C H A P T E R

4

Optimum Design Concepts

Optimality Conditions

Section 4.2 Review of Some Basic Calculus Concepts

4.1———————————————————————————————————

Answer True or False.

1. A function can have several local minimum points in a small neighborhood of x^*. *True*
2. A function cannot have more than one global minimum point. *False*
3. The value of the function having global minimum at several points must be the same. *True*
4. A function defined on an open set cannot have a global minimum. *False*
5. The gradient of a function $f(x)$ at a point is normal to the surface defined by the level surface $f(x)$ = constant. *True*
6. Gradient of a function at a point gives a local direction of maximum decrease in the function. *False*
7. The Hessian matrix of a continuously differentiable function can be asymmetric. *False*
8. The Hessian matrix for a function is calculated using only the first derivatives of the function. *False*
9. Taylor series expansion for a function at a point uses the function value and its derivatives. *True*
10. Taylor series expansion can be written at a point where the function is discontinuous. *False*
11. Taylor series expansion of a complicated function replaces it with a polynomial function at the point. *True*
12. Linear Taylor series expansion of a complicated function at a point is only a good local approximation for the function. *True*
13. A quadratic form can have first-order terms in the variables. *False*
14. For a given $\mathbf{x}$, the quadratic form defines a vector. *False*
15. Every quadratic form has a symmetric matrix associated with it. *True*
16. A symmetric matrix is positive definite if its eigenvalues are nonnegative. *False*
17. A matrix is positive semidefinite if some of its eigenvalues are negative and others are nonnegative. *False*
18. All eigenvalues of a negative definite matrix are strictly negative. *True*
19. The quadratic form appears as one of the terms in Taylor's expansion of a function. *True*
20. A positive definite quadratic form must have positive value for any $\mathbf{x} \neq \mathbf{0}$. *True*

4.2

Write the Taylor series expansion for the following function up to quadratic terms.

cos$x$ about the point $x^* = \pi/4$

Solution

$f(x) = \cos x;\ f(\pi/4) = \cos(\pi/4) = 1/\sqrt{2};\ \ f'(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2};$

$f''(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2};\ \ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2 \cos x$

$= (1/\sqrt{2}) - (1/\sqrt{2})(x - \pi/4) + 0.5(-1/\sqrt{2})(x - \pi/4)^2 = 1.0444 - 0.15175x - 0.35355x^2$

4.3

Write the Taylor series expansion for the following function up to quadratic terms.

cos$x$ about the point $x^* = \pi/3$

Solution

$f(x) = \cos x;\ f(\pi/3) = \cos(\pi/3) = 1/2;\ f'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2;$

$f''(\pi/3) = -\cos(\pi/3) = -1/2;\ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2$

$\cos x = (1/\sqrt{2}) - (\sqrt{3}/2)(x - \pi/3) + 0.5(-1/2)(x - \pi/3)^2 = 1.1327 - 0.34243x - 0.25x^2$

4.4

Write the Taylor series expansion for the following function up to quadratic terms.

sin$x$ about the point $x^* = \pi/6$

Solution

$f(x) = \sin x;\ f(\pi/6) = \sin(\pi/6) = 1/2;\ f'(\pi/6) = \cos(\pi/6) = -\sqrt{3}/2;$

$f''(\pi/3) = -\cos(\pi/3) = -1/2;\ \ \bar{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2$

$\sin x = 1/2 - (\sqrt{3}/2)(x - \pi/6) + 0.5(-1/2)(x - \pi/6)^2 = -0.02199 + 1.12783x - 0.25x^2$

4.5

Write the Taylor series expansion for the following function up to quadratic terms.

sin x about the point $x^* = \pi/4$

Solution

$$f(x) = \sin x;\ \ f\ (\pi/4) = \sin\ (\pi/4) = 1/\sqrt{2};\ f'(\pi/4) = \cos\ (\pi/4) = -1/\sqrt{2};$$

$$f''(\pi/4) = -\sin\ (\pi/4) = -1/\sqrt{2};\ \ \overline{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2 \sin x$$

$$= 1/\sqrt{2} + (1/\sqrt{2})(x - \pi/4) + 0.5(-1/\sqrt{2})(x - \pi/4)^2 = (1 + x - \pi/4 - x^2/2 + \pi x/4 - \pi^2/32)/\sqrt{2}$$

$$= \left[(1 - \pi/4 - \pi^2/32) + (1 + \pi/4)x - x^2/2\right]/\sqrt{2} = 0.06634 + 1.2625x - 0.35355x^2$$

4.6

Write the Taylor series expansion for the following function up to quadratic terms.

e^x about the point $x^*=0$

Solution

$$f(x) = e^x; f(x) = f'(x) = f''(x) = e^x; f(0) = f'(0) = f''(0) = 1$$

$$\overline{f}(x) = f(x^*) = +f'(x^*) = (x - x^*) + 0.5 f''(x)(x - x^*)^2$$

$$e^x = e^0 + e^0(x - 0) + 0.5e^0(x - 0)^2 = 1 + x + 0.5x^2$$

4.7

Write the Taylor series expansion for the following function up to quadratic terms.

e^xabout the point $x^*=2$

Solution

$$f(x) = e^x,\ x^* = 2;\ f'(x) = e^x;\ f''(x) = e^x;\ f(x^*) = f'(x^*) = f''(x^*) = e^2 = 7.389$$

$$\overline{f}(x) = f(x^*) + f'(x^*)(x - x^*) + 0.5 f''(x^*)(x - x^*)^2$$

$$e^x = 7.389 + 7.389(x - 2) + 0.5(7.389)(x - 2)^2 = 7.389 - 7.389x + 3.6945x^2$$

4.8

Write the Taylor series expansion for the following function up to quadratic terms.

$f(x_1, x_2) = 10x_1^4 - 20x_1^2x_2 + 10x_2^2 + x_1^2 - 2x_1 + 5$ about the point (1,1). Compare approximate and exact values of the function at the point (1.2,0.8).

Solution

$$f(x_1, x_2) = 10x_1^4 - 20x_1^2x_2 + 10x_2^2 + x_1^2 - 2x_1 + 5;\ x^* = (1, 1)$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 40x_1^3 - 40x_1x_2 + 2x_1 - 2 \\ -20x_1^2 + 20x_2 \end{bmatrix};\ \ \mathbf{H}(x_1, x_2) = \begin{bmatrix} 120x_1^2 - 40x_2 + 2 & -40x_1 \\ -40x_1 & 20 \end{bmatrix}$$

$$\overline{f}(x_1, x_2) = f(x^*) + \nabla f^T(x^*)(x - x^*) + 0.5(x - x^*)^T \mathbf{H}(x^*)(x - x^*)$$

$$f(x^*) = 4;\ \nabla f(x^*) = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ \ \mathbf{H}(x^*) = \begin{bmatrix} 82 & -40 \\ -40 & 20 \end{bmatrix}$$

$$\overline{f}(x_1, x_2) = 4 + \tfrac{1}{2}\begin{bmatrix} (x_1 - 1) \\ (x_2 - 1) \end{bmatrix}^T \begin{bmatrix} 82 & -40 \\ -40 & 20 \end{bmatrix}\begin{bmatrix} (x_1 - 1) \\ (x_2 - 1) \end{bmatrix} = 41x_1^2 - 42x_1 - 40x_1x_2 + 20x_2 + 10x_2^2 + 15$$

$$f(1.2, 0.8) = 8.136;\ \ \overline{f}(1.2, 0.8) = 7.64;\ \ \text{Error} = f - \overline{f} = 0.496$$

Determine the nature of the following quadratic forms.

4.9

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 + 4x_1x_2 + 2x_1x_3 - 7x_2^2 - 6x_2x_3 + 5x_3^2$

Solution

$$F(\mathbf{x}) = x_1^2 + 4x_1x_2 + 2x_1x_3 - 7x_2^2 - 6x_2x_3 + 5x_3^2$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -7 & -3 \\ 1 & -3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};\ \ \mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -7 & -3 \\ 1 & -3 & 5 \end{bmatrix}$$

Principal Minors: $M_1 = 1 > 0$; $M_2 = (-7) - (2)(2) = -11 < 0$; $M_3 = -69 < 0$. Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, **A** is **indefinite**, so is the quadratic form.

4.10

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = 2x_1^2 + 2x_2^2 - 5x_1x_2$

Solution

$$F(\mathbf{x}) = 2x_1^2 + 2x_2^2 - 5x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -2.5 \\ -2.5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -2.5 \\ -2.5 & 2 \end{bmatrix}; \quad \text{Principal Minors: } \mathrm{M}_1 = 2 > 0; \quad \mathrm{M}_2 = -2.25 < 0$$

Since $M_1 > 0$ and $M_2 < 0$, **A** is indefinite, so the quadratic form is **indefinite**.

4.11

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 + x_2^2 + 3x_1x_2$

Solution

$F(\mathbf{x}) = x_1^2 + x_2^2 + 3x_1x_2$; (note that the factor of 0.5 does not affect the form of the matrix)

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1.5 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \ \mathbf{A} = \begin{bmatrix} 1 & 1.5 \\ 1.5 & 1 \end{bmatrix}; \ \text{Eigenvalue problem:}$$

$$\begin{vmatrix} 1-\lambda & 1.5 \\ 1.5 & 1-\lambda \end{vmatrix} = 0 \ ; \ (1-\lambda)(1-\lambda) - 1.5^2 = 0; \ \lambda^2 - 2\lambda - 1.25 = 0; \ \lambda_1 = -0.5, \ \lambda_2 = 2.5$$

The matrix and the quadratic form are **indefinite** since one eigenvalue is positive and the other negative.

4.12

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = 3x_1^2 + x_2^2 - x_1x_2$

Solution

$$F(\mathbf{x}) = 3x_1^2 + x_2^2 - x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 1 \end{bmatrix}; \quad \text{Principal Minors: } M_1 = 3 > 0; \quad M_2 = 2.75 > 0$$

Since $M_1 > 0$ and $M_2 > 0$, so the quadratic form is **positive definite**.

4.13

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 - x_2^2 + 4x_1x_2$

Solution

$$F(\mathbf{x}) = x_1^2 - x_2^2 + 4x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$; Principal Minors: $M_1 = 1 > 0$; $M_2 = -5 < 0$

Since $M_1 > 0$ and $M_2 < 0$, the matrix is **indefinite** and so is the quadratic form.

4.14

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 - x_2^2 + x_3^2 - 2x_2x_3$

Solution

$$F(\mathbf{x}) = x_1^2 - x_2^2 + x_3^2 - 2x_2x_3$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Principal Minors: $M_1 = 1 > 0$, $M_2 = -1 < 0$, $M_3 = |\mathbf{A}| = 1(-1-1) = -2 < 0$

Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, so the quadratic form is **indefinite**.

4.15

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 - 2x_1x_2 + 2x_2^2$

Solution

$$F(\mathbf{x}) = x_1^2 - 2x_1x_2 + 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$; Principal Minors: $M_1 = 1 > 0$, $M_2 = 1 > 0$

Since $M_1 > 0$ and $M_2 > 0$, the quadratic form is **positive definite**.

4.16

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 - x_1x_2 - x_2^2$

Solution

$$F(\mathbf{x}) = x_1^2 - x_1x_2 - x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & -0.5 \\ -0.5 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5 \\ -0.5 & -1 \end{bmatrix};\ \text{Principal Minors: } M_1 = 1 > 0,\ M_2 = -1.25 < 0$$

Since $M_1 > 0$ and $M_2 < 0$, the quadratic form is **indefinite**.

4.17

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 + 2x_1x_3 - 2x_2^2 + 4x_3^2 - 2x_2x_3$

Solution

$$F(\mathbf{x}) = x_1^2 + 2x_1x_3 - 2x_2^2 + 4x_3^2 - 2x_2x_3$$

$$= \left[\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\right]\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & 4 \end{bmatrix};\text{Principal Minors: } M_1 = 1 > 0,\ M_2 = -2 < 0, M_3 = -7 < 0$$

Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, the quadratic form is **indefinite**.

4.18

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = 2x_1^2 + x_1x_2 + 2x_2^2 + 4x_3^2 - 2x_1x_3$

Solution

$$F(\mathbf{x}) = 2x_1^2 + x_1x_2 + 2x_2^2 + 4x_3^2 - 2x_1x_3$$

$$= \left[\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\right]\begin{bmatrix} 2 & 0.5 & -1 \\ 0.5 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0.5 & -1 \\ 0.5 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix};\text{Principal Minors: } M_1 = 2 > 0,\ M_2 = 3.75 > 0, M_3 = 9.25 > 0$$

Since $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$, the quadratic form is **positive definite**.

4.19

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = x_1^2 + 2x_2x_3 + x_2^2 + 4x_3^2$

Solution

$$F(\mathbf{x}) = x_1^2 + 2x_2x_3 + x_2^2 + 4x_3^2 = \begin{bmatrix} [x_1 & x_2 & x_3] \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}; \text{Principal Minors: } M_1 = 1 > 0,\ M_2 = 1 > 0, M_3 = 3 > 0$$

Since $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$, the quadratic form is **positive definite**.

4.20 Det

Determine the nature of the following quadratic form.

$F(\boldsymbol{x}) = 4x_1^2 + 2x_1x_3 - x_2^2 + 4x_3^2$

Solution

$$F(\mathbf{x}) = 4x_1^2 + 2x_1x_3 - x_2^2 + 4x_3^2 = \begin{bmatrix} [x_1 & x_2 & x_3] \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}; \text{Principal Minors: } M_1 = 4 > 0,\ M_2 = -4 < 0, M_3 = -15 < 0$$

Since $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$, the quadratic form is **indefinite**.

Section 4.4 Optimality Conditions: Unconstrained Problems

4.21

Answer True or False.

1. If the first-order necessary condition at a point is satisfied for an unconstrained problem, it can be a local maximum point for the function. *True*
2. A point satisfying first-order necessary conditions for an unconstrained function may not be a local minimum point. *True*
3. A function can have a negative value at its maximum point. *True*
4. If a constant is added to a function, the location of its minimum point is changed. *False*
5. If a function is multiplied by a positive constant, the location of the function's minimum point is unchanged. *True*
6. If curvature of an unconstrained function of a single variable at the point x^* is zero, then it is a local maximum point for the function. *False*
7. The curvature of an unconstrained function at its local minimum point is negative. *False*
8. The Hessian of an unconstrained function at its local minimum point must be positive definite. *False*
9. The Hessian of an unconstrained function at its minimum point is negative definite. *False*
10. If the Hessian of unconstrained function is indefinite at a candidate point, the point may be a local maximum or minimum. *False*

4.22

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f\left(x_1, x_2\right) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$$

<u>Solution</u>

$$f\left(x_1, x_2\right) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7$$

The gradient and Hessian of $f\left(\mathbf{x}\right)$ are

$$\nabla f = \begin{bmatrix} 6x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}.$$

Setting gradient to zero gives **x** = (0, 0) as the only candidate minimum point.
Principal Minors of the Hessian: $M_1 = 6 > 0$, $M_2 = 20 > 0$. Since $M_1 > 0$ and $M_2 > 0$, the Hessian is positive definite. Therefore, the point (0, 0) is a local minimum point $\left(f = 7\right)$.

4.23

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3$

Solution

$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3;$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

Setting gradient to zero gives $\mathbf{x}$ = (0, 0) as the only candidate minimum point.

Eigenvalue test: $\left|\mathbf{H} - \lambda\mathbf{I}\right| = (2-\lambda)(2-\lambda) - 16 = 0;\ \lambda_1 = 6,\ \lambda_2 = -2$

Therefore, the Hessian is indefinite and second order necessary condition is violated. The stationary point (0, 0) is an inflection point.

4.24

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$

Solution

$f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2;$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 3x_1^2 + 12x_2^2 + 10x_1 \\ 24x_1x_2 + 4x_2 + 3 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 6x_1 + 10 & 24x_2 \\ 24x_2 & 24x_1 + 4 \end{bmatrix}$$

Setting the gradient to zero gives a nonlinear system of equations. Using Newton-Raphson method or any nonlinear equation solver, we find two solutions, as

$\mathbf{x}^{*1} = (-3.332, 0.0395);\quad \mathbf{x}^{*2} = (-0.398, 0.5404)$

$$\mathbf{H}(\mathbf{x}^{*1}) = \begin{bmatrix} -9.992 & 0.948 \\ 0.948 & -75.968 \end{bmatrix};\ \mathrm{M}_1 = -9.992 < 0,\ \mathrm{M}_2 = 758.17 > 0$$

$\mathbf{H}(\mathbf{x}^{*1})$ is negative definite. Therefore $\mathbf{x}^{*1} = (-3.332, 0.0395)$ is a local maximum point.

$$\mathbf{H}(\mathbf{x}^{*2}) = \begin{bmatrix} 7.612 & 12.970 \\ 12.970 & -5.552 \end{bmatrix};\ \mathrm{M}_1 = 7.612,\ \mathrm{M}_2 = -210.483$$

$\mathbf{H}(\mathbf{x}^{*2})$ is indefinite. Therefore $\mathbf{x}^{*2} = (-0.398, 0.5404)$ is an inflection point.

4.25

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 5x_1 - x_1^2 x_2/16 + x_2^2/4x_1$$

Solution

$$f(x_1, x_2) = 5x_1 - x_1^2 x_2/16 + x_2^2/4x_1$$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\nabla f = \begin{bmatrix} 5 - x_1 x_2/8 - x_2^2/4x_1^2 \\ -x_1^2/16 + x_2/2x_1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} -x_2/8 + x_2^2/2x_1^3 & -x_1/8 - x_2/2x_1^2 \\ -x_1/8 - x_2/2x_1^2 & 1/2x_1 \end{bmatrix}$$

When ∇f is set to zero the second equation gives $x_2 = x_1^3/8$.

Substituting into the first equation, we get

$5 - x_1^4/64 - x_1^4/256 = 0,\ (5/256)x_1^4 = 5;\ x_1 = \pm 4$.

For $x_1 = 4, x_2 = 8$, and $x_1 = -4, x_2 = -8$.

For the first point $(4, 8)$

$$\mathbf{H} = \begin{bmatrix} -8/8 + 64/2(64) & -4/8 - 8/2(16) \\ -4/8 - 8/2(16) & 1/2(4) \end{bmatrix} = \begin{bmatrix} -1/2 & -3/4 \\ -3/4 & -1/8 \end{bmatrix};$$

$M_1 = -1/2 < 0,\ M_2 = -5/8 < 0$

Since **H** is indefinite, the second order necessary condition is violated. Thus, point $(4, 8)$ is an **inflection** point.

For the second point $(-4, -8)$,

$$\mathbf{H} = \begin{bmatrix} -(-8)/8 + 64/2(-64) & -(-4)/8 - (-8)/2(16) \\ -(-4)/8 - (-8)/2(16) & 1/2(-4) \end{bmatrix} = \begin{bmatrix} 1/2 & 3/4 \\ 3/4 & -1/8 \end{bmatrix};.$$

$M_1 = 1/2 > 0, M_2 = -5/8 < 0$

Since **H** is indefinite, the second order necessary condition is violated. Thus, point $(-4, -8)$ is an **inflection** point.

4.26

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$f(x) = \cos x$

Solution

$f(x) = \cos x$

The necessary condition gives $f'(x) = -\sin x = 0$

The solution of necessary condition gives: $x = n\pi,\ n = 0,\ \pm 1,\ \pm 2, \ldots$

$f''(x) = -\cos x$; For $x = (2n+1)\pi,\ n = 0, \pm 1, \pm 2, \ldots,$

$f''(x) = -\cos[(2n+1)\pi] = 1 > 0.$

Thus, $x = (2n+1)\pi,\ n = 0, \pm 1, \pm 2, \ldots,$ are local minimum points $(f = -1)$.

For $x = 2n\pi,\ n = 0,\ \pm 1,\ \pm 2, \ldots\ f''(x) = -\cos(2n\pi) = -1 < 0.$

Thus, $x = 2n\pi,\ n = 0,\ \pm 1,\ \pm 2, \ldots$. are local maximum points $(f = 1)$.

4.27

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

Solution

$f(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$

The gradient and Hessian of $f(\mathbf{x})$ are

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Solution of necessary conditions of $\tilde{\mathbf{N}}f = \mathbf{0}$ gives $\mathbf{x}^* = (0,\ 0)$. The Hessian at $\mathbf{x}^*$ is positive definite since $M_1 = 2 > 0$ and $M_2 = 3 > 0$. Thus $(0, 0)$ is a local minimum point $(f = 0)$.

4.28

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$f(x) = x^2 e^{-x}$

Solution

$f(x) = x^2 e^{-x}$

The necessary conditions gives $f'(x) = 2xe^{-x} - x^2 e^{-x} = 0$; or $2x - x^2 = 0$.

Therefore, x = 0, 2 are the stationary points.

$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2 e^{-x} = \left(x^2 - 4x + 2\right)e^{-x}$

$f''(0) = 2 > 0$. Therefore, $x = 0$ is a local minimum point. f* = 0.

$f''(2) = -0.27067 < 0$. Therefore, $x = 2$ is a local maximum point. f* = 0.541.

4.29

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$f(x_1, x_2) = x_1 + 10/(x_1 x_2) + 5x_2$

Solution

$f(x_1, x_2) = x_1 + 10/(x_1 x_2) + 5x_2$

The necessary condition gives:

$\partial f/\partial x_1 = 1 - 10/\left(x_1^2 x_2\right) = 0;\ \ \partial f/\partial x_2 = -10/\left(x_1 x_2^2\right) + 5 = 0;$ or $x_1^2 x_2 = 10,\ \ 5x_1 x_2^2 = 10$

These equations give $x_1 = 5x_2$. Substituting the equation, we obtain $x_2 = 0.7368$. Therefore, $\mathbf{x}^* = \left(3.684,\ 0.7368\right)$ is a stationary point. Hessian is given as

$$\mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} 20/\left(x_1^3 x_2\right) & 10/\left(x_1^2 x_2^2\right) \\ 10/\left(x_1^2 x_2^2\right) & 20/\left(x_1 x_2^3\right) \end{bmatrix};\quad \mathbf{H} = \begin{bmatrix} 0.5429 & 1.3572 \\ 1.3572 & 13.572 \end{bmatrix}\quad \begin{matrix} \mathrm{M}_1 = 0.5429 > 0 \\ \mathrm{M}_2 = 5.526 > 0 \end{matrix}$$

Since Hessian is positive definite, $\mathbf{x}^* = \left(3.684,\ 0.7368\right)$ is a local minimum point $(f = 11.0521)$.

4.30

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = x_1^2 - 2x_1 + 4x_2^2 - 8x_2 + 6$$

Solution

$$f(x_1, x_2) = x_1^2 - 2x_1 + 4x_2^2 - 8x_2 + 6$$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 - 2 \\ 8x_2 - 8 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (1,1)$. For the Hessian **H**, $M_1 = 2 > 0$, $M_2 = 16 > 0$; so it is positive definite, and $(1,1)$ is a local minimum point $(f = 1)$.

4.31

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$$

Solution

$$f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 6x_1 - 2x_2 \\ -2x_1 + 10x_2 + 8 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (-2/7,\ -6/7)$. For the Hessian, $M_1 = 6 > 0$, $M_2 = 56 > 0$, so it is positive definite, and the point $(-2/7, -6/7)$ is a local minimum point $(f = -\frac{24}{7})$.

4.32

The annual operating cost U for an electrical line system is given by the following expression

$$U = \frac{(21.9 \times 10^7)}{V^2 C} + (3.9 \times 10^6)C + 1000V$$

where V = line voltage in kilovolts and C = line conductance in mhos. Find stationary points for the function, and determine V and C to minimize the operating cost.

