SAMPLE EXAMS

SAMPLE EXAM 1: MID –TERM EXAM

for

First Course on Optimization (Time allowed = 2 hrs)

1. A proposal to provide on-site power generation for a hospital is to be evaluated. The proposal is to purchase an initial equipment package now costing $750,000. This package has a life of 25 years and no salvage value. Maintenance costs are $18000 per year. Ten years from now, an additional equipment package costing $200,000 is needed for a planned expansion. The life of this package is 15 years and the salvage value is $15000. Maintenance and operating costs are an additional $18000 per year. What is the present worth of this proposal? Assume 8% interest compounded annually.
(15 points)

2. Minimize $f(\mathbf{x}) = x_1^2 + x_2^2$

   Subject to $x_1^2 - 9x_2 \geq 0$; $x_1 \geq 4$

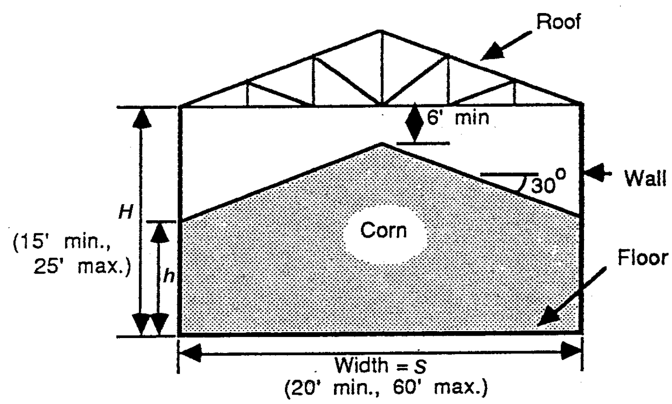
   $x_1 + x_2 \leq 10$; $x_2 \geq 0$

   a. Show the feasible region on the graph sheet.
   b. State whether this region is (circle your choice)
      (1) open or closed
      (2) bounded or unbounded
      (3) convex or not convex?
   c. Determine $\mathbf{x}^*$ graphically and label this point.
   (15 points)

3. The cross section of a corn storage building is shown below. At least 1000 ft^3 of corn is to be stored per foot of length of the building. Formulate the problem to provide the least cost of building per unit volume of corn stored. Neglect costs associated with the ends of the building.
   (20 points)

   <u>Cost of building per foot of length</u>

Roof	$3S^2$
Walls	$2H^2$ each
Floor	$5S$
Ends	Neglect



4. Answer True or False

(i) _____ The number of independent equality constraints can be larger than the number of design variables for a design optimization problem.

(ii) _____ The number of "≤ type" constraints must be less than the number of design variables for a design problem.

(iii) _____ A function can have more than one global minimum point.

(iv) _____ Gradient of a function at a point is normal to the level surface defined by the function.

(v) _____ A point satisfying first order necessary conditions may not be a minimum point for the function.

(vi) _____ If Hessian of the cost function for unconstrained problem is positive definite at a candidate point, the point is a local maximum for the function.

(vii) _____ A point satisfying KKT necessary conditions for a general optimum design problem can be a maximum point for the cost function.

(viii) _____ In optimum design problems, "≥ type" constraints cannot be treated.

(ix) _____ A linear equality constraint always defines a nonconvex region.

(x) _____ The minimum point for a constrained optimization problem gives a stationary value for the Lagrange function.

(10 points)

5. The equilibrium state for an engineering system can be obtained by minimizing the following function:

$$f(x_1, x_2) = 132.6\, x_1^2 - 53.1\, x_1 x_2 + 207.6\, x_2^2 - 1.732\, x_1 - x_2$$

(i) Determine equilibrium state for the system

(ii) Check sufficient conditions and discuss global optimality of the solution.

(15 points)

6. An optimum design problem is formulated as

minimize $f(x_1, x_2) = 2\sqrt{2}x_1 + x_2$; subject to:

$$\frac{1}{x_1} + \frac{1}{x_1 + \sqrt{2}x_2} - 2 \le 0; \quad \frac{1}{x_1 + \sqrt{2}x_2} - 1 \le 0; \quad x_1 \ge 0; \quad x_2 \ge 0$$

(i) Write KKT necessary conditions.

(ii) Ignoring the constraints $x_1 \ge 0,\ x_2 \ge 0$, define the cases to be solved for the candidate minimum points
(iii) Find solution for any two cases defined in (ii).
(25 points)

SAMPLE EXAM 2: MID-TERM EXAM

for

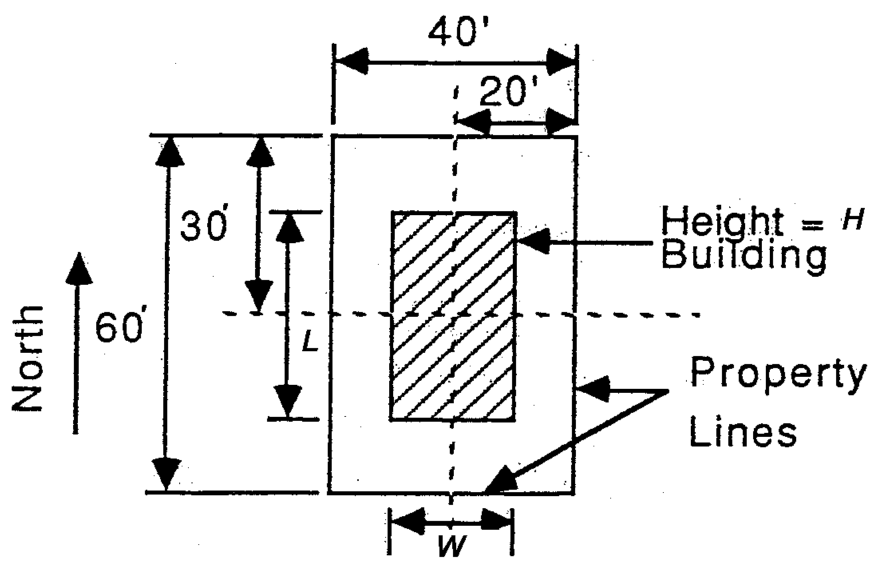
First Undergraduate Course (Time allowed = 90 minutes)

1. A company is considering the purchase of a new piece of testing equipment which is expected to produce \$8,000 additional profit at the end of the first year of operation; this amount will decrease by \$500 per year for each additional year of ownership. The equipment costs \$18,000 and will have a salvage value of \$3,000 after 4 years of use. The same equipment could also be rented for \$6,500 per year, payable in advance on the first day of each rental year. The interest rate is 25% compounded annually. Use a present worth analysis to compare the alternatives. What is your recommendation to the company - purchase, rent, neither? (15 points)

2. A rectangular storage structure is to be constructed for which the following costs apply:

	Cost, \$/ft^2	
Roof	3.00 % W	(e.g., \$30/ft^2 if W = 10 ft)
Walls	2.00 % H	
Floor	5.00	

The structure is to be placed at the center of a plot of land measuring 40 ft % 60 ft, as shown below, with 5 ft minimum clearance to the property line. Because of solar collectors on the adjacent property to the north, the tank must not cast a noon-day shadow outside the property line when the sun is at an angle of 70^{o} with the horizontal. At least 15,000 ft^3 of volume is required (neglect the thickness of the walls and roof). Formulate the problem to determine the dimensions for the least structure cost per unit volume. Express the formulation in the standard form. It is not necessary to normalize the constraints. Do not solve. (15 points)



3. An optimum design problem is formulated as

minimize $f(\mathbf{x}) = (x_1 - 3)^2 + (x_2 - 3)^2$

subject to: (I) $2x_1 - 4x_2 \geq -4$

(II) $-x_1^2 + x_2 \leq 0$

(III) $x_1 \leq 2$

(IV) $x_1 \geq 0$

(V) $x_2 \geq 0$

a. Plot and label the constraints and identify the feasible region.
b. Plot iso-cost lines and identify the optimum point.
c. Identify active constraints at the optimum point; give values of design variables and the cost function at the optimum point.
(15 points)

4. An unconstrained optimum design problem is formulated as

minimize $f(a, b) = a + \dfrac{25}{ab} + 5b$

Check whether or not $a = 5$ and $b = 1$ is a minimum point for the problem.
(8 points)

5. Check for convexity of the following design problem:

maximize $f(\mathbf{x}) = -x_1^2 - x_2^2 + 3x_1 - 3x_2 + x_1x_2$

subject to the constraints

$$2x_1 - 4x_2 \geq -4$$
$$x_1^2 - x_2 \geq 2$$
$$x_1 \leq 2$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

(7 points)

6. Answer True or False

(i) _____ The number of independent equality constraints can be greater than the number of design variables for a design optimization problem.

(ii) _____ The number of "≤ type" constraints must be less than the number of design variables for a design problem.

(iii) _____ A function can have more than one global minimum point.

(iv) _____ Gradient of a function at a point is normal to the level surface defined by the function.

(v) _____ A point satisfying first order necessary conditions is always a minimum point for the function.

(vi) _____ If Hessian of the cost function for unconstrained problem is positive definite at a candidate point, the point is a local maximum for the function.

