 0$ gives $\mathbf{d} = c\left(1,\ -t/R\right)^T$, $c \ne 0$. Hence $Q = \mathbf{d}^T\, \tilde{\mathbf{N}}^2 L\, \mathbf{d} = 0$. Thus, the sufficient condition is not satisfied.

3. The effect of variations in constraint limits on cost function.

$\Delta f = -u_1 e_1 = 0.03263 e_1$

5.74

Referring to the formulation in Exercise 3.48, we have

Minimize $f = 1.57\times10^{-5} A(h^2+5.625\times10^{5})^{1/2}$,

subject to $g_1 = (250)\left(h^2 + 5.625\times10^{5}\right)^{\frac{1}{2}}(100/h + 0.2309)/A - 250 \le 0$;

$g_2 = 50000\left(h^2 + 5.625\times10^{5}\right)^{\frac{1}{2}}(100/h + 0.2309)/A - 250 \le 0$. The optimum solution obtained by the graphical method is $A^* = 390$, $h^* = 500$, $f^* = 5.5$ where g_1 and g_7 are active.

Check for necessary conditions. Since only g_1 and g_7 are active, we can set $u_2 = u_3 = u_4 = u_6 = u_8 = 0$.

$$L = 1.57\times10^{-5} A\left(h^2 + 5.625\times10^{5}\right)^{\frac{1}{2}} + u_1\left[(250)\left(h^2 + 5.625\times10^{5}\right)^{\frac{1}{2}}(100/h + 0.2309)/A - 250\right]$$

$$+ u_7(500 - h)$$

The KKT necessary conditions are:

$$\partial L/\partial A = 1.57\times10^{-5}\left(h^2 + 5.625\times10^{-5}\right)^{\frac{1}{2}} - u_1\left[(250)\left(h^2 + 5.625\times10^{5}\right)^{\frac{1}{2}}(100/h + 0.2309)/A^2\right] = 0$$

$$\partial L/\partial h = 1.57\times10^{-5} Ah\left(h^2 + 5.625\times10^{5}\right)^{-\frac{1}{2}} + u_1(250)$$

$$\left[h\left(h^2 + 5.625\times10^{5}\right)^{-\frac{1}{2}}(100/h + 0.2309)\Big/A + \left(h^2 + 5.625\times10^{5}\right)^{\frac{1}{2}}\left(-100/h^2\right)\Big/A\right] - u_7 = 0$$

$g_i \le 0$, $u_i g_i = 0$, $u_i \ge 0$; $i = 1$ to 8

Substituting the optimum values, we obtain $u_1 = 2.21610\times10^{-2} > 0, u_7 > 0$, $u_7 = 1.67\times10^{-3}$ (o.k.). All other conditions are satisfied. Thus, the point ($A^* = 390$, $h^* = 500$) satisfies the necessary condition.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function

$$\Delta f = -u_1 e_1 - u_7 e_7 = -2.216\times10^{-2} e_1 - 1.67\times10^{-3} e_7$$

5.75

Referring to the formulation in Exercise 3.49, we have

Minimize $f = 1.57 \times 10^{-5} A\left(0.25s^2 + 10^6\right)^{0.5}$,

Subject to $g_1 = (250)\left(0.25s^2 + 10^6\right)^{0.5}(0.1 + 346.4/s)/A - 250 \le 0$;

$g_2 = \left(10^6 + 0.25s^2\right)^{0.5}(25 + 86602.5/s) - \left(2.072617 \times 10^7\right)A^2 / \left(10^6 + 0.25s^2\right) \le 0$;

$g_3 = (50)\left(0.25s^2 + 10^6\right)^{1.5} / \left(60.62As^2\right) - (50) \le 0$;

$g_4 = (50)\left(0.25s^2 + 10^6\right)^{1.5} / \left(4.2 \times 10^8 A\right) - (50) \le 0$; $g_5 = 100 - A \le 0$; $g_6 = A - 5000 \le 0$;

$g_7 = 500 - s \le 0$; $g_8 = s - 4000 \le 0$. The optimum solution obtained by the graphical method is A^* = 415, s^* = 1480, f^* = 8.1 where g_1 is active.