Solution

$U = \left(21.9\times10^7\right)/\left(V^2C\right)+\left(3.9\times10^6\right)C+1000V$

The gradient and Hessian are given as

$$\nabla U = \begin{bmatrix} -43.8\times10^7/\left(V^3C\right)+1000 \\ -21.9\times10^7/\left(V^2C^2\right)+3.9\times10^6 \end{bmatrix}; \quad \mathbf{H}(V,C) = 10^7\begin{bmatrix} 131.4/\left(V^4C\right) & 43.8/\left(V^3C^2\right) \\ 43.8/\left(V^3C^2\right) & 43.8/\left(V^3C^2\right) \end{bmatrix};$$

$M_1 = 131.4\times10^7/\left(V^4C\right)$, $M_2 = 3836.88\times10^{14}/\left(V^6C^4\right) > 0$

The necessary condition of $\nabla U = 0$ gives two stationary points as

$\mathbf{x}^{*1} = \left(2.417643\times10^2,\ 3.099542\times10^{-2}\right)$ and $\mathbf{x}^{*2} = \left(-2.417643\times10^2,\ -3.099542\times10^{-2}\right)$.

At $\mathbf{x}^{*1}$, $M_1 > 0$ and $M_2 > 0$ as $C > 0$, or

$$\mathbf{H}\left(\mathbf{x}^{*1}\right) = \begin{bmatrix} 12.408782 & 3.226283\times10^4 \\ 3.226283\times10^4 & 2.516501\times10^8 \end{bmatrix}; \quad M_1 = 12.408782 > 0; \quad M_2 = 2.0817816\times10^9 > 0$$

Since Hessian at $\mathbf{x}^{*1}$ is positive definite, the point x^{*1} is a local minimum point.

$U = \left(\mathbf{x}^{*1}\right) = 4.835286\times10^5$, which is the minimum operating cost. At $\mathbf{x}^{*2}$,

$$\mathbf{H}\left(\mathbf{x}^{*2}\right) = \begin{bmatrix} -12.408782 & -3.226283\times10^4 \\ -3.226283\times10^4 & -2.516501\times10^8 \end{bmatrix}; \quad \begin{matrix} M_1 = -12.408782 < 0 \\ M_2 = 2.0817816 > 0 \end{matrix}$$

Since Hessian at $\mathbf{x}^{*2}$ is negative definite, the point $\mathbf{x}^{*2}$ is a local maximum point.

4.33

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2$$

Solution

$$f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1 - 2x_1x_2$$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 - 4 - 2x_2 \\ 4x_2 - 2x_1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (8, 4)$. For the Hessian **H**, $M_1 = 2 > 0$, $M_2 = 4 > 0$; so it is positive definite, and $(8, 4)$ is a local minimum point $(f = 0)$.

4.34

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 12x_1^2 + 22x_2^2 - 1.5x_1 - x_2$$

Solution

$$f(x_1, x_2) = 12x_1^2 + 22x_2^2 - 1.5x_1 - x_2$$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 - 1.5 \\ 44x_2 - 1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 44 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = \left(0.75, \frac{1}{44}\right)$. For the Hessian **H**, $M_1 = 2 > 0$, $M_2 = 88 > 0$; so it is positive definite, and $\left(0.75, \frac{1}{44}\right)$ is a local minimum point $(f = 5.6136)$.

4.35

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 7x_1^2 + 12x_2^2 - x_1$$

Solution

$$f\left(x_1, x_2\right) = 7x_1^2 + 12x_2^2 - x_1$$

The gradient and Hessian are given as

$$\nabla f = \begin{bmatrix} 14x_1 - 1 \\ 24x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 14 & 0 \\ 0 & 24 \end{bmatrix}.$$

Solution of $\nabla f = 0$ gives $\mathbf{x}^* = \left(\frac{1}{14}, 0\right)$. For the Hessian $\mathbf{H}$, $M_1 = 14 > 0$, $M_2 = 336 > 0$; so it is positive definite, and $\left(\frac{1}{14}, 0\right)$ is a local minimum point $\left(f = -0.035714\right)$.

4.36

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 12x_1^2 + 21x_2^2 - x_2$$

Solution

$$f\left(x_1, x_2\right) = 12x_1^2 + 21x_2^2 - x_2$$

The gradient and Hessian are given as

$$\nabla f = \begin{bmatrix} 12x_1 \\ 42x_2 - 1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 12 & 0 \\ 0 & 42 \end{bmatrix}.$$

Solution of $\nabla f = 0$ gives $\mathbf{x}^* = \left(0, \frac{1}{42}\right)$. For the Hessian $\mathbf{H}$, $M_1 = 12 > 0$, $M_2 = 504 > 0$; so it is positive definite, and $\left(0, \frac{1}{42}\right)$ is a local minimum point $\left(f = -0.0119\right)$.

4.37

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 25x_1^2 + 20x_2^2 - 2x_1 - x_2$$

Solution

$$f\left(x_1, x_2\right) = 25x_1^2 + 20x_2^2 - 2x_1 - x_2$$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 50x_1 - 2 \\ 40x_2 - 1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 50 & 0 \\ 0 & 40 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = \left(\frac{1}{25}, \frac{1}{40}\right)$. For the Hessian **H**, $M_1 = 50 > 0$, $M_2 = 2000 > 0$; so it is positive definite, and $\left(\frac{1}{25}, \frac{1}{40}\right)$ is a local minimum point $\left(f = -0.0525\right)$.

4.38

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$$

Solution

$$f\left(x_1, x_2, x_3\right) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3$$

The gradient and Hessian are given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 + 2x_2 \\ 4x_2 + 2x_1 + 2x_3 \\ 4x_3 + 2x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (0, 0, 0)$. For the Hessian **H**, $M_1 = 2 > 0$, $M_2 = 24 > 0$; so it is positive definite, and $(0, 0, 0)$ is a local minimum point $\left(f = 0\right)$.

4.39

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 8x_1^2 + 8x_2^2 - 80\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 80\sqrt{x_1^2 + x_2^2 + 20x_2 + 100} - 5x_1 - 5x_2$$

Solution

$f\left(x_1, x_2\right) = 8x_1^2 + 8x_2^2 - 80\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 80\sqrt{x_1^2 + x_2^2 + 20x_2 + 100} - 5x_1 - 5x_2$;

The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 16x_1 - 80x_1\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 80x_1\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 5 \\ 16x_2 - 40(2x_2 - 20)\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 40(2x_2 + 20)\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 5 \end{bmatrix}; \quad .$$

Solution of $\tilde{\mathbf{N}}f$ = 0 and the hessian would be solved numerically using a program such as Mathematica or MATLAB.

4.40

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 9x_1^2 + 9x_2^2 - 100\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 64\sqrt{x_1^2 + x_2^2 + 16x_2 + 64} - 5x_1 - 41x_2$$

Solution

$f\left(x_1, x_2\right) = 9x_1^2 + 9x_2^2 - 100\sqrt{x_1^2 + x_2^2 - 20x_2 + 100} - 64\sqrt{x_1^2 + x_2^2 + 20x_2 + 100} - 5x_1 - 41x_2$; The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 18x_1 - 100x_1\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 64x_1\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 5 \\ 18x_2 - 50(2x_2 - 20)\left(x_1^2 + x_2^2 - 20x_2 + 10\right)^{-\frac{1}{2}} - 32(2x_2 + 20)\left(x_1^2 + x_2^2 + 20x_2 + 10\right)^{-\frac{1}{2}} - 41 \end{bmatrix}; \quad .$$

Solution of $\tilde{\mathbf{N}}f$ = 0 and the hessian would be solved numerically using a program such as Mathematica or MATLAB.

4.41

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Solution

$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1^2)$; The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} -400x_1(x_2 - x_1^2) - 2 + 2x_1 \\ 200(x_2 - x_1^2) \end{bmatrix}; \quad .$$

Solution of $\tilde{\mathbf{N}}f = 0$ and the hessian would be solved numerically using a program such as Mathematica or MATLAB.

4.42

Find stationary points for the following function (use a numerical method such as the Newton-Raphson method, or a software package like Excel, MATLAB, and Mathematica, if needed). Also determine the local minimum, local maximum, and inflection points for the function (infection points are those stationary points that are neither minimum nor maximum).

$$f(x_1, x_2, x_3, x_4) = (x_1 - 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4$$

Solution

$f(x_1, x_2, x_3, x_4) = (x_1 - 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4$; The gradient is given as

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 2x_1 - 4 - 2x_2 \\ 4x_2 - 2x_1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution of $\tilde{\mathbf{N}}f = 0$ gives $\mathbf{x}^* = (8, 4)$. For the Hessian $\mathbf{H}$, $M_1 = 2 > 0$, $M_2 = 4 > 0$; so it is positive definite, and $(8, 4)$ is a local minimum point $(f = 0)$.

Section 4.5 Necessary Conditions: Equality Constrained Problem

4.43

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 - 4 = 0$

Solution

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$;

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + v(x_1 + x_2 - 4)$;

The necessary conditions give

$\partial L/\partial x_1 = 8x_1 - 5x_2 - 8 + v = 0$; $\partial L/\partial x_2 = 6x_2 \ -5x_1 + v = 0$; $\mathrm{h} = x_1 + x_2 - 4 = 0$

The solution of these equations is $x_1 = 13/6, x_2 = 11/6, v = -1/6$.

Therefore, (13/6, 11/6) is a KKT point; $f = -25/3$

Check for regularity: $\nabla \mathrm{h} = (1, 1)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible points is satisfied.

The problem is solved graphically in Exercise 3.12. The graph shows that the stationary point is actually a local as well as a global minimum point for the function. The problem is also solved graphically in Exercise 4.97.

4.44

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 - 4 = 0$

Solution

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$;

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8x_1 + v(x_1 + x_2 - 4)$;

The necessary conditions give

$\partial L/\partial x_1 = -8x_1 + 5x_2 + 8 + v = 0$; $\partial L/\partial x_2 = -6x_2 \ +5x_1 + v = 0$; $\mathrm{h} = x_1 + x_2 - 4 = 0$

The solution of these equations is $x_1 = 13/6, x_2 = 11/6, v = 1/6$.

Therefore, (2.166667, 1.833333) is a KKT point; $F = -25/3$

Check for regularity: $\nabla \mathrm{h} = (1, 1)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible points is satisfied.

The problem is solved graphically in Exercise 3.12. The graph shows that the stationary point is not a local maximum point for the function. There is no local maximum point; the function is actually unbounded. The problem is also solved graphically in Exercise 4.98.

4.45

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 + 1)^2$
subject to $2x_1 + 3x_2 - 4 = 0$

Solution

Minimize $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 + 1)^2$

subject to h= $2x_1 + 3x_2 - 4 = 0$

$L = (x_1 - 2)^2 + (x_2 + 1)^2 + v(2x_1 + 3x_2 - 4);$

The KKT necessary conditions give

$\partial L/\partial x_1 = 2(x_1 - 2) + 2v = 0;\ \partial L/\partial x_2 = 2(x_2 + 1) + 3v = 0;\ \ \mathrm{h} = 2x_1 + 3x_2 - 4 = 0$

The solution of these equations is $x_1 = 32/13, x_2 = -4/13, v = -6/13.$

Therefore, (32/13, -4/13) is a KKT point; $f = 9/13$

Check for regularity: $\nabla \mathrm{h}$ = (2, 3). Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible point is satisfied.

The problem is solved graphically in Exercise 4.99.

4.46

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x_1, x_2) = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13$
subject to $x_1 - 3x_2 + 3 = 0$

Solution

Minimize $f(x_1, x_2) = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13$
subject to h $= x_1 - 3x_2 + 3 = 0$
$L = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13 + v(x_1 - 3x_2 + 3)$
The KKT necessary conditions give

$$\frac{\partial \mathrm{L}}{\partial x_1} = 8x_1 - 4 + v = 0, \qquad \frac{\partial \mathrm{L}}{\partial x_2} = 18x_2 + 6 - 3v = 0, \qquad \frac{\partial \mathrm{L}}{\partial v} = x_1 - 3x_2 + 3 = 0$$

The solution of these equations is $x_1 = -0.4,\ x_2 = 2.6/3,\ v = 7.2$
Therefore, (-0.4, 2.6/3) is a KKT point; $f = 27.2$
Check for regularity: $\nabla \mathrm{h}$ = (1,-3). Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible points is satisfied.
The problem is solved graphically in Exercise 4.100.

4.47

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x) = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2$
subject to $2x_1 + 3x_2 - 1 = 0$
$x_1 + x_2 + 2x_3 - 4 = 0$

Solution

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2$

subject to $h_1 = 2x_1 + 3x_2 - 1$; $h_2 = x_1 + x_2 + 2x_3 - 4$

$L = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2 + v_1(2x_1 + 3x_2 - 1) + v_2(x_1 + x_2 + 2x_3 - 4)$;

The KKT necessary conditions give

$\partial L/\partial x_1 = 2(x_1 - 1) + 2v_1 + v_2 = 0$; $\partial L/\partial x_2 = 2(x_2 + 2) + 3v_1 + v_2 = 0$; $\partial L/\partial x_3 = 2(x_3 - 2) + 2v_2 = 0$;

$h_1 = 2x_1 + 3x_2 - 1 = 0$; $h_2 = x_1 + x_2 + 2x_3 - 4 = 0$

The solution of these equations is

$x_1 = 1.71698, x_2 = -0.81132, x_3 = 1.547170057$

$v_1 = -0.943396132, v_2 = 0.452829943$

Therefore, (1.71698066, -0.811320724, 1.547170057) is a KKT point; $f = 2.1318$

Check for regularity: Gradients of the constraints are linearly independent; therefore the point is a regular point of the feasible set.

The problem is solved graphically in Exercise 4.101.

4.48

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 - 16 = 0$

Solution

Minimize $f(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4$ subject to $\mathrm{h} = x_1^2 + x_2^2 + 2x_1 - 16 = 0$

$L(x_1, x_2, v) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4 + v\left(x_1^2 + x_2^2 + 2x_1 - 16\right)$

$\partial L/\partial x_1 = 18x_1 + 18x_2 + 2vx_1 + 2v = 0;\ \partial L/\partial x_2 = 18x_1 + 26x_2 + 2vx_1 = 0$**;**

$\mathrm{h} = x_1^2 + x_2^2 + 2x_1 - 16 = 0$

These equations are nonlinear, which can be solved numerically. Using any nonlinear equation solver, we can find the following KKT points:

1. $x_1 = 1.5088, x_2 = 3.2720, v = -17.151503, f = 244.528$ (local maximum)
2. $x_1 = 2.5945, x_2 = -2.0198, v = -1.4390, f = 15.291$ (local minimum)
3. $x_1 = -3.630, x_2 = -3.1754, v = -23.2885, f = 453.154$ (local maximum)
4. $x_1 = -3.7322, x_2 = 3.0879, v = -2.1222, f = 37.877$ (local minimum)

Check for regularity: $\nabla \mathrm{h} = \left(2x_1 + 2,\ 2x_2\right)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible points is satisfied for each KKT point. The problem is solved graphically in Exercise 3.13. The local optimality of the four points can be observed in the figure there. The problem is also solved graphically in Exercise 4.102.

4.49

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 = 0$

Solution

Minimize $f = (x_1 - 1)^2 + (x_2 - 1)^2$ subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 + x_2 - 4)$; the KKT necessary conditions are

$\partial L/\partial x_1 = 2(x_1 - 1) + v = 0;\ \partial L/\partial x_2 = 2(x_2 - 1) + v = 0;\ \mathrm{h} = x_1 + x_2 - 4 = 0$

Solution of these equations is $x_1 = 2, x_2 = 2, v = -2$. Therefore, $(2, 2)$ is a KKT point; $f = 2$.

Check for regularity: $\nabla \mathrm{h} = (1, 1)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible points is satisfied.

The problem is solved graphically in Exercise 4.103.

4.50

Consider the following problem with equality constraints:
Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 = 0$
$x_1 - x_2 - 2 = 0$
1. Is it a valid optimization problem? Explain.
2. Explain how you would solve the problem? Are necessary conditions needed to find the optimum solution?

Solution

Minimize $f = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $x_1 + x_2 - 4 = 0$, and $x_1 - x_2 - 2 = 0$

1. It is not a valid optimization problem because there is only one feasible point of the constraint set; solution of the two linear equalities.

2. Solving the constraint equations, we get $x_1 = 3$, $x_2 = 1$, $f(3,1) = 4$.

Necessary conditions are not needed for this case since a unique solution has been found by solving the constraint equations. If Lagrange multipliers for the constraints are needed, then we need to write the necessary conditions and solve for them.
The problem is also solved graphically in Exercise 4.104.

4.51

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 = 4$

Solution

Minimize $f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$ subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 + v(x_1 + x_2 - 4)$; the KKT necessary conditions are

$\partial L/\partial x_1 = 8x_1 - 5x_2 + v = 0;\ \partial L/\partial x_2 = 6x_2 - 5x_1 + v = 0;\ \mathrm{h} = x_1 + x_2 - 4 = 0$

Solution of these equations is $x_1 = 11/6, x_2 = 13/6, v = -23/6$.

Therefore, $(11/6, 13/6)$ is a KKT point; $f^* = -1/3$.

Check for regularity: $\nabla \mathrm{h} = (1, 1)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible point is satisfied.
The problem is also solved graphically in Exercise 4.105.

4.52

Find points satisfying the necessary conditions for the following problem; check if they are optimum points using the graphical method (if possible).

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 = 4$

Solution

Minimize $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8$ subject to $\mathrm{h} = x_1 + x_2 - 4 = 0$

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8 + v\left(x_1 + x_2 - 4\right)$

The KKT necessary conditions are

$\partial L/\partial x_1 = -8x_1 + 5x_2 + v = 0;\ \partial L/\partial x_2 = -6x_2 + 5x_1 + v = 0;\ \mathrm{h} = x_1 + x_2 - 4 = 0$

Solution of these equations is $x_1 = 11/6,\ x_2 = 13/6,\ v = 23/6$.

Therefore, $\left(11/6, 13/6\right)$ is a KKT point; F = -1/3.

Check for regularity: $\nabla \mathrm{h} = \left(1, 1\right)$. Since $\nabla \mathrm{h}$ is the only vector, regularity of feasible point is satisfied.

The problem is also solved graphically in Exercise 4.106.

Section 4.6 Necessary Conditions for a General Constrained Problem

4.53

Answer True or False

1. A rectangular point of the feasible region is defined as a point where the cost function gradient is independent of the gradients of active constraints. *False*
2. A point satisfying KKT conditions for a general optimum design problem can be a local max-point for the cost function. *True*
3. At the optimum point, the number of active independent constraints is always more than the number of design variables. *False*
4. In the general optimum design problem formulation, the number of independent equality constraints must be "≤" to the number if design variables. *True*
5. In the general optimum design problem formulation, the number of inequality constraints cannot exceed the number of design variables. *False*
6. At the optimum point, Lagrange multipliers for the "≤ type" inequality constraints must be nonnegative. *True*
7. At the optimum point, the Lagrange multipliers for the "≤ type" constraints can be zero. *True*
8. While solving an optimum design problem by KKT conditions, each case defined by the switching conditions can have multiple solutions. *True*
9. In optimum design problem formulation, "≥ type" constraints cannot be treated. *False*
10. Optimum design points for constrained optimization problems give stationary value to the Lagrange function with respect to design variables. *True*
11. Optimum design points having at least one active constraint give stationary value to the cost function. *False*
12. At a constrained optimum design point that is regular, the cost function gradient is linearly dependent on the gradients of the active constraints functions. *True*
13. If a slack variable has zero value at the optimum, the inequality constraint is inactive. *False*
14. Gradients of inequality constraints that are active at the optimum point must be zero. *False*
15. Design problems with equality constraints have the gradient of the cost function as zero at the optimum point. *False*

4.54

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 \le 4$

Solution

Minimize $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8$ subject to $g = x_1 + x_2 - 4 \le 0$

$L = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8 + u\left(x_1 + x_2 - 4 + s^2\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = -8x_1 + 5x_2 + u = 0;\ \partial L/\partial x_2 = -6x_2 + 5x_1 + u = 0;$

$\partial L/\partial u = x_1 + x_2 - 4 + s^2 = 0; \partial L/\partial s = 2us = 0$

Case 1. $u = 0$; gives a KKT point as (0, 0); $F^* = -8$.
Case 2. $s = 0$ (or g = 0); gives a KKT point as (11/6, 13/6); $u^* = 23/6,\quad F^* = -1/3$.
Check for regularity: $\nabla g = (1, 1)$. Since there is only one constraint, regularity is satisfied.
The problem is also solved graphically in Exercise 4.107.

4.55

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$
subject to $x_1 + x_2 \le 4$

Solution

Minimize $f = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$ subject to $g = x_1 + x_2 - 4 \le 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8 + u\left(x_1 + x_2 - 4 + s^2\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = 8x_1 - 5x_2 + u = 0;\ \partial L/\partial x_2 = 6x_2 - 5x_1 + u = 0;$

$\partial L/\partial u = x_1 + x_2 - 4 + s^2 = 0; \partial L/\partial s = 2us = 0$

Case 1. $u = 0$; gives a KKT point as (0, 0); $f\left(\mathbf{x}^*\right) = -8$.

Case 2. $s = 0$; gives no candidate point. ($u < 0$)

Check for regularity: $\nabla g = (1, 1)$. Since there is only one constraint, regularity is satisfied.

The problem is also solved graphically in Exercise 4.108.

4.56

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 \leq 4$

Solution

Minimize $f = -4x_1^2 - 3x_2^2 + 5x_1x_2 + 8x_1$ subject to $g = x_1 + x_2 - 4 \leq 0$

$L = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + u\left(x_1 + x_2 - 4 + s^2\right)$; the KKT necessary conditions are

$\partial L/\partial x_1 = -8x_1 + 5x_2 + 8 + u = 0;\ \partial L/\partial x_2 = -6x_2 + 5x_1 + u = 0;$

$\partial L/\partial u = x_1 + x_2 - 4 + s^2 = 0; \partial L/\partial s = 2us = 0$

Case 1. $u = 0$; gives a KKT point as (48/23, 40/23); $F\left(\mathbf{x}^*\right) = -192/23$=8.348.

Case 2. $s = 0$; gives a KKT point as (13/6, 11/6); $u^* = 1/6$, $F\left(\mathbf{x}^*\right) = -8.33333$.

Check for regularity: $\tilde{\mathrm{N}}\mathrm{g} = (1, 1)$. Since there is only one constraint, regularity is satisfied.

The problem is also solved graphically in Exercise 4.109.

4.57

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 \geq 4$
$x_1 - x_2 - 2 = 0$

Solution

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $\mathrm{h} = x_1 - x_2 - 2 = 0;\ \mathrm{g} = -x_1 - x_2 + 4 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 - x_2 - 2) + u\left(-x_1 - x_2 + 4 + s^2\right)$

$\partial L/\partial x_1 = 2(x_1 - 1) + v - u = 0;\quad \partial L/\partial x_2 = 2(x_2 - 1) - v - u = 0$

$\mathrm{h} = x_1 - x_2 - 2 = 0;\ \text{-}x_1 - x_2 + 4 + s^2 = 0;\quad us = 0,\ \ u \geq 0.$

Case 1. $u = 0$; no candidate minimum.

Case 2. $s = 0$; gives $(3, 1)$ as a KKT point with $v = -2, u = 2, f = 4$.

Since $\tilde{\mathrm{N}}\mathrm{h} = (1, -1)$ and $\tilde{\mathrm{N}}\mathrm{g} = (-1, \text{-}1)$ are linearly independent, regularity is satisfied.

The problem is also solved graphically in Exercise 4.110.

4.58

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 = 4$
$x_1 - x_2 - 2 \geq 0$

Solution

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $h = x_1 + x_2 - 4 = 0$; $g = -x_1 + x_2 + 2 \leq 0$.

$$L = (x_1 - 1)^2 + (x_2 - 1)^2 + v(x_1 + x_2 - 4) + u(-x_1 + x_2 + 2 + s^2)$$

$$\partial L/\partial x_1 = 2(x_1 - 1) + v - u = 0; \quad \partial L/\partial x_2 = 2(x_2 - 1) + v + u = 0$$

$$h = x_1 + x_2 - 4 = 0; \ -x_1 + x_2 + 2 + s^2 = 0; \quad us = 0, \quad u \geq 0.$$

Case 1. $u = 0$; no candidate minimum.

Case 2. $s = 0$; gives $(3, 1)$ as a KKT point with $v = -2, u = 2, f = 4$.

