CHAPTER

11

More on Numerical Methods for Unconstrained Optimum Design

Section 11.1 More on Step Size Determination

11.1

$\alpha^* = 1.42857E+00$; $f^* = 7.71429E + 00$; No. of function evaluations = 11

11.2

Since the function $f(\alpha)$ is a quadratic function, the quadratic fitting curve is the function itself.
One iteration is required to find α^*, because that function $f(\alpha)$ is a quadratic function.
$f'(\alpha) = 14\alpha - 20 = 0$; $\alpha^* = 10/7$; $f(10/7) = 7.71429$

11.3

discontinuous, non-smooth, and non-unimodal functions

11.4

$\mathbf{c} = \tilde{\mathbf{N}} f(\mathbf{x}) = (-1 + x_2,\ x_1 + 2x_2) = (3, 10)$ $\mathbf{c} \bullet \mathbf{d} = -13 < 0$; $\mathbf{d}$ is a direction of descent.

$\mathbf{x}^{(0)} + \alpha\mathbf{d} = (2 - \alpha, 4 - \alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}) = 2\alpha^2 - 13\alpha + 32$.

(i) Since $f(\alpha)$ is a quadratic function, the quadratic polynomial interpolation is the function itself.
$f'(\alpha) = 4\alpha - 13 = 0$;
$\alpha^* = 13/4$ and $a_2 = 2 > 0$, so $\alpha^* = 13/4$ is a minimum point.

(ii) When $f(\alpha) = 15$, $\alpha = 1.81386$ or 4.68614

Section 11.4 Search Direction Determination: Newton's Method

11.9

1. F 2. F 3. F 4. F

11.10

1. $\mathbf{x}^{(0)} = (1, 1)$, k = 0, $\varepsilon = 0.001$
2. $\mathbf{c}^{(0)} = (2x_1 - 4 - 2x_2, 4x_2 - 2x_1) = (-4, 2)$, $\|\mathbf{c}^{(0)}\| = \sqrt{20} > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (3, 1)$, $(\mathbf{c}^{(0)} \bullet \mathbf{d}^{(0)}) = -10 < 0$. The descent condition is satisfied.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (1 + 3\alpha, 1 + \alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) = 5\alpha^2 - 10\alpha - 3$;
   $f'(\alpha) = 10\alpha - 10 = 0$; $\alpha_0 = 1$; $\boldsymbol{x}^{(1)} = (4, 2)$
6. k = k + 1 and go to step 2.

Note: $\mathbf{c}^{(1)} = (0, 0)$, $\|\mathbf{c}^{(1)}\| = 0$, so $\boldsymbol{x}^{(1)}$ is the optimum point.

11.11

1. $\boldsymbol{x}^{(0)} = (1, 1)$, k = 0, $\varepsilon = 0.001$
2. $\mathbf{c}^{(0)} = (24.192x_1 - 1.7321, 43.008x_2 - 1) = (22.4599, 42.008)$, $\|\mathbf{c}^{(0)}\| = 47.6 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 24.192 & 0 \\ 0 & 43.008 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-0.928402, -0.976749)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (1 - 0.928402\alpha, 1 - 0.976749\alpha)$;
   $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) = 12.096(1 - 0.928402\alpha)^2 + 21.504(1 - 0.976749\alpha)^2$
   $- 1.7321(1 - 0.928402\alpha) - (1 - 0.976749\alpha)$;
   $f'(\alpha) = 61.88311\alpha - 61.88309 = 0$, $\alpha_0 = 1$; $\mathbf{x}^{(1)} = (0.071598, 0.023251)$
6. k = k + 1, go to step 2.

Note: $\mathbf{c}^{(1)} = (0, 0)$, $\|\mathbf{c}^{(1)}\| = 0$, $\mathbf{x}^{(1)}$ is the optimum point.

11.12

1. $\boldsymbol{x}^{(0)} = (2, 1)$, k = 0, $\varepsilon = 0.001$
2. $\mathbf{c}^{(0)} = (13.966x_1 - 1, 24.83x_2) = (26.932, 24.83)$, $\|\mathbf{c}^{(0)}\| = 36.63 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 13.966 & 0 \\ 0 & 24.83 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-1.928396, -1)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (2 - 1.928396\alpha, 1 - \alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) = 6.983(2 - 1.928396\alpha)^2 +$
   $12.415(1 - \alpha)^2 - (2 - 1.928396\alpha)$; $f'(\alpha) = 76.765520\alpha - 76.76556 = 0$, $\alpha_0 = 1$;
   $\mathbf{x}^{(1)} = (0.071604, 0)$
6. k = k + 1, go to step 2.

Note: $\mathbf{c}^{(1)} = (0, 0)$, $\|\mathbf{c}^{(1)}\| = 0$, $\mathbf{x}^{(1)}$ is the optimum point.