(vii) _____ A point satisfying KKT necessary conditions for a general optimum design problem can be a maximum point for the cost function.
(viii) _____ In optimum design problem formulation, "≥ type" constraints cannot be treated.
(ix) _____ A nonlinear equality constraint always defines a nonconvex region.
(x) _____ The minimum point for a constrained optimization problem gives a stationary value for the Lagrange function.
(15 points)

7. A circular tank that is closed at both ends is to be fabricated to have a volume of 250π m^3. The fabrication cost is found to be proportional to the surface area of the sheet metal needed for fabrication of the tank and is \$400/m^2. The tank is to be housed in a shed with sloping roof which limits the height of the tank by the relation $H \le 8D$ where H is the height and D is the diameter of the tank. The problem is formulated as

minimize $f = 400(2\pi D^2/4 + \pi DH)$; subject to the constraints:

$$\frac{\pi D^2 H}{4} = 250\pi; \;\; H \le 8D; \;\; H \ge 0, D \ge 0$$

a. Write KKT necessary conditions for the design problem
b. Identify all the cases for solution of the KKT necessary conditions
c. Check if the following case gives a solution that satisfies all the KKT necessary conditions:
$H \le 8D$ as active;
Lagrange multipliers for the other two inequalities as zero.
(20 points)

8. An optimum design problem is formulated as

minimize $f(x_1, x_2) = -x_1 - 2x_2$

subject to
$$g_1(x_1, x_2) = 0.1x_1 + 0.4x_2 - 225 \le 0$$
$$g_2(x_1, x_2) = x_1 + x_2 - 800 \le 0$$
$$g_3(x_1, x_2) = \frac{x_1}{600} + \frac{x_2}{1200} - 1.0 \le 0$$

The point $x_1 = \frac{950}{3}$, $x_2 = \frac{1450}{3}$ satisfies the KKT necessary conditions. The Lagrange multipliers for the constraints are: 10/3, 2/3, 0.
a. What is the effect on the cost function if the right hand side of constraint one is changed to 5?
b. What is the effect on the cost function if the right hand side of constraint two is changed to−50?
(5 points)

SAMPLE EXAM 3: MIDTERM EXAM

for

First Undergraduate Course (Time allowed = 70 minutes)

Problem 1 (25 Points)

An engineering design optimization problem is formulated as follows:

$$\text{Optimize} \quad f = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$$

Subject to

$$g_1 : x_1 + 3x_2 \le 6$$

$$g_2 : 5x_1 + 2x_2 \le 10$$

$$g_3 : -x_1 \le 0$$

$$g_4 : -x_2 \le 0$$

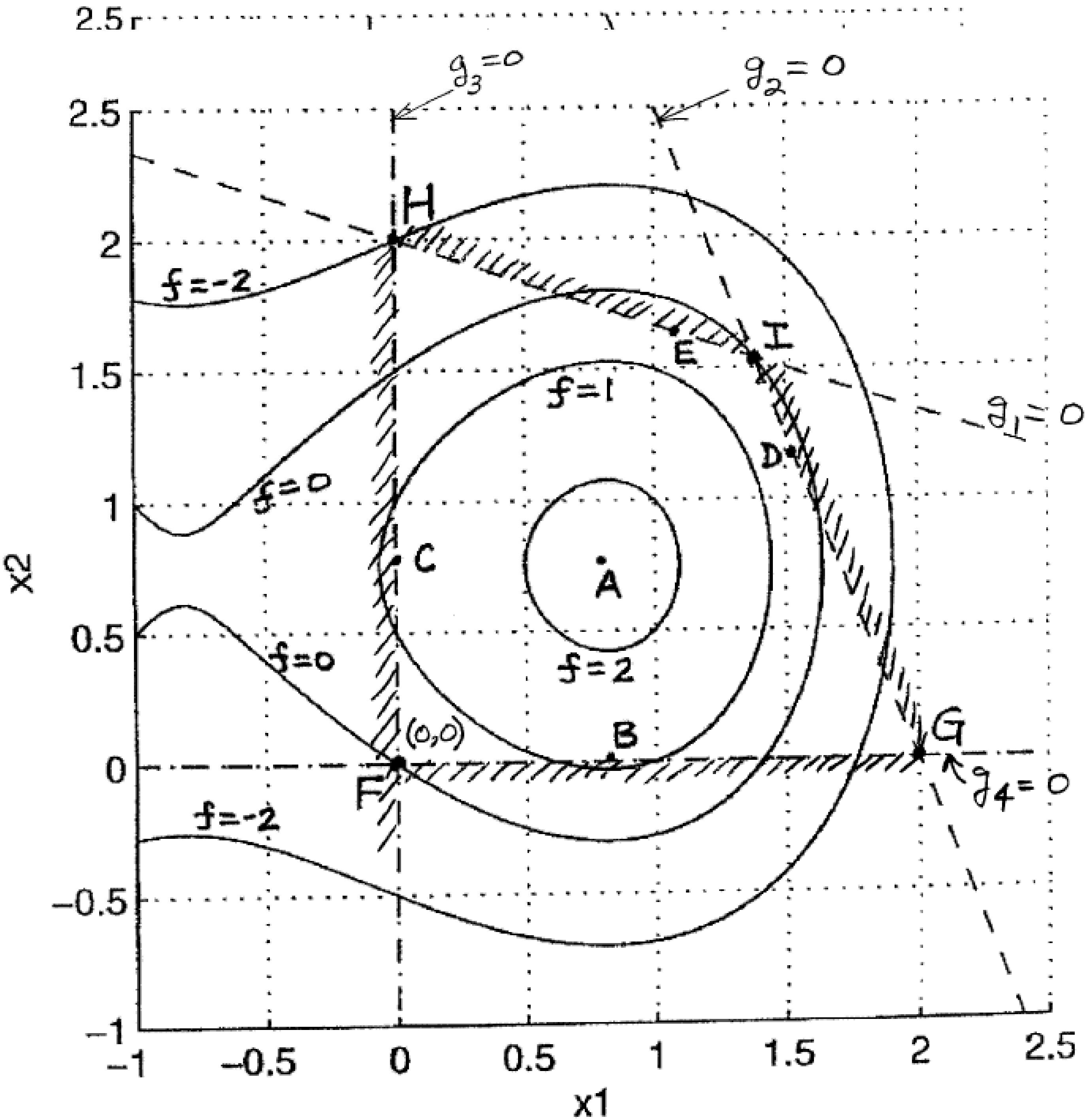
A graphical representation for the problem is shown in the figure. The *dashed lines* show the constraint boundaries and solid lines show the cost function contours. ANSWER THE FOLLOWING QUESTIONS.

(a) Label each constraint on the graph as g_1 to g_4. (8)

(b) Hatch-out the infeasible side of each constraint. What is the feasible region for the problem? (5)
FGIH

(c) For each of the following points, identify whether the point is a local minimum, local maximum or neither of the two by circling your selection: (12)

Point A:	Min	Max	Neither
Point B:	Min	Max	Neither
Point F:	Min	Max	Neither
Point G:	Min	Max	Neither
Point H:	Min	Max	Neither
Point I:	Min	Max	Neither



$g_3=0$
$g_2=0$
$g_1=0$
$g_4=0$
H
I
E
D
C
A
B
G
F
(0,0)
f=-2
f=0
f=1
f=2
f=0
f=-2
x2
x1

Problem 2 (25 Points)

An engineering design problem is formulated as the following unconstrained optimization problem:

$$\text{minimize} \quad f(x, y, z) = x^2 + y^2 + 2y + 2z^3 - 4zx - 8z$$

(i) **(8 Points)** Write gradient vector and Hessian matrix for the function.

$$\nabla f = \begin{bmatrix} 2x - 4z \\ 2y + 2 \\ 6z^2 - 4x - 8 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 12z \end{bmatrix}$$

(ii) **(7 Points)** Check if (2, -1, 1) is a stationary point for the function?

The gradient vector should be zero.

$$\nabla f = \begin{bmatrix} 2 \times 2 - 4 \\ -2 + 2 \\ 6 - 8 - 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} \neq \mathbf{0}$$

The gradient is not zero, so it is not a stationary point.

(iii) **(10 Points)** Check if the point (4, -1, 2) is a local minimum point for the function.

$$\nabla f = \begin{bmatrix} 8-8 \\ -2+2 \\ 24-16-8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The gradient is zero, so it is a stationary point. Check the form of the Hessian.

$$\mathbf{H} = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 24 \end{bmatrix}$$

$M_1 = 2 > 0$

$M_2 = 4 > 0$

$M_3 = (2)(2)(24)+(-4)(8) = 64 > 0$

Hessian is positive definite, so (4, -1, 2) is a local minimum point because it satisfies both second order necessary and sufficient conditions.

Problem 3 (25 Points)

An engineering design problem is formulated as

minimize $f(x_1, x_2) = 2x_1^2 + x_2^2 - 11x_1 - 4.5x_2 + 2$

subject to $g_1:\ 10 \geq 2x_1 + 5x_2$ (1)

$g_2:\ 2x_1 + x_2 \leq 8$ (2)

The point (2.5, 1) has been identified as a local minimum point. Using constraint variation sensitivity result, determine the change in the minimum value for the cost function if the constant 10 in the first constraint (g_1) is changed to 9.5?