Since Ñ$h = (1, 1)$ and Ñ$g = (-1, 1)$ are linearly independent, regularity is satisfied.

The problem is also solved graphically in Exercise 4.111.

4.59

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 \geq 4$
$x_1 - x_2 \geq 2$

Solution

Minimize $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $g_1 = -x_1 - x_2 + 4 \leq 0$; $g_2 = -x_1 + x_2 + 2 \leq 0$.

$$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(-x_1 - x_2 + 4 + s_1^2) + u_2(-x_1 + x_2 + 2 + s_2^2)$$

$$\partial L/\partial x_1 = 2(x_1 - 1) - u_1 - u_2 = 0; \quad \partial L/\partial x_2 = 2(x_2 - 1) - u_1 + u_2 = 0$$

$$-x_1 - x_2 + 4 + s_1^2 = 0; \ -x_1 + x_2 + 2 + s_2^2 = 0; \quad u_1 s_1 = 0, u_2 s_2 = 0, \quad u_1, u_2 \geq 0.$$

Case 1. $u_1 = 0$, $u_2 = 0$; no candidate minimum.

Case 2. $u_1 = 0$, $s_2 = 0$; no candidate minimum.
Case 3. $s_1 = 0$, $u_2 = 0$; no candidate minimum.

Case 4. $s_1 = 0$, $s_2 = 0$; gives $(3, 1)$ as a KKT point with $u_1 = 2, u_2 = 2, f = 4$.

Since Ñ$g_1 = (-1, -1)$ and Ñ$g_2 = (-1, 1)$ are linearly independent, regularity is satisfied.

The problem is also solved graphically in Exercise 4.112.

4.60

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x, y) = (x-4)^2 + (y-6)^2$
subject to $12 \geq x + y$
$x \geq 6, y \geq 0$

Solution

Minimize $f(x,y) = (x-4)^2 + (y-6)^2$;

subject to $g_1 = x + y - 12 \leq 0$;

$g_2 = -x + 6 \leq 0$;

$g_3 = -y \leq 0$

$$L = (x-4)^2 + (y-6)^2 + u_1\left(x + y - 12 + s_1^2\right) + u_2\left(-x + 6 + s_2^2\right) + u_3\left(-y + s_3^2\right)$$

$\partial L/\partial x = 2(x-4) + u_1 - u_2 = 0$;

$\partial L/\partial y = 2(y-6) + u_1 - u_3 = 0$

$x + y - 12 + s_1^2 = 0$;

$-x + 6 + s_2^2 = 0$;

$-y + s_3^2 = 0$

$u_1 s_1 = 0$, $u_2 s_2 = 0$, $u_3 s_3 = 0$

$u_1, u_2, u_3 \geq 0$.

Case 1. $u_1 = u_2 = u_3 = 0$; no candidate minimum.

Case 2. $u_1 = u_2 = 0, s_3 = 0$; no candidate minimum.

Case 3. $u_1 = u_3 = 0, s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_2 = 4$, $s_1 = 0$, $s_3 = 6$, $f = 4$.

Case 4. $u_2 = u_3 = 0, s_1 = 0$; no candidate minimum.

Case 5. $u_1 = 0, s_2 = s_3 = 0$; no candidate minimum.

Case 6. $u_2 = 0, s_1 = s_3 = 0$; no candidate minimum.

Case 7. $u_3 = 0, s_1 = s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_1 = 0, u_2 = 4$, $s_3 = 6, f = 4$.

Case 8. $s_1 = s_2 = s_3 = 0$; no candidate minimum.

The problem is also solved graphically in Exercise 4.113.

4.61

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$
subject to $x_1 + 3x_2 \le 6$
$5x_1 + 2x_2 \le 10$
$x_1, x_2 \ge 0$

Solution

Minimize $f(\mathbf{x}) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$;

subject to $g_1 = x_1 + 3x_2 - 6 \le 0$;

$g_2 = 5x_1 + 2x_2 - 10 \le 0$; $g_3 = -x_1 \le 0$; $g_4 = -x_2 \le 0$;

$$L = \left(2x_1 + 3x_2 - x_1^3 - 2x_2^2\right) + u_1\left(x_1 + 3x_2 - 6 + s_1^2\right) + u_2\left(5x_1 + 2x_2 - 10 + s_2^2\right)$$
$$+ u_3\left(-x_1 + s_3^2\right) + u_4\left(-x_2 + s_4^2\right)$$

$$\partial L/\partial x_1 = 2 - 3x_1^2 + u_1 + 5u_2 - u_3 = 0;$$
$$\partial L/\partial x_2 = 3 - 4x_2 + 3u_1 + 2u_2 - u_4 = 0;$$
$$x_1 + 3x_2 - 6 + s_1^2 = 0;\ 5x_1 + 2x_2 - 10 + s_2^2 = 0;\ -x_1 + s_3^2 = 0;\ -x_2 + s_4^2 = 0;$$

$u_i s_i = 0$; $u_i \ge 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$ .There are two possible solution points: $(-0.816, 0.75)$ and $(0.816, 0.75)$. For $(-0.816, 0.75)$, $g_3 = 0.816 > 0$ (violation). For $(0.816, 0.75)$, $g_1 = -2.934 < 0$, $g_2 = -4.42 < 0$, $g_3 = -0.816 < 0$, $g_4 = -0.75 < 0$. All the KKT conditions are satisfied; therefore (0.816,0.75) is a KKT point ($f = 2.214$).

Case 2. $u_1 = u_2 = u_3 = 0, s_4 = 0$. $g_4 = 0 \rightarrow x_2 = 0$; $x_1 = \pm 0.816$, $u_4 = 3 > 0$. $x_1 = -0.816 \rightarrow g_3 > 0$ (violated). At $x_1 = 0.816$ and $x_2 = 0$, $g_1 = -5.184 < 0$, $g_2 = -5.92 < 0$. All the KKT conditions are satisfied; therefore $(0.816, 0)$ is a KKT point ($f = 1.0887$).

Case 3. $u_1 = u_2 = u_4 = 0, s_3 = 0$. KKT point: $(0, 0.75)$, $u_3 = 2$, $f = 1.125$.

Case 4. $u_1 = u_3 = u_4 = 0, s_2 = 0$. Candidate points: $(-9.8407, 1.2317)$ and $(1.5073, 1.2317)$; first point violates g_3; the second point is a KKT point with $u_2 = 0.9632$; $f = 0.251$.

Case 5. $u_2 = u_3 = u_4 = 0, s_1 = 0$. Candidate points: $(-1.821, 1.655)$ and $(1.0339, 1.655)$; first point violates g_3; second is a KKT point with $u_1 = 1.2067$; $f = 0.4496$.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; $(0, 0)$ is a KKT point with $u_3 = 2$ and $u_4 = 3$; $f = 0$.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives $(2, 0)$ as a KKT point with $u_2 = 2$, $u_4 = 7$; $f = -4$.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0;$ gives no candidate point.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0;$ gives $(0, 2)$ as a KKT point with $u_1 = 5/3, u_3 = 11/3$, $f = -2$.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0;$ gives $(1.386, 1.538)$ as a KKT point with $u_1 = 0.633,\ u_2 = 0.626;\ f = -0.007388$.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0;$ gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0;$ gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

From the above investigation, Cases 1, 2, 3, 4, 5, 6, 7, 10, 11 generate KKT points.

Check for regularity: For cases 1, 2, 3, 4 and 5, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 7, $\nabla g_2 = (5, 2)$, $\nabla g_4 = (0, -1)$. Since ∇g_2 and ∇g_4 are linearly independent, regularity is satisfied. For case 10, $\nabla g_1 = (1, 3)$, $\nabla g_3 = (-1, 0)$. Since ∇g_1 and ∇g_3 are linearly independent, regularity is satisfied. For case 11, $\nabla g_1 = (1, 3)$, $\nabla g_2 = (5, 2)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.
The problem is also solved graphically in Exercise 4.114.

4.62

Find points satisfying KKT necessary conditions for the following problem; check if they are optimum points using the graphical method for two variable problems.

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 \leq 4$

Solution

Minimize $f(\mathbf{x}) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$; subject to $x_1 + x_2 - 4 \leq 0$.

$L(\mathbf{x}, u) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + u(x_1 + x_2 - 4 + s^2)$

$\partial L/\partial x_1 = 8x_1 - 5x_2 - 8 + u = 0$; $\partial L/\partial x_2 = 6x_2 - 5x_1 + u = 0$;

Case 1. $u = 0$; gives a KKT point as (48/23, 40/23); $f(\mathbf{x}^*) = -192/23$.

Case 2. $s = 0$ (or g = 0); gives no candidate point ($u = -1/6$).

Check for regularity: $\nabla g = (1, 1)$. Since there is only one constraint, regularity is satisfied.

The problem is also solved graphically in Exercise 4.115.

4.63

Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$
subject to $x_1 + x_2 \geq 4$

Solution

Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$; subject to $g = -x_1 - x_2 + 4 \leq 0$.

$L = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6 + u(-x_1 - x_2 + 4 + s^2)$

$\partial L/\partial x_1 = 2x_1 - 4 - u = 0$; $\partial L/\partial x_2 = 2x_2 - 2 - u = 0$

$-x_1 - x_2 + 4 + s^2 = 0$; $us = 0$, $u \geq 0$

Case 1. $u = 0$; gives no candidate point $s^2 = -1$.

Case 2. $s = 0$; gives $(2.5, 1.5)$ as a KKT point with $u = 1$ and $f = 1.5$. Since only one constraint is active, regularity is satisfied.

The problem is also solved graphically in Exercise 4.116.

4.64

Minimize $f(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$
subject to $x_1 + 2x_2 \le 10$
$4x_1 - 3x_2 \le 20$
$x_i \ge 0; i = 1,2$

Solution

Minimize $f(\mathbf{x}) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$, subject to $g_1 = x_1 + 2x_2 - 10 \le 0$,
$g_2 = 4x_1 - 3x_2 - 20 \le 0$, $g_3 = -x_1 \le 0$, $g_4 = -x_2 \le 0$.

There are 16 cases, but only the case $u_1 = u_3 = u_4 = 0$, $s_2 = 0$ yields a solution:
$(6.3, 1.733)$, $u_2 = 0.8$, $f = -56.901$.
Since only one constraint is active, regularity is satisfied.
The problem is also solved graphically in Exercise 4.117

4.65

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \le 0$

Solution

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$, subject to $g_1 = x_1 + x_2 - 4 \le 0$

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) + u_1 = 0$;

$\partial L/\partial x_2 = 2(x_2 - 1) + u_1 = 0$;

$x_1 + x_2 - 4 + s_1^2 = 0$; $u_1 s_1 = 0$; $u_1 \ge 0$

Case 1. $u_1 = 0$, gives $(1,1)$ as a KKT point ($f = 0$). Since no constraint is active, regularity is satisfied.

Case 2. $s_1 = 0$; gives no candidate point.

The problem is also solved graphically in Exercise 4.118

4.66

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \leq 0$
$x_1 - x_2 - 2 \leq 0$

Solution

Minimize $f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $g_1 = x_1 + x_2 - 4 \leq 0$; $g_2 = x_1 - x_2 - 2 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2) + u_2(x_1 - x_2 - 2 + s_2^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) + u_1 + u_2 = 0$; $\partial L/\partial x_2 = 2(x_2 - 1) + u_1 - u_2 = 0$

$x_1 + x_2 - 4 + s_1^2 = 0$; $x_1 - x_2 - 2 + s_2^2 = 0$;

$u_1 s_1 = 0$, $u_2 s_2 = 0$; $u_1 \geq 0$, $u_2 \geq 0$

Case 1. $u_1 = 0$, $u_2 = 0$; gives $(1, 1)$ as a KKT point, $f = 0$

Case 2. $u_1 = 0$, $s_2 = 0$; no candidate minimum.

Case 3. $s_1 = 0$, $u_2 = 0$; no candidate minimum.

Case 4. $s_1 = 0$, $s_2 = 0$; no candidate minimum.

Only the first case gives a solution that satisfies all the KKT necessary conditions. Since no constraint is active, regularity is satisfied.
The problem is also solved graphically in Exercise 4.119.

4.67

Minimize$f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \leq 0$
$2 - x_1 \leq 0$

Solution

Minimize $f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to: $g_1 = x_1 + x_2 - 4 \leq 0$; $g_2 = 2 - x_1 \leq 0$.

$L = (x_1 - 1)^2 + (x_2 - 1)^2 + u_1(x_1 + x_2 - 4 + s_1^2) + u_2(2 - x_1 + s_2^2)$

$\partial L/\partial x_1 = 2(x_1 - 1) + u_1 - u_2 = 0$; $\partial L/\partial x_2 = 2(x_2 - 1) + u_1 = 0$

$x_1 + x_2 - 4 + s_1^2 = 0$; $2 - x_1 + s_2^2 = 0$; $u_1 s_1 = 0$, $u_2 s_2 = 0$, $u_1 \geq 0$, $u_2 \geq 0$

$u_1 s_1 = 0$; $u_2 s_2 = 0$; $u_1 \geq 0$, $u_2 \geq 0$

Case 1. $u_1 = 0, u_2 = 0$; no candidate minimum point $(s_2^2 < 0)$.

Case 2. $u_1 = 0, s_2 = 0$; gives $(2, 1)$ as a KKT point with $u_2 = 2$, $f = 1$.

Case 3. $s_1 = 0, u_2 = 0$; no candidate minimum $(u_1 < 0)$.

Case 4. $s_1 = s_2 = 0$; no candidate minimum $(u_1 < 0)$.

Case 2 yields a KKT point. This point is regular since there is only one active constraint.
The problem is also solved graphically in Exercise 4.120.

4.68

Minimize $f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 \geq 16$

Solution

Minimize $f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$, subject to $g_1 = 16 - (x_1^2 + x_2^2 + 2x_1) \leq 0$.

$L = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4 + u_1(16 - x_1^2 - x_2^2 - 2x_1 + s_1^2)$

$\partial L/\partial x_1 = 18x_1 - 18x_2 - 2u_1x_1 - 2u_1 = 0;\ \partial L/\partial x_2 = -18x_1 + 26x_2 - 2u_1x_2 = 0$

$x_1 + x_2 - 4 + s_1^2 = 0; 2 - x_1 + s_2^2 = 0$; $u_1s_1 = 0$, $u_2s_2 = 0$, $u_1 \geq 0$, $u_2 \geq 0$

$-x_1^2 - x_2^2 - 2x_1 + 16 + s_1^2 = 0; u_1s_1 = 0, u_1 \geq 0$

Case 1. $u_1 = 0$; no candidate minimum $(s_1^2 < 0)$.

Case 2. $s_1 = 0$; Solving the nonlinear system of equations, we get the following KKT points:

$(2.5945,\ 2.0198)$, $u_1 = 1.4390$, $f = 15.291$; $(-3.630, 3.1754)$, $u_1 = 23.2885$, $f = 215.97$;
$(1.5088, -3.2720)$, $u_1 = 17.1503$, $f = 244.53$; $(-3.7322, -3.0879)$, $u_1 = 2.1222$, $f = 37.877$.

Since only one constraint is active, regularity is satisfied.
The problem is also solved graphically in Exercise 4.121.

4.69

Minimize $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
subject to $x_1 + x_2 \leq 4$
$x_1 - 3x_2 = 1$

Solution

Minimize $f(x) = (x_1 - 3)^2 + (x_2 - 3)^2$; subject to $h = x_1 - 3x_2 - 1 = 0$; $g = x_1 + x_2 - 4 \leq 0$.

$L = (x_1 - 3)^2 + (x_2 - 3)^2 + v \leq (x_1 - 3x_2 - 1) + u(x_1 + x_2 - 4 + s^2)$

$\partial L/\partial x_1 = 2(x_1 - 3) + v + u = 0$; $\partial L/\partial x_2 = 2(x_2 - 3) + 3v + u = 0$

$h = x_1 - 3x_2 - 1 = 0$; $x_1 + x_2 - 4 + s^2 = 0$; $us = 0$, $u \geq 0$.

Case 1. $u = 0$; no candidate minimum $(s^2 < 0)$.

Case 2. $s = 0$; gives $(3.25, 0.75)$ as a KKT point with $v = -1.25, u = 0.75$, $f = 5.125$.

Since $\nabla h = (1, -3)$ and $\nabla g = (1, 1)$ are linearly independent, regularity is satisfied.

The problem is also solved graphically in Exercise 4.122.

4.70

Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$
subject to $x_1 + x_2 \le 3$

Solution

Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$, subject to $g = x_1 + x_2 - 3 \le 0$.

$L = x_1^3 - 16x_1 + 2x_2 - 3x_2^2 + u(x_1 + x_2 - 3 + s^2)$

$\partial L/\partial x_1 = 3x_1^2 - 16 + u = 0$; $\partial L/\partial x_2 = 2 - 6x_2 + u = 0$; $x_1 + x_2 - 3 + s^2 = 0$; $ug = 0$, $u \ge 0$.

Case 1. $u = 0$; gives $(4/\sqrt{3}, 1/3)$, $f = -24.3$ and $(-4/\sqrt{3}, 1/3)$, $f = 24.967$, as KKT points.

Case 2. $s = 0$; gives $(0, 3)$, $u = 16$, $f = -21$; $(2, 1)$, $u = 4$, $f = -25$, as KKT points.

For both cases, there is only one active constraint; so, regularity is satisfied.
The problem is also solved graphically in Exercise 4.123.

4.71

Minimize $f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$
subject to $x_1^2 - x_2^2 + 8x_2 \le 16$

Solution

Minimize $f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$, subject to $g = x_1^2 - x_2^2 + 8x_2 - 16 \le 0$

$L = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2 + u(x_1^2 - x_2^2 + 8x_2 - 16 + s^2)$

$\partial L/\partial x_1 = 6x_1 - 2x_2 + 2ux_1 = 0$

$\partial L/\partial x_2 = -2x_1 + 10x_2 + 8 - 2ux_2 + 8u = 0.$

$x_1^2 - x_2^2 + 8x_2 - 16 + s^2 = 0$

$us = 0$

$s^2, u \ge 0$

Case 1. $u = 0$;
KKT conditions reduce to

$$6x_1 - 2x_2 = 0 \ and \ -2x_1 + 10x_2 + 8 = 0$$

This gives $(-2/7, -6/7)$ as a KKT point $(f = -24/7)$. We calculate the slack variable from the constraint equation as

$s^2 = -x_1^2 + x_2^2 - 8x_2 + 16 = -\left(-\frac{2}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 - 8\left(-\frac{6}{7}\right) + 16 = 4.849 > 0;$

so the point is a feasible point

This is an unconstrained case, so check the form of Hessian of the cost function:

$\nabla f = \begin{bmatrix} 6x_1 - 2x_2 \\ -2x_1 + 10x_2 + 8 \end{bmatrix}$

$\nabla^2 f(-\frac{2}{7}, -\frac{6}{7}) = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}$

$M_1 = 6 > 0, M_2 = 60 - 4 = 56 > 0;$ Positive definite

The Hessian of cost function is positive definite; therefore, Hessian of the Lagrangian is positive definite. Second order necessary and sufficiency conditions are satisfied for a local minimum of f; the point is a local minimum with $f(x^*) = \frac{-24}{7}$.

Case 2. $s = 0$; no candidate minima $(u < 0)$.

We solve the following KKT conditions for x_1, x_2, and u

$$6x_1 - 2x_2 + 2ux_1 = 0$$

$$-2x_1 + 10x_2 + 8 - 2ux_2 + 8u = 0.$$

$$x_1^2 - x_2^2 + 8x_2 - 16 = 0$$

The Excel Solver gives a solution for these equations as

$$x_1 = 2.8, x_2 = 1.2, and\ u = -2.57 < 0\ a\ violation$$

Therefore this case does not give any KKT point.

Variables		
x	2.79999986681174	
y	1.20000004255272	
u	-2.57142869777053	
s	0	
Equations		Value
6x-2y+2ux	=6*x-2*y+2*u*x	0
10y-2x-2uy+8u+8	=10*y-2*x-2*u*y+8*u+8	0
x*x-y*y+8y-16+s*s	=x*x-y*y+8*y-16+s*s	0
us	=u*s	0
s*s	=s*s	>=0
u	=u	>=0

For case 1, since there is no active constraint, the regularity is satisfied.
The problem is also solved graphically in Exercise 4.124.

4.72

Minimize $f(x, y) = (x-4)^2 + (y-6)^2$
subject to $x + y \le 12$
$x \le 6$
$x, y \ge 0$

Solution

Minimize $f(x,y) = (x-4)^2 + (y-6)^2$; subject to $g_1 = x + y - 12 \le 0$;
$g_2 = x - 6 \le 0$; $g_3 = -x \le 0$; $g_4 = -y \le 0$;
$L = \left((x-4)^2 + (y-6)^2\right) + u_1\left(x + y - 12 + s_1^2\right) + u_2\left(x - 6 + s_2^2\right)$
$+u_3\left(-x + s_3^2\right) + u_4\left(-y + s_4^2\right)$
$\partial L/\partial x = 2(x-4) + u_1 + u_2 - u_3 = 0$; $\partial L/\partial y = 2(y-6) + u_1 - u_4 = 0$;
$x + y - 12 + s_1^2 = 0$; $x - 6 + s_2^2 = 0$; $-x + s_3^2 = 0$; $-y + s_4^2 = 0$;
$u_i s_i = 0$; $u_i \ge 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives $(4, 6)$ as a KKT point ; $f = 0$.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives no candidate point.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives no candidate point.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives no candidate point.

Case 6. $u_1 = u_2 = 0,\ s_3 = s_4 = 0$; gives no candidate point.

Case 7. $u_1 = u_3 = 0,\ s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0,\ s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0,\ s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0,\ s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0,\ s_1 = s_2 = 0$; gives no candidate point.

Case 12. $u_1 = 0,\ s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0,\ s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0,\ s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0,\ s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: Only the first case gives a solution that satisfies all the KKT necessary conditions. Since no constraint is active, regularity is satisfied.
The problem is also solved graphically in Exercise 4.125.

4.73

Minimize $f(x, y) = (x-8)^2 + (y-8)^2$
subject to $x + y \leq 12$
$x \leq 6$
$x, y \geq 0$

Solution

Minimize $f(x,y) = (x-8)^2 + (y-8)^2$; subject to $g_1 = x + y - 12 \leq 0$;
$g_2 = x - 6 \leq 0$; $g_3 = -x \leq 0$; $g_4 = -y \leq 0$;

$L = \left((x-8)^2 + (y-8)^2\right) + u_1\left(x + y - 12 + s_1^2\right) + u_2\left(x - 6 + s_2^2\right)$
$+ u_3\left(-x + s_3^2\right) + u_4\left(-y + s_4^2\right)$

$\partial L/\partial x = 2(x-8) + u_1 + u_2 - u_3 = 0$; $\partial L/\partial y = 2(y-8) + u_1 - u_4 = 0$;

$x + y - 12 + s_1^2 = 0$; $x - 6 + s_2^2 = 0$; $-x + s_3^2 = 0$; $-y + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \geq 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives no candidate point.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives no candidate point.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives no candidate point.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives $(6, 6)$ as a KKT point with $u_1 = 4$; $f = 8$.

Case 6. $u_1 = u_2 = 0,\ s_3 = s_4 = 0$; gives no candidate point.

Case 7. $u_1 = u_3 = 0,\ s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0,\ s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0,\ s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0,\ s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0,\ s_1 = s_2 = 0$; gives $(6, 6)$ as a KKT point with $u_1 = 4$; $f = 8$.

Case 12. $u_1 = 0,\ s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0,\ s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0,\ s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0,\ s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For case 5, there is only one active constraint, so regularity is satisfied. For case 11, $\tilde{N}g_1 = (1, 1)$, $\tilde{N}g_2 = (1, 0)$. Since $\tilde{N}g_1$ and $\tilde{N}g_2$ are linearly independent regularity is satisfied. The problem is also solved graphically in Exercise 4.126.