1. Check for necessary conditions.

The only active constraint is g_1, so we can set $u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$.

$$L = 1.57 \times 10^{-5} A\left(0.25s^2 + 10^6\right)^{\frac{1}{2}} + u_1\left[(250)\left(0.25s^2 + 10^6\right)^{0.5}(0.1 + 346.4/s)/A - 250\right]$$

The KKT necessary conditions are:

$$\partial L/\partial A = 1.57 \times 10^{-5}\left(0.25s^2 + 10^6\right)^{0.5} - u_1\left[(250)\left(0.25s^2 + 10^6\right)^{0.5}(0.1 + 346.4/s)/A^2\right] = 0$$

$$\partial L/\partial s = 1.57 \times 10^{-5} A(0.25s)\left(0.25s^2 + 10^6\right)^{-0.5} + u_1(250)\Big[0.25s\left(0.25s^2 + 10^6\right)^{-0.5}$$

$$(0.1 + 346.4/s)/A - \left(0.25s^2 + 10^6\right)^{0.5}\left(346.4/s^2\right)/A\Big] = 0$$

$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i$ = 1 to 8

Substituting the optimum values, we obtain u_1 = 0.03252 > 0, u_1 = 0.03304 > 0 (o.k.). The difference is acceptable since the optimum values are obtained from graphical method which has only limited accuracy. For the following calculation, we assume $u_1 \doteq 0.0328$. All the other conditions are also satisfied. Thus, the point (A^* = 415, s^* = 1480) satisfies the necessary conditions.

2. Check for sufficient condition.

$$\nabla^2 L = 10^{-6}\begin{bmatrix} 9.535 & 9.420 \\ 9.420 & 5.097 \end{bmatrix};\ \ \nabla g_1 = (250)\begin{bmatrix} -2.413 \times 10^{-3} \\ -2.346 \times 10^{-4} \end{bmatrix}$$

$\nabla g_1^T \mathbf{d} = 0$ gives $\mathbf{d} = c(1, -10.29)$, $c \ne 0$. Hence $Q = \mathbf{d}^T\left(\nabla^2 L\right)\mathbf{d} = 4.41 \times 10^{-4} c^2 > 0$. Thus, the sufficient condition is satisfied.

3. The effect of variations in constraint limits on cost function

$\Delta f = -u_1 e_1 = -0.0328 e_1$

5.76

Referring to the formulation in Exercise 3.50, we have

Minimize $f = 7.85\times10^{-3}\left(2\sqrt{2}\,A_1 + A_2\right)$,

subject to $g_1 = 30618.6/A_1 + 17677.65\Big/\left(A_1 + \sqrt{2}A_2\right) - 150 \le 0$; $g_2 = 35355.3\Big/\left(A_1 + \sqrt{2}A_2\right) - 150 \le 0$; $g_3 = 50 - A_1 \le 0$; $g_4 = A_1 - 5000 \le 0$; $g_5 = 50 - A_2 \le 0$; $g_6 = A_2 - 5000 \le 0$. The optimum solution obtained by the graphical method is $A_1^* = 300$, $A_2^* = 50$, $f^* = 7.0$ where g_1 and g_5 are active.

1. Check for necessary conditions. Since only g_5 are active, we can set $u_2 = u_3 = u_4 = u_6 = 0$.

$$L = 7.85\times10^{-3}\left(2\sqrt{2}A_1 + A_2\right) + u_1\left[30618.6\Big/A_1 + 17677.65\Big/\left(A_1 + \sqrt{2}A_2\right) - 150\right] + u_5\left(50 - A_2\right)$$

The KKT necessary conditions are:

$$\partial L/\partial A_1 = 7.85\times10^{-3}\left(2\sqrt{2}\right) + u_1\left[-30618.6\Big/A_1^2 - 17677.65\Big/\left(A_1 + \sqrt{2}A_2\right)^2\right] = 0$$

$$\partial L/\partial A_2 = 7.85\times10^{-3} + u_1\left[-17677.65\left(\sqrt{2}\right)\Big/\left(A_1 + \sqrt{2}A_2\right)^2\right] - u_5 = 0$$

$$g_i \le 0,\; u_i g_i = 0,\; u_i = 0;\; i = 1 \text{ to } 6$$

Substituting the optimum values, we obtain $u_1 = 0.0473 > 0$, $u_5 = 0 \ge 0$ (o.k.). All the other conditions in (3) are also satisfied. Thus, the point $\left(A_1^* = 300,\; A_2^* = 50.\right)$ satisfies the necessary conditions.