11.13

1. $\mathbf{x}^{(0)} = (1, 2)$, k = 0, ε = 0.001
2. $\mathbf{c}^{(0)} = (24.192x_1,\ 43.008x_2 - 1) = (24.192, 85.016)$, $\|\mathbf{c}^{(0)}\| = 88.391 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 24.192 & 0 \\ 0 & 43.008 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\,\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-\,1, -\,1.9767485)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (1-\alpha, 2 - 1.9767485\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) = 12.096(1-\alpha)^2 + 21.504(2 - 1.9767485\alpha)^2 - (2 - 1.9767485\alpha)$; $f'(\alpha) = 180.15125\alpha - 180.15125 = 0$, $\alpha_0 = 1$;
   $\mathbf{x}^{(1)} = (0, 0.0232515)$
6. k = k + 1, go to step 2.

Note: $\mathbf{c}^{(1)} = (0, 0)$, $\|\mathbf{c}^{(1)}\| = 0$, $\mathbf{x}^{(1)}$ is the optimum point.

11.14

1. $\mathbf{x}^{(0)} = (3, 1)$, k = 0, ε = 0.001
2. $\mathbf{c}^{(0)} = (50x_1 - 2, 40x_2 - 1) = (148, 39)$, $\|\mathbf{c}^{(0)}\| = 153.05 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 50 & 0 \\ 0 & 40 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\,\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-\,2.96, -\,0.975)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (3 - 2.96\alpha, 1 - 0.975\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) = 25(3 - 2.96\alpha)^2 + 20(1 - 0.975\alpha)^2 - 2(3 - 2.96\alpha) - (1 - 0.975\alpha)$; $f'(\alpha) = 476.105\alpha - 476.105 = 0$, $\alpha_0 = 1$;
   $\mathbf{x}^{(1)} = (0.04, 0.025)$
6. k = k + 1, go to step 2.

Note: $\mathbf{c}^{(1)} = (0, 0)$, $\|\mathbf{c}^{(1)}\| = 0$, $\mathbf{x}^{(1)}$ is the optimum point.

11.15

1. $\mathbf{x}^{(0)} = (1, 1, 1)$, k = 0, ε = 0.001
2. $\mathbf{c}^{(0)} = (2x_1 + 2x_2, 4x_2 + 2x_1 + 2x_3, 4x_3 + 2x_2) = (4, 8, 6)$, $\|\mathbf{c}^{(0)}\| = 10.77 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\,\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-1, -1, -1)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (1-\alpha, 1-\alpha, 1-\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)}) = 9(1-\alpha)^2$;
   $f'(\alpha) = -\,18(1-\alpha) = 0$,
   $\alpha_0 = 1$; $\mathbf{x}^{(1)} = (0, 0, 0)$
6. k = k + 1, go to step 2.

Note: $\mathbf{c}^{(1)} = (0, 0, 0)$, $\|\mathbf{c}^{(1)}\| = 0$, $\mathbf{x}^{(1)}$ is the optimum point.

11.16

1. $\mathbf{x}^{(0)} = (4, 6)$, $k = 0$, $\varepsilon = 0.001$
2. $\mathbf{c}^{(0)} = (-16.971393, 69.957143)$, $\|\mathbf{c}^{(0)}\| = 72 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 4.363556 & -5.929724 \\ -5.929724 & 8.643596 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-104.936473, -80.082574)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (4 - 104.936473\alpha, 6 - 80.082574\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)})$
   $= 8(4 - 104.936473\alpha)^2 + 8(6 - 80.082574\alpha)^2 - 5(10 - 185.019047\alpha)$
   $- 80\sqrt{(4-104.936473\alpha)^2 + (6-80.082574\alpha)^2 - 20(6-80.082574\alpha) + 100}$
   $- 80\sqrt{(4-104.936473\alpha)^2 + (6-80.082574\alpha)^2 + 20(6-80.082574\alpha) + 100}$;
   Equal Interval Search gives $\alpha_0 = 0.063913$; $\mathbf{x}^{(1)} = (-2.70680, 0.88168)$
6. $k = k + 1$, go to step 2.

11.17

1. $\mathbf{x}^{(0)} = (5, 2)$, $k = 0$, $\varepsilon = 0.001$
2. $\mathbf{c}^{(0)} = (3.378436, 22.556490)$, $\|\mathbf{c}^{(0)}\| = 22.8 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 5.798076 & -2.474302 \\ -2.474302 & 13.877611 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-1.381415, -1.871686)$, $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (5 - 1.381415\alpha, 2 - 1.871686\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)})$
   $= 9(5 - 1.381415\alpha)^2 + 9(2 - 1.871686\alpha)^2 - 5(5 - 1.381415\alpha) - 41(2 - 1.871686\alpha)$
   $- 100\sqrt{(5-1.381415\alpha)^2 + (2-1.871686\alpha)^2 - 20(2-1.871686\alpha) + 100}$
   $- 64\sqrt{(5-1.381415\alpha)^2 + (2-1.871686\alpha)^2 + 16(2-1.871686\alpha) + 64}$;
   Equal Interval Search gives $\alpha_0 = 0.889257$; $\mathbf{x}^{(1)} = (3.771567, 0.335589)$
6. $k = k + 1$, go to step 2.