4.74

Maximize $F(x,y)=(x-4)^2+(y-6)^2$
subject to $x+y\le 12$
$6\ge x$
$x,y\ge 0$

Solution

Minimize $f(x,y)=-(x-4)^2-(y-6)^2$; subject to $g_1=x+y-12\le 0$;

$g_2=x-6\le 0$; $g_3=-x\le 0$; $g_4=-y\le 0$;

$$L=\left(-(x-4)^2-(y-6)^2\right)+u_1\left(x+y-12+s_1^2\right)+u_2\left(x-6+s_2^2\right)$$
$$+u_3\left(-x+s_3^2\right)+u_4\left(-y+s_4^2\right)$$

$\partial L/\partial x=-2(x-4)+u_1+u_2-u_3=0$;

$\partial L/\partial y=-2(y-6)+u_1-u_4=0$;

$x+y-12+s_1^2=0$;

$x-6+s_2^2=0$;

$-x+s_3^2=0$;

$-y+s_4^2=0$;

$u_is_i=0$; $u_i\ge 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1=u_2=u_3=u_4=0$; gives $(4, 6)$ as a KKT point ; $F=0$.

Case 2. $u_2=u_3=u_4=0,\ s_1=0$; gives $(5, 7)$ as a KKT point with u_1 = 2; $F=2$.

Case 3. $u_1=u_3=u_4=0,\ s_2=0$; gives $(6, 6)$ as a KKT point with u_2 = 4; $F=4$. Note that for this case, s_1 is also 0. Therefore this is an abnormal case where both u_1 and s_1 are zero.

Case 4. $u_1=u_2=u_4=0,\ s_3=0$; gives $(0, 6)$ as a KKT point with u_3 =8; $F=16$.

Case 5. $u_1=u_2=u_3=0,\ s_4=0$; gives $(4, 0)$ as a KKT point with u_4 =12; $F=36$.

Case 6. $u_3=u_4=0,\ s_1=s_2=0$; gives $(6, 6)$ as a KKT point with u_1 = 0, u_2 = 4; $F=4$. Note that for this case, s_1 is also 0. Therefore this is an abnormal case where both u_1 and s_1 are zero.

Case 7. $u_1=u_4=0,\ s_2=s_3=0$; gives no candidate point.

Case 8. $u_1=u_2=0,\ s_3=s_4=0$; gives $(0, 0)$ as a KKT point with u_3 = 8 and u_4 = 12; $F=52$.

Case 9. $u_2=u_3=0,\ s_1=s_4=0$; gives no candidate point.

Case 10. $u_2=u_4=0,\ s_1=s_3=0$; gives $(0, 12)$ as a KKT point with u_1 = 12, u_3 = 20; $F=52$.

Case 11. $u_1=u_3=0,\ s_2=s_4=0$; gives $(6, 0)$ as a KKT point with u_2 = 4, u_4 = 12; $F=40$.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0;$ gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0;$ gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 1, 2, 3, 4 and 5, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 7, $\nabla g_2 = (1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_2 and ∇g_4 are linearly independent, regularity is satisfied. For case 10, $\nabla g_1 = (1, 1)$, $\nabla g_3 = (-1, 0)$. Since ∇g_1 and ∇g_3 are linearly independent, regularity is satisfied. For case 11, $\nabla g_1 = (1, 1)$, $\nabla g_2 = (1, 0)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.
The problem is also solved graphically in Exercise 4.127.

4.75

Maximize $F(r,t) = (r-8)^2 + (t-8)^2$
subject to $10 \geq r + t$
$t \leq 5$
$r, t \geq 0$

Solution

Minimize $f(r,t) = -(r-8)^2 - (t-8)^2$; subject to $g_1 = r + t - 10 \leq 0$;

$g_2 = t - 5 \leq 0$; $g_3 = -r \leq 0$; $g_4 = -t \leq 0$;

$L = \left(-(r-8)^2 - (t-8)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(t - 5 + s_2^2\right)$

$+u_3\left(-r + s_3^2\right) + u_4\left(-t + s_4^2\right)$

$\partial L/\partial r = -2(r-8) + u_1 - u_3 = 0$; $\partial L/\partial t = -2(t-8) + u_1 + u_2 - u_4 = 0$;

$r + t - 10 + s_1^2 = 0$; $t - 5 + s_2^2 = 0$; $-r + s_3^2 = 0$; $-t + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \geq 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives no candidate point.

Case 2. $u_1 = u_2 = u_3 = 0,\ s_4 = 0$; gives $(8, 0)$ as a KKT point with u_4 = 16; $F = 64$.

Case 3. $u_1 = u_2 = u_4 = 0,\ s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0,\ s_2 = 0$; gives no candidate point.

Case 5. $u_2 = u_3 = u_4 = 0,\ s_1 = 0$; gives no candidate point.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; gives $(0, 0)$ as a KKT point with u_3 = 16, u_4 = 16; F =128.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0$; gives $(10, 0)$ as a KKT point with u_1 = 4, u_4 = 20; F =68.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0$; gives no candidate point.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0$; gives no candidate point.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 2, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 9, $\nabla g_1 = (1, 1)$, $\nabla g_4 = (0, -1)$. Since ∇g_1 and ∇g_4 are linearly independent, regularity is satisfied.
The problem is also solved graphically in Exercise 4.128.

4.76

Maximize $F(r,t) = (r-3)^2 + (t-2)^2$
subject to $10 \geq r + t$
$t \leq 5$
$r, t \geq 0$

Solution

Minimize $f(r,t) = -(r-3)^2 - (t-2)^2$
subject to $g_1 = r + t - 10 \leq 0;$

$g_2 = t - 5 \leq 0; \quad g_3 = -r \leq 0; \quad g_4 = -t \leq 0;$

$L = \left(-(r-3)^2 - (t-2)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(t - 5 + s_2^2\right) + u_3\left(-r + s_3^2\right) + u_4\left(-t + s_4^2\right)$

$\partial L/\partial r = -2(r-3) + u_1 - u_3 = 0 \qquad$ (a)

$\partial L/\partial t = -2(t-2) + u_1 + u_2 - u_4 = 0 \qquad$ **(*b*)**

$r + t - 10 + s_1^2 = 0; \quad t - 5 + s_2^2 = 0; \quad -r + s_3^2 = 0; \quad -t + s_4^2 = 0; \qquad$ (c)

$u_i s_i = 0;\ u_i \geq 0;$ i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0;$

Equations (a) and (b) give $(3, 2)$ as a KKT point ; F = 0. Equations (c) give all $s_i^2 > 0, i = 1\ to\ 4$. Therefore it is a feasible point

Case 2. $u_2 = u_3 = u_4 = 0,\ s_1 = 0;$

Equations (a) to (c) give $(5.5, 4.5)$ as a KKT point with u_1 = 5; $F = 12.5$.

Case 3. $u_1 = u_3 = u_4 = 0,\ s_2 = 0;$

Equations (a) to (c) give $(3, 5)$ as a KKT point with u_2 = 6; $F = 9$.

Case 4. $u_1 = u_2 = u_4 = 0,\ s_3 = 0;$

Equations (b) and (c) give $(0, 2)$ as a KKT point with u_3 = 6 from Eq. (a); $F = 9$.

Case 5. $u_1 = u_2 = u_3 = 0,\ s_4 = 0;$

Equations (a) and (c) give $(3, 0)$ as a KKT point with u_4 = 4 from Eq. (b); $F = 4$.

Case 6. $u_3 = u_4 = 0,\ s_1 = s_2 = 0;$

Equations (a) and (c) give $(5, 5)$ as a KKT point with u_1 = 4, u_2 = 2; F =13.

Case 7. $u_1 = u_4 = 0,\ s_2 = s_3 = 0;$

Equations (a) and (c) give $(0, 5)$ as a KKT point with u_2 = 6, u_3 = 6; F =18 .

Case 8. $u_1 = u_2 = 0,\ s_3 = s_4 = 0;$

Equations (a) and (c) give $(0, 0)$ as a KKT point with u_3 = 6, u_4 = 4; F =13.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0;$

Equations (a) and (c) give $(10, 0)$ as a KKT point with u_1 = 14, u_4 = 18; F =53.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0;$

Equations (a) and (c) give (0, 10); not a KKT point since u_1 = 16, u_2 = 22 and $s_2^2 = -5$.

Case 11. $u_1 = u_3 = 0, s_2 = s_4 = 0;$

Equations (c) give t =5 and t = 0 which is an inconsistency.

Case 12. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 13. $u_1 = 0, s_2 = s_3 = s_4 = 0;$ gives no candidate point.

Case 14. $u_2 = 0, s_1 = s_3 = s_4 = 0;$ gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 1, 2, 3, 4 and 5, there is only one active constraint, so regularity is satisfied. For case 6, $\nabla g_3 = (-1, 0)$, $\nabla g_4 = (0, -1)$. Since ∇g_3 and ∇g_4 are linearly independent, regularity is satisfied. For case 8, $\nabla g_2 = (0, 1)$, $\nabla g_3 = (-1, 0)$. Since ∇g_2 and ∇g_3 are linearly independent, regularity is satisfied. For case 9, $\nabla g_1 = (1, 1)$, $\nabla g_4 = (0, -1)$. Since ∇g_1 and ∇g_4 are linearly independent, regularity is satisfied. For case 11, $\nabla g_1 = (1, 1)$, $\nabla g_2 = (0, 1)$. Since ∇g_1 and ∇g_2 are linearly independent regularity is satisfied.

The problem is also solved graphically in Exercise 4.129.

4.77

Maximize $F(r,t) = (r-8)^2 + (t-8)^2$
subject to $r + t \le 10$
$t \ge 0$
$r \le 0$

Solution

Minimize $f(r,t) = -(r-8)^2 - (t-8)^2$; subject to $g_1 = r + t - 10 \le 0$;
$g_2 = r \le 0$; $g_3 = -t \le 0$;
$L = \left(-(r-8)^2 - (t-8)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(-t + s_2^2\right)$
$+u_3\left(r + s_3^2\right)$
$\partial L/\partial r = -2(r-8) + u_1 + u_3 = 0$; $\partial L/\partial t = -2(t-8) + u_1 + u_2 = 0$;
$r + t - 10 + s_1^2 = 0$; $-t + s_2^2 = 0$; $r + s_3^2 = 0$;
$u_i s_i = 0$; $u_i \ge 0$; i = 1 to 3 (there are 8 cases).

Case 1. $u_1 = u_2 = u_3 = 0$; no candidate minimum.
Case 2. $u_1 = u_2 = 0, s_3 = 0$; no candidate minimum.
Case 3. $u_1 = u_3 = 0, s_2 = 0$; no candidate minimum.
Case 4. $u_2 = u_3 = 0, s_1 = 0$; no candidate minimum.
Case 5. $u_1 = 0, s_2 = s_3 = 0$; no candidate minimum.
Case 6. $u_2 = 0, s_1 = s_3 = 0$; no candidate minimum.
Case 7. $u_3 = 0, s_1 = s_2 = 0$; no candidate minimum.
Case 8. $s_1 = s_2 = s_3 = 0$; no candidate minimum.
The problem is also solved graphically in Exercise 4.130.

4.78

Maximize $F(r,t) = (r-3)^2 + (t-2)^2$
subject to $10 \geq r + t$
$t \geq 5$
$r, t \geq 0$

Solution

Minimize $f(r,t) = -(r-3)^2 - (t-2)^2$; subject to $g_1 = r + t - 10 \leq 0$;

$g_2 = -t + 5 \leq 0$; $g_3 = -r \leq 0$; $g_4 = -t \leq 0$;

$L = \left(-(r-3)^2 - (t-2)^2\right) + u_1\left(r + t - 10 + s_1^2\right) + u_2\left(-t + 5 + s_2^2\right)$

$+u_3\left(-r + s_3^2\right) + u_4\left(-t + s_4^2\right)$

$\partial L/\partial r = -2(r-3) + u_1 - u_3 = 0$; $\partial L/\partial t = -2(t-2) + u_1 - u_2 - u_4 = 0$;

$r + t - 10 + s_1^2 = 0$; $-t + 5 + s_2^2 = 0$; $-r + s_3^2 = 0$; $-t + s_4^2 = 0$;

$u_i s_i = 0$; $u_i \geq 0$; i = 1 to 4 (there are 16 cases).

Case 1. $u_1 = u_2 = u_3 = u_4 = 0$; gives no candidate point.

Case 2. $u_1 = u_2 = u_3 = 0, s_4 = 0$; gives no candidate point.

Case 3. $u_1 = u_2 = u_4 = 0, s_3 = 0$; gives no candidate point.

Case 4. $u_1 = u_3 = u_4 = 0, s_2 = 0$; gives $(3, 5)$ as a KKT point with $u_3 = 6$; $F = 9$.

Case 5. $u_2 = u_3 = u_4 = 0, s_1 = 0$; gives no candidate point.

Case 6. $u_1 = u_2 = 0, s_3 = s_4 = 0$; gives no candidate point.

Case 7. $u_1 = u_3 = 0, s_2 = s_4 = 0$; gives no candidate point.

Case 8. $u_1 = u_4 = 0, s_2 = s_3 = 0$; gives no candidate point.

Case 9. $u_2 = u_3 = 0, s_1 = s_4 = 0$; gives no candidate point.

Case 10. $u_2 = u_4 = 0, s_1 = s_3 = 0$; gives $(0, 10)$ as a KKT point with $u_1 = 16$, $u_3 = 22$; $F = 73$.

Case 11. $u_3 = u_4 = 0, s_1 = s_2 = 0$; gives no candidate point.

Case 12. $u_1 = 0, s_2 = s_3 = s_4 = 0$; gives no candidate point.

Case 13. $u_2 = 0, s_1 = s_3 = s_4 = 0$; gives no candidate point.

Case 14. $u_3 = 0, s_1 = s_2 = s_4 = 0$; gives no candidate point.

Case 15. $u_4 = 0, s_1 = s_2 = s_3 = 0$; gives no candidate point.

Case 16. $s_1 = s_2 = s_3 = s_4 = 0$; gives no candidate point.

Check for regularity: For cases 4, there is only one active constraint, so regularity is satisfied. For case 10, $\tilde{N}g_1 = (1, 1)$, $\tilde{N}g_3 = (-1, 0)$. Since $\tilde{N}g_1$ and $\tilde{N}g_3$ are linearly independent regularity is satisfied.

The problem is also solved graphically in Exercise 4.31

4.79

Consider the problem of designing the "can" formulated in Section 2.2. Write KKT conditions and solve them. Interpret the necessary conditions at the solution point graphically.

Solution

Minimize $f(D, H) = \pi DH + \pi D^2/2$,

subject to $g_1 = 400 - \pi D^2 H/4 \le 0$;

$g_2 = 3.5 - D \le 0$;

$g_3 = D - 8 \le 0$;

$g_4 = 8 - H \le 0$;

$g_5 = H - 18 \le 0$

$$L = \pi DH + \pi D^2/2 + u_1(400 - \pi D^2 H/4 + s_1^2) + u_2(3.5 - D + s_2^2) + u_3(D - 8 + s_3^2)$$
$$+ u_4(8 - H + s_4^2) + u_5(H - 18 + s_5^2)$$

$$\partial L/\partial D = \pi H + \pi D - \pi DHu_1/2 - u_2 + u_3 = 0; \quad \partial L/\partial H = \pi D - \pi D^2 u_1/4 - u_4 + u_5 = 0;$$

$$g_i + s_i^2 = 0, \quad u_i s_i = 0, \quad u_i \ge 0; \quad i = 1 \text{ to } 5.$$

The switching conditions yield 32 cases because we have five inequality constraints. But only one case gives solution; others do not satisfy all the KKT conditions. The case is $u_2 = u_3 = u_5 = 0$ and $s_1 = s_4 = 0$. Solving this, we get $H = 8$, $D = 7.98$, $u_1 = 0.5$, $u_4 = 0.063$; $f = 300.6$ cm^2. This solution can be verified graphically. It is seen that $(7.98, 8)$ is the minimum point.

4.80

A minimum weight tubular column design problem is formulated in Section 2.7 using mean radius R and thickness t as design variables. Solve the KKT conditions for the problem imposing an additional constraint $R/t \le 50$ for the following data: P = 50kN, l = 5.0m, E = 210GPa, σ_a=250MPa and ρ =7850kg/m^3. Interpret the necessary conditions at the solution point graphically.

Solution

Use kilograms, Newtons and meters as units: P = 50 kN = 5×10^4 N, l = 5.0 m, E = 210 GPa = 2.1×10^{11} N/m^2, σ_a = 250 MPa = 2.5×10^8 N/m^2, ρ = 7850 kg/m^3.
Referring to Formulation 1 in Section 2.7, we have

$$f(R,t) = 2\rho l\pi Rt = 2(7850)(5.0)(\pi)Rt = 246{,}615.02Rt\text{, kg}$$

$$g_1 = P/2\pi Rt - s_a = 7.957.75/Rt - 2.5\times10^8 \le 0\text{; (stress constraint)}$$

$$g_2 = P - \pi^3 ER^3t/4l^2 = (5\times10^4) - 6.5113\times10^{10}\ R^3t \le 0\text{; (buckling constraint)}$$

$$g_3 = R/t - 50 \le 0\text{; (ratio constraint);}\quad g_4 = -R \le 0;\quad g_5 = -t \le 0$$

$$\begin{aligned}L = {} & (246{,}615.02)Rt + u_1\left(7957.75/Rt - 2.5\times10^8 + s_1^2\right) + u_2\left(5\times10^4 - 6.5113\times10^{10}R^3t + s_2^2\right)\\ & + u_3\left(R/t - 50 + s_3^2\right) + u_4\left(-R + s_4^2\right) + u_5\left(-t + s_5^2\right)\end{aligned}$$

$$\partial L/\partial R = (246{,}615.02)t - u_1\left(7.95775\times10^3/R^2t\right) - 1.95339\times10^{11}R^2tu_2 + u_3/t - u_4 = 0$$

$$\partial L/\partial t = (246{,}615.02)R - u_1\left(7.95775\times10^3/Rt^2\right) - 6.511315\times10^{10}R^3u_2 - \left(R/t^2\right)u_3 - u_5 = 0$$

There are 32 cases from the switching conditions. We shall examine one case which yields a solution, i.e., $u_1 = u_4 = u_5 = 0, s_2 = s_3 = 0.$ Solving this case, we get

$$R = 7.871686\times10^{-2}, t = 1.574337\times10^{-3}, u_2 = 3.056\times10^{-4} \text{ and } u_3 = 0.3038.$$

At this point g_1, g_4, g_5 are < 0. All the KKT conditions are satisfied, so $\left(7.871686\times10^{-2}, 1.574337\times10^{-3}\right)$ is a candidate minimum point.
At this point, buckling and ratio constraints are active and the cost function is f = 30.56 kg.

4.81

A minimum weight tubular column design problem is formulated in Section 2.7 using outer radius R_o and inner radius R_i as design variables. Solve the KKT conditions for the problem imposing an additional constraint $0.5(R_o + R_i)/(R_o - R_i) \le 50$. Use the same data as in Exercise 4.80. Interpret the necessary conditions at the solution point graphically.

Solution

Referring to the Formulation 2 in Section 2.7 and Exercise 4.80, we have

$$f(R_o, R_i) = \pi\rho l\left(R_o^2 - R_i^2\right) = \pi(7850)(5)\left(R_o^2 - R_i^2\right) = \left(1.2331\times10^5\right)\left(R_o^2 - R_i^2\right)$$

$$g_1 = P/\pi\left(R_o^2 - R_i^2\right) - \sigma_a = 15915.49\big/\left(R_o^2 - R_i^2\right) - 2.5\times10^8 \le 0; \text{(stress constraint)}$$

$$g_2 = P - \pi^3 E\left(R_o^4 - R_i^4\right)\big/16l^2 = 50000 - 1.62783\times10^{10}\left(R_o^4 - R_i^4\right) \le 0; \text{(buckling)}$$

$$g_3 = 0.5\left(R_o + R_i\right)\big/\left(R_o - R_i\right) - 50 \le 0;\ \text{(ratio)}$$

$$g_4 = -R_o \le 0; \quad g_5 = -R_i \le 0$$

$$L = \left(1.2331\times10^5\right)\left(R_o^2 - R_i^2\right) + u_1\left[15915.49\big/\left(R_o^2 - R_i^2\right) - 2.5\times10^8 + s_1^2\right]$$
$$+ u_2\left[50000 - \left(1.62783\times10^{10}\right)\left(R_o^4 - R_i^4\right) + s_2^2\right] + u_3\left(0.5\left(R_o + R_i\right)\big/\left(R_o - R_i\right) - 50 + s_3^2\right)$$
$$+ u_4\left(-R_o + s_4^2\right) + u_5\left(-R_i + s_5^2\right)$$

$$\partial L/\partial R_o = \left(2.4662\times10^5\right)R_o - \left(2R_o u_1\right)(15915.49)\big/\left(R_o^2 - R_i^2\right)^2$$
$$-4\left(1.62783\times10^{10}\right)R_o^3 u_2 - u_3 R_i\big/\left(R_o - R_i\right)^2 - u_4 = 0$$

$$\partial L/\partial R_i = \left(2.4662\times10^5\right)R_o - \left(2R_o u_1\right)(15915.49)\big/\left(R_o^2 - R_i^2\right)^2$$
$$+ 4\left(1.62783\times10^{10}\right)R_i^3 u_2 + u_3 R_o\big/\left(R_o - R_i\right)^2 - u_5 = 0$$

$$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\quad i = 1 \text{ to } 5$$

There are 32 cases because there are five inequality constraints. The case which yields a solution is $u_1 = u_4 = u_5 = 0, s_2 = s_3 = 0.$ Solving this case, we get

$$R_i = 7.792774\times10^{-2}, R_o = 7.950204\times10^{-2}, u_2 = 3.056\times10^{-4}, u_3 = 0.3055.$$

Checking for feasibility: we get $g_1, g_4, g_5 < 0$. All the KKT conditions are satisfied.

So $\left(7.950204\times10^{-2},\ 7.792774\times10^{-2}\right)$ is a candidate minimum point. At this point, buckling and ratio constraints are active and the cost function is 30.561 kg.

4.82

An engineering design problem is formulated as
Minimize $f(x_1, x_2) = x_1^2 + 320x_1x_2$
Subject to $\frac{1}{60x_2}x_1 - 1 \le 0$

$$1 - \frac{1}{3600}x_1(x_1 - x_2) \le 0$$

$$x_1, x_2 \ge 0$$

Write KKT necessary condition and solve for the candidate minimum designs. Verify the solutions graphically. Interpret the KKT conditions on the graph for the problem.

