CHAPTER

16
Global Optimization Concepts and Methods

16. 1

Calculate a global minimum point for the problem (See Branin and Hoo, 1972)
Minimize

$$f(\mathbf{x}) = \left(4 - 2.1x_1^2 + \frac{1}{3}x_1^4\right)x_1^2 + x_1x_2 + (-4 + 4x_2^2)x_2^2$$

subject to

$$-3 \le x_1 \le 3$$
$$-2 \le x_2 \le 2$$

Solution:
Six local minima:

$\boldsymbol{x}^{(0)} = (0, 0)$, $f^{(0)} = 0$; $\boldsymbol{x}^*$= (-0.0898, 0.712), ($f_G^* = -1.0316$)

$\boldsymbol{x}^{(0)} = (1, -1)$, $f^{(0)} = 1.233$; $\boldsymbol{x}^*$= (0.0898, -0.712,), ($f_G^* = -1.0316$)

$\boldsymbol{x}^{(0)} = (-3, -2)$, $f^{(0)} = 162.9$; $\boldsymbol{x}^*$= (1.703, -0.796), ($f^* = -0.215$)

$\boldsymbol{x}^{(0)} = (1000, 1000)$, $f^{(0)} = 3.33E + 17$; $\boldsymbol{x}^*$= (3, 2), ($f^* = 162.9$)

$\boldsymbol{x}^{(0)} = (-1000, 1000)$, $f^{(0)} = 3.33E + 17$; $\boldsymbol{x}^*$= (-3, 2), ($f^* = 150.9$)

$\boldsymbol{x}^{(0)} = (10000, -10)$, $f^{(0)} = 3.33E + 23$; $\boldsymbol{x}^*$= (3, -2), ($f^* = 150.9$)

There are six local minima and two global minima. The global minima are:

$\boldsymbol{x}^*$= (-0.898, 0.712), $\boldsymbol{x}^*$= (0.712, -0.898), ($f_G^* = -1.0316$)

If the “Multistart” option is turned “on” in the Excel Solver, all the starting points converge to one of the two global minimum points. The following Excel sheet gives a snapshot for one of the global solutions:

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.1.xlsx]Q16.1
Report Created: 3/23/2016 9:22:32 AM
Result: Solver has converged to the current solution. All Constraints are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.031 Seconds.
Iterations: 9 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$F$9	Objective function: Min Sum of LHS	0	-1.031628453

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	0	-0.089842217	Contin
$E$8	Variable value x2	0	0.712656332	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$8	Variable value x1	-0.089842217	$D$8<=3	Not Binding	3.089842217
$D$8	Variable value x1	-0.089842217	$D$8>=-3	Not Binding	2.910157783
$E$8	Variable value x2	0.712656332	$E$8<=2	Not Binding	1.287343668
$E$8	Variable value x2	0.712656332	$E$8>=-2	Not Binding	2.712656332

16.2 ____

Calculate a global minimum point for the problem (See Lucidi and Piccioni, 1989)

Minimize

$$f(\mathbf{x}) = \frac{\pi}{n}\left\{10\sin^2(\pi x_1) + \sum_{i=1}^{n-1}\left[(x_i - 1)^2(1 + 10\sin^2(\pi\, x_{i+1}))\right] + (x_n - 1)^2\right\}$$

subject to

$$-10 \le x_i \le 10; \quad i = 1 \text{ to } 5$$

Solution:

10^n local minima. Depending on the starting point, many local minima are found. When the "multistart" option is turned "on", all starting points converge to the following global minimum:

Global minimum: (for n=2), $\boldsymbol{x}^{(0)} = (10, 10, 1, 1, 1)$, $f^{(0)} = 0$; $\boldsymbol{x}^{*}$= (1, 1, 1, 1, 1), ($f_G^* = 0$).