11.18

1. $\mathbf{x}^{(0)} = (5, 2)$, k = 0, ε = 0.001
2. $\mathbf{c}^{(0)} = \{-400x_1(x_2 - x_1^2) - 2(1 - x_1), 200(x_2 - x_1^2)\} = (46008, -4600)$; $\|\mathbf{c}^{(0)}\| = 46237 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 29202 & -2000 \\ -2000 & 200 \end{bmatrix}$, $\mathbf{H}^{(0)}$ is positive definite.
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-0.00086938, 22.991306)$; $\mathbf{d}^{(0)}$ is a descent direction.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (5 - 0.00086938\alpha, 2 + 22.991306\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)})$
   $= 100[(2 + 22.991306\alpha) - (5 - 0.00086938\alpha)^2]^2 + (-4 + 0.00086938\alpha)^2$;
   Equal interval search gives $\alpha \cong 1.0$; $\mathbf{x}^{(1)} = (4.99913, 24.99085)$
6. k = k+1, go to step 2.

11.19

1. $\mathbf{x}^{(0)} = (1, 2, 3, 4)$, k = 0, ε = 0.001
2. $\mathbf{c}^{(0)} = \{2(x_1 - 10x_2) + 40(x_1 - x_4)^3, -20(x_1 - 10x_2) + 4(x_2 - 2x_3)^3, 10(x_3 - x_4) - 8(x_2 - 2x_3)^3,$
   $-10(x_3 - x_4) - 40(x_1 - x_4)^3\} = (-1118, 124, 502, 1090)$, $\|\mathbf{c}^{(0)}\| = 1645 > \varepsilon$
3. $\mathbf{H}^{(0)} = \begin{bmatrix} 1082 & -20 & 0 & -1080 \\ -20 & 392 & -384 & 0 \\ 0 & -384 & 778 & -10 \\ -1080 & 0 & -10 & 1090 \end{bmatrix}$
4. $\mathbf{d}^{(0)} = -\mathbf{H}^{(0)^{-1}}\mathbf{c}^{(0)} = (-1.70175, -2.070175, -1.70175, -2.70175)$; $\mathbf{c}^{(0)}\bullet\mathbf{d}^{(0)} < 0$; $\mathbf{d}^{(0)}$ satisfies the descent condition.
5. $\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)} = (1 - 1.70175\alpha, 2 - 2.070175\alpha, 3 - 1.70175\alpha, 4 - 2.70175)$;
   Equal interval search gives $\alpha = 1.333088$; $\mathbf{x}^{(1)} = (-1.26859, -0.75973, 0.73141, 0.39833)$
6. k = k + 1, go to step 2.

11.20

Solved using computer program to implement modified Newton's algorithm:

(10.52) $\mathbf{x}^* = (4, 2)$, $f^* = -8.0$, NIT = 1, NFE = 67;
(10.53) $\mathbf{x}^* = (0.071598, 0.023251)$, $f^* = -0.073633$, NIT = 1, NFE = 67;
(10.54) $\mathbf{x}^* = (0.071602, 0.0)$, $f^* = -0.035801$, NIT = 1, NFE = 67;
(10.55) $\mathbf{x}^* = (0.0, 0.023251)$, $f^* = -0.011626$, NIT = 1, NFE = 67;
(10.56) $\mathbf{x}^* = (0.04, 0.025)$, $f^* = -0.0525$, NIT = 1, NFE = 67;
(10.57) $\mathbf{x}^* = (0, 0, 0)$, $f^* = 0$, NIT = 1, NFE = 67;
(10.58) $\mathbf{x}^* = (4.144658, 0.361558)$, $f^* = -1616.18353$, NIT = 4, NFE = 258;
(10.59) $\mathbf{x}^* = (3.732927, 0.341201)$, $f^* = -1526.55649$, NIT = 2, NFE = 106;
(10.60) $\mathbf{x}^* = (1.0, 1.0)$, $f^* = 0$, NIT = 17, NFE = 1292;
(10.61) $\mathbf{x}^* = (-0.003242, -0.000305, 0.006046, 0.006056)$, $f^* = 0$, NIT = 10, NFE = 727.