2. Check for sufficient condition

$$\tilde{\mathbf{N}}^2 L = \begin{bmatrix} 1.541\times10^{-4} & 9.322\times10^{-6} \\ 9.322\times10^{-6} & 1.318\times10^{-5} \end{bmatrix}$$

$$\tilde{\mathbf{N}}g_1 = (150)\begin{bmatrix} -204.1\Big/A_1^2 - 117.9\Big/\left(A_1 + \sqrt{2}A_2\right)^2 \\ -117.9\left(\sqrt{2}\right)\Big/\left(A_1 + \sqrt{2}A_2\right)^2 \end{bmatrix} = (150)\begin{bmatrix} -3.126\times10^{-3} \\ -1.213\times10^{-3} \end{bmatrix}$$

$\tilde{\mathbf{N}}g_1^T\,\mathbf{d} = 0$ gives $\mathbf{d} = c\left(1,\ -2.58\right)^T$, $c \ne 0$. Hence $Q = \mathbf{d}^{\mathrm{T}}\left(\tilde{\mathbf{N}}^2 L\right)\mathbf{d} = 1.937\times10^{-4}c^2 > 0$, if c $\ne 0$. Thus, sufficient condition is satisfied.

3. The effect of variations in constraint limits on cost function

$\Delta f = -u_1 e_1 = -0.0473 e_1$.

5.77

Referring to the formulation in Exercise 3.51, we have

Minimize f = 153.7R t,

subject to $g_1 = \dfrac{4.045\times10^6}{R^3t+Rt^3/4}+\dfrac{5.257\times10^3\left(2R+t\right)}{R^3t+Rt^3/4}+\dfrac{1.649\times10^{10}\left(2R+t\right)}{\left(R^3t+Rt^3/4\right)^2}-1\le 0;$

$g_2 = 70-2R+t\le 0;\ \ g_3 = 2R/t-91\le 0;\ \ g_4 = 4.4794\times10^7\big/\left(R^3t+Rt^3/4\right)-20\le 0;$

$g_5 = R-0.5t-250\le 0;\ \ g_6 = 35-R+0.5t\le 0;\ \ g_7 = t-40\le 0;\ \ g_8 = 1-t\le 0.$

The optimum solution obtained by the graphical method is R^* = 130, t^* =2.86, f^* = 57000 where g_1 and g_3 are active.

1. Check for necessary conditions.

Since only g_1 and g_3 are active, we can set $u_2 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$.

$$L = 153.7R\,t + u_1\left[\frac{4.045\times10^6}{R^3t+Rt^3/4}+\frac{5.257\times10^3\left(2R+t\right)}{R^3t+Rt^3/4}+\frac{1.649\times10^{10}\left(2R+t\right)}{\left(R^3t+Rt^3/4\right)^2}-1\right]+u_3\left(\frac{2R}{t}-91\right)$$

The KKT necessary conditions are:

$$\frac{\partial L}{\partial R} = 153.7t+u_1\Bigg[\frac{-4.045\times10^6\left(3R^2t+t^3/4\right)}{\left(R^3t+Rt^3/4\right)^2}+\frac{5.257\times10^3\left(2\right)}{R^3t+Rt^3/4}-\frac{5.257\times10^3\left(2R+t\right)\left(3R^2t+t^3/4\right)}{\left(R^3t+Rt^3/4\right)^2}$$

$$+\frac{1.649\times10^{10}\left(2\right)}{\left(R^3t+Rt^3/4\right)}-\frac{1.649\times10^{10}\left(2R+t\right)\left(R^3t+Rt^3/4\right)\left(3R^2t+t^3/4\right)}{\left(R^3t+Rt^3/4\right)^2}\Bigg] + u_3\left(2/t\right)=0 \qquad (1)$$

$$\frac{\partial L}{\partial t}=153.7R+u_1\Bigg[\frac{-4.045\times10^6\left(3R^2t+t^3/4\right)}{\left(R^3t+Rt^3/4\right)^2}+\frac{5.257\times10^3}{\left(R^3t+Rt^3/4\right)}-\frac{5.257\times10^3\left(2R+t\right)\left(R^3t+3Rt^2/4\right)}{\left(R^3t+Rt^3/4\right)}$$

$$+\frac{1.649\times10^{10}}{\left(R^3t+Rt^3/4\right)^2}-\frac{1.649\times10^{10}\left(2R+t\right)\left(2\right)\left(R^3t+Rt^3/4\right)\left(R^3+3Rt^2/4\right)}{\left(R^3t+Rt^3/4\right)^4}\Bigg]+u_3\left(-2R/t^2\right)=0 \qquad (2)$$

$$g_i\le 0,\ u_ig_i=0,\ u_i\ge 0;\quad i=1 \text{ to } 8 \qquad (3)$$

Substituting the optimum values into (1) and (2), we get $u_1 = 2.817\times10^4 > 0, u_3 =$ 294.07 > 0 (o.k.). All the other conditions are also satisfied. Therefore, the point $\left(R^* = 130, t^* = 2.86\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function.