(5 Points) Write problem in the standard form:

minimize $f(x_1, x_2) = 2x_1^2 + x_2^2 - 11x_1 - 4.5x_2 + 2$

subject to $g_1:\ 2x_1 + 5x_2 - 10 \leq 0$; $g_2:\ 2x_1 + x_2 - 8 \leq 0$

(5 Points) Calculate cost function and check constraints at (2.5, 1):

$$f = 2(2.5)^2 + 1 - 11(2.5) - 4.5(1) + 2 = -16.5$$
$$g_1 = 2(2.5) + 5 - 10 = 0\ (active)$$
$$g_2 = 2(2.5) + 1 - 8 = -2 < 0\ (inactive)$$

(8 Points) Calculate Lagrange multiplier for g_1:

$$L = 2x_1^2 + x_2^2 - 11x_1 - 4.5x_2 + 2 + u_1\left(2x_1 + 5x_2 - 10\right)$$

$$\frac{\partial L}{\partial x_1} = 4x_1 - 11 + 2u_1 = 0; \quad u_1 = 0.5$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 4.5 + 5u_1 = 0; \quad u_1 = 0.5$$

(7 Points) Calculate change in cost

$$\Delta f = -u_1\left(\Delta e_1\right); \quad \Delta e_1 = e_{1new} - e_{1old} = 9.5 - 10 = -0.5$$
$$\Delta f = -0.5(-0.5) = 0.25$$

The cost function will increase by 0.25.

Problem 4 (25 Points)

A trucking company wants to purchase several new trucks. It has \$2 million to spend. The investment should yield a maximum of trucking capacity for each day in tonnes×kilometers. Data for the three available truck models are given in the table below; i.e., truck load capacity, average speed, crew required/shift, hours of operations for three shifts, and the cost of each truck. There are some limitations on the operations that need to be considered. The labor market is such that the company can hire at the most 150 persons to operate the trucks. Garage and maintenance facilities can handle at the most 25 trucks. How many trucks of each type should the company purchase? Formulate the design optimization problem. DO NOT SOLVE THE PROBLEM.

Truck model	Truck load capacity (tonnes)	Average truck speed (km/h)	Crew required per shift	No. of hours of operations per day (3 shifts)	Cost of each truck (\$)
A	10	55	1	18	40,000
B	20	50	2	18	60,000
C	18	50	2	21	70,000

(a) Identify and clearly define the design variables for the problem. (6)

A = Number of Type A to be purchased

B = Number of Type B to be purchased

C = Number of Type C to be purchased

(b) Describe in words the objective function for the problem. (2)

Objective function: maximize the trucking capacity, tonnes × kilometers

(c) Write an expression for the objective function. (5)

$$Capacity = A(10\times55\times18) + B(20\times50\times18) + C(18\times50\times21)$$

$$= 9900A + 18000B + 18900C$$

(d) Clearly describe each constraint in words. (4)

The total cost of trucks should not exceed \$2 mllion

Number of drivers hired cannot exceed 150

Maintenance facility cannot handle more than 25 trucks

Number of trucks purchased of each type must be positive or at the most zero

(e) Develop an expression for each constraint. (8)

Available capital constraint:	$40000A + 60000B + 70000C \le 2{,}000{,}000$
Limitation of drivers:	$3(A + 2B + 2C) \le 150$
Limitation of maintenance facility:	$A + B + C \le 25$
Nonnegativity of variables:	$A, B, C \ge 0$

SAMPLE EXAM 4: FINAL EXAM

for

First Undergraduate Course (Time allowed = 2 hrs)

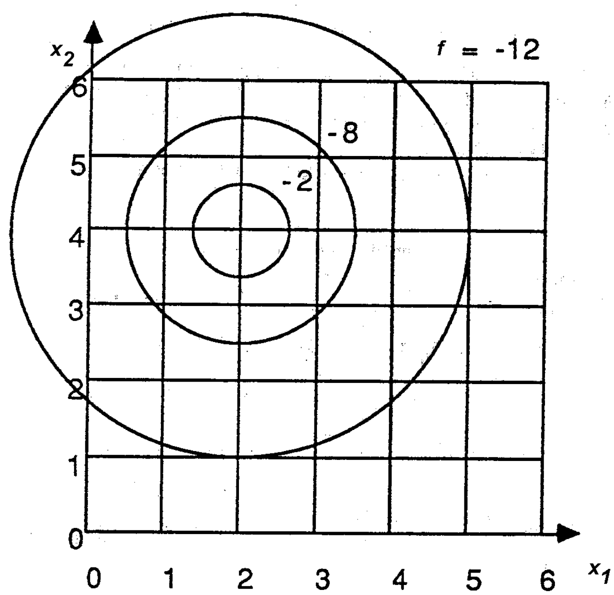
1. An airlines company is considering the purchase of new jet passenger airplanes. Pertinent data on the planes under consideration are listed in the table below. A maximum of \$150 million is available for the purchase. Pilots are available to fly up to 30 new airplanes and the available area of maintenance facilities is 900,000 ft^2. For overall fleet balance, at least a third of the planes purchased should be short range airplanes. Formulate this problem to maximize the annual profit resulting from the purchase. Ignore the fact that the number of airplanes must be an integer.
   Express the formulation in standard form, but do not normalize. DO NOT SOLVE.
   (10 points)

Type	Purchase Cost \$million/plane	Net annual profit including Capital recovery costs \$1000/plane	Area needed for maintenance ft^2/plane	Minimum Number Required
A Long Range	6.7	420	40,000	1
B Medium Range	5.0	300	30,000	1
C Short Range	3.5	230	24,000	3

2. A function $f(\mathbf{x})$ is to be minimized subject to the constraints

   $$x_1 - x_2 \le 1; \quad 6x_1 + 5x_2 \le 30\,; \quad x_1, x_2 \ge 0$$

   A plot of several constant values of $f(\mathbf{x})$ are shown.
   a) Identify on the plot all of the local minimum points for the constrained problem.
   b) Identify the global minimum point.
   (10 points)



3. An optimum design problem is given as:

minimize $f(x_1, x_2) = 3(x_1-1)^2 + (x_2-2)^2$

subject to $3x_1 + 2x_2 - 6 \le 0$

$(x_1 + 1)^2 + (x_2 + 1)^2 - 9.61 = 0$

a) Determine whether or not the point (−1, 2.1) satisfies the necessary conditions for a minimum point.

b) The point (1.846, 0.231) is a local minimum point. Determine the approximate minimum value of $f(\mathrm{x})$ in this region if the equality constraint is changed to

$(x_1 + 1)^2 + (x_2 + 1)^2 - 9 = 0$

(10 points)

4. Maximize $z = 6x_1 + 4x_2$

subject to $3x_1 + 5x_2 \le 15$

$2x_1 + x_2 \ge 2$

$x_1 - 2x_2 = -2$

$x_1 \ge 0$

x_2 is unrestricted in sign

a) Set up the initial Simplex tableau for the given LP problem. Define any variables that you add.

b) Complete one iteration and circle the pivot element for the second iteration, but do not proceed further.

c) List the values of every variable and function

(i) at the initial point

(ii) after the first iteration

(15 points)

5. Given the following LP problem (10 points)

minimize $f = 9x_1 + 3x_2 + 2x_3$

subject to g_1: $2x_1 - 3x_2 + x_3 \geq 5$

g_2: $-x_1 - 2x_2 + 2x_3 \leq 2$

$x_1,\ x_2 \geq 0$

The final tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	
	1	− 0·8	0	− 0·4	−0·2	0·4	1.6
	0	− 1·2	1	− 0·2	0·4	0·2	1.8
	0	13	0	4	1	− 4	$f - 18$

where x_4 = surplus variable for g_1

x_5 = slack variable for g_2

x_6 = artificial variable for g_1

a) What are the optimal **x** and $f(\mathbf{x})$. Identify the basic variables and the active constraints.
b) List the Lagrange multipliers for the two constraints
c) Is the solution unique? Explain
d) The RHS of g_2 is increased from 2 to 5.
   (i) Does the basis change?
   (ii) If not, what is the new $f(\mathbf{x})$?
e) What are the upper and lower bounds on the cost coefficient of x_1 such that the optimal point **x** does not change?