Section 11.5 Search Direction Determination: Quasi – Newton Methods

11.21

1. T 2. T 3. T 4. T 5. T 6. F

Note: First iterations for Exercises 11.22 – 11.31, and 11.33 – 11.42 are the same as the steepest descent method of Exercises 10.52 – 10.61, respectively.

11.22

DFP Iteration 2: $\mathbf{x}^{(1)} = (2, 0.5)$

2. $\mathbf{c}^{(1)} = (-1, -2)$, $\|\mathbf{c}^{(1)}\| = \sqrt{5} > \varepsilon$, so continue
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.84 & 0.38 \\ 0.38 & 0.41 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (1.6, 1.2)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (2 + 1.6\alpha, 0.5 + 1.2\alpha)$;
$f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)}) = (2 + 1.6\alpha)^2 + 2(0.5 + 1.2\alpha)^2 - 4(2 + 1.6\alpha) - 2(2 + 1.6\alpha)(0.5 + 1.2\alpha)$;
$f'(\alpha) = 3.2\alpha - 4 = 0$, $\alpha_1 = 1.25$
5. $\mathbf{x}^{(2)} = (4, 2)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$, $\mathbf{x}^{(2)}$ is the optimum point.

11.23

DFP iteration 2: $\mathbf{x}^{(1)} = (0.42151, -0.08198)$

2. $\mathbf{c}^{(1)} = (8.4651, -4.5259)$, $\|\mathbf{c}^{(1)}\| = 9.6 > \varepsilon$, so continue
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.9228 & -0.2651 \\ -0.2651 & 0.1030 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-9.0111, 2.710)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.42151 - 9.0111\alpha, -0.08198 + 2.71\alpha)$;
$f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)}) = 12.096(0.42151 - 9.0111\alpha)^2 + 21.504(-0.08198 + 2.71\alpha)^2$
$- 1.7321(0.42151 - 9.0111\alpha) - (-0.08198 + 2.71\alpha)$;
$f'(\alpha) = 2280.2436\alpha - 88.5445 = 0$, $\alpha_1 = 0.03881$
5. $\mathbf{x}^{(2)} = (0.07160, 0.02325)$
6. $\mathbf{c}^{(2)} \approx (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.24

DFP iteration 2: $\mathbf{x}^{(1)} = (0.57939, -0.30973)$

2. $\mathbf{c}^{(1)} = (7.0912, -7.6915)$, $\|\mathbf{c}^{(1)}\| = 10.5 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.7573 & -0.4183 \\ -0.4183 & 0.2955 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-8.5875, 5.239)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.57939 - 8.5875\alpha, -0.30973 + 5.239\alpha)$; $f(\alpha) = f(\mathbf{x}^{(0)} + \alpha\mathbf{d}^{(0)})$
$= 6.983(0.57939 - 8.5875\alpha)^2 + 12.415(-0.30973 + 5.239\alpha)^2 - (0.57939 - 8.5875\alpha)$;
$f'(\alpha) = 1711.4369\alpha - 101.1915 = 0$, $\alpha_1 = 0.059127$
5. $\mathbf{x}^{(2)} = (0.0716, 0.0)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.25

DFP iteration 2: $\mathbf{x}^{(1)} = (0.41844, -0.04373)$

2. $\mathbf{c}^{(1)} = (10.1229, -2.8807)$, $\|\mathbf{c}^{(1)}\| = 10.5 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.9768 & -0.1497 \\ -0.1497 & 0.04722 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-10.3196, 1.6518)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.41844 - 10.3196\alpha, -0.04373 + 1.6518\alpha)$; $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$
$= 12.096(0.41844 - 10.3196\alpha)^2 + 21.504(-0.04373 + 1.6518\alpha)^2 - (-0.04373 + 1.6518\alpha)$;
$f'(\alpha) = 2693.651\alpha - 109.2227 = 0$, $\alpha_1 = 0.040548$
5. $\mathbf{x}^{(2)} = (0, 0.02325)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.26

DFP iteration 2: $\mathbf{x}^{(1)} = (0.0010536, 0.20974)$

2. $\mathbf{c}^{(1)} = (-1.9473, 7.3895)$, $\|\mathbf{c}^{(1)}\| = 7.6 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.0615 & -0.19685 \\ -0.19685 & 0.95877 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (1.5744, -7.4681)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.0010536 + 1.5744\alpha, 0.20974 - 7.4681\alpha)$;
$f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)}) = 25(0.0010536 + 1.5744\alpha)^2 + 20(0.20974 - 7.4681\alpha)^2$
$- 2(0.0010536 + 1.5744\alpha) - (0.20974 - 7.4681\alpha)$;
$f'(\alpha) = 2354.8375\alpha - 58.2521 = 0$, $\alpha_1 = 0.024737$
5. $\mathbf{x}^{(2)} = (0.04, 0.025)$
6. $\mathbf{c}^{(2)} = (0,0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.27