$\Delta f = -u_1e_1-u_3e_3 = -2.817\times10^4e_1-294.07e_3$.

5.78

Referring to the formulation in Exercise 3.52, we have

Minimize $f = 6.126\left(d_o^2 - d_i^2\right)$, subject to $g_1 = \dfrac{1.42603\times10^8 d_o}{\left(d_o^4 - d_i^4\right)} - 1.65\times10^4 \le 0$;

$$g_2 = \frac{40743.7\left(d_o^2 + d_o d_i + d_i^2\right)}{\left(d_o^4 - d_i^4\right)} - 5000 \le 0; \quad g_3 = \frac{3.719\times10^6}{\left(d_o^4 - d_i^4\right)} - 10 \le 0;$$

$$g_4 = \frac{\left(d_o + d_i\right)}{\left(d_o - d_i\right)} - 60 \le 0; \quad g_5 = \frac{\left(d_o - d_i\right)}{2} - 2 \le 0; \quad g_6 = 0.5 - \left(d_o + d_i\right)\big/2 \le 0;$$

$$g_7 = d_o - 50 \le 0; \quad g_8 = 5 - d_o \le 0; \quad g_9 = d_i - 45 \le 0; \quad g_{10} = 4 - d_i \le 0.$$

The optimum solution obtained by the graphical method is $d_o^* = 41.6, \mathrm{d}_i^* = 40.2,\ f^* = 680$ where g_3 and g_4 are active.

1. Check for necessary conditions.

Since only g_3 and g_4 are active, we can set $u_1 = u_2 = u_5 = u_6 = u_7 = u_8 = u_9 = u_{10} = 0$.

$$L = 6.126\left(d_o^2 - d_i^2\right) + u_3\left(\frac{3.719\times10^6}{\left(d_o^4 - d_i^4\right)} - 10\right) + u_4\left[\frac{\left(d_o + d_i\right)}{\left(d_o - d_i\right)} - 60\right]$$

The KKT necessary conditions are:

$$\frac{\partial L}{\partial d_o} = 12.252 d_o + u_3\left[\frac{-3.719\times10^6(4)d_o^3}{\left(d_o^4 - d_i^4\right)^2} - 10\right] + u_4\left[\frac{\left(d_o - d_i\right) - \left(d_o + d_i\right)}{\left(d_o - d_i\right)^2}\right] = 0$$

$$\frac{\partial L}{\partial d_i} = -12.252 d_i + u_3\left[\frac{3.719\times10^6(4)d_i^3}{\left(d_o^4 - d_i^4\right)^2} - 10\right] + u_4\left[\frac{\left(d_o - d_i\right) + \left(d_o + d_i\right)}{\left(d_o - d_i\right)^2}\right] = 0$$

$$g_i \le 0, \quad u_i g_i = 0, \quad u_i \ge 0; \quad i = 1 \text{ to } 10$$

Substituting the optimum values, we get $u_3 = 35.71 > 0, u_4 = 6.067 > 0$ (o.k.). All other conditions are satisfied. Thus, the point $\left(d_o^* = 41.6, d_i^* = 40.2\right)$ satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations in constraint limits on cost function.

$$\Delta f = -u_3 e_3 - u_4 e_4 = -35.71 e_3 - 6.067 e_4.$$

5.79

Referring to the formulation in Exercise 3.53, we have

Minimize $f = 5.027t(d_o - t)$,

subject to $g_1 = \dfrac{1.688\times10^9}{t(d_o-t)\left[d_o^2+(d_o-2t)^4\right]} + \dfrac{4.098\times10^7 d_o}{d_o^4-(d_o-2t)^4} + \dfrac{1.435\times10^{17} d_o}{\left[d_o^4-(d_o-2t)^4\right]^2} - 1 \le 0$

$g_2 = d_o/t - 9 \le 0;\ g_3 = 2.466\times10^{13}\Big/\left[d_o^4-(d_o-2t)^4\right] - 100 \le 0;$

$g_4 = 250 - d_o \le 0;\ g_5 = d_o - 1500 \le 0;\ g_6 = 5 - t \le 0;$

$g_7 = t - 100 \le 0;$

The optimum solution obtained by the graphical method is $d_o^* \doteq 1310,\ t^* \doteq 14.2,\ f^* = 92500$ where g_2 and g_3 are active.

1. Check for necessary conditions.

Since only g_2 and g_3 are active, we can set $u_1 = u_2 = u_4 = u_5 = u_6 = u_7 = 0$.