6. The minimum of $f(\mathbf{x})$ is desired. Starting from the point $\mathbf{x}^{(0)} = (1,1)$, the first iteration of the Steepest Descent method gives the point $\mathbf{x}^{(1)} = (2.111, 0.444)$.

$$f(\mathbf{x}) = 2x_1^2 + x_2^2 - 8x_1$$

a) Determine the search direction at point $\mathbf{x}^{(1)}$ using:
   1. Steepest Descent
   2. Conjugate Gradient
   3. Newton's Method
b) Establish whether $\mathbf{d} = (1.5, -1)$ is a descent direction starting from $\mathbf{x}^{(1)}$.
c) Starting from $\mathbf{x}^{(0)} = (1, 1)$, find $\mathbf{x}^{(1)}$ using $\mathbf{d} = (1, -1)$.
(15 points)

7. An optimum design problem is formulated as

find y_1 and y_2 to minimize : $f(y_1, y_2) = y_1(y_1 + 6y_2)$

subject to the constraints : $h_1 = \dfrac{y_1^2 y_2}{200} - 1.0 = 0$; $g_1 = 1 - y_2/2y_1 \le 0$; $g_2 = y_2/3y_1 - 1.0 \le 0$

$$g_3 = y_2/20 - 1.0 \le 0\ ; \quad g_4 = -y_1 \le 0\ ; \quad g_5 = -y_2 \le 0$$

At a given design point (4, 8),

a) define the linear programming subproblem using 50% move limits

b) define the quadratic programming subproblem. (15 points)

8. An optimum design problem is formulated in a normalized form as

minimize $f = 200\, x_1 + 100\, x_2$

subject to g_1: $1.0 - \dfrac{x_1 x_2}{10000} \le 0$; g_2: $\dfrac{x_1}{100} - 1.0 \le 0$

g_3: $\dfrac{x_2}{200} - 1.0 \le 0$; g_4: $-\dfrac{x_1}{50} + \dfrac{x_2}{100} \le 0$

At the design point (90, 120), the solution of QP subproblem is given as

d = (−30, 30); **u** = (15000, 0, 0, 0)

Calculate an appropriate step size and the improved design in the given direction. Let initial $R = 1.0$ and $\gamma = 0.5$.

(15 points)

SAMPLE EXAM 5: FINAL EXAM

for

First Undergraduate Course (Time allowed = 2 hrs)

1. The circular tube shown below is to be designed for two conditions:

   when P = 40 kN, axial stress $\sigma \le \sigma_y$

   when P = 0, deflection due to weight $\delta_{max} \le 0.001l$

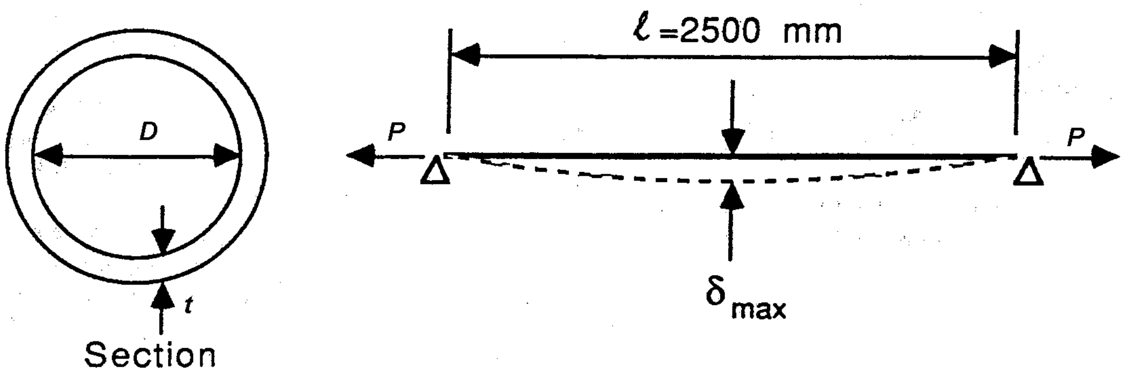
   Limits for dimensions are t = 2.0 to 15.0 mm, D = 15.0 to 125.0 mm, and $\dfrac{D}{t} \le 20$. The design is to be optimized for minimum weight.

   Material Properties: σ_y = 165 MPa ; ρ = 80 kN/m^3 ; E = 210 GPA

   Member Properties: Area $A = \pi Dt$; Weight $W = \rho Al$; $I = \pi D^3 t/8$

   Constraint Relations: $\sigma = P/A$; $\delta_{max} = \dfrac{5Wl^3}{384EI}$ when P = 0

   Formulate the optimal design problem. Express the results in standard normalized form. (10 points)



2. Given the following optimum design problem:

   minimize $f(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$

   subject to $x_1 + 2x_2 \le 10$

   $4x_1 - 3x_2 - 20 \le 0$

   $x_1 \ge 0$

   $x_2 \ge 0$

   Check if point x_1 = 70/11 and x_2 = 20/11 is a candidate local minimum point.
   (10 points)

2. Solve the following problem using the Simplex method up to the point that the first feasible solution is found. List the values of all the variables at each iteration.
(15 points)
Minimize $f = x_1 - 2x_2$
subject to g_1: $2x_1 - x_2 \geq 2$
g_2: $x_1 + x_2 \geq 3$
g_3: $x_2 \leq 2$
$x_1, x_2 \geq 0$

4. Given the following LP problem:
minimize $f = 2x_1 - 4x_2$
subject to g_1: $10x_1 + 5x_2 \leq 15$
g_2: $4x_1 + 10x_2 \leq 36$
$x_1, x_2 \geq 0$

Slack variables for g_1 and g_2 and x_3 and x_4 respectively. The final tableau is

x_1	x_2	x_3	x_4	b
2	1	1/5	0	3
−16	0	−2	1	6
10	0	4/5	0	$f+12$

a. Determine optimal values of f and **x**.
b. Determine Lagrange multipliers for g_1 and g_2.
c. Determine the right-hand-side ranges for g_1 and g_2.
d. What is the smallest value that f can have, with the current basis, if the right hand side of g_1 is changed? What is the right hand side of g_1 for this case?
(10 points)

5. Consider the problem : minimize $f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$
a. Find steepest descent direction at the point (2,1).
b. Find Newton direction at the point (2,1). (15 points)

6. Consider the problem : minimize $f(x_1, x_2) = 5x_1 - x_1^2\, x_2/16 + x_2^2/4x_1$
Determine if the direction (2, 5) is that of descent for $f(x_1, x_2)$ at the point (1, 2). (5 points)

7. Consider the problem : minimize $f(x_1,x_2) = 250 \% 10^3\, x_1 x_2$

subject to the constraints

$$x_1 x_2 \geq 6.4 \% 10^{-3}$$

$$x_1^3 x_2 \leq 1.6 \% 10^{-4}$$

$$x_1 \geq 0,\ x_2 \geq 0$$

a. Define the problem in the standard form normalizing the constraints.
b. Linearize the problem at the point (0.4, 0.01) and define a linear subproblem with proper move limits.
c. Define the quadratic programming subproblem at the point (0.4, 0.01).
(20 points)

8. The minimum weight design of a closed and thin-walled cylindrical pressure vessel subjected to stress, strain and volume constraints can be formulated as

minimize $x^2 + 320\, xy$

subject to $\dfrac{x}{60y} - 1 \leq 0$

$$1 - \frac{x(x-y)}{3600} \leq 0$$

$$x \geq 0,\ y \geq 0$$

where x is the radius and y is the thickness. At a design point of $x = 40$, $y = 0.5$, the search direction for the descent function is given as $\mathbf{d} = (25.6, 0.45)$. The Lagrange multipliers for the constraints are $\mathbf{u} = (4880, 19400)$. Choose $\gamma = 0.5$ and calculate the step size for the constrained steepest descent method at the given point.
(15 points)

SAMPLE EXAM 6: FINAL EXAM

for

First Undergraduate Course (Time allowed = 75 minutes)

1. (15 points) An engineering design problem is formulated as follows:

Maximize $z = -2x_1 + 3x_2$

subject to

$$6 - 3x_1 + 2x_2 \geq 0$$
$$-x_1 + x_2 - 4 \geq 0$$
$$x_1 \text{ is free in sign; } x_2 \geq 0$$

(i) Transcribe the problem to the standard LP format. (7)

(2) Minimize $f = 2(x_1^+ - x_1^-) - 3x_2;$ $\quad x_1 = x_1^+ - x_1^-$ (2)

(3)
$$3(x_1^+ - x_1^-) - 2x_2 + x_3 = 6; \quad x_3 = slack\ var\,iable$$
$$-(x_1^+ - x_1^-) + x_2 - x_4 + x_5 = 4; \quad x_4 = surplus\ var\,iable$$
$$x_1^+, x_1^-, x_2 \geq 0; \quad x_5 = artificia;\ var\,iable$$

(ii) Set up the initial tableau for the problem so that the Simplex method can be used to solve the problem. *DO NOT SOLVE THE PROBLEM*. (6)

(3) Artificial cost, $w = x_5 = 4 + x_1^+ - x_1^- - x_2 + x_4$

(3)

	x_1^+	x_1^-	x_2	x_3	x_4	x_5	**b**
x_3	3	-3	-2	1	0	0	6
x_5	-1	1	1	0	-1	1	4
Cost	2	-2	-3	0	0	0	f - 0
Artificial cost	1	-1	1	0	1	0	w - 4

(iii) What are the initial values for the basic and nonbasic variables? (2)

Basic: $x_3 = 6$, $x_5 = 4$; all others are nonbasic – have zero value

2. (16 points) A linear design optimization problem is to be solved using the Simplex method. The initial tableau for the problem is given below, where x_3 is a slack variable, x_4 is a surplus variable, and x_5 is an artificial variable.