DFP iteration 2: $\mathbf{x}^{(1)} = (0.38298, -0.23404, 0.074468)$

2. $\mathbf{c}^{(1)} = (0.2979, -0.02128, -0.1702)$, $\|\mathbf{c}^{(1)}\| = 0.344 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.9032 & -0.2132 & -0.1648 \\ -0.2132 & 0.5310 & -0.3624 \\ -0.1648 & -0.3624 & 0.7200 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-0.302, 0.0131, 0.164)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.38298 - 0.302\alpha, -0.23404 + 0.0131\alpha, 0.074468 + 0.164\alpha)$;
$f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)}) = (0.38298 - 0.302\alpha)^2 + 2(-0.23404 + 0.0131\alpha)^2 + 2(0.074468 +$
$0.164\alpha)^2 + 2(-0.23404 + 0.0131\alpha)(0.457448 - 0.138\alpha)$;
$f'(\alpha) = 0.2834472\alpha - 0.118152431 = 0$, $\alpha_1 = 0.416841$
5. $\mathbf{x}^{(2)} = (0.2571, -0.2286, 0.1428)$
6. $\mathbf{c}^{(2)} = (0.057, -0.1146, 0.114)$, $\|\mathbf{c}^{(2)}\| = 0.1714$; $\mathbf{s}^{(1)} = \alpha_1\mathbf{d}^{(1)} = (-0.1259, 0.00546, 0.06636)$;
$\mathbf{y}^{(1)} = \mathbf{c}^{(2)} - \mathbf{c}^{(1)} = (-0.2409, -0.09332, 0.2842)$

11.28

DFP iteration 2: $\mathbf{x}^{(1)} = (5.338509, 0.482583)$

2. $\mathbf{c}^{(1)} = (4.9740, 1.20667)$, $\|\mathbf{c}^{(1)}\| = 5.1 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.91192 & 0.27162 \\ 0.27162 & 0.16695 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-4.86364, -1.55249)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (5.338509 - 4.86364\alpha, 0.482583 - 1.55249\alpha)$; $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$;
   Golden section search gives $\alpha_1 = 0.19904$
5. $\mathbf{x}^{(2)} = (4.37045, 0.173575)$
6. $\mathbf{c}^{(2)} = (0.840, -2.631)$, $\|\mathbf{c}^{(2)}\| = 2.76$

11.29

DFP iteration 2: $\mathbf{x}^{(1)} = (4.751894, 0.343496)$

2. $\mathbf{c}^{(1)} = (4.70785, -0.70513)$, $\|\mathbf{c}^{(1)}\| = 4.76 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.99836 & 0.06772 \\ 0.06772 & 0.07508 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-4.65238, -0.26588)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (4.751894 - 4.65238\alpha, 0.343496 - 0.26588\alpha)$; $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$;
   Golden section search gives $\alpha_1 = 0.21813$
5. $\mathbf{x}^{(2)} = (3.73707, 0.28550)$
6. $\mathbf{c}^{(2)} = (0.0497, -0.869)$, $\|\mathbf{c}^{(2)}\| = 0.87$

11.30

DFP iteration 2: $\mathbf{x}^{(1)} = (-1.4411, 2.644)$

2. $\mathbf{c}^{(1)} = (292.730, 101.806)$, $\|\mathbf{c}^{(1)}\| = 309.93 > \varepsilon$
3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.011 & 0.102 \\ 0.102 & 0.99 \end{bmatrix}$; $\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (-13.604242, -130.6464)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (-1.4411 - 13.604242\alpha, 2.644 - 130.6464\alpha)$; $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$;
   Golden section search gives $\alpha_1 = 0.0346748$.
5. $\mathbf{x}^{(2)} = (-1.912824, -1.886134)$
6. $\mathbf{c}^{(2)} = (14.59567, 5.338001)$, $\|\mathbf{c}^{(2)}\| = 15.54$

11.31

DFP iteration 2: $\mathbf{x}^{(1)} = (3.07233, 1.77015, 2.06949, 1.97957)$

2. $\mathbf{c}^{(1)} = (22.93732,\ 239.414,\ 107.238,\ -53.095)$, $\|\mathbf{c}^{(1)}\| = 269 > \varepsilon$

3. $\mathbf{A}^{(1)} = \begin{bmatrix} 0.53219 & -0.047504 & 0.16177 & 0.46871 \\ -0.047504 & 0.99522 & 0.016446 & 0.047591 \\ 0.16177 & 0.016446 & 0.94407 & -0.16209 \\ 0.46871 & 0.047591 & -0.16209 & 0.53038 \end{bmatrix}$;