$L = 5.027t(d_o - t) + u_2\left[d_o/t - 92\right] + u_3\left[2.466\times10^{13}\Big/\left(d_o^4-(d_o-2t)^4\right) - 100\right]$

The KKT necessary conditions are:

$\partial L/\partial d_o = 5.027 + u_2/t - u_3\left[4\left(2.466\times10^{13}\right)\left(d_o^3-(d_o-2t)^3\right)\Big/\left(d_o^4-(d_o-2t)^4\right)^2\right. = 0$

$\partial L/\partial t = 5.027 + (d_o - 2t) + u_2\left(-d_o/t^2\right) - u_3\left[8\left(2.466\times10^{13}\right)(d_o-2t)^3\Big/\left(d_o^4-(d_o-2t)^4\right)2\right] = 0$

$g_i \le 0,\ u_i g_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 7$

Substituting the optimum values, we get $u_2 = 5.08\times10^2 > 0$, $u_3 = 4.624\times10^2 > 0$ (o.k.). All the other conditions are also satisfied. Thus, the point satisfies the necessary conditions.

2. Check for sufficient condition. Use the same argument as in Exercise 5.53.

3. The effect of variations of constraint limits on cost function.

$\Delta f = -u_1e_1 - u_2e_2 = -5.082\times10^2 e_2 - 4.624\times10^2 e_3.$

5.80

The formulation is given in Exercise 3.54.
Optimum solution and check of necessary and sufficient conditions can be found in Exercise 5.48.

Referring to the formulation of Exercise 3.54, we have

Minimize $f=\left(6.59734-10^{-3}\right)D^2\left(H^2+4800\right)^{1/2}$, subject to

$$g_1=\left(2.546475\times10^4\right)\left(H^2+4800\right)^{\frac{1}{2}}\Big/\left(D^2H\right)-1.5\times10^4\le 0;$$

$$g_2=\left(2.0\times10^4\right)\left(H^2+4800\right)^{\frac{1}{2}}\Big/H-\left(1.816774\times10^6\right)D^4\Big/\left(H^2+4800\right)\le 0;$$

$$g_3=H-500\le 0;\ \ g_4=50-H\le 0;\ \ g_5=D-50\le 0;\ \ g_6=0.5-D\le 0.$$

The optimum solution obtained by the graphical method is $H^*=50, D^*\doteq 3.42, f^*\doteq 6.6$, where g_2 and g_4 are active.

1. Check for necessary conditions. Since only g_2 and g_4 are active, we can set $u_1=u_3=u_5=u_6=0$ and $s_2=s_4=0$.

$$L=\left(6.59734\times10^{-3}\right)D^2\left(H^2+4800\right)^{\frac{1}{2}}+$$
$$u_2\left[\left(2.0\times10^4\right)\left(H^2+4800\right)^{\frac{1}{2}}\Big/H-\left(1.816774\times10^6\right)D^4\Big/\left(H^2+4800\right)\right]+u_4\left(50-H\right)$$

The KKT necessary conditions are

$$\partial L/\partial D=0.0131947D\left(H^2+4800\right)^{\frac{1}{2}}+u_2\left[\left(-7.267096\times10^6\right)D^3\Big/\left(H^2+4800\right)\right]=0 \tag{1}$$

$$\partial L/\partial H=\left(6.59734\times10^{-3}\right)D^2H\left(H^2+4800\right)^{\frac{1}{2}}+u_2\left[-\left(2.0\times10^4\right)\left(H^2+4800\right)^{\frac{1}{2}}\Big/H^2\right.$$
$$\left.+\left(2.0\times10^4\right)\left(H^2+4800\right)^{\frac{1}{2}}+\left(3.633548\times10^6\right)D^4H\Big/\left(H^2+4800\right)^2\right]-u_4=0 \tag{2}$$

$$g_i+s_i^2=0,\ \ u_is_i=0,\ \ u_i\ge 0;\ \ i = 1 \text{ to } 6 \tag{3}$$

Substituting the optimum value into (1) and (2), we get $u_2=9.68\times10^{-5}>0,\ u_4=4.68\times10^{-2}>0$ (o.k.)

2. Check for sufficient condition. Since the number of active constraints is equal to the number of design variables, the point $\left(H^*=50, D^*=3.42\right)$ is indeed an isolated local minimum point. The sufficient condition is deemed satisfied at this point.

The Lagrange multipliers of active constraints are $u_2=9.68\times10^{-5}$, $u_4=4.68\times10^{-2}$.

Therefore, $\Delta f=-u_2e_2-u_4e_4=-9.68\times10^{-5}e_2-4.68\times10^{-2}e_4$.