Solution

Minimize $f(\mathbf{x}) = x_1^2 + 320x_1x_2$, subject to $g_1 = x_1/60x_2 - 1 \le 0;$

$g_2 = 1 - x_1(x_1 - x_2)/3600 \le 0; \quad g_3 = -x_1 \le 0; \quad g_4 = -x_2 \le 0$

$L = (x_1^2 + 320x_1x_2) + u_1(x_1/60x_2 - 1 + s_1^2) + u_2(1 - x_1(x_1 - x_2)/3600 + s_2^2) + u_3(-x_1 + s_3^2) + u_4(-x_2 + s_4^2)$

$\partial L/\partial x_1 = 2x_1 + 320x_2 + u_1/60x_2 - u_2(2x_1 - x_2)/3600 - u_3 = 0$

$\partial L/\partial x_2 = 320x_1 - u_1x_1/60x_2^2 + u_2\, x_1/3600 - u_4 = 0;$

$g_i + s_i^2 = 0,\ u_is_i = 0,\ u_i \ge 0; \quad i = 1 \text{ to } 4$

There are 16 cases because there are four inequality constraints. A case which yields the solution is identified as $s_1, s_2 = 0;\ u_3, u_4 = 0$. The solution is

$x_1 = 60.50634,\ x_2 = 1.008439,\ u_1 = 19529,\ u_2 = 229.9,\ f = 23{,}186.4.$

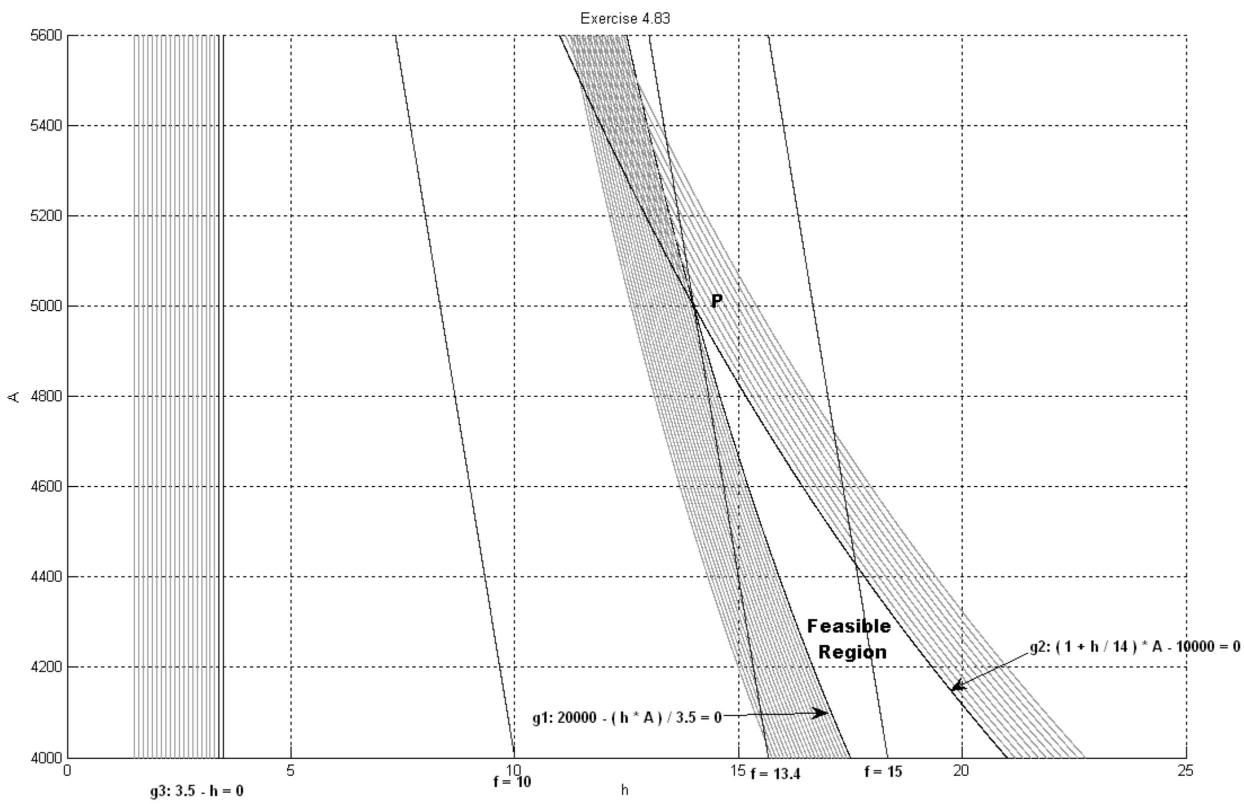
All the KKT conditions are satisfied. The solution can be verified graphically. It shows that the point obtained using KKT conditions is indeed a minimum point.

Formulate and solve the following problems graphically. Verify the KKT conditions at the solution point and show gradients of the cost function and active constraints on the graph.

4.83

A 100 ×100m lot is available to construct a multistory office building. At least 20,000 m^2 total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21m, and the area for parking outside the building must be at least 25 percent of the floor area. It has been decided to fix the height of each story at 3.5m. The cost of the building in millions of dollars is estimated at 0.6 h +0.001 A, where A is the cross-sectional area of the building per floor and h is the height of the building. Formulate the minimum cost design problem.

Solution



Note : $g4 = -A$ is not shown on the graph.

According to the graphical solution, the point P (14, 5000) is minimum point.

Referring to the formulation in Exercise 2.1 we have

Minimize $f = 0.6h + 0.001A$, subject to: $g_1 = 20{,}000 - hA/3.5 \le 0$;

$g_2 = (1 + h/14)A - 10{,}000 \le 0$; $g_3 = 3.5 - h \le 0$; $g_4 = h - 21 \le 0$; $g_5 = -A \le 0$

$L = (0.6h + 0.001A) + u_1\left(20{,}000 - hA/3.5 + s_1^2\right) + u_2\left[(1 + h/14)A - 10{,}000 + s_2^2\right]$

$+u_3\left(3.5 - h + s_3^2\right) + u_4\left(h - 21 + s_4^2\right) + u_5\left(-A + s_5^2\right)$

$\partial L/\partial h = 0.6 - u_1(A/3.5) + u_2(A/14) - u_3 + u_4 = 0$

$\partial L/\partial A = 0.01 - u_1(h/3.5) + u_2(1 + h/14) - u_5 = 0$

$g_i + s_i^2 = 0$, $u_i s_i = 0$, $u_i \ge 0$; $i = 1$ to 5; $s_i^2 \ge 0$, Regularity is satisfied.

There are 32 cases because we have five inequality constraints. The case which yields a solution is $u_3 = u_4 = u_5 = 0, s_1 = s_2 = 0.$ The solution is h = 14, A = 5000, $u_1 = 5.9\times 10^{-4}$, $u_2 = 6.8\times 10^{-4}$, f = 13.4 mil. dollars. The solution can be verified graphically. It is seen that the point obtained using the KKT conditions is indeed a minimum point.

MATLAB Code for Exercise 4.83

```
clear all
[h,A]=meshgrid(0:0.5:25, 4000:0.5:5600);
f=0.6*h+0.001*A;
g1=20000-(h.*A)/3.5;
g2=(1+h/14).*A-10000;
g3=3.5-h;
g4=-A;
cla reset
axis ([0 25 4000 5600])
xlabel('h'),ylabel('A')
title('Exercise 4.83')
hold on

cv1=[0:100:2000];
const1=contour(h,A,g1,cv1,'g');
cv1=[0 0.1];
const1=contour(h,A,g1,cv1,'k');

cv2=[0:50:500];
const2=contour(h,A,g2,cv2,'g');
cv2=[0 0.1];
const2=contour(h,A,g2,cv2,'k');

cv3=[0:0.1:2];
const3=contour(h,A,g3,cv3,'g');
cv3=[0 0.1];
const3=contour(h,A,g3,cv3,'k');

cv4=[0:1:5];
const4=contour(h,A,g4,cv4,'g');
cv4=[0 0.1];
const4=contour(h,A,g4,cv4,'k');

fv=[10 13.4 15];
fs=contour(h,A,f,fv,'b');
a=[14];
b=[5000];
plot(a,b,'.k');

grid
hold off
```

4.84

A refinery has two crude oils:

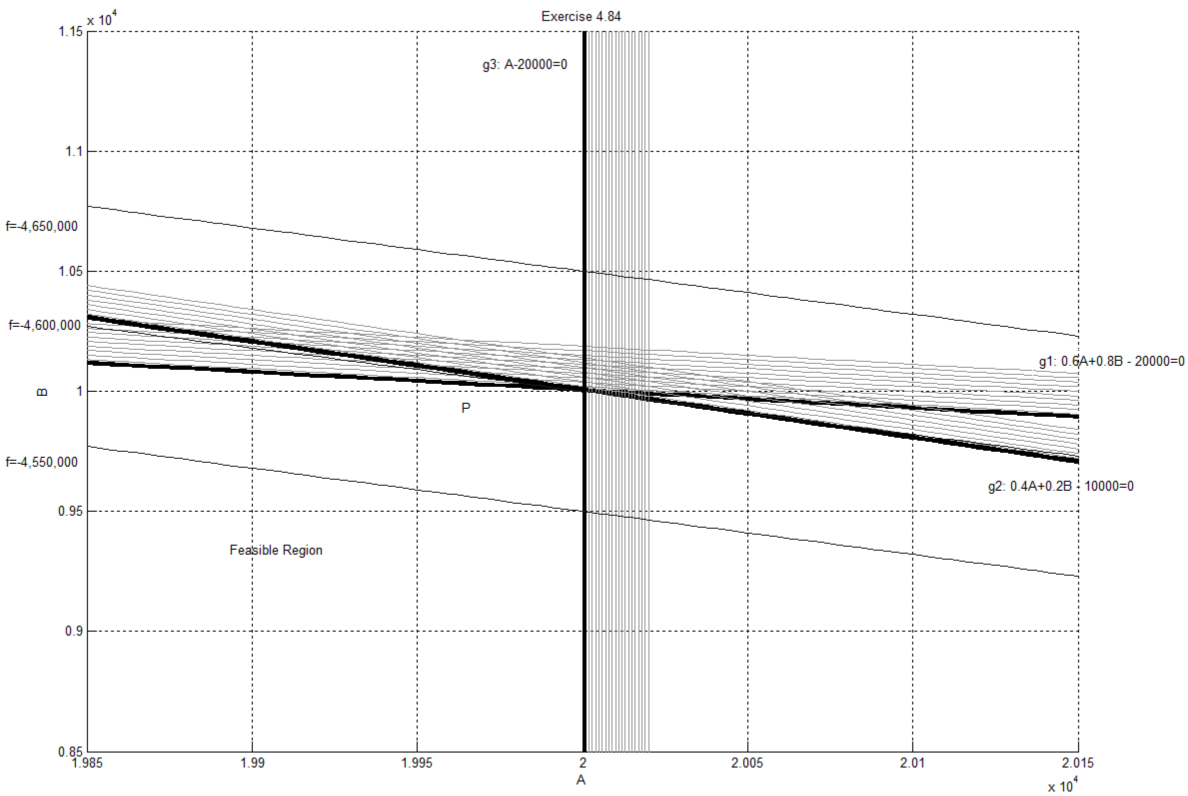
1. Crude *A* costs $120/barrel (bbl) and 20,000bbl are available.
2. Crude *B* costs $150/bbl and 30,000 are available.

The company manufactures gasoline and lube oil from the crudes. Yield and sale price barrel of the product and markets are shown in Table E2.2. How much crude oils should the company use to maximize its profit? Formulate the optimum design problem.

Table E2.2 Data for Refinery Operation

Product	Yield/bbl Crude *A*	Yield/bbl Crude *B*	Sale Price per bbl ($)	Market (bbl)
Gasoline	0.6	0.8	200	20,000
Lube oil	0.4	0.2	450	10,000

Solution



Note : $g4 = B - 30000, g5 = -A$ and $g6 = -B$ are not shown on the graph.

According to the graphical solution, the point P (20000, 10000) is the minimum point with f^* = -4,600,000.

Referring to the formulation in Exercise 2.2, we have
Minimize $f = -180A - 100B$
subject to:

$g_1 = 0.6A + 0.8B - 20{,}000 \le 0$, (gasoline market)

$g_2 = 0.4A + 0.2B - 10{,}000 \le 0$ (lube oil market)

$g_3 = A - 20{,}000 \le 0$

$g_4 = B - 30{,}000 \le 0$

$g_5 = -A \le 0; \quad g_6 = -B \le 0$

$L = (-180A - 100B) + u_1(0.6A + 0.8B - 20{,}000) + s_1^2 + u_2(0.4A + 0.2B - 10{,}000 + s_2^2) +$
$u_3(A - 20{,}000 + s_3^2) + u_4(B - 30{,}000 + s_4^2) + u_5(-A + s_5^2) + u_6(-B + s_6^2)$

$\partial L/\partial A = -180 + 0.6u_1 + 0.4u_2 + u_3 - u_5 = 0$

$\partial L/\partial B = -100 + 0.8u_1 + 0.2u_2 + u_4 - u_6 = 0$

$g_i + s_i^2 = 0, \quad u_i s_i = 0, \quad u_i \ge 0; \quad i = 1 \text{ to } 6$

There are 64 cases because we have six inequality constraints. We shall examine three cases. Two of them yield solutions.

Case 1. $u_2 = u_4 = u_5 = u_6 = 0, s_1 = s_3 = 0$. The solution is given as A = 20,000, B = 10,000, u_1 = 125, u_3 = 105; s_2 = 0, so g_2 is also active.

Case 2. $u_3 = u_4 = u_5 = u_6 = 0, s_1 = s_2 = 0$. The solution is given as A = 20,000, B = 10,000, u_1 = 20, u_2 = 420; s_3 = 0, so g_3 is also active.

Case 3. $u_1 = u_4 = u_5 = u_6 = 0, s_2 = s_3 = 0$. The solution is given as A = 20,000, B = 10,000, u_2 = 500, $u_3 = -20$ (violation); so this case does not give a solution.

The candidate minimum point derived in Case 1 and Case 2 can be verified graphically. It is seen that $(20000, 10000)$ is the optimum point. At this optimum point, g_1, g_2 and g_3 (constraints on gasoline and lube oil markets, and limit on crude A) are all active. The optimum cost is $f = -4{,}600{,}000.$. The optimum point is irregular, since there are three active constraints and two design variables. The Lagrange multipliers are not unique.

MATLAB Code for Exercise 4.84

```
clear all
[A,B]=meshgrid(19850:10:20150,8500:10:11500);
f=-180*A-100*B;
g1=0.6*A+0.8*B-20000;
g2=0.4*A+0.2*B-10000;
g3=A-20000;
g4=B-30000;
g5=-A;
g6=-B;
cla reset
axis auto
xlabel('A'),ylabel('B')
title('Exercise 4.84')
hold on

cv1=[0:15:150];
const1=contour(A,B,g1,cv1,'g');
cv1=[0:1:10];
const1=contour(A,B,g1,cv1,'k');

cv2=[0:4:30];
const2=contour(A,B,g2,cv2,'g');
cv2=[0:0.5:3];
const2=contour(A,B,g2,cv2,'k');

cv3=[0:1:20];
const3=contour(A,B,g3,cv3,'g');
cv3=[0:0.2:1];
const3=contour(A,B,g3,cv3,'k');

cv4=[0:1:200];
const4=contour(A,B,g4,cv4,'g');
cv4=[0 5];
const4=contour(A,B,g4,cv4,'k');

cv5=[0:1:5];
const3=contour(A,B,g5,cv5,'g');
cv5=[0];
const5=contour(A,B,g5,cv5,'k');

cv6=[0:1:5];
const4=contour(A,B,g6,cv6,'g');
cv6=[0];
const6=contour(A,B,g6,cv6,'k');

fv=[-4550000 -4600000 -4650000];
fs=contour(A,B,f,fv,'b');
a=[20000];
b=[10000];
plot(a,b,'.k');

grid
hold off
```

4.85

Design a beer bug, shown in Fig. E2.3, to hold as much beer as possible. The height and radius of the mug should be not more than 20 cm. The mug must be at least 5cm in radius. The surface area of the sides must not be greater than 900 cm^2 (ignore the area of the bottom of the mug and ignore the mug handle – see figure). Formulate the optimum design problem.

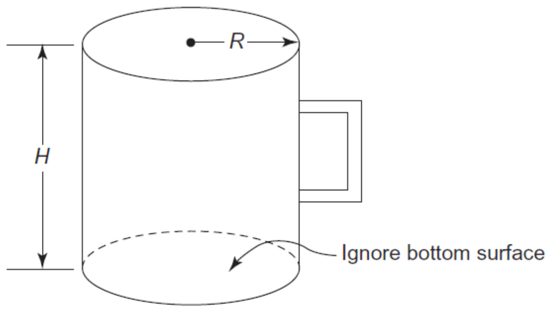
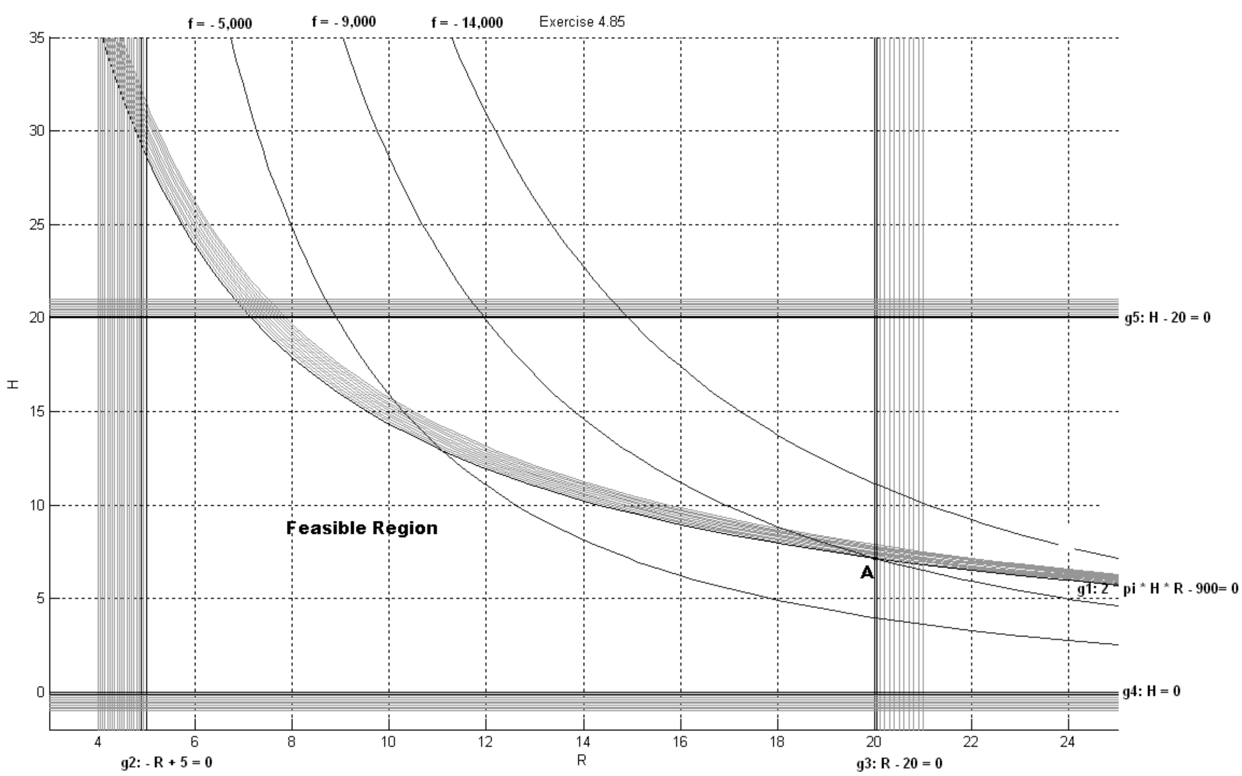


FIGURE E2.3 Beer mug.

Solution



According to the graphical solution, the point A (20, 7.161973) is minimum point.

Referring to the formulation in Exercise 2.3, we have 32 cases because we have five inequality constraints. The case which yields a solution is identified as $u_2 = u_4 = u_5 = 0,\ s_1 = s_3 = 0$. The solution is R = 20, H = 7.161973, u_1 = 10, u_3 = 450, $f = -9000$. The solution can be verified graphically. It is seen that $(20, 7.161973)$ is a minimum point where g_1 (surface area constraint) and g_3 (max. radius constraint) are active.

MATLAB Code for Exercise 4.85

```
clear all
[R,H]=meshgrid(3:1:25,-2:1:35);
f=-pi*R.^2.*H;
g1=2*pi*R.*H-900;
g2=-R+5;
g3=R-20;
g4=-H;
g5=H-20;
cla reset
axis ([3 25 -2 35])
xlabel('R'),ylabel('H')
title('Exercise 4.85')
hold on

cv1=[0:9:90];
const1=contour(R,H,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(R,H,g1,cv1,'k');

cv2=[0:0.05:1];
const2=contour(R,H,g2,cv2,'g');
cv2=[0 0.1];
const2=contour(R,H,g2,cv2,'k');

cv3=[0:0.1:1];
const3=contour(R,H,g3,cv3,'g');
cv3=[0 0.05];
const3=contour(R,H,g3,cv3,'k');

cv4=[0:0.1:1];
const4=contour(R,H,g4,cv4,'g');
cv4=[0 0.1];
const4=contour(R,H,g4,cv4,'k');

cv5=[0:0.1:1];
const3=contour(R,H,g5,cv5,'g');
cv5=[0 0.1];
const5=contour(R,H,g5,cv5,'k');

fv=[-14000 -9000 -5000];
fs=contour(R,H,f,fv,'b');
a=[20];
b=[7.161973];
plot(a,b,'.k');

grid
hold off
```

4.86

A company is redesigning its parallel flow heat exchanger of length *l* to increase its heat transfer. An end view of the units is shown in Fig. E2.4. There are certain limitations on the design problem. The smallest available conducting tube has a radius of 0.5 cm and all tubes must be of the same size. Further, the total cross sectional area of all the tubes cannot exceed 2000cm^2 to ensure adequate space inside the outer shell. Formulate the problem to determine the number of tubes and the radius of each tube to maximize the surface are of the tubes in the exchanger.

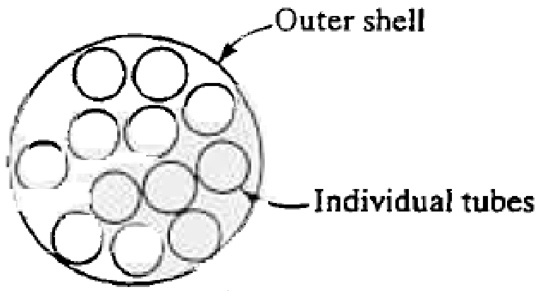
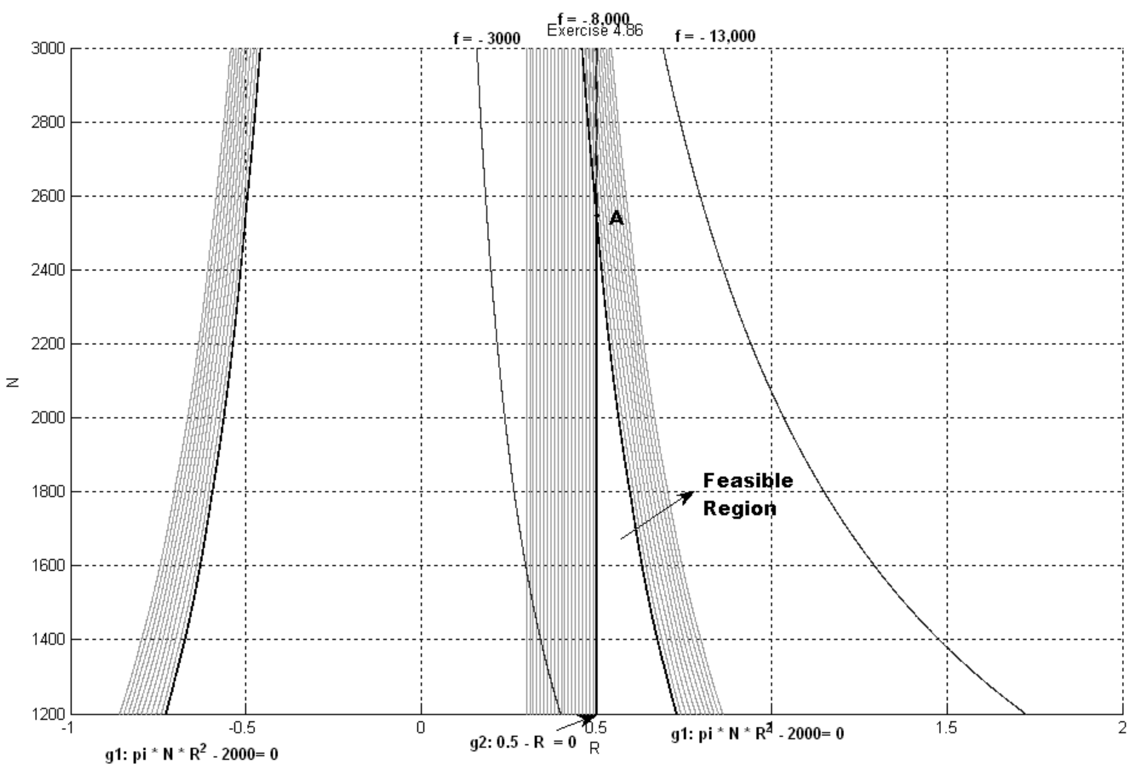


FIGURE E2.4 Cross section of heat exchanger.