$\mathbf{d}^{(1)} = -\mathbf{A}^{(1)}\mathbf{c}^{(1)} = (6.7043, -236.4163, -117.4936,\ 23.3972)$

4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (3.07233 + 6.7043\alpha, 1.77015 - 236.4163\alpha, 2.06949 - 117.4936\alpha,$ $1.97957 + 23.3972\alpha)$;

Golden section search for the minimum of $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$ gives $\alpha_1 = 0.006124$

5. $\mathbf{x}^{(2)} = (3.11339, 0.32226, 1.34991, 2.12286)$

6. $\mathbf{c}^{(2)} = (38.66, -51.58, 99.79, -31.14)$, $\|\mathbf{c}^{(2)}\| = 123$

11.32

Solved using computer program to implement the Davidon-Fletcher-Powell method

(10.52) $\mathbf{x}^* = (4, 2)$, $f^* = -8$, NIT = 3, NFE = 79;
(10.53) $\mathbf{x}^* = (0.071596, 0.023251)$, $f^* = -0.073633$, NIT = 4, NFE = 89;
(10.54) $\mathbf{x}^* = (0.071602, 0.0)$, $f^* = -0.035801$, NIT = 4, NFE = 88;
(10.55) $\mathbf{x}^* = (0.0, 0.023251)$, $f^* = -0.011626$, NIT = 4, NFE = 91;
(10.56) $\mathbf{x}^* = (0.04, 0.25)$, $f^* = -0.0525$, NIT = 4, NFE = 91;
(10.57) $\mathbf{x}^* = (0, 0, 0)$, $f^* = 0$, NIT = 4, NFE = 105;
(10.58) $\mathbf{x}^* = (4.144685, 0.361554)$, $f^* = -1616.18353$, NIT = 5, NFE = 127;
(10.59) $\mathbf{x}^* = (3.732882, 0.34114)$, $f^* = -1526.55649$; NIT = 5, NFE = 121;
(10.60) $\mathbf{x}^* = (1, 1)$, $f^* = 0$, NIT = 23, NFE = 915;
(10.61) $\mathbf{x}^* = (0.001137, 0.000136, -0.006215, -0.006148)$, $f^* = 0$, NIT = 14, NFE = 436.

11.33

BFGS iteration 2: $\mathbf{x}^{(1)} = (2, 0.5)$

2. $\mathbf{c}^{(1)} = (-1, -2)$, $\|\mathbf{c}^{(1)}\| = \sqrt{5} > \varepsilon$, so continue

3. $\mathbf{H}^{(1)} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (2, 1.5)$

4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (2 + 2\alpha, 0.5 + 1.5\alpha)$; $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$; $f'(\alpha) = 0$, $\alpha_1 = 1.0$

5. $\mathbf{x}^{(2)} = (4, 2)$

6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.34

BFGS iteration 2: $\mathbf{x}^{(1)} = (0.42151, -0.08198)$

2. $\mathbf{c}^{(1)} = (8.4651, -4.5259)$, $\|\mathbf{c}^{(1)}\| = 9.6 > \varepsilon$
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 4.1268 & 10.7269 \\ 10.7269 & 37.2728 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-9.3771, 2.8201)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.42151 - 9.3771\alpha, -0.08198 + 2.8201\alpha)$;
   $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$; $f'(\alpha) = 0$ gives $\alpha_1 = 0.037316$
5. $\mathbf{x}^{(2)} = (0.07160, 0.02325)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.35

BFGS iteration 2: $\mathbf{x}^{(1)} = (0.57939, -0.30973)$

2. $\mathbf{c}^{(1)} = (7.0912, -7.6915)$, $\|\mathbf{c}^{(1)}\| = 10.5 > \varepsilon$
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 6.021 & 8.6176 \\ 8.6176 & 15.4828 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-9.2879, 5.6664)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.57939 - 9.2879\alpha, -0.30973 + 5.6664\alpha)$;
   $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$; $f'(\alpha) = 0$ gives $\alpha_1 = 0.054668$
5. $\mathbf{x}^{(2)} = (0.0716, 0.0)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.36

BFGS iteration 2: $\mathbf{x}^{(1)} = (0.41844, -0.04373)$

2. $\mathbf{c}^{(1)} = (10.1229, -2.8807)$, $\|\mathbf{c}^{(1)}\| = 10.5 > \varepsilon$
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 1.979 & 6.3209 \\ 6.3209 & 41.209 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-10.4659, 1.6752)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.41844 - 10.4659\alpha, -0.04373 + 1.6752\alpha)$;
   $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$; $f'(\alpha) = 0$ gives $\alpha_1 = 0.039981$
5. $\mathbf{x}^{(2)} = (0, 0.02325)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.37