	X_1	x_2	x_3	x_4	x_5	**b**
x3	-2	1	1	0	0	4
x5	1 (3)	1	0	-1	1	2
Cost	-4	-2	0	0	0	f - 0
Artificial cost	-1 (2)	-1	0	1	0	w - 2
x3	0	3	1	-2	2	8
x1	1	1	0	-1	1	2
Cost	0	2	0	-4	4	f+8
Art. (5)	0	0	0	0	1	w-0

ANSWER THE FOLLOWING QUESTIONS.

(i) Identify basic and nonbasic variables for the initial tableau. (2)

Basic: x3, x5; Nonbasic: x1, x2, x4

(ii) Complete (ONLY) one iteration of the Simplex method to obtain the next tableau. (10)

(iii) Does your tableau give an optimum solution for the linear design problem? YES/NO (1)

(iv) Does your tableau indicate end of Phase I of the Simplex method? YES/NO (1)

(v) Does your tableau indicate the problem to be infeasible? YES/NO (1)

(vi) Does your tableau indicate the problem to be unbounded? YES/NO (1)

3. (15 points) A linear engineering design problem is formulated as follows:

Minimize $f = 10x_1 + 6x_2 + 2x_3 - 6x_4$
subject to

$$-x_1 + x_2 + x_3 - x_4 \geq 1$$
$$3x_1 + x_2 - x_3 - 3x_4 \geq 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The final Simplex tableau for the problem is given as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_2	0	1	0.5	-1.5	-0.75	-0.25	0.75	0.25	1.25
x_1	1	0	-0.5	-0.5	0.25	-0.25	-0.25	0.25	0.25
Cost	0	0	4	8	2	4	-2	-4	f - 10
Artificial	0	0	0	0	0	0	1	1	w - 0

x_5 and x_6 are surplus variables for constraint #1 and 2, respectively. x_7 and x_8 are artificial variables for constraint #1 and 2, respectively.

ANSWER THE FOLLOWING QUESTIONS. (1.5) each

i	x_1 is a nonbasic variable	YES/NO
ii	Value of x_2 at the optimum point is 0.25	YES/NO
iii	Value of x_6 is zero at the optimum point	YES/NO
iv	Value of the cost function at the optimum is 10	YES/NO
v	Constraint #1 is inactive at the optimum point	YES/NO
vi	Constraint #2 is active at the optimum point	YES/NO
vii	Lagrange multiplier for constraint #1 is 2	YES/NO
viii	Lagrange multiplier for constraint #2 is -4	YES/NO
ix	The problem has multiple optimum points	YES/NO
x	Cost coefficient of x_3 can be increased to 4 without affecting the optimum cost function	YES/NO

4. (20 points) An engineering design problem is formulated in terms of the design variables *x* and *y* as follows:

Minimize $f = x\left(\frac{1}{2}x - y^2\right)$

Subject to

$$h_1: \; y(x + 4y) = 10$$

$$g_1: \; -2x^2 + \tfrac{1}{3}y \le 0$$

$$g_2: \; -y - 4 \le 0$$

(i) Evaluate cost and constraint functions at the point (2,3). (4)

$$f = 2\left(\tfrac{1}{2} \times 2 - 3^2\right) = -16$$
$$h_1 := 3(2 + 4 \times 3) - 10 = 32$$
$$g_1 = -2 \times 2^2 + \tfrac{1}{3} \times 3 = -7$$
$$g_2 = -3 - 4 = -7$$

(ii) Evaluate gradients of the cost and constraint functions at the point (2,3). (8)

$$\nabla f = \begin{bmatrix} x - y^2 \\ -2xy \end{bmatrix} = \begin{bmatrix} -7 \\ -12 \end{bmatrix}; \qquad \nabla h_1 = \begin{bmatrix} y \\ x + 8y \end{bmatrix} = \begin{bmatrix} 3 \\ 26 \end{bmatrix}$$

$$\nabla g_1 = \begin{bmatrix} -4x \\ 1/3 \end{bmatrix} = \begin{bmatrix} -8 \\ 1/3 \end{bmatrix}; \qquad \nabla g_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(iii) Define the quadratic programming subproblem at the point (2,3). (8)

Minimize

$$\bar{f} = -16 - 7d_1 - 12d_2 + \tfrac{1}{2}\left(d_1^2 + d_2^2\right)$$

subject to

$$32 + 3d_1 + 26d_2 = 0$$

$$-7 - 8d_1 + \tfrac{1}{3}d_2 \le 0$$

$$-7 - d_2 \le 0$$

5. (20 Points) An engineering design problem is formulated as the following unconstrained problem:

Minimize $f = 8x_1^2 + 2x_1x_2 + 5x_2^2 - 2x_1 + 3x_2$

Starting from a point (-1, -1), one iteration of the steepest descent method has been completed yielding a new point as (0.21, 0.45).

(i) Calculate the steepest descent direction at the point (-1, -1). 6

$$\mathbf{c} = \nabla f = \begin{bmatrix} 16x_1 + 2x_2 - 2 \\ 2x_1 + 10x_2 + 3 \end{bmatrix} = \begin{bmatrix} -20 \\ -9 \end{bmatrix}$$

Steepest descent direction

$$\mathbf{d} = -\mathbf{c} = \begin{bmatrix} 20 \\ 9 \end{bmatrix}$$

(ii) Calculate the steepest descent direction at the point (0.21, 0.45). 2

$$\mathbf{c}^{(1)} = \nabla f = \begin{bmatrix} 16x_1 + 2x_2 - 2 \\ 2x_1 + 10x_2 + 3 \end{bmatrix} = \begin{bmatrix} 2.26 \\ 7.92 \end{bmatrix}$$

(iii) Calculate the conjugate gradient direction at the point (0.21, 0.45). 8

$$\mathbf{d}^{(1)} = -\mathbf{c}^{(1)} + \beta_0 \mathbf{d}^{(0)} = -\begin{bmatrix} 2.26 \\ 7.92 \end{bmatrix} + 0.141 \begin{bmatrix} 20 \\ 9 \end{bmatrix} = \begin{bmatrix} 0.82 \\ -6.65 \end{bmatrix}; \;\; \beta_1 = \frac{67.8}{481} = 0.141$$

(iv) Show that the conjugate gradient direction calculated in part (iii) is a direction of descent for the cost function. 4

$$\mathbf{d}^{(1)} \bullet -\mathbf{c}^{(1)} = 1.8532 - 52.668 < 0$$

The descent condition is satisfied.

6. (14 points) An engineering design problem is formulated in terms of the design variables x, y, and z as follows:

Minimize $f = 2xz^2$

subject to

$$g_1:\ (4\times10^{-4})(5zy + 2z^2) - 1 \le 0$$

$$g_2:\ 1 - 2.5\times10^{-5}(x+y)z^2 \le 0$$

The quadratic programming subproblem has been defined and solved at the point (50, 10, 25). The Lagrange multipliers for the constraints are obtained as (20000, 70000). Using the initial value for the penalty parameter R_0 as 1, calculate the value of the descent function Φ at the given point.

SOLUTION:

$$f = 2\times50\times25^2 = 31250 \quad (2)$$

$$g_1 = (4\times10^{-4})(5\times25\times10 + 2\times25^2) - 1 = 0$$

$$g_2 = 1 - 2.5\times10^{-5}(50+10)\times25^2 = 0.0625$$

(4)

$$r = 20000 + 70000 = 90000 \quad (2)$$

$$R = max(1{,}90000) = 90000 \quad (2)$$

$$V = max\{0;\ 0.0.0625\} = 0.0625 \quad (2)$$

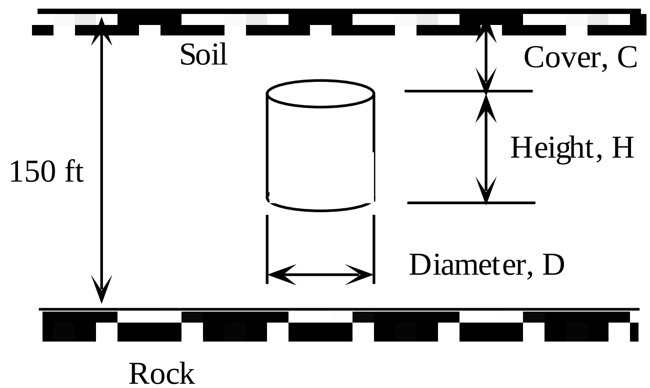
$$\Phi = f + RV = 62500 + 0.0625\times90000 = 68125 \quad (2)$$

SAMPLE EXAM 7: PROBLEM FORMULATION

for

First Undergraduate Course (Time allowed = 20 minutes)

A chemical manufacturer plans to construct an underground tank for liquid propane storage, as shown in the figure. The required volume of the cylindrical tank of diameter D and height H (ignore thickness of the tank wall) must be at least 400,000 ft^3. A sketch of the tank is shown below along with the cost data for construction and operations. The depth of the cover C must be at least 8 ft. The depth of the excavation cannot exceed 150 ft because rock is encountered at that level and it is uneconomical to dig through the rock. Another requirement for the problem is that the diameter to height ratio cannot exceed 1.2. The objective is to minimize the total cost of construction and operations of the tank facility. Also the annual cost of operations of the tank should not exceed $400,000.