Solution



According to the graphical solution, the point A (0.5, 2546.5) is minimum point.
Referring to the formulation in Exercise 2.4, we have for unit length l = 1:
Minimize $f = -2\pi l NR = -2\pi NR$,
subject to

$$g_1 = 0.5 - R \le 0$$

$$g_2 = \pi NR^2 - 2000 \le 0$$

$$g_3 = -N \le 0$$

$$L = -2\pi NR + u_1\left(0.5 - R + s_1^2\right) + u_2\left(\pi NR^2 - 200 + s_2^2\right) + u_3\left(-N + s_3^2\right)$$

$$\partial L/\partial N = -2\pi R + u_2\, \pi R^2 - u_3 = 0$$

$$\partial L/\partial R = -2\pi N - u_1 + 2\pi R u_2 = 0$$

$$g_i + s_i^2 = 0,\quad u_i s_i = 0,\quad u_i \ge 0;\quad i = 1 \text{ to } 3$$

There are 4 cases. We shall examine one case which yields a solution. The case is $s_1 = s_2 = 0$ and u_3 = 0 giving R = 0.5, N = 2546.5, u_1 = 16000, u_2 = 4, $f = -8000$. This solution can be verified graphically. It can be seen that the point obtained using KKT conditions is a minimum point and the cross-sectional area constraint and radius limit constraint are active. Practical optimum can be taken as N = 2546 and R = 0.5.

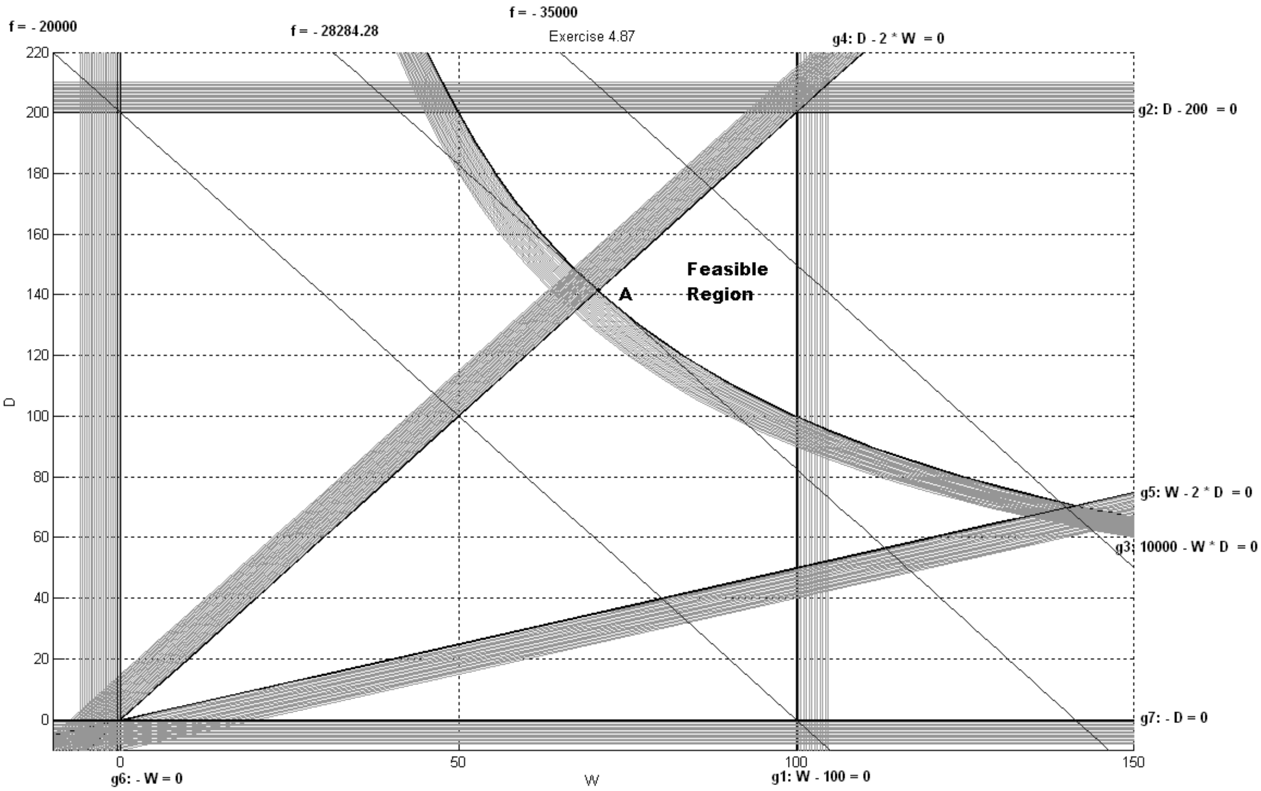
MATLAB Code

```
clear all
[R,N]=meshgrid(-1:0.1:2, 1200:1:3000);
f=-2*pi*1*N.*R;
g1=0.5-R;
g2=pi*N.*R.^2-2000;
g3=-N;
cla reset
axis ([-1 2 1200 3000])
xlabel('R'),ylabel('N')
title('Exercise 4.86')
hold on

cv1=[0:0.01:0.2];
const1=contour(R,N,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(R,N,g1,cv1,'k');

cv2=[0:80:800];
const2=contour(R,N,g2,cv2,'g');
cv2=[0 15];
const2=contour(R,N,g2,cv2,'k');

cv3=[0:0.1:1];
const3=contour(R,N,g3,cv3,'g');
cv3=[0 0.1];
const3=contour(R,N,g3,cv3,'k');

fv=[-13000 -8000 -3000];
fs=contour(R,N,f,fv,'b');
a=[0.5];
b=[2546.5];
plot(a,b,'.k');

grid
hold off
```

4.87

Proposals for a parking ramp having been defeated, we plan to build parking lot in the downtown urban renewal section. The cost of land is $200W + 100D$, where W is the width along the street and D the depth of the lot in meters. The available width along the street is 100m, while the maximum depth available is 200 m. We want to have at least 10,000 m^2 in the lot. To avoid unsightly lots, the city requires that the longer dimension of any lot be no more than twice the shorter dimension. Formulate the minimum cost design problem.

Solution



According to the graphical solution, the point A (70.7107, 141.4214) is the minimum point.

Referring to the formulation in Exercise 2.5, we have
Minimize $f = 200W + 100D$
subject to

$g_1 = W - 100 \le 0$

$g_2 = D - 200 \le 0$

$g_3 = 10{,}000 - WD \le 0$

$g_4 = D - 2W \le 0$

$g_5 = W - 2D \le 0$

$g_6 = -W \le 0; \; g_7 = -D \le 0$

$$L = 200W + 100D + u_1\left(W - 100 + s_1^2\right) + u_2\left(D - 200 + s_2^2\right) + u_3\left(10{,}000 - WD + s_3^2\right) + u_4\left(D - 2W + s_4^2\right)$$
$$+ u_5\left(W - 2D + s_5^2\right) + u_6\left(-W + s_6^2\right) + u_7\left(-D + s_7^2\right)$$

$$\partial L/\partial W = 200 + u_1 - u_3 D - 2u_4 + u_5 - u_6 = 0; \quad \partial L/\partial D = 100 + u_2 - u_3 W + u_4 - 2u_5 - u_7 = 0$$

There are in all 128 cases because there are 7 inequality constraints. We shall examine one case which yields a solution. The case is identified as $u_1 = u_2 = u_5 = u_6 = u_7 = 0$, giving the solution as $W = 70.7107,\; D = 141.4214,\; u_3 = 1.41421,\; u_4 = 0,\; f = 28284.28$;all other constraints are inactive. The solution can be verified graphically. It is seen from the graph that the point obtained using KKT conditions is a minimum point. It is also revealed from the graph that Lagrange multiplier of g_4 must be zero since the gradient of cost function and g_3 are along the same line but in opposite directions.

<u>MATLAB Code for Exercise 4.87</u>

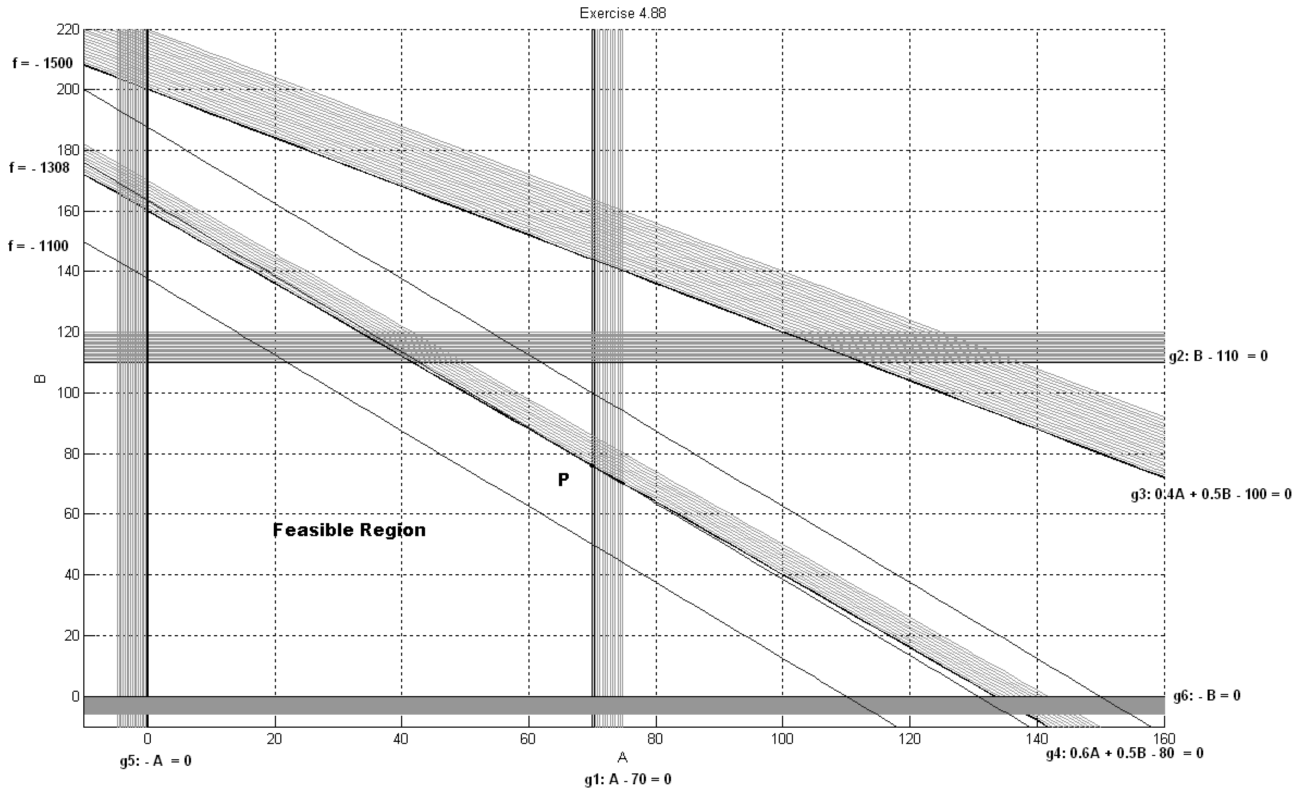
```
clear all
[W,D]=meshgrid(-10:5:150 , -10:10:220);
f=200*W+100*D;
g1=W-100;
g2=D-200;
g3=10000-W.*D;
g4=D-2*W;
g5=W-2*D;
g6=-W;
g7=-D;
cla reset
axis ([-10 150 -10 220])
xlabel('W'),ylabel('D')
title('Exercise 4.87')
hold on
cv1=[0:0.4:5];
const1=contour(W,D,g1,cv1,'g');
cv1=[0 0.3];
const1=contour(W,D,g1,cv1,'k');

cv2=[0:0.5:10];
```

```
const2=contour(W,D,g2,cv2,'g');
cv2=[0 0.2];
const2=contour(W,D,g2,cv2,'k');

cv3=[0:50:1000];
const3=contour(W,D,g3,cv3,'g');
cv3=[0 20];
const3=contour(W,D,g3,cv3,'k');

cv4=[0:0.5:15];
const4=contour(W,D,g4,cv4,'g');
cv4=[0 0.2];
const4=contour(W,D,g4,cv4,'k');

cv5=[0:1:20];
const5=contour(W,D,g5,cv5,'g');
cv5=[0 0.5];
const5=contour(W,D,g5,cv5,'k');

cv6=[0:0.3:6];
const6=contour(W,D,g6,cv6,'g');
cv6=[0 0.5];
const6=contour(W,D,g6,cv6,'k');

cv7=[0:0.5:8];
const7=contour(W,D,g7,cv7,'g');
cv7=[0 0.5];
const7=contour(W,D,g7,cv7,'k');

fv=[20000 28284.28 35000];
fs=contour(W,D,f,fv,'b');
a=[70.7107];
b=[141.4214];
plot(a,b,'.k');

grid
hold off
```

4.88

A manufacturer sells products A and B. Profit from A is \$10/kg and from B \$8/k g. Available raw materials for the products are: 100 kg of C and 80 kg of D. To produce 1 kg of A, 0.4 kg of C and 0.6kg of D are needed. To produce 1 kg of B, 0.5kg of C and 0.5kg of D are needed. The markets for the products are 70kg for A and 110kg for B. How much A and B should be produced to maximize profit? Formulate the design optimization problem.

Solution



According to the graphical solution, the point P (70, 76) is minimum point.
Referring to the formulation in Exercise 2.6, we have

Minimize $f = -10A - 8B$, subject to: $g_1 = A - 70 \le 0;\ g_2 = B - 110 \le 0;\ g_3 = 0.4A + 0.5B - 100 \le 0;$

$g_4 = 0.6A + 0.5B - 80 \le 0;\ g_5 = -A \le 0;\ g_6 = -B \le 0$

$L = (-10A - 8B) + u_1(A - 70 + s_1^2) + u_2(B - 110 + s_2^2) + u_3(0.4A + 0.5B - 100 + s_3^2)$

$u_4(0.6A + 0.5B - 80 + s_4^2) + u_5(-A + s_5^2) + u_6(-B + s_6^2)$

$\partial L/\partial A = -10 + u_1 + 0.4u_3 + 0.6u_4 - u_5 = 0;\ \partial L/\partial B = -8 + u_2 + 0.5u_3 + 0.5u_4 - u_6 = 0$

$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0,\ i = 1 \text{ to } 6$

There are 64 cases because there are 6 inequality constraints. The case which yields a solution is given as $u_2 = u_3 = u_5 = u_6 = 0, s_1 = s_4 = 0$. The solution is A = 70, B = 76, u_1 = 0.4, u_4 = 16, $f = -1308$. This solution can be verified graphically. It is seen from the graph that the solution is a minimum point and limits on product A and raw material D are active.

MATLAB Code for Exercise 4.88

```
clear all
[A,B]=meshgrid(-10:5:160 , -10:5:220);
f=-10*A-8*B;
g1=A-70;
g2=B-110;
g3=0.4*A+0.5*B-100;
g4=0.6*A+0.5*B-80;
g5=-A;
g6=-B;

cla reset
axis ([-10 160 -10 220])
xlabel('A'),ylabel('B')
title('Exercise 4.88')
hold on
cv1=[0:0.4:5];
const1=contour(A,B,g1,cv1,'g');
cv1=[0 0.3];
const1=contour(A,B,g1,cv1,'k');

cv2=[0:0.5:10];
const2=contour(A,B,g2,cv2,'g');
cv2=[0 0.2];
const2=contour(A,B,g2,cv2,'k');

cv3=[0:0.5:10];
const3=contour(A,B,g3,cv3,'g');
cv3=[0 0.1];
const3=contour(A,B,g3,cv3,'k');

cv4=[0:0.5:5];
const4=contour(A,B,g4,cv4,'g');
cv4=[0 0.1];
const4=contour(A,B,g4,cv4,'k');

cv5=[0:0.3:5];
const5=contour(A,B,g5,cv5,'g');
cv5=[0 0.1];
const5=contour(A,B,g5,cv5,'k');

cv6=[0:0.3:6];
const6=contour(A,B,g6,cv6,'g');
cv6=[0 0.1];
const6=contour(A,B,g6,cv6,'k');

fv=[-1500 -1308 -1100];
fs=contour(A,B,f,fv,'b');
a=[70];
b=[76];
plot(a,b,'.k');

grid
hold off
```

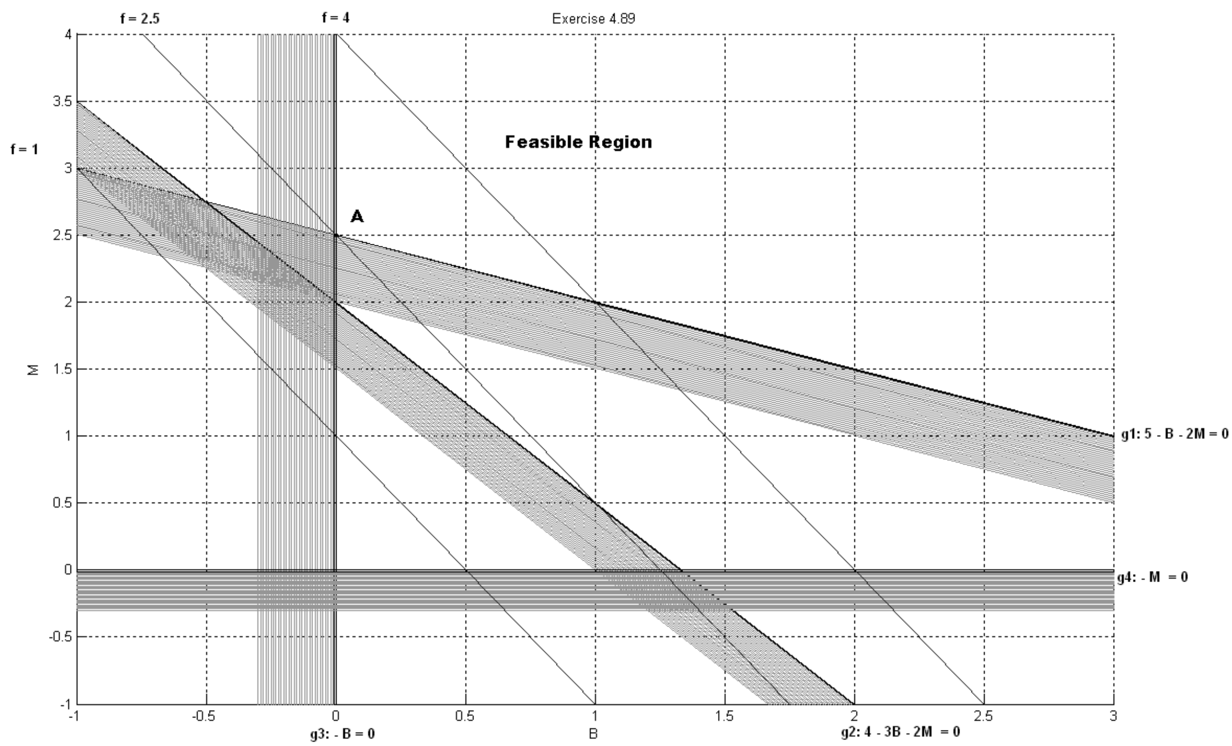
4.89

Design a diet of bread and milk to get at least 5 units of vitamin A and 4 units of vitamin B each day. The amount of vitamins A and B in 1kg of each food and the cost per kilogram of food are given in Table E2.7. Formulate the design optimization problem so that we get at least the basic requirements of vitamins at the minimum cost.

Table E2.7 Data for the Diet Problem

Vitamin	Bread	Milk
A	1	2
B	3	2
Cost/kg	2	1

Solution



According to the graphical solution, the point A (0, 2.5) is minimum point.
Referring to the formulation in Exercise 2.7, we have

Minimize $f = 2B + M$, subject to: $g_1 = 5 - B - 2M \le 0;\ \ g_2 = 4 - 3B - 2M \le 0;$

$g_3 = -B \le 0;\ \ g_4 = -M \le 0$

$L = (2B + M) + u_1(5 - B - 2M + s_1^2) + u_2(4 - 3B - 2M + s_2^2) + u_3(-B + s_3^2) + u_4(-M + s_4^2)$

$\partial L/\partial B = 2 - u_1 - 3u_2 - u_3 = 0;\ \ \partial L/\partial M = 1 - 2u_1 - 2u_2 - u_4 = 0;\ g_i + s_i^2 = 0,\ \ u_i s_i = 0,\ \ u_i \ge 0;\ \ i = 1 \text{ to } 4$

There are in all 16 cases because there are 4 inequality constraints. The case which yields a solution is given as $u_2 = u_4 = 0, s_1 = s_3 = 0$. The solution is B = 0, M = 2.5, u_1 = 0.5, u_3 = 1.5, f = 2.5. It is seen from the graph for the problem that the solution is a minimum point and the Vitamin A constraint and minimum bread constraint are active.