The following Excel sheet gives a snapshot for the global solution:

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.2(n=2).xlsx]Q16.2(n=2)
Report Created: 3/24/2016 3:38:18 PM
Result: Solver converged in probability to a global solution.
Solver Engine

Engine: GRG Nonlinear
Solution Time: 0.14 Seconds.
Iterations: 0 Subproblems: 12

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.000001, Population Size 500, Random Seed 0, Derivatives Central, Multistart, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$I$9	Objective function: Min Sum of LHS	254.4690049	1.3599E-21

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	10	1	Contin
$E$8	Variable value x2	10	1	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$8	Variable value x3	1	$F$8<=10	Not Binding	9
$F$8	Variable value x3	1	$F$8>=-10	Not Binding	11
$G$8	Variable value x4	1	$G$8<=10	Not Binding	9
$G$8	Variable value x4	1	$G$8>=-10	Not Binding	11
$H$8	Variable value x5	1	$H$8<=10	Not Binding	9
$H$8	Variable value x5	1	$H$8>=-10	Not Binding	11
$D$8	Variable value x1	1	$D$8<=10	Not Binding	9
$D$8	Variable value x1	1	$D$8>=-10	Not Binding	11
$E$8	Variable value x2	1	$E$8<=10	Not Binding	9
$E$8	Variable value x2	1	$E$8>=-10	Not Binding	11

For n=3, there are many local minma. The global minimum is given below when the “multistart” option is used in Excel Solver.

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.2(n=3).xlsx]Q16.2(n=3)
Report Created: 3/24/2016 3:55:26 PM
Result: Solver converged in probability to a global solution.
Solver Engine

Engine: GRG Nonlinear
Solution Time: 0.203 Seconds.
Iterations: 0 Subproblems: 14

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.000001, Population Size 500, Random Seed 0, Derivatives Central, Multistart, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$I$9	Objective function: Min Sum of LHS	112.050138	5.6895E-18

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	-8	1	Contin
$E$8	Variable value x2	-4	1	Contin
$F$8	Variable value x3	2	1	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$G$8	Variable value x4	1	$G$8<=10	Not Binding	9
$G$8	Variable value x4	1	$G$8>=-10	Not Binding	11
$H$8	Variable value x5	1	$H$8<=10	Not Binding	9
$H$8	Variable value x5	1	$H$8>=-10	Not Binding	11
$D$8	Variable value x1	1	$D$8<=10	Not Binding	9
$D$8	Variable value x1	1	$D$8>=-10	Not Binding	11
$E$8	Variable value x2	1	$E$8<=10	Not Binding	9
$E$8	Variable value x2	1	$E$8>=-10	Not Binding	11
$F$8	Variable value x3	0.999999998	$F$8<=10	Not Binding	9
$F$8	Variable value x3	0.999999998	$F$8>=-10	Not Binding	11

16.3 *Calculate a global minimum point for the problem* (See Walster et al., 1984)

Minimize

$$f(\boldsymbol{x}) = \sum_{i=1}^{11}\left(a_i - \frac{x_1(b_i^2 + b_i x_2)}{b_i^2 + b_i x_3 + x_4}\right)^2$$

subject to

$-2 \le x_i \le 2;\quad i = 1 \text{ to } 4$

where the coefficients (a_i, b_i) (i=1 to 11) are given as follows: (0.1957, 4), (0.1947, 2), (0.1735, 1), (0.16, 0.5), (0.0844, 0.25), (0.0627, 1/6), (0.0456, 0.125), (0.0342, 0.1), (0.0323, 1/12), (0.0235, 1/14), (0.0246, 0.0625).

Solution:

Many local minima.

Global minimum: $\boldsymbol{x}^{(0)} = (2, 2, 2, 2)$, $f^{(0)} = 6.4072$; $\boldsymbol{x}^*$= (0.1928, 0.1908, 0.1231, 0.1358), ($f_G^* = 3.0749E+5$).

The following Excel sheet gives a snapshot for the global solution obtained by using the "Multistart" option in Excel Solver:

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.3.xlsx]Q16.3 (with + sign)
Report Created: 3/30/2016 4:02:45 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine

Engine: GRG Nonlinear
Solution Time: 0.093 Seconds.
Iterations: 48 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001
Convergence 0.00000000001, Population Size 1000, Random Seed 0, Derivatives Central, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$H$9	Objective function: Min Sum of LHS	6.407202424	0.00030749