Construction Costs:

Excavation: Assume that the shape of excavation is cylindrical with diameter D and depth (C + H). Cost of excavation = $540 per cubic yard.

Tank top and bottom: Cost = $100 per ft^2 of surface area.

Tank wall: Cost = $[0.10 D(C+H)] per ft^2 of wall area.

Annual Operating Costs:

The annual operating cost depends on the total surface area of the tank and the depth of the cover, and is given as [200(total surface area of the tank) - 6000C] $.

The design variables for the problem are identified as:

C = depth of the cover, ft
D = diameter of the tank, ft
H = height of the tank, ft

CIRCLE YOUR CHOICE (each question carries 5 points).

1. The cost of excavation is given as
   A. $\$135\pi D^2(C+H)$
   B. $\$5\pi D^2(C+H)$
   C. $\$135\pi D^2 H$
   D. $\$5\pi D^2 H$

2. Cost of the tank's bottom is given as
   A. $\$25\pi D^2$
   B. $\$100\pi D^2$
   C. $\$50\pi D^2$
   D. $\$100\pi D$

3. Cost of tank wall is given as
   A. $\$0.1\pi D(C+H)H$
   B. $\$0.1\pi D(C+H)D$
   C. $\$0.1\pi D^2 H(C+H)$
   D. $\$0.1DH(C+H)$

4. The constraint for the volume is given as
   A. $\pi D^2 H \geq 400{,}000$
   B. $\pi D^2 H \leq 400{,}000$
   C. $\pi D^2 H = 1{,}600{,}000$
   D. $\pi D^2 H \geq 1{,}600{,}000$

5. The constraint on the depth of excavation is given as
   A. $C+H=150$
   B. $H \leq 150$
   C. $C+H \leq 150$
   D. $C+H \geq 150$

6. The annual operating cost constraint is given as
   A. $100\pi D[H+D]-6000C \leq 400{,}000$
   B. $100\pi D[2H+D]-6000C = 400{,}000$
   C. $100\pi D[H+D]-6000C = 400{,}000$
   D. $100\pi D[2H+D]-6000C \leq 400{,}000$

SAMPLE EXAM 8: GRAPHICAL OPTIMIZATION

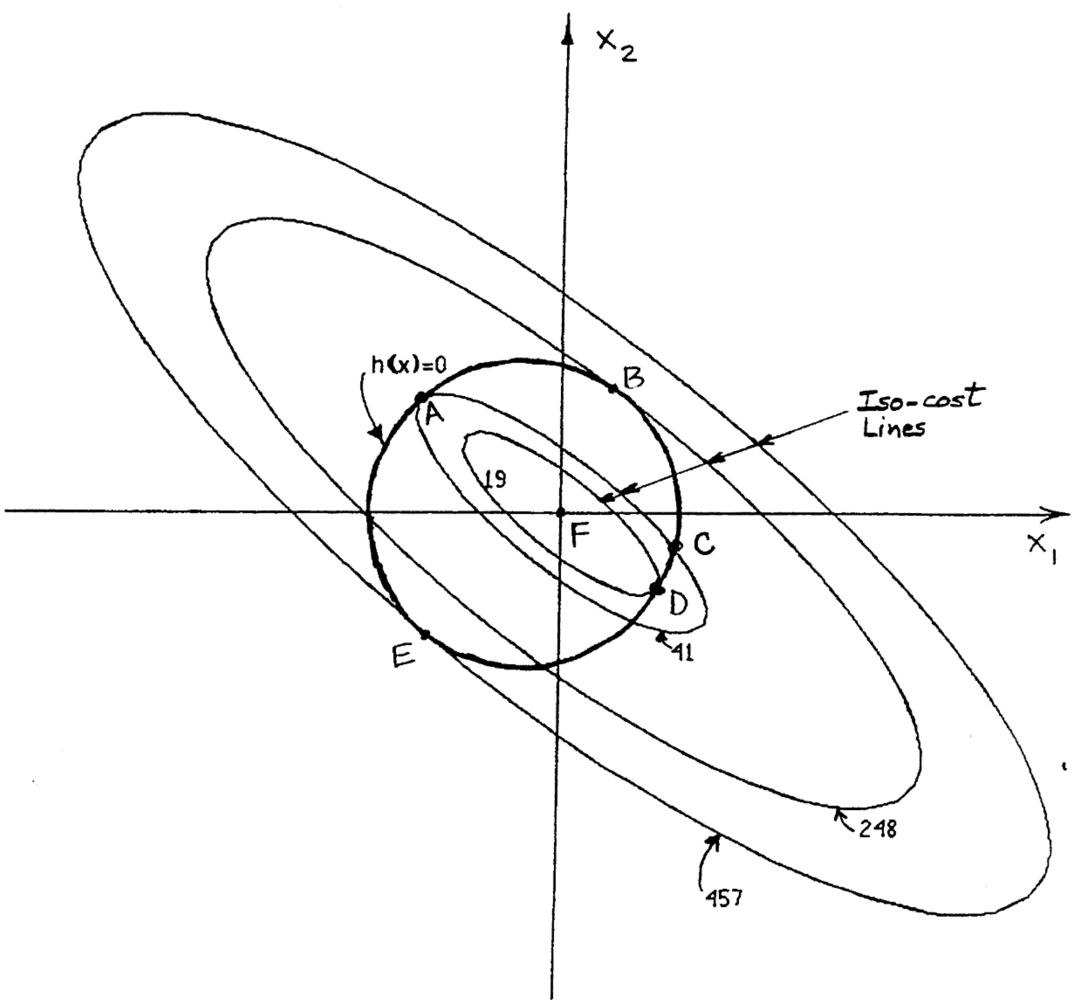
for

First Undergraduate Course (Time allowed = 20 minutes)

The graphical representation for a nonlinear design optimization problem with one equality constraint is given below. The equality constraint represents a circle, identified as $h(\mathbf{x}) = 0$ in the figure. Four iso-cost curves (ellipses) with cost function values of 19, 41, 248 and 457 are shown in the figure.

Circle True or False (5 points each)

(i)	Point C is a local maximum point.	True False
(ii)	Point A is a local minimum point.	True False
(iii)	Point F is a local maximum point.	True False
(iv)	Point B is a local minimum point	True False
(v)	Point E is a local maximum point when the equality constraint is changed to an inequality such that the region outside the circle is infeasible.	True False
(vi)	Point D is a local minimum point when the equality constraint is changed to an inequality such that the region outside the circle is infeasible.	True False



SAMPLE EXAM 9: OPTIMALITY CONDITIONS

for

First Undergraduate Course (Time allowed = 30 minutes)

1. (10 Points) Hessian (2×2 matrix) of a function at its stationary point is given as

$$\mathbf{H} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$$

Circle your choice:

A. **H** is positive definite

B. **H** is positive semidefinite

C. **H** is negative definite

D. **H** is negative semidefinite

E. **H** is indefinite

2. (10 Points) An engineering design problem is formulated as

maximize $F(x_1, x_2) = -x_1^2 - 2x_2^2 + 5x_1 + 12x_2 - 2$

subject to $5 \geq 5x_1 + 2x_2$ (1)

$x_1 + 2x_2 \leq 8$ (2)

The Lagrangian function for the problem is defined as (**Circle your choice**):

A. $L = x_1^2 + 2x_2^2 - 5x_1 - 12x_2 + 2 + u_1\left(5x_1 + 2x_2 - 5 + s_1^2\right) + u_2\left(x_1 + 2x_2 - 8 + s_2^2\right)$

B. $L = -x_1^2 - 2x_2^2 + 5x_1 + 12x_2 - 2 + u_1\left(5x_1 + 2x_2 - 5 + s_1^2\right) + u_2\left(x_1 + 2x_2 - 8 + s_2^2\right)$

C. $L = x_1^2 + 2x_2^2 - 5x_1 - 12x_2 + 2 + u_1\left(5 - 5x_1 - 2x_2 + s_1^2\right) + u_2\left(x_1 + 2x_2 - 8 + s_2^2\right)$

D. $L = x_1^2 + 2x_2^2 - 5x_1 - 12x_2 + 2 - u_1\left(5x_1 + 2x_2 - 5 + s_1^2\right) - u_2\left(x_1 + 2x_2 - 8 + s_2^2\right)$

3. (30 Points) An engineering design problem is formulated as the following unconstrained optimization problem:

minimize $f(x, y) = 2x^3 + y^2 - 4xy - 8x$

Circle YES or NO for the following questions:

A. Point (1, -1/2) is a stationary point for the function *f*. YES NO

B. Point (2, 4) is a local minimum point for the function *f*. YES NO

C. Point (-2/3, -4/3) is a local maximum point for the function *f*. YES NO

4. An Optimization problem is formulated as

minimize $f(r,t) = -(r-3)^2 - (t-2)^2$

subject to $g_1 = r + t - 10 \le 0$

$g_2 = t - 5 \le 0$

The Lagrangian for the problem is defined as

$$L = -(r-3)^2 - (t-2)^2 + u_1(r + t - 10 + s_1^2) + u_2(t - 5 + s_2^2)$$

Circle True or False (5 points each)