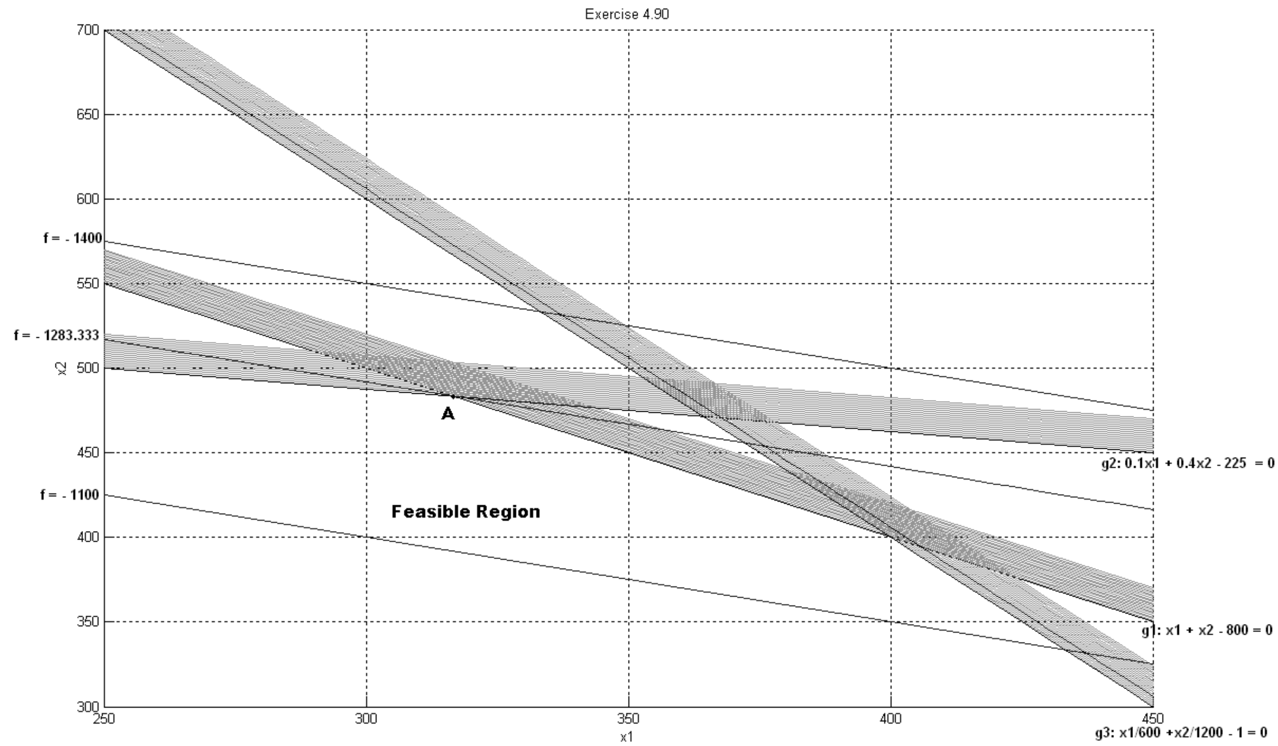
MATLAB Code for Exercise 4.89

```
clear all
[B,M]=meshgrid(-1:1:3 , -1:1:4);
f=2*B+M;
g1=5-B-2*M;
g2=4-3*B-2*M;
g3=-B;
g4=-M;

cla reset
axis ([-1 3 -1 4])
xlabel('B'),ylabel('M')
title('Exercise 4.89')
hold on
cv1=[0:0.03:1];
const1=contour(B,M,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(B,M,g1,cv1,'k');

cv2=[0:0.03:1];
const2=contour(B,M,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(B,M,g2,cv2,'k');

cv3=[0:0.01:0.3];
const3=contour(B,M,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(B,M,g3,cv3,'k');

cv4=[0:0.01:0.3];
const4=contour(B,M,g4,cv4,'g');
cv4=[0 0.01];
const4=contour(B,M,g4,cv4,'k');

fv=[1 2.5 4];
fs=contour(B,M,f,fv,'b');
a=[0];
b=[2.5];
plot(a,b,'.k');

grid
hold off
```

4.90

Enterprising chemical engineering students have set up a still in a bathtub. They can produce 225 bottles of pure alcohol each week. They bottle two products from alcohol: (i) wine, 20 proof, and (ii) whiskey, 80 proof. Recall that pure alcohol is 200 proof. They have an unlimited supply of water but can only obtain 800 empty bottles per week because of stiff competition. The weekly supply of sugar is enough for either 600 bottles of wine or 1200 bottles of whiskey. They make \$1.00 profit on each bottle of wine and \$2.00 profit on each bottle of whiskey. They can sell whatever they produce. How many bottles of wine and whisky should they produce each week to maximize profit? Formulate the design optimization problem. (created by D. Levy)

Solution



Note : $g_4 = -x_1,\ g_5 = -x_2$ are not shown on the graph.
According to the graphical solution, the point A (316.667, 483.333) is minimum point.
Referring to the formulation in Exercise 2.8, we have
Minimize $f = -x_1(\text{wine}) - 2x_2(\text{whiskey})$, subject to $g_1 = x_1 + x_2 - 800 \le 0$; $g_2 = 0.1x_1 + 0.4x_2 - 225 \le 0$

$$g_3 = x_1/600 + x_2/1200 - 1 \le 0;\quad g_4 = -x_1 \le 0;\quad g_5 = -x_2 \le 0.$$

$$L = -x_1 - 2x_2 + u_1\left(x_1 + x_2 - 800 + s_1^2\right) + u_2\left(0.1x_1 + 0.4x_2 - 225 + s_2^2\right)$$

$$+u_3\left(x_1/600 + x_2/1200 - 1 + s_3^2\right) + u_4\left(-x_1 + s_4^2\right) + u_5\left(-x_2 + s_5^2\right)$$

$$\partial L/\partial x_1 = -1 + u_1 + 0.1u_2 + u_3/600 - u_4 = 0;\ \partial L/\partial x_2 = -2 + u_1 + 0.4u_2 + u_3/1200 - u_5 = 0$$

$$g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\ i = 1 \text{ to } 5$$

There are 32 cases because there are 5 inequality constraints. The case which yields a solution is identified as $u_3 = u_4 = u_5 = 0, s_1 = s_2 = 0$. The solution is

$x_1 = 316.667,\ x_2 = 483.333, u_1 = 2/3, u_2 = 1/12000,\ f = -1283.333$.

All other constraints are inactive. It is seen from the graph for the problem that the solution is indeed a minimum point and the constraints of bottle and alcohol supply are active.

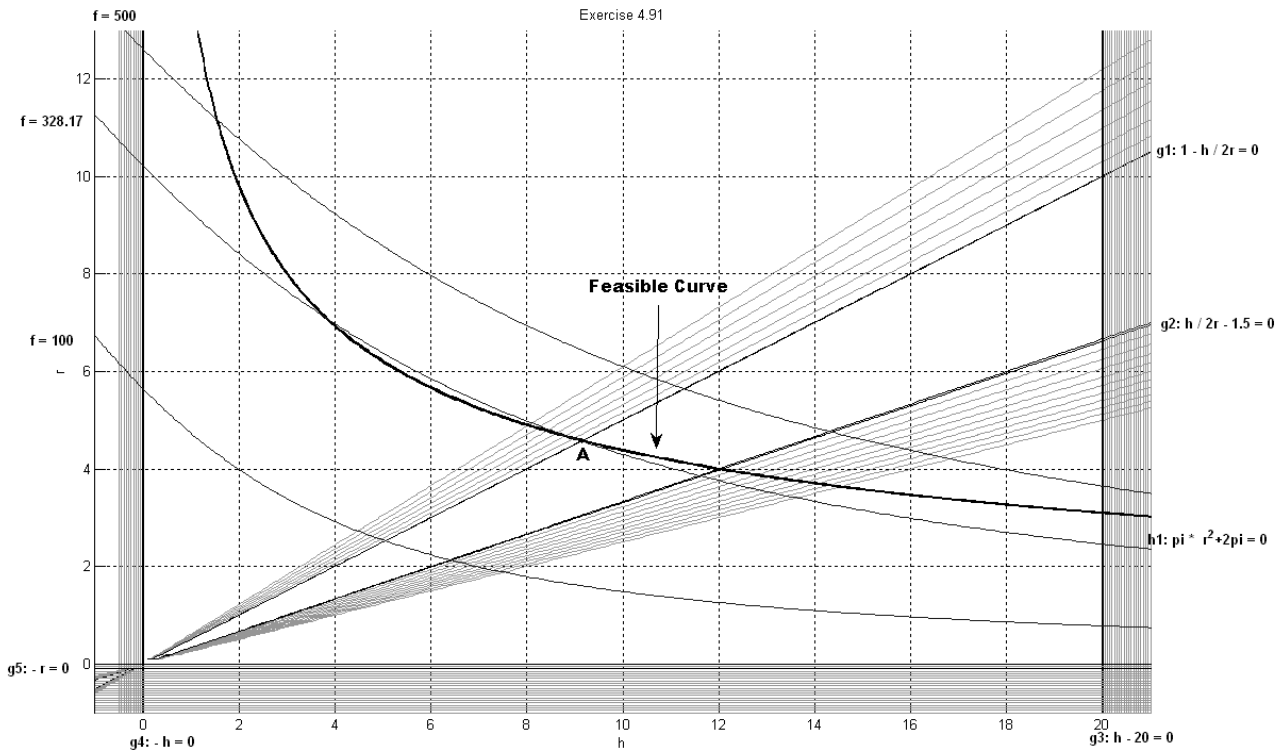
MATLAB Code for Exercise 4.90

```
clear all
[x1,x2]=meshgrid(250:10:450 , 300:10:700);
f=-x1-2*x2;
g1=x1+x2-800;
g2=0.1*x1+0.4*x2-225;
g3=x1/600+x2/1200-1;
g4=-x1;
g5=-x2;

cla reset
axis ([250 450 300 700])
xlabel('x1'),ylabel('x2')
title('Exercise 4.90')
hold on

cv1=[0:1:20];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x1,x2,g1,cv1,'k');

cv2=[0:0.5:8];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.005];
const2=contour(x1,x2,g2,cv2,'k');

cv3=[0:0.001:0.02];
const3=contour(x1,x2,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x1,x2,g3,cv3,'k');

cv4=[0:0.01:0.3];
const4=contour(x1,x2,g4,cv4,'g');
cv4=[0 0.01];
const4=contour(x1,x2,g4,cv4,'k');

cv5=[0:0.3:5];
const5=contour(x1,x2,g5,cv5,'g');
cv5=[0 0.1];
const5=contour(x1,x2,g5,cv5,'k');

fv=[-1400 -1283.333 -1100];
fs=contour(x1,x2,f,fv,'b');
a=[316.667];
b=[483.333];
plot(a,b,'.k');

grid
hold off
```

4.91

Design a can closed at the end using the smallest area of sheet metal for a specified interior volume of 600 m^3. The can is a right circular cylinder with interior height h and radius r. The ratio of height to diameter must not be less than 1.0 and not greater than 1.5. The height cannot be more than 20 cm. Formulate the design optimization problem.

Solution



According to the graphical solution, the point A (9.14156, 4.57078) is minimum point. Referring to the formulation in Exercise 2.9, we have

Minimize $f = \pi r^2 + 2\pi rh$, subject to $h_1 = \pi r^2 h - 600 = 0$; $g_1 = 1 - h/2r \le 0$; $g_2 = h/2r - 1.5 \le 0$;

$g_3 = h - 20 \le 0$; $g_4 = -h \le 0$; $g_5 = -r \le 0$

$$L = \pi r^2 + 2\pi rh + v_1(\pi r^2 h - 600) + u_1(1 - h/2r + s_1^2) + u_2(h/2r - 1.5 + s_2^2)$$

$$+ u_3(h - 20 + s_3^2) + u_4(-h + s_4^2) + u_5(-r + s_5^2)$$

$$\partial L/\partial r = 2\pi r + 2\pi h + 2\pi rhv_1 + u_1(h/2r^2) - u_2(h/2r^2) - u_5 = 0$$

$$\partial L/\partial h = 2\pi r + \pi r^2 v_1 - u_1/2r + u_2/2r + u_3 - u_4 = 0$$

$$h_1 = \pi r^2 h - 600 = 0; \quad g_i + s_i^2 = 0, \quad u_i s_i = 0, \quad u_i \ge 0; \quad i = 1 \text{ to } 5$$

There are in all 32 cases because there are 5 inequality constraints. The case which yields a solution is identified as $u_2 = u_3 = u_4 = u_5 = 0, s_1 = 0$. The solution is h = 9.14156, r = 4.57078, $v_1 = -0.364635$, u_1 = 43.7562, f = 328.17. All other constraints are satisfied.

It is seen from the graph for the problem that the solution is a minimum point and the constraint on minimum height/diameter ratio is active.

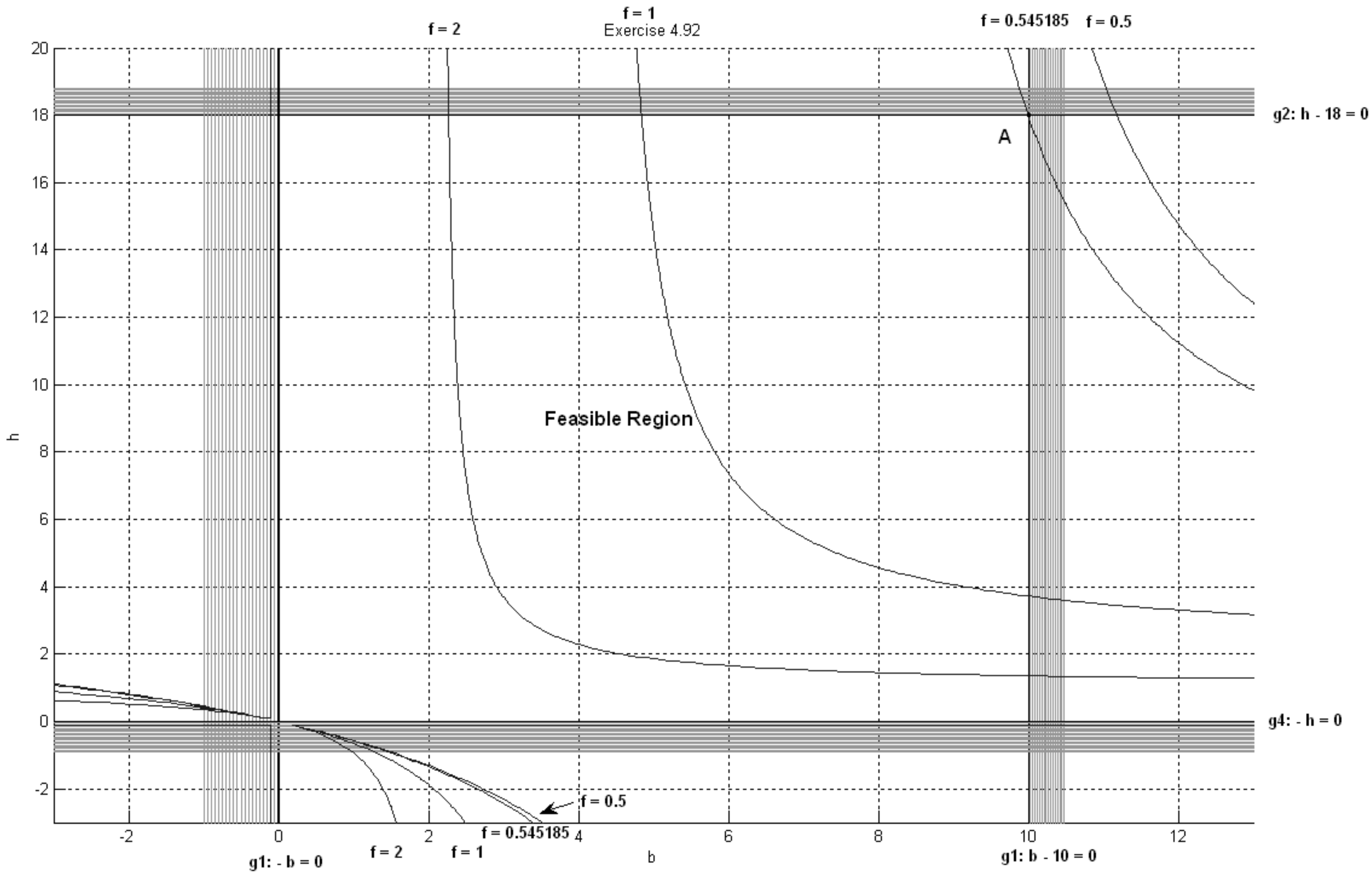
MATLAB Code for Exercise 4.91

```
clear all
[h,r]=meshgrid(-1:0.1:21 , -1:0.1:13);
f=pi*r.^2+2*pi*r.*h;
g1=1-(h./(2*r));
g2=h./(2*r)-1.5;
g3=h-20;
g4=-h;
g5=-r;

h1=pi*r.^2.*h-600;

cla reset
axis ([-1 21 -1 13])
xlabel('h'),ylabel('r')
title('Exercise 4.91')
hold on
cv1=[0:0.03:0.2];
const1=contour(h,r,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(h,r,g1,cv1,'k');

cv2=[0:0.05:0.5];
const2=contour(h,r,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(h,r,g2,cv2,'k');

cv3=[0:0.05:1];
const3=contour(h,r,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(h,r,g3,cv3,'k');

cv4=[0:0.05:0.5];
const4=contour(h,r,g4,cv4,'g');
cv4=[0 0.01];
const4=contour(h,r,g4,cv4,'k');

cv5=[0:0.05:3];
const5=contour(h,r,g5,cv5,'g');
cv5=[0 0.1];
const5=contour(h,r,g5,cv5,'k');

cv6=[0:0.1:10];
const6=contour(h,r,h1,cv6,'k');

fv=[100 328.17 500];
fs=contour(h,r,f,fv,'b');
a=[9.14156];
b=[4.57078];
plot(a,b,'.k');

grid
hold off
```

4.92

Design a shipping container closed at both ends with dimensions $b \times b \times h$ to minimize the ratio: (round-trip cost of shipping the container only) / (one-way cost of shipping the contents only).
Use the following data:
Mass of the container/surface area: 80 kg/ m^2
Maximum b: 10 m
Maximum h: 18 m
One-way shipping cost, full of empty :$18/kg gross mass
Mass of the contents: 150 kg/ m^3
Formulate the design optimization problem.

Solution



According to the graphical solution, the point A (10, 18) is the minimum point.
Referring to the formulation in Exercise 2.10, we have : Minimize $f = (32/15)(1/h + 2/b)$ subject to

$g_1 = b - 10 \le 0$; $g_2 = h - 18 \le 0$; $g_3 = -b \le 0$; $g_4 = -h \le 0$

$$L = (32/15)(1/h + 2/b) + u_1\left(b - 10 + s_1^2\right) + u_2\left(h - 18 + s_2^2\right) + u_3\left(-b + s_3^2\right) + u_4\left(-h + s_4^2\right)$$

$$\partial L/\partial b = (32/15)\left(-2/b^2\right) + u_1 - u_3 = 0; \ \partial L/\partial h = (32/15)\left(-1/h^2\right) + u_2 - u_4 = 0$$

$$g_i + s_i^2 = 0, \ u_i s_i = 0, \ u_i \ge 0, \ i = 1 \text{ to } 4$$

There are 16 cases because there are 4 inequality constraints. The case which yields a solution is given as $u_3 = u_4 = 0, s_1 = s_2 = 0$. The solution is $b = 10$, $h = 18$, $u_1 = 0.4267$, $u_2 = 0.00658$, $f = 0.545185$. It is seen from the graph for the problem that the solution is indeed a minimum point and the constraints on maximum *b* and *h* are active.

MATLAB Code for Exercise 4.92

```
clear all
[b,h]=meshgrid(-3:0.1:15 , -3:0.1:23);
f=(32/15)*(1./h+2./b);
g1=b-10;
g2=h-18;
g3=-b;
g4=-h;

cla reset
axis ([-3 13 -3 20])
xlabel('b'),ylabel('h')
title('Exercise 4.92')
hold on
cv1=[0:0.03:0.5];
const1=contour(b,h,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(b,h,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(b,h,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(b,h,g2,cv2,'k');

cv3=[0:0.05:1];
const3=contour(b,h,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(b,h,g3,cv3,'k');

cv4=[0:0.05:0.9];
const4=contour(b,h,g4,cv4,'g');
cv4=[0 0.01];
const4=contour(b,h,g4,cv4,'k');

 fv=[0.5 0.545185 1 2];

fs=contour(b,h,f,fv,'b');
a=[10];
b=[18];
plot(a,b,'.k');

grid
hold off
```

4.93

Certain mining operations require an open top rectangular container to transport materials. The data for the problem are:

Construction costs:

sides: \$50/m^2

ends: \$60/m^2

bottom: \$90/m^2

Salvage value: 25 percent of the construction cost

Useful life: 20 years

Yearly maintenance: \$12/m^2 of outside surface area

Minimum volume needed: 150 m^3

Interest rate: 12 percent per annum

Formulate the problem of determining the container dimensions for minimum present cost.

Solution

Referring to the formulation in Exercise 2.11, we have

Design Variables: dimensions of the container; b = width, m; d = depth, m; h = height, m

Cost Function: minimize total present cost;

$$\begin{aligned} \textit{Present Cost} = &\left[2dh(50) + 2bh(60) + bd(90)\right]\left[1 - 0.25\left(sppwf\ (0.12,\ 20)\right)\right] \\ &+ (2dh + 2bh + bd)(12)\left(uspwf\ (0.12,\ 20)\right) \end{aligned}$$

$sppwf(i, n) = (1 + i)^{-n}$ = single payment present worth factor

$uspwf(i, n) = \frac{1}{i}[1 - (1 + i)^{-n}]$ = uniform series present worth factor

$i = 0.12, n = 20$

$sppwf\ (0.12,\ 20) = (1 + 0.12)^{-20} = 0.103667$

$uspwf\ (0.12,\ 20) = \left[1 - (1 + 0.12)^{-20}\right] / 0.12 = 7.46944$

$$\textit{Present Cost} = (100dh + 120bh + 90bd)(1 - 0.25 \times 0.103667) + (2dh + 2bh + bd)(12)(7.46944)$$

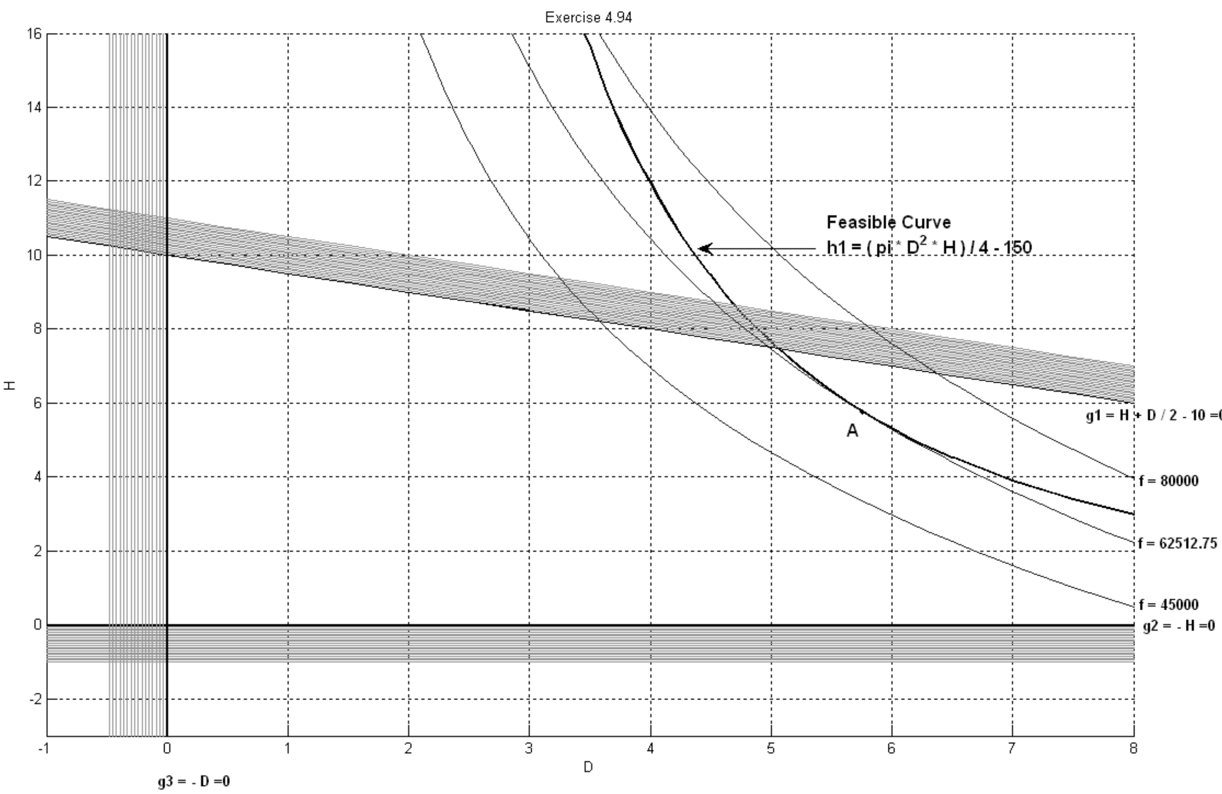
$$\textit{Present Cost} = 276.6749dh + 296.15656bh + 177.30077bd$$

Constraints: $bdh \geq 150$; $b \geq 0$, $d \geq 0$, $h \geq 0$

4.94

Design a circular tank closed at both ends to have a volume of 250 m^3. The fabrication cost in proportional to the surface area of the sheet metal and is \$400/m^2. The tank is to be housed in a shed with a sloping roof. Therefore, height *H* of the tank is limited by the relation $H \le 10 - D/2$, where *D* is the diameter of the tank. Formulate the minimum cost design problem.

Solution



According to the graphical solution, the point A (5.758823, 5.758826) is minimum point. Referring to the formulation in Exercise 2.12, we have

Minimize $f = 400\left(\pi D^2/2 + \pi DH\right)$, subject to: $h_1 = \pi D^2 H/4 - 150 = 0$; $g_1 = H + D/2 - 10 \le 0$;

$g_2 = -H \le 0$; $g_3 = -D \le 0$

$$L = 400\left(\pi D^2/2 + \pi DH\right) + v_1\left(\pi D^2 H/4 - 150\right) + u_1\left(H + D/2 - 10 + s_1^2\right) + u_2 + \left(-H + s_2^2\right) + u_3\left(-D + s_3^2\right)$$

$$\partial L/\partial D = 400\pi D + 400\pi H + \pi DHv_1/2 + u_1/2 - u_3 = 0$$

$$\partial L/\partial H = 400\pi D + \pi D^2 v_1/4 + u_1 - u_2 = 0$$

$$h_1 = \pi D^2 H/4 - 150 = 0;\quad g_i + s_i^2 = 0,\ u_i s_i = 0,\ u_i \ge 0;\quad i = 1 \text{ to } 3$$

There are 8 cases because there are 3 inequality constraints. The case which yields a solution is $u_1 = u_2 = u_3 = 0$. The solution is $D = 5.758823$, $H = 5.758826$, $v_1 = -277.834$, $f = 62512.75$. It is seen from the graph for the problem that the solution is indeed a minimum point and no inequality constraint is active.