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	2	0.19283363	Contin
$E$8	Variable value x2	2	0.19083444	Contin
$F$8	Variable value x3	2	0.12311725	Contin
$G$8	Variable value x4	2	0.13576531	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$8	Variable value x1	0.192833632	$D$8<=2	Not Binding	1.80716637
$D$8	Variable value x1	0.192833632	$D$8>=-2	Not Binding	2.19283363
$E$8	Variable value x2	0.190834439	$E$8<=2	Not Binding	1.80916556
$E$8	Variable value x2	0.190834439	$E$8>=-2	Not Binding	2.19083444
$F$8	Variable value x3	0.123117248	$F$8<=2	Not Binding	1.87688275
$F$8	Variable value x3	0.123117248	$F$8>=-2	Not Binding	2.12311725
$G$8	Variable value x4	0.135765308	$G$8<=2	Not Binding	1.86423469
$G$8	Variable value x4	0.135765308	$G$8>=-2	Not Binding	2.13576531

When the negative sign is used in the numerator of the cost function, same global solution is obtained except that the sign of the x_2 value is negative. The following spreadsheet gives a snapshot of the Excel Answer sheet for this case.

$$f(\boldsymbol{x}) = \sum_{i=1}^{11}\left(a_i - \frac{x_1(b_i^2 - b_i x_2)}{b_i^2 + b_i x_3 + x_4}\right)^2$$

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.3(with negative sign).xlsx]Q16.3(with neg sign)
Report Created: 3/30/2016 4:13:08 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: GRG Nonlinear
Solution Time: 0.109 Seconds.
Iterations: 46 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001
Convergence 0.0000000001, Population Size 1000, Random Seed 0, Derivatives Central, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$H$9	Objective function: Min Sum of LHS	1.394555876	0.00030749

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	2	0.19283168	Contin
$E$8	Variable value x2	2	-0.19086527	Contin
$F$8	Variable value x3	2	0.12311808	Contin
$G$8	Variable value x4	2	0.13578014	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$8	Variable value x1	0.192831678	$D$8<=2	Not Binding	1.80716832
$D$8	Variable value x1	0.192831678	$D$8>=-2	Not Binding	2.19283168
$E$8	Variable value x2	-0.19086527	$E$8<=2	Not Binding	2.19086527
$E$8	Variable value x2	-0.19086527	$E$8>=-2	Not Binding	1.80913473
$F$8	Variable value x3	0.123118083	$F$8<=2	Not Binding	1.87688192
$F$8	Variable value x3	0.123118083	$F$8>=-2	Not Binding	2.12311808
$G$8	Variable value x4	0.135780135	$G$8<=2	Not Binding	1.86421986
$G$8	Variable value x4	0.135780135	$G$8>=-2	Not Binding	2.13578014

16.4

Calculate a global minimum point for the problem (See Evtushenko, 1974)

Minimize

$$f(\mathbf{x}) = -\left[\sum_{i=1}^{6} \frac{1}{6} \ \sin 2\pi \left(x_i + \frac{i}{5}\right)\right]^2$$

subject to

$0 \le x_i \le 1; \quad i = 1 \text{ to } 6$

Solution:

Many local minima.

Two global minima are found:

(1) $\boldsymbol{x}^{(0)} = (1,1,1,1,1,1)$, $f^{(0)} = -0.025$; $\boldsymbol{x}^{*}$= (0.05, 0.85, 0.65, 0.45, 0.25, 0.05), $(f_G^* = -1)$.

(2) $\boldsymbol{x}^{(0)} = (0,0,0,0,0,0)$, $f^{(0)} = -0.025$; $\boldsymbol{x}^{*}$= (0.55, 0.35, 0.15, 0.95, 0.75, 0.55), $(f_G^* = -1)$.

The following Excel sheet shows a snapshot for one of the global solutions:

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.4.xlsx]Q16.4
Report Created: 3/25/2016 9:04:53 AM
Result: Solver converged in probability to a global solution.
Solver Engine

Engine: GRG Nonlinear
Solution Time: 7.114 Seconds.
Iterations: 0 Subproblems: 350

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 500, Random Seed 0, Derivatives Central, Multistart, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$J$9	Objective function: Min Sum of LHS	-0.025125236	-1