(i)	This problem has 2 KKT solution cases	True False
(ii)	At the point (3, 2), constraint g_1 is active	True False
(iii)	At the point (3, 2), the Lagrange multiplier u_1 for constraint g_1 is zero	True False
(iv)	At the point (3, 2), constraint g_2 is active	True False
(v)	At the point (3, 2), the Lagrange multiplier u_2 for constraint g_2 is positive	True False
(vi)	At the point (5.5, 4.5), constraint g_1 is active	True False
(vii)	At the point (5.5, 4.5), constraint g_2 is active	True False
(viii)	At the point (5.5, 4.5), the Lagrange multiplier u_1 for constraint g_1 is 5	True False
(ix)	At the point (5.5, 4.5), the Lagrange multiplier u_2 for constraint g_2 is zero	True False
(x)	The point (5, 5) is determined to be a local minimum point. The Lagrange multiplier u_2 for constraint g_2 could be negative	True False

Note: Point (3, 2) implies r = 3 and t = 2

Karush-Kuhn-Tucker (KKT) Optimality Conditions

The Problem:

Minimize $f(\mathbf{x})$

Subject to

$h_i(\mathbf{x}) = 0; \quad i = 1 \text{ to } p$

$g_j(\mathbf{x}) \le 0; \quad j = 1 \text{ to } m$

KKT Optimality Conditions:

1. *Define the Lagrangian*

$$L = f + \sum_{i=1}^{p} v_i h_i + \sum_{j=1}^{m} u_j \left(g_j + s_j^2\right)$$

2. *The Gradient Conditions*

$$\frac{\partial L}{\partial x_k} = 0; \quad k = 1 \text{ to } n$$

$$\frac{\partial L}{\partial v_i} = 0 = h_i; \quad i = 1 \text{ to } p$$

$$\frac{\partial L}{\partial u_j} = 0 = \left(g_j + s_j^2\right); \quad j = 1 \text{ to } m$$

3. *The Switching Conditions*

$$\frac{\partial L}{\partial s_j} = 0 = 2u_j s_j; \quad j = 1 \text{ to } m$$

4. *The Feasibility Check for Inequality Constraints*

$$s_j^2 \ge 0; \ (\text{or } \ g_j \le 0; \quad j = 1 \text{ to } m)$$

5. *Non-negativity of Lagrange Multipliers for Inequalities*

$$u_j \ge 0; \quad j = 1 \text{ to } m$$

6. *Regularity Check*

   Gradients of active constraints should be linearly independent
   In such a case the LM for the constraints are unique.

SAMPLE EXAM 10: LINEAR PROGRAMMING

for

First Undergraduate Course (Time allowed = 40 minutes)

1. (10 points) An engineering design problem is formulated as a linear programming (LP) problem:

Maximize $z = x_1 - 2x_2$

subject to

$$10 \geq -3x_1 + 2x_2$$
$$2x_1 - x_2 \geq 6$$
$$x_2 \geq 0$$

ANSWER THE FOLLOWING QUESTIONS FOR THE LP.

i	Standard definition of the LP problem requires minimization of the function	YES/NO
ii	x_1 must be decomposed as $x_1 = x_1^+ - x_1^-$; *with* $x_1^+, x_1^- \geq 0$ to transcribe the problem to the standard form	YES/NO
iii	x_2 must be decomposed as $x_2 = x_2^+ - x_2^-$; *with* $x_2^+, x_2^- \geq 0$ to transcribe the problem to the standard form	YES/NO
iv	Right sides of the LP constraints in the standard form are free in sign	YES/NO
v	Constraint #1 requires introduction of a slack variable to transcribe it into the standard form	YES/NO
vi	The problem requires Two-Phase Simplex method for its solution	YES/NO
vii	Constraint #2 requires introduction of a surplus variable to transcribe it into the standard form	YES/NO
viii	Constraint #1 requires addition of an artificial variable	YES/NO
ix	Solution of the problem requires use of an artificial cost function in the Simplex procedure	YES/NO
x	In Phase II of the Simplex iterations, the artificial cost function determines the pivot element	YES/NO

2. (20 points) A linear design optimization problem is to be solved using the Two-Phase Simplex method. The initial tableau for Phase I of the problem is given below, where x_3 is a slack variable, x_4 is a surplus variable, and x_5 is an artificial variable.

	x_1	x_2	x_3	x_4	x_5	**b**
x3	-2	1	1	0	0	4
x5	1	1	0	-1	1	2
Cost	-4	-2	0	0	0	f - 0
Artificial cost	-1	-1	0	1	0	w - 2
x3	0	3	1	-2	2	8
x1	1	1	0	-1	1	2
Cost	0	2	0	-4	4	f+8
Art.	0	0	0	0	1	w-0

(i) *ANSWER THE FOLLOWING QUESTIONS.* (10 points)

i	a21 can be the pivot element for this tableau	YES/NO
ii	a12 can be the pivot element for this tableau	YES/NO
iii	x5 is a basic variable having value of 2	YES/NO
iv	x1 is a basic variable having a value of 4	YES/NO
v	x4 is a nonbasic variable	YES/NO
vi	Artificial cost function value is 2 for the initial tableau	YES/NO
vii	If a12 is selected as the pivot element, x3 will become a nonbasic variable	YES/NO
viii	a11 can be selected as a pivot element	YES/NO
ix	x3 is a basic variable in the initial tableau	YES/NO
x	If a22 is selected as the pivot element, it will make x1 a basic variable	YES/NO

(ii) Choosing a21 as the pivot element, complete the Simplex iteration in the blank tableau shown above. (10 Points)

3. (10 points) A linear design optimization problem is to be solved using the Simplex method. During the Simplex iteration, the following tableau is obtained, where x_3 is a slack variable, x_4 is a surplus variable, and x_5 is an artificial variable.

	x_1	x_2	x_3	x_4	x_5	**b**
x2	0	1	3	-2	2	8
x1	1	0	1	-1	1	2
Cost	0	0	2	-4	4	f+8
Art. Cost	0	0	0	0	1	w-0

ANSWER THE FOLLOWING QUESTIONS. (1 Point each)

(i) x1 is a nonbasic variable YES/NO

(ii) x5 is a nonbasic variable YES/NO

(iii) Does the tableau give an optimum solution for the linear design problem? YES/NO

(iv) Does the tableau indicate end of Phase I of the Simplex method? YES/NO

(v) Does the tableau indicate the problem to be unbounded? YES/NO

(vi) x2 is a basic variable YES/NO

(vii) Value of x2 in the tableau is 2 YES/NO

(viii) Value of the cost function at the current point is -8 YES/NO

(ix) x3 is the pivot column for the next iteration YES/NO

(x) In Phase II, x5 column can become a pivot column YES/NO

4. (10 points) A linear engineering design problem is formulated as follows:

Minimize $f = 10x_1 + 6x_2 + 2x_3 - 6x_4$

subject to

$$-x_1 + x_2 + x_3 - x_4 \ge 1$$

$$3x_1 + x_2 - x_3 - 3x_4 \ge 2$$

$$x_1, x_2, x_3, x_4 \ge 0$$

The final Simplex tableau for the problem is given as follows:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_2	0	1	0.5	-1.5	-0.75	-0.25	0.75	0.25	1.25
x_1	1	0	-0.5	-0.5	0.25	-0.25	-0.25	0.25	0.25
Cost	0	0	4	8	2	4	-2	-4	f - 10
Artificial	0	0	0	0	0	0	1	1	w - 0

x_5 and x_6 are surplus variables for constraints #1 and #2, respectively. x_7 and x_8 are artificial variables for constraints #1 and #2, respectively.