BFGS iteration 2: $\mathbf{x}^{(1)} = (0.0010536, 0.20974)$

2. $\mathbf{c}^{(1)} = (-1.9473, 7.3895)$, $\|\mathbf{c}^{(1)}\| = 7.6 > \varepsilon$
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 47.4335 & 9.7394 \\ 9.7394 & 3.0402 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (1.5783, -7.4867)$
4. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (0.0010536 + 1.5783\alpha, 0.20974 - 7.4867\alpha)$; $f(\alpha) = f(\mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)})$;
   $f'(\alpha) = 0$ gives $\alpha_1 = 0.024675$
5. $\mathbf{x}^{(2)} = (0.04, 0.025)$
6. $\mathbf{c}^{(2)} = (0, 0)$, $\|\mathbf{c}^{(2)}\| = 0$

11.38

BFGS iteration 2: $\mathbf{x}^{(1)} = (0.38298, -0.23404, 0.074468)$

2. $\mathbf{c}^{(1)} = (0.2979, -0.02128, -0.1702)$, $\|\mathbf{c}^{(1)}\| = 0.344 > \varepsilon$, so continue.
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 1.6280 & 1.3837 & 1.0697 \\ 1.3837 & 4.0440 & 2.3522 \\ 1.0697 & 2.3522 & 2.8173 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-0.3020, 0.01313, 0.1641)$
4. Step size $\alpha_1 = 0.416749$
5. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha_1\mathbf{d}^{(1)} = (0.2571, -0.2286, 0.1429)$
6. $\mathbf{c}^{(2)} = (0.0571, -0.1143, 0.1143)$, $\|\mathbf{c}^{(2)}\| = 0.17$

11.39

BFGS iteration 2: $\mathbf{x}^{(1)} = (5.338509, 0.482583)$

2. $\mathbf{c}^{(1)} = (4.9740, 1.20667)$, $\|\mathbf{c}^{(1)}\| = 5.1 > \varepsilon$
3. $\mathbf{H}^{(1)} =$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-4.8882, -1.56033)$
4. Step size $\alpha_1 = 0.198041$
5. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha_1\mathbf{d}^{(1)} = (4.37046, 0.173574)$
6. $\mathbf{c}^{(2)} = (0.84, -2.631)$, $\|\mathbf{c}^{(2)}\| = 2.76$

11.40

BFGS iteration 2: $\mathbf{x}^{(1)} = (4.751894, 0.343496)$

2. $\mathbf{c}^{(1)} = (4.70785, -0.70513)$, $\|\mathbf{c}^{(1)}\| = 4.76 > \varepsilon$
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 1.02432 & -0.956 \\ -0.956 & 14.186 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-4.85502, -0.27747)$
4. Step size $\alpha_1 = 0.209025$
5. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha_1\mathbf{d}^{(1)} = (3.73707, 0.2855)$
6. $\mathbf{c}^{(2)} = (0.05, -0.869)$, $\|\mathbf{c}^{(2)}\| = 0.87$

11.41

BFGS iteration 2: $\mathbf{x}^{(1)} = (-1.4411, 2.644)$

2. $\mathbf{c}^{(1)} = (292.730, 101.806)$, $\|\mathbf{c}^{(1)}\| = 309.93 > \varepsilon$
3. $\mathbf{H}^{(1)} = \begin{bmatrix} 7025.16 & -721.44 \\ -721.44 & 75.30 \end{bmatrix}$; $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (-11.1791, -108.454)$
4. Step size $\alpha = 0.0419697$
5. $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} + \alpha\mathbf{d}^{(1)} = (-1.910283, -1907780)$
6. $\mathbf{c}^{(2)} = (-3.908053, -0.5005836)$, $\|\mathbf{c}^{(2)}\| = 3.94$

11.42

BFGS iteration 2: $\mathbf{x}^{(1)} = (3.07233, 1.77015, 2.06949, 1.97957)$

2. $\mathbf{c}^{(1)} = (22.93732, 239.414, 107.238, -53.095)$, $\|\mathbf{c}^{(1)}\| = 269 > \varepsilon$

3. $$\mathbf{H}^{(1)} = \begin{bmatrix} 260.120 & 26.310 & -89.6075 & -259.6225 \\ 26.31 & 3.6506 & -9.1085 & -26.3584 \\ -89.6075 & -9.1085 & 31.9827 & 89.7825 \\ -259.6225 & -26.3584 & 89.7825 & 261.1256 \end{bmatrix};$$

   $\mathbf{H}^{(1)}\mathbf{d}^{(1)} = -\mathbf{c}^{(1)}$, $\mathbf{d}^{(1)} = (6.883, -242.722, -120.628, 24.021)$

4. Step size $\alpha_1 = 0.0059653$

5. $\mathbf{x}^{(2)} = (3.11339, 0.32224, 1.34991, 2.12286)$

6. $\mathbf{c}^{(2)} = (38.656, -51.580, 99.792, -31.144)$, $\|\mathbf{c}^{(2)}\| = 123$