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	1	0.050000002	Contin
$E$8	Variable value x2	1	0.849999999	Contin
$F$8	Variable value x3	1	0.650000001	Contin
$G$8	Variable value x4	1	0.449999999	Contin
$H$8	Variable value x5	1	0.249999999	Contin
$I$8	Variable value x6	1	0.049999998	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$8	Variable value x1	0.050000002	$D$8<=1	Not Binding	0.949999998
$D$8	Variable value x1	0.050000002	$D$8>=0	Not Binding	0.050000002
$E$8	Variable value x2	0.849999999	$E$8<=1	Not Binding	0.150000001
$E$8	Variable value x2	0.849999999	$E$8>=0	Not Binding	0.849999999
$F$8	Variable value x3	0.650000001	$F$8<=1	Not Binding	0.349999999
$F$8	Variable value x3	0.650000001	$F$8>=0	Not Binding	0.650000001
$G$8	Variable value x4	0.449999999	$G$8<=1	Not Binding	0.550000001
$G$8	Variable value x4	0.449999999	$G$8>=0	Not Binding	0.449999999
$H$8	Variable value x5	0.249999999	$H$8<=1	Not Binding	0.750000001
$H$8	Variable value x5	0.249999999	$H$8>=0	Not Binding	0.249999999
$I$8	Variable value x6	0.049999998	$I$8<=1	Not Binding	0.950000002
$I$8	Variable value x6	0.049999998	$I$8>=0	Not Binding	0.049999998

16. 5

Minimize (Exercise 3.14)

$f(\mathbf{x}) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$

subject to

$$\frac{1}{6}x_1 + \frac{1}{2}x_2 - 1.0 \le 0$$

$$\frac{1}{2}x_1 + \frac{1}{5}x_2 - 1.0 \le 0$$

$x_1, \ x_2 \ge 0$

Solution:

Four local minima:

$\boldsymbol{x}^{(0)} = (0.5, 0.5)$, $f^{(0)} = 1.875$; $\boldsymbol{x}^{*}= (0, 0)$, $(f^* = 0)$

$\boldsymbol{x}^{(0)} = (1, 1)$, $f^{(0)} = 2$; $\boldsymbol{x}^{*}= (1.384, 1.538)$, $(f^* = -0.003)$

$\boldsymbol{x}^{(0)} = (0, 10)$, $f^{(0)} = -170$; $\boldsymbol{x}^{*}= (0, 2)$, $(f^* = -2)$

$\boldsymbol{x}^{(0)} = (10, 0)$, $f^{(0)} = -980$; $\boldsymbol{x}^{*}= (2, 0)$, $(f_G^* = -4)$

Global minimum: $\boldsymbol{x}^{*}= (2, 0)$, $(f_G^* = -4)$

The following Excel sheet gives a snapshot for the global solution:

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0 Seconds.

Iterations: 1 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$F$9	Objective function: Min Sum of LHS	-980	-4

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	10	2	Contin
$E$8	Variable value x2	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$10	Constraint 1 Sum of LHS	0.33334	$F$10<=1	Not Binding	0.66666
$F$11	Constraint 2 Sum of LHS	1	$F$11<=1	Binding	0
$D$8	Variable value x1	2	$D$8>=0	Not Binding	2
$E$8	Variable value x2	0	$E$8>=0	Binding	0

16.6 ____

Calculate a global minimum point for the problem (See Problem 25 in Hock and Schittkowski, 1981)

Minimize

$$f(\mathbf{x}) = \sum_{i=1}^{99} f_i^2(\mathbf{x})$$

$$f_i(\mathbf{x}) = -\frac{i}{100} + \exp\left(-\frac{1}{x_1}(u_i - x_2)^{x_3}\right)$$

$$u_i = 25 + \left[-50\ln(0.01i)\right]^{2/3}; \quad i = 1 \text{ to } 99$$

subject to

$$0.1 \le x_1 \le 100,\ \ 0.0 \le x_2 \le 25.6,\ \ 0.0 \le x_3 \le 5$$

Solution:

Many local minima.

Global minimum: $\boldsymbol{x}^{(0)} = (30, 15, 3)$, $f^{(0)} = 32.835$; $\boldsymbol{x}^{*}$= (50, 25, 1.5), ($f_G^* = 0$).

The following Excel sheet gives a snapshot for the global solution:

Microsoft Excel 15.0 Answer Report
Worksheet: [Q16.6.xlsx]Q1
Report Created: 3/25/2016 2:49:31 PM
Result: Solver converged in probability to a global solution.
Solver Engine

Engine: GRG Nonlinear
Solution Time: 68.921 Seconds.
Iterations: 0 Subproblems: 263

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Convergence 0.0001, Population Size 500, Random Seed 0, Derivatives Central, Multistart, Require Bounds
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$G$9	Objective function: Min Sum of LHS	32.835	1.30386E-16