ANSWER THE FOLLOWING QUESTIONS (2 Points each).

i	Value of x_2 at the optimum point is 0.25	YES/NO
ii	Value of x_6 is zero at the optimum point	YES/NO
iii	Value of x_1 is 0.25 at the optimum point	YES/NO
iv	Value of x_5 is 2 at the optimum point	YES/NO
v	Value of the cost function at the optimum is 10	YES/NO
vi	Constraint #1 is inactive at the optimum point	YES/NO
vii	Constraint #2 is active at the optimum point	YES/NO
viii	Lagrange multiplier for constraint #1 is 2	YES/NO
ix	Lagrange multiplier for constraint #2 is -4	YES/NO
x	The problem has multiple optimum points	YES/NO

SAMPLE EXAM 11: NONLINEAR PROGRAMMING

for

First Undergraduate Course (Time allowed = 45 minutes)

INDICATE YOUR ANSWER TO EACH QUESTION IN THE FOLLOWING TABLE

Question #1: Part (i) (3 points)	A	B	C	
Question #1: Part (ii) (3 points)	A	B	C	
Question #1: Part (iii) (3 points)	A	B	C	
Question #2 (8 points)	A	B	C	
Question #3 (8 points)	A	B		
Question #4 (9 points)	A	B	C	D
Question #5 (8 points)	A	B	C	D
Question #6 (8 points)	A	B	C	D
Total: 50 points				

1. (9 points) An engineering design problem, formulated in terms of the design variables x and y, has following two constraints:

$$h_1 : \ y(x+4y)=10$$
$$g_1 : \ -2x^2 + \tfrac{1}{3}y \le 0$$

(iv) At the design point (2, 3), constraint g_1 is (*Mark your selection*):

(A) Active

(B) Violated

(C) Inactive

SOLUTION: $g_1 = -2 \times 2^2 + \frac{1}{3} \times 3 = -7 < 0;$ inactive

(v) At the design point (2, 1), constraint h_1 is (*Mark your selection*):

(A) Active

(B) Violated

(C) Inacive

SOLUTION: $h_1 := 1(2 + 4 \times 1) - 10 = -4 \neq 0;$ *violation*

(vi) Gradient of the constraint function h_1 at the point (2, 3) is given as (*Mark your selection*):

(A) $\nabla h_1 = \begin{bmatrix} 2 \\ 26 \end{bmatrix}$ (B) $\nabla h_1 = \begin{bmatrix} 3 \\ 26 \end{bmatrix}$ (C) $\nabla h_1 = \begin{bmatrix} 3 \\ 14 \end{bmatrix}$

SOLUTION: $\nabla h_1 = \begin{bmatrix} y \\ x + 8y \end{bmatrix} = \begin{bmatrix} 3 \\ 26 \end{bmatrix}$

2. **(8 points)** The cost function for an engineering design problem, formulated in terms of the design variables *x* and *y*, is given as:

$$\text{Minimize } f = x\left(x - 2y^2\right)$$

subject to some constraints. At the point (2, 2), the cost function for the quadratic programming (QP) subproblem is given as (*Mark your selection*):

$$(A)\ \bar{f} = -12 - 4d_1 - 8d_2 + 0.5(d_1^2 + d_2^2)$$
$$(B)\ \bar{f} = -12 - 4d_1 - 16d_2 - 0.5(d_1^2 + d_2^2)$$
$$(C)\ \bar{f} = -12 - 4d_1 - 16d_2 + 0.5(d_1^2 + d_2^2)$$

SOLUTION:

$$f(2{,}2) = 2(2 - 2 \times 2 \times 2) = -12$$
$$\nabla f = \begin{bmatrix} 2x - 2y^2 \\ -4xy \end{bmatrix} = \begin{bmatrix} 2 \times 2 - 2 \times 2 \times 2 \\ -4 \times 2 \times 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -16 \end{bmatrix}$$
$$\bar{f} = -12 - 4d_1 - 16d_2 + 0.5(d_1^2 + d_2^2)$$

3. (8 Points) An unconstrained cost function for a design problem is formulated as:

$$\text{Minimize } f = 3x_1^2 + x_1x_2 + 2x_2^2 - x_1 + x_2$$

At the design point (-1, 2), the direction **d** = (3, 5) is a direction of descent for the cost function (*Mark your selection*):

(A) Yes

(B) No

SOLUTION: The gradient of the cost function is given as

$$\mathbf{c} = \nabla f = \begin{bmatrix} 6x_1 + x_2 - 1 \\ x_1 + 4x_2 + 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$$
$$\mathbf{c} \cdot \mathbf{d} = -5 \times 3 + 8 \times 5 = 25 > 0 \ \ NOT\ A\ DESCENT\ DIRECTION$$

4. **(9 points)** An engineering design problem is formulated in terms of the design variables *x* and *y* as:

$$\text{Minimize } f = x\left(x - 2y^2\right)$$

subject to the constraints.

$$h_1:\ \frac{1}{10}\left(xy + 3y^2\right) - 1 = 0$$

$$g_1:\frac{1}{100}(-6x^2 + y) \le 0$$

$$g_2:\ -\frac{1}{4}y - 1 \le 0$$

During the line search along a search direction, a design point is obtained as (1.345, 2.160). The descent function value (Φ) at this point using the penalty parameter value of 100 is most nearly (*Mark your selection*):

(A) 58.28

(B) 79.76

(C) 143.26

(D) -164.74

SOLUTION: The problem functions at the point (1.345, 2.160) are evaluated as

$$f(1.345, 2.16) = 1.345(1.345 - 2 \times 2.16^2) = -1.345 \times 7.9962 = -10.741439$$

$$h_1(1.345, 2.16) = \tfrac{1}{10}(1.345 \times 2.16 + 3 \times 2.16^2) - 1$$

$$= \tfrac{1}{10}(2.9052 + 13.9968) - 1 = \frac{16.902}{10} - 1 = 0.6902$$

$$g_1(1.345, 2.16) = \tfrac{1}{100}(-6 \times 1.345^2 + 2.16) = \tfrac{1}{100}(-10.85415 + 2.16) = -0.0869415$$

$$g_2(1.345, 2.16) = -\tfrac{1}{4}(2.16) - 1 = -0.54 - 1 = -1.54$$

$$V(1.345, 2.16) = max(0.6902, -0.0869415, -1.54) = 0.6902$$

$$\Phi(1.345, 2.16) = -10.741439 + 100 \times 0.6902 = 58.28$$

5. (8 Points) An engineering design problem is formulated as the following unconstrained problem:

Minimize $f = 8x_1^2 + 2x_1x_2 + 5x_2^2 - 2x_1 + 3x_2$

Starting from a design point (-1, -1), one iteration of the steepest descent method has been completed yielding a new design point as (0.21, 0.45).

The value of the scale parameter β_1 for the conjugate gradient direction is most nearly (*Mark your selection*):

(A) -0.141 (B) 0.317 (C) 0.141 (D) 7.094

SOLUTION: Gradient at the starting design point (-1, -1) is

$$\mathbf{c}^{(0)} = \nabla f = \begin{bmatrix} 16x_1 + 2x_2 - 2 \\ 2x_1 + 10x_2 + 3 \end{bmatrix} = \begin{bmatrix} -20 \\ -9 \end{bmatrix}$$

Gradient at the new design point (0.21, 0.45) is

$$\mathbf{c}^{(1)} = \nabla f = \begin{bmatrix} 16x_1 + 2x_2 - 2 \\ 2x_1 + 10x_2 + 3 \end{bmatrix} = \begin{bmatrix} 16 \times 0.21 + 2 \times 0.45 - 2 \\ 2 \times 0.21 + 10 \times 0.45 + 3 \end{bmatrix} = \begin{bmatrix} 3.36 + 0.9 - 2 \\ 0.42 + 4.5 + 3 \end{bmatrix} = \begin{bmatrix} 2.26 \\ 7.92 \end{bmatrix}$$

The value of the parameter β_1 at the new design point (0.21, 0.45) is

$$\beta_1 = \left(\frac{\left\| \mathbf{c}^{(1)} \right\|}{\left\| \mathbf{c}^{(0)} \right\|} \right)^2$$

$$\beta_1 = \frac{2.26^2 + 7.92^2}{20^2 + 9^2} = \frac{5.1076 + 62.7264}{400 + 81} = \frac{67.8}{481} = 0.141$$

6. (8 Points) An engineering design problem is formulated as the following unconstrained problem:

Minimize $f = 7x_1^2 + 12x_2^2 - x_1$

Starting from a design point (2, 1), one iteration of the steepest descent method has been completed yielding a new design point as (0.58, -0.31).

The conjugate direction at the new design point if the parameter $\beta_1 = 0.1$ is used, is most nearly (*Mark your selection*):

$(A)\ \begin{bmatrix} -4.42 \\ 9.84 \end{bmatrix}$ $(B)\ \begin{bmatrix} 4.42 \\ -9.84 \end{bmatrix}$ $(C)\ \begin{bmatrix} -7.12 \\ 7.44 \end{bmatrix}$ $(D)\ \begin{bmatrix} -9.82 \\ 5.04 \end{bmatrix}$

SOLUTION: Gradient at the starting design point (2, 1) is

$$\mathbf{c} = \nabla f = \begin{bmatrix} 14x_1 - 1 \\ 24x_2 \end{bmatrix} = \begin{bmatrix} 27 \\ 24 \end{bmatrix}$$

Gradient at the new design point (0.58, -0.31) is

$$\mathbf{c}^{(1)} = \nabla f = \begin{bmatrix} 14x_1 - 1 \\ 24x_2 \end{bmatrix} = \begin{bmatrix} 7.12 \\ -7.44 \end{bmatrix}$$

Calculate the conjugate gradient direction at the new design point (0.58, -0.31) with $\beta_1 = 0.1$.

$$\mathbf{d}^{(1)} = -\mathbf{c}^{(1)} + \beta_0 \mathbf{d}^{(0)} = -\begin{bmatrix} 7.12 \\ -7.44 \end{bmatrix} + 0.1\begin{bmatrix} -27 \\ -24 \end{bmatrix} = \begin{bmatrix} -7.12 - 2.7 \\ 7.44 - 2.4 \end{bmatrix} = \begin{bmatrix} -9.82 \\ 5.04 \end{bmatrix}$$