11.43

Solved using computer program to implement the BFGS method (Result from - 11.32- Davidon-Fletcher-Powell method)

(10.52) $\mathbf{x}^* = (4, 2)$, $f^* = -8$;
(10.53) $\mathbf{x}^* = (0.071596, 0.023251)$, $f^* = -0.073633$;
(10.54) $\mathbf{x}^* = (0.071602, 0.0)$, $f^* = -0.035801$;
(10.55) $\mathbf{x}^* = (0.0, 0.023251)$, $f^* = -0.011626$;
(10.56) $\mathbf{x}^* = (0.04, 0.25)$, $f^* = -0.0525$;
(10.57) $\mathbf{x}^* = (0, 0, 0)$, $f^* = 0$;
(10.58) $\mathbf{x}^* = (4.144685, 0.361554)$, $f^* = -1616.18353$;
(10.59) $\mathbf{x}^* = (3.732882, 0.34114)$, $f^* = -1526.55649$;
(10.60) $\mathbf{x}^* = (1, 1)$, $f^* = 0$;
(10.61) $\mathbf{x}^* = (0.001137, 0.000136, -0.006215, -0.006148)$, $f^* = 0$.

Section 11.6 Engineering Applications of Unconstrained Methods

11.44

$$P(x_1, x_2) = \frac{E}{4L^3}\, s^2 x_1^2 \left(\frac{A_1 + A_2}{2}\right) + \frac{E}{L^3}\, h^2 x_2^2 \left(\frac{A_1 + A_2}{2}\right) + \frac{E}{L^3}\, s\, h\, x_1 x_2 \left(\frac{A_1 - A_2}{2}\right)$$

$$- W x_1 \cos\theta - W x_2 \sin\theta$$

$$\frac{\partial P}{\partial x_1} = \frac{E}{2L^3}\, s^2 x_1 \left(\frac{A_1 + A_2}{2}\right) + \frac{E}{L^3}\, s\, h\, x_2 \left(\frac{A_1 - A_2}{2}\right) - W\cos\theta = 0$$

$$\frac{\partial P}{\partial x_2} = \frac{2E}{L^3}\, h^2 x_2 \left(\frac{A_1 + A_2}{2}\right) + \frac{E}{L^3}\, s\, h\, x_1 \left(\frac{A_1 - A_2}{2}\right) - W\sin\theta = 0$$

$$\frac{Eh^2}{L^3} \left(\frac{4A_1A_2}{A_1 + A_2}\right) x_2 = W\sin\theta - 2\left(\frac{A_1 - A_2}{A_1 + A_2}\right)\frac{h}{s} W\cos\theta;$$

$$\frac{Es^2}{L^3} \left(\frac{A_1A_2}{A_1 + A_2}\right) x_1 = W\cos\theta - \left(\frac{A_1 - A_2}{A_1 + A_2}\right) \frac{s}{2h}\, W\sin\theta$$

$$x_1 = \left[W\cos\theta - \left(\frac{A_1 - A_2}{A_1 + A_2}\right) \frac{s}{2h}\, W\sin\theta\right] / \left[\frac{Es^2}{L^3} \left(\frac{A_1A_2}{A_1 + A_2}\right)\right] = 3.7754 \text{ mm}$$

$$x_2 = \left[W\sin\theta - 2\left(\frac{A_1 - A_2}{A_1 + A_2}\right)\frac{h}{s} W\cos\theta\right] / \left[\frac{Eh^2}{L^3} \left(\frac{4A_1A_2}{A_1 + A_2}\right)\right] = 2.2835 \text{ mm}$$

11.45

Same as Exercise 11.44, $x_1 = 2.2213$ mm, $x_2 = 1.8978$ mm

11.46

$x^{(0)} = 0.0$, $x^* = 0.619084$, $F^* = 0.0$, NFE = 22

11.47

$x^{(0)} = 10.0$, $x^* = 9.424753$, $F^* = 0.0$, NFE = 26

11.48

$x^{(0)} = 2.0$, $x^* = 1.570807$, $F^* = 0.0$, NFE = 26

11.49

$x^{(0)} = -4.0$, $x^* = 1.496045$, $F^* = 0.0$, NFE = 26

11.50

$\mathbf{x}^{(0)} = (4.0, 1.0)$, $\mathbf{x}^* = (3.667328, 0.739571)$, $F_1^2 + F_2^2 = 0.000037$, NFE = 93

11.51

$\mathbf{x}^{(0)} = (10.0, 10.0)$, $\mathbf{x}^* = (4.000142, 7.999771)$, $F_1^2 + F_2^2 = 0.0$, NFE = 76