MATLAB Code for Exercise 4.94

```
clear all
[D,H]=meshgrid(-1:0.5:8 , -3:0.5:16);
f=400*((pi*D.^2.)/2+pi*D.*H);
h1=(pi*D.^2.*H)/4-150;
g1=H+D/2-10;
g2=-H;
g3=-D;

cla reset
axis ([-1 8 -3 16])
xlabel('D'),ylabel('H')
title('Exercise 4.94')
hold on

cv0=[0:0.01:0.9];
const0=contour(D,H,h1,cv0,'k');

cv1=[0:0.05:1];
const1=contour(D,H,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(D,H,g1,cv1,'k');

cv2=[0:0.05:1];
const2=contour(D,H,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(D,H,g2,cv2,'k');

cv3=[0:0.03:0.5];
const3=contour(D,H,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(D,H,g3,cv3,'k');

fv=[45000 62512.75 80000];
fs=contour(D,H,f,fv,'b');
a=[5.758823];
b=[5.758826];
plot(a,b,'.k');

grid
hold off
```

4.95

Design the steel framework shown in Fig. E2.13 at a minimum cost. The cost of a horizontal member in one direction is \$20w and in the other direction it is \$30d. The cost of a vertical column is \$50h. The frame must enclose a total volume of at least 600 m^3. Formulate the design optimization problem.

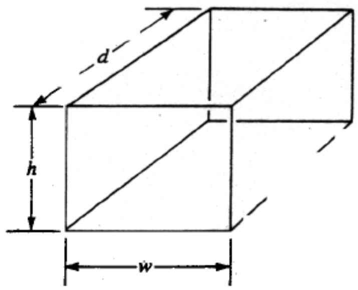


FIGURE E2.13 Steel frame.

Solution

Referring to the formulation in Exercise 2.13, we have

Design Variables: w = width of the frame, m; d = depth of the frame, m; h = height of the frame, m

Cost Function: minimize the cost; $Cost = 80w + 120d + 200h$

Constraint: $wdh \geq 600$ (volume constraint); $w,d,h \geq 0$

4.96

Two electric generators are interconnected to provide total power to meet the load. Each generator's cost is a function of the power output, as shown in Fig. E2.14. All costs and power are expressed on a per unit basis. The total power needed is at least 60 units. Formulate a minimum cost design problem to determine the power outputs P_1 and P_2.

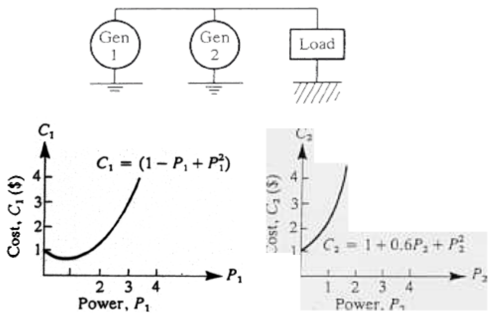
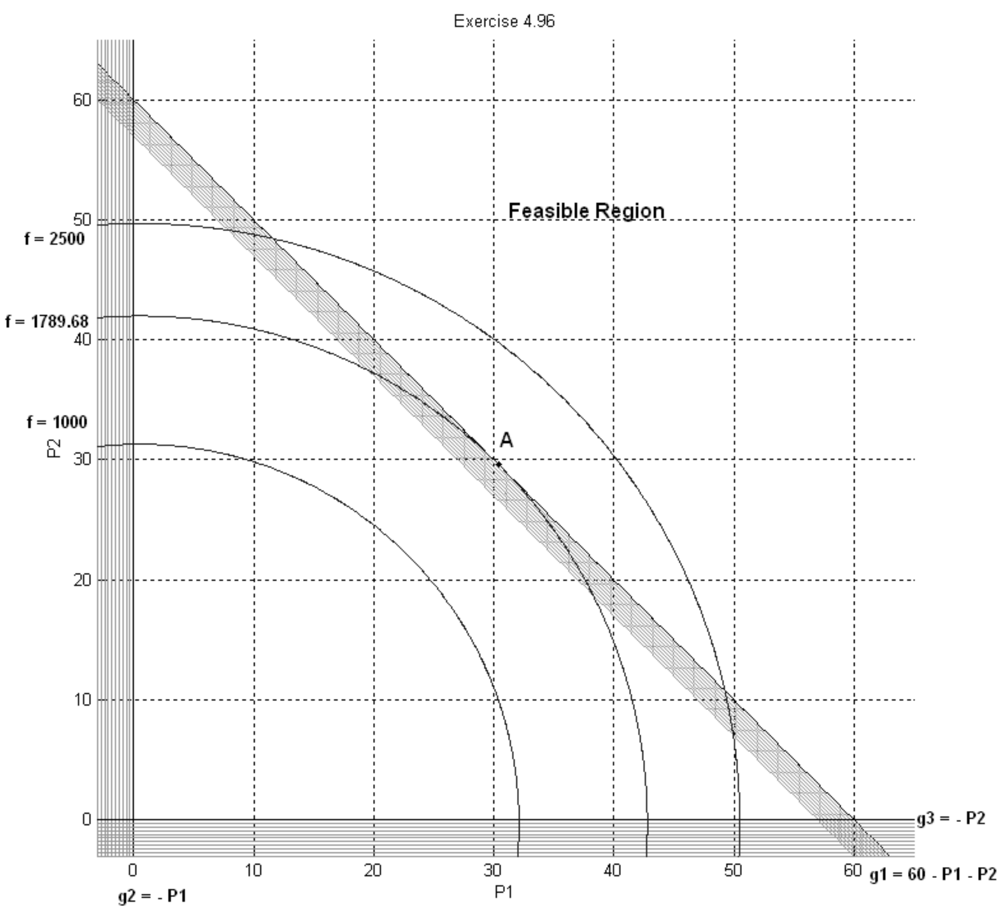


FIGURE E2.14 Power generator.

Solution



According to the graphical solution, the point A (30, 29) is minimum point.
Referring to the formulation in Exercise 2.14, we have

Minimize $f = \left(1 - P_1 + P_1^2\right) + \left(1 + 0.6P_2 + P_2^2\right)$

subject to

$g_1 = 60 - P_1 - P_2 \le 0$

$g_2 = -P_1 \le 0$

$g_3 = -P_2 \le 0$

$$L = \left(1 - P_1 + P_1^2\right) + \left(1 + 0.6P_2 + P_2^2\right) + u_1\left(60 - P_1 - P_2 + s_1^2\right) + u_2\left(-P_1 + s_2^2\right) + u_3\left(-P_2 + s_3^2\right)$$

$$\partial L/\partial P_1 = -1 + 2P_1 - u_1 - u_2 = 0;\quad \partial L/\partial P_2 = 0.6 + 2P_2 - u_1 - u_3 = 0$$

$$g_i + s_i^2 = 0,\quad u_i s_i = 0,\quad u_i \ge 0;\quad i = 1 \text{ to } 3$$

There are 8 cases because there are 3 inequality constraints. The case which yields a solution is given as $u_2 = u_3 = 0, s_1 = 0$. The solution is $P_1 = 30.4, P_2 = 29.6, u_1 = 59.8,\ f = 1789.68$. It is seen from the graph for the problem that the solution is indeed a minimum point and the power need constraint is active.

MATLAB Code for Exercise 4.96

```
clear all
[D,H]=meshgrid(-1:0.5:8 , -3:0.5:16);
f=400*((pi*D.^2.)/2+pi*D.*H);
h1=(pi*D.^2.*H)/4-150;
g1=H+D/2-10;
g2=-H;
g3=-D;

cla reset
axis ([-1 8 -3 16])
xlabel('D'),ylabel('H')
title('Exercise 4.96')
hold on

cv0=[0:0.01:0.9];
const0=contour(D,H,h1,cv0,'k');

cv1=[0:0.05:1];
const1=contour(D,H,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(D,H,g1,cv1,'k');

cv2=[0:0.05:1];
const2=contour(D,H,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(D,H,g2,cv2,'k');

cv3=[0:0.03:0.5];
const3=contour(D,H,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(D,H,g3,cv3,'k');
```

```
fv=[45000 62512.75 80000];
fs=contour(D,H,f,fv,'b');
a=[5.758823];
b=[5.758826];
plot(a,b,'.k');

grid
hold off
```

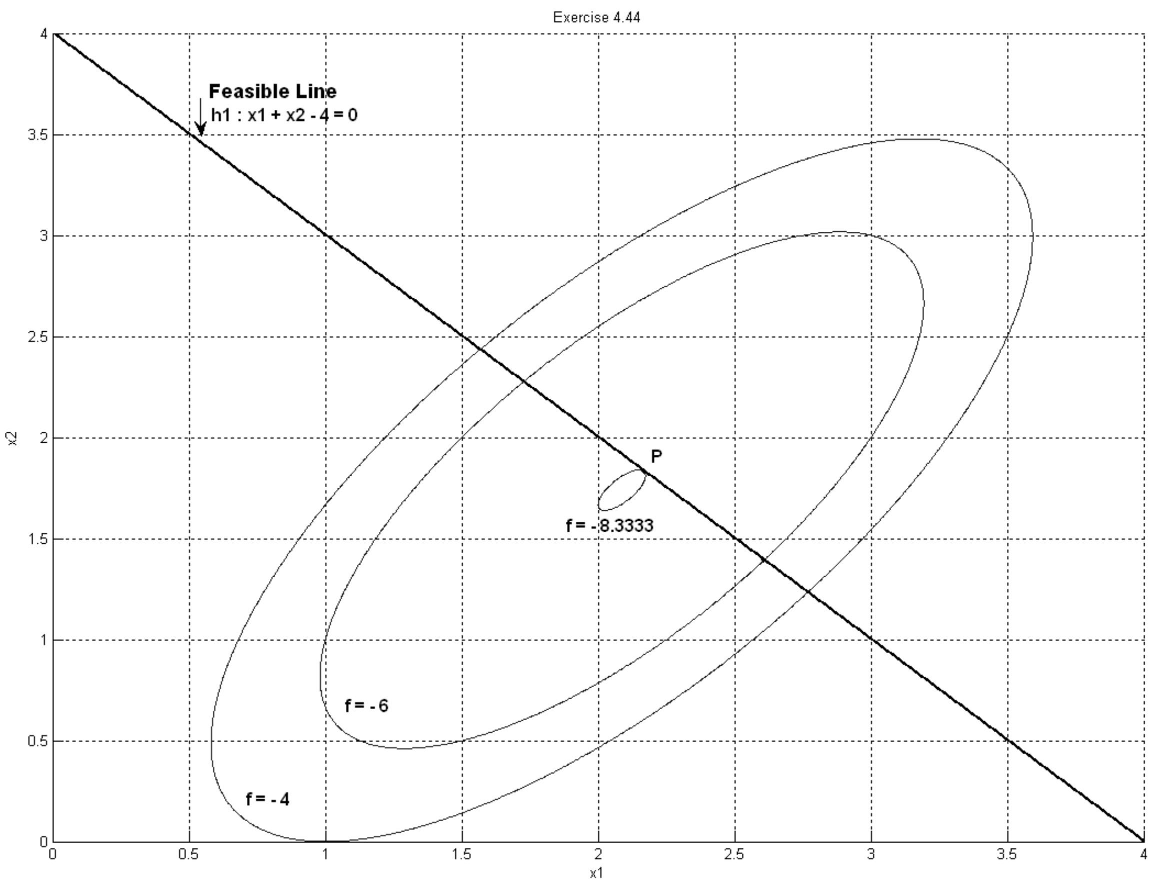
Section 4.7 Physical Meaning of Lagrange Multipliers

Solve the following problems graphically, verify the KKT necessary conditions for the solution points and study the effect on the cost function of changing the boundary of the active constraint(s) by one unit.

4.97

Exercise 4.43
Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 - 4 = 0$

Solution



Referring to Exercise 4.43, the point satisfying the KKT necessary conditions is
$x_1 = 2.166667,\ x_2 = 1.833333, \nu = -0.166667, f = -8.3333$

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 8x_1 - 5x_2 - 8 \\ 6x_2 - 5x_1 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

At optimum point P (2.16667, 1.83333)
$\nabla f(2.16667, 1.83333) = \begin{bmatrix} 8(2.16667) - 5(1.83333) - 8 \\ 6(1.83333) - 5(2.16667) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
These vectors are along the same line.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial b} = -\nu^* = -(-0.166667)$$

If we set b = 1, the new value of cost function will be approximately
$f^* = -8.3333 - (-0.16667)(1) = -8.16663$

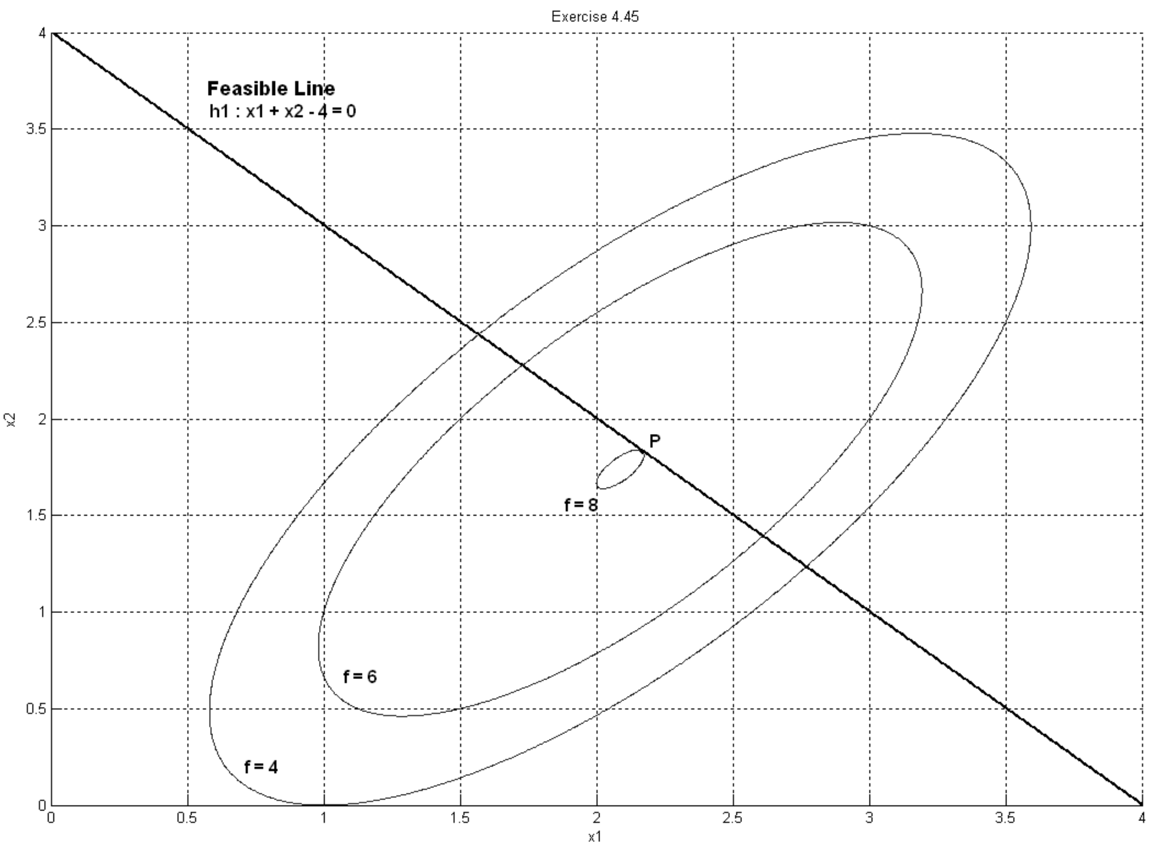
MATLAB Code for Exercise 4.97

```
clear all
axis equal
[x1,x2]=meshgrid(-5:0.005:5, -5:0.005:5);
f=4*x1.^2+3*x2.^2-5*x1.*x2-8*x1;
h1=x1+x2-4;
cla reset
axis ([0 4 0 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.43')
hold on
cv1=[0:0.001:0.01];
const1=contour(x1,x2,h1,cv1,'k');
fv=[-8.3333 -6 -4 ];
fs=contour(x1,x2,f,fv,'b');

a=[2.16667 2.60763];
b=[1.83333 1.39237];
plot(a,b,'.k')

grid
hold off
```

4.98

Exercise 4.44
Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
subject to $x_1 + x_2 - 4 = 0$

Solution



It is seen in the graph that there is no finite minimum point for *f*. In otherwords, $f^* = -\infty, or\ F^* = \infty$.

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

Referring to Exercise 4.44, the point satisfying the KKT necessary conditions is
$x_1 = 2.16667, x_2 = 1.83333\ , \nu = 0.16667$

$$\nabla L = \begin{bmatrix} -8x_1 + 5x_2 + 8 + \nu \\ -6x_2 + 5x_1 + \nu \end{bmatrix}$$

$$\nabla^2 L = \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix}$$

$M_1 = -8 < 0, M_2 = 48 - 25 = 23 > 0;$ Negative definite

The Hessian of cost function is negative definite. So, this is not a convex problem. This also violates the second order necessary condition for a local minimum point.

Sufficiency check

$$\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\nabla h^T \cdot d = [1 \quad 1]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 1 \cdot d_1 + 1 \cdot d_2 = 0$$

$$d_1 = -d_2 = c$$

$d = (c, -c) \quad (c \neq 0$ is an arbitry constant$)$

$$Q = d^T \cdot \nabla^2 L \cdot d = [c \quad -c]\begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix}\begin{bmatrix} c \\ -c \end{bmatrix} = -24c^2 < 0 \ (c \neq 0)$$

The sufficient condition is NOT satisfied , so $x_1 = 2.16667$, $x_2 = 1.83333$ is NOT isolated minimum. Since $Q < 0$, second order necessary condition is violated ,so the point P cannot be a local minimum point.

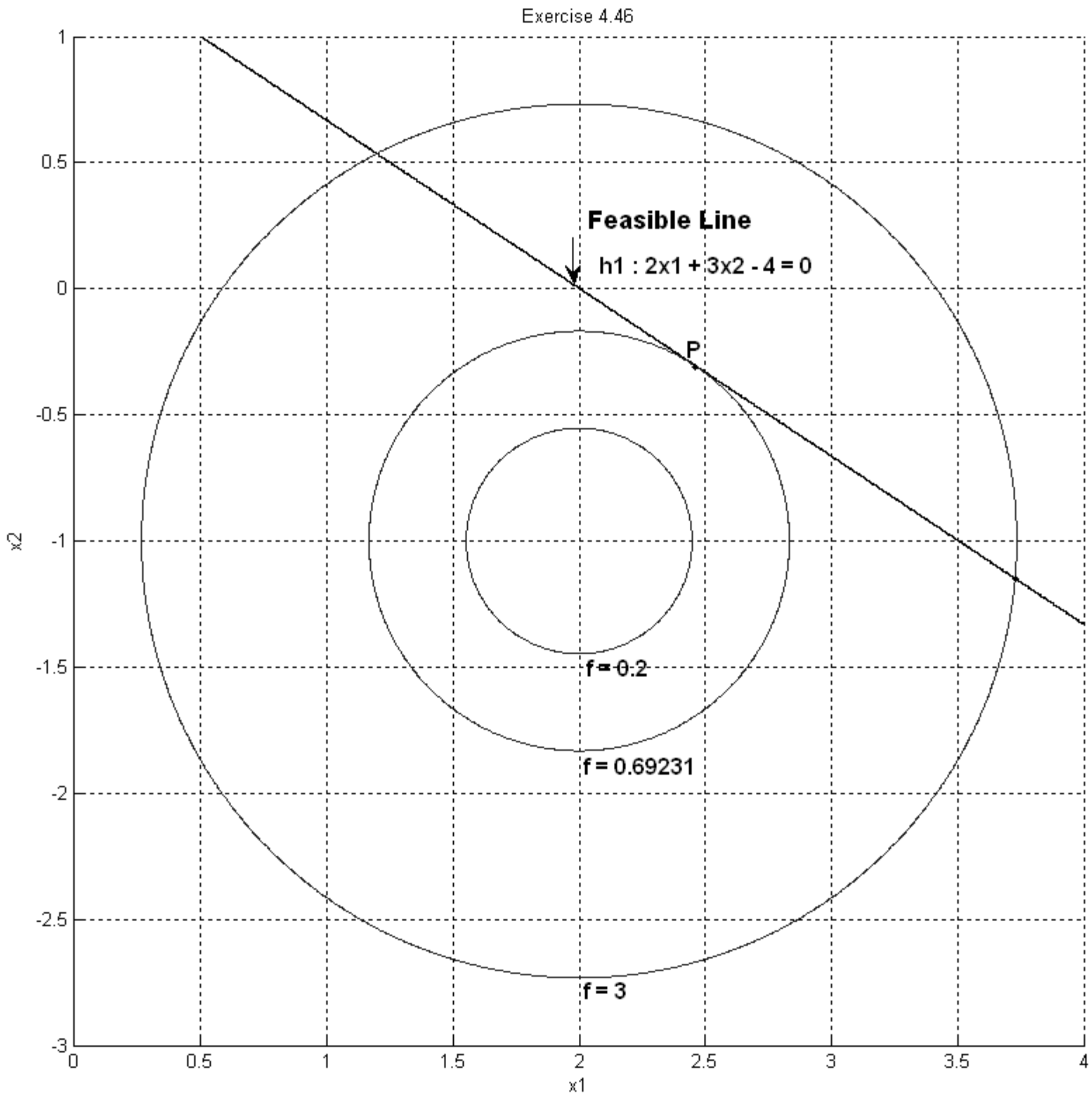
<u>MATLAB Code for Exercise 4.98</u>

```
clear all
axis equal
[x1,x2]=meshgrid(-5:0.005:5, -5:0.005:5);
f=-1*(4*x1.^2+3*x2.^2-5*x1.*x2-8*x1);
h1=x1+x2-4;
cla reset
axis ([0 4 0 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.44')
hold on
cv1=[0:0.001:0.01];
const1=contour(x1,x2,h1,cv1,'k');
fv=[8.3333 6 4];
fs=contour(x1,x2,f,fv,'b');

a=[2.16667];
b=[1.83333];
plot(a,b,'.k')

grid
hold off
```

4.99

Exercise 4.45
Minimize $f(x_1, x_2) = (x_1 - 2)^2 + (x_2 + 1)^2$
subject to $2x_1 + 3x_2 - 4 = 0$

Solution



Referring to Exercise 4.45, the point satisfying the KKT necessary conditions is
$x_1 = 2.46154,\ x_2 = -0.307692, \mathrm{v} = -0.46154, \qquad f = 0.69231$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 that the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 2) \\ 2(x_2 + 1) \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

At optimum point P (2.46154, -0.307692)
$\nabla f(2.46154, -0.307692) = \begin{bmatrix} 2(2.46154 - 2) \\ 2(-0.307692 + 1) \end{bmatrix} = \begin{bmatrix} 0.92308 \\ 1.38162 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
These vectors are shown at point P in above figure. Note that they are along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial b} = -v^* = -(-0.46154)$$

If we set b = 1, the new value of cost function will be approximately
$f^* = 0.69231 — (-0.46154)(1) = 1.15385$

<u>MATLAB Code for Exercise 4.99</u>

```
clear all
axis equal
[x1,x2]=meshgrid(0:0.005:4, -3:0.005:1);
f=(x1-2).^2+(x2+1).^2;
h1=2*x1+3*x2-4;

cla reset
axis equal
axis ([0 4 -3 1])
xlabel('x1'),ylabel('x2')
title('Exercise 4.45')
hold on

cv1=[0:0.001:0.01];
const1=contour(x1,x2,h1,cv1,'k');

fv=[0.2 0.69231 3];
fs=contour(x1,x2,f,fv,'b');

a=[2.46154 3.72551];
b=[-0.307692 -1.15034];
plot(a,b,'.k');

grid
hold off
```