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	30	49.99999708	Contin
$E$8	Variable value x2	15	25.00000012	Contin
$F$8	Variable value x3	3	1.499999984	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$8	Variable value x1	49.99999708	$D$8<=100	Not Binding	50.00000292
$D$8	Variable value x1	49.99999708	$D$8>=0.1	Not Binding	49.89999708
$E$8	Variable value x2	25.00000012	$E$8<=25.6	Not Binding	0.599999881
$E$8	Variable value x2	25.00000012	$E$8>=0	Not Binding	25.00000012
$F$8	Variable value x3	1.499999984	$F$8<=5	Not Binding	3.500000016
$F$8	Variable value x3	1.499999984	$F$8>=0	Not Binding	1.499999984

16.7

Calculate a global minimum point for the problem (See Problem 47 in Hock and Schittkowski, 1981)

Minimize

$$f(\boldsymbol{x}) = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^4 + (x_4 - x_5)^4$$

subject to

$$x_1 + x_2^2 + x_3^3 - 3 = 0$$
$$x_2 - x_3^2 + x_4 - 1 = 0$$
$$x_1 x_5 - 1 = 0$$

Solution:

Many local minima.

Global minimum: $\boldsymbol{x}^{(0)} = (2, \sqrt{2}, -1, 2 - \sqrt{2}, 0.5\ )$, $f^{(0)} = 12.498$; $\boldsymbol{x}^{*}$= (1, 1, 1, 1, 1), ($f_G^* = 0$).

The following Excel sheet gives a snapshot for the global solution:

Microsoft Excel 15.0 Answer Report

Worksheet: [Q16.7.xlsx]Q1

Report Created: 3/22/2016 4:37:54 PM

Result: Solver found a solution. All Constraints and optimality conditions are satisfied.

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.265 Seconds.

Iterations: 47 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.0000001

Convergence 0.000001, Population Size 100, Random Seed 0, Derivatives Central

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$I$9	Objective function: Min Sum of LHS	12.49806374	2.46644E-16

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	2	1.000000005	Contin
$E$8	Variable value x2	1.414	0.999999993	Contin
$F$8	Variable value x3	-1	1.000000002	Contin
$G$8	Variable value x4	0.586	1.000000012	Contin
$H$8	Variable value x5	0.5	0.999999995	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$I$10	Constraint 1 Sum of LHS	-3.47322E-09	$I$10=0	Binding	0
$I$11	Constraint 2 Sum of LHS	-4.53926E-11	$I$11=0	Binding	0
$I$12	Constraint 3 Sum of LHS	-3.10718E-10	$I$12=0	Binding	0

16.8

Calculate a global minimum point for the problem (See Problem 59 in Hock and Schittkowski, 1981)

Minimize

$$f(x) = -75.196 + b_1x_1 + b_2x_1^3 - b_3x_1^4 + b_4x_2 - b_5x_1x_2 + b_6x_2x_1^2 + b_7x_1^4x_2 - b_8x_2^2 + c_1x_2^3 - c_2x_2^4 + \frac{28.106}{x_2+1} + c_3x_1^2x_2^2 + c_4x_1^3x_2^2 - c_5x_1^3x_2^3 - c_6x_1x_2^2 + c_7x_1x_2^3 + 2.8673\exp\left(\frac{x_1x_2}{2000}\right) - c_8x_1^3x_2 - 0.12694x_1^2$$

subject to

$$x_1x_2 - 700 \geq 0$$

$$x_2 - x_1^2/125 \geq 0$$

$$(x_2 - 50)^2 - 5(x_1 - 55) \geq 0$$

$$0 \leq x_1 \leq 75, \quad 0 \leq x_2 \leq 65$$

where the parameters (b_i, c_i) (i=1 to 8) are given as (3.8112E+00, 3.4604E−03), (2.0567E−03, 1.3514E−05), (1.0345E−05, 5.2375E−06), (6.8306E+00, 6.3000E−08), (3.0234E−02, 7.0000E−10), (1.2814E−03, 3.4050E−04), (2.2660E−07, 1.6638E−06), (2.5645E−01, 3.5256E−05).

Solution:

Six local minima:

$\boldsymbol{x}^{(0)} = (10, 10)$, $f^{(0)} = 1.39$; $\boldsymbol{x}^*$= (46.387, 52.217), ($f^* = -6.74$)

$\boldsymbol{x}^{(0)} = (100000000, -100)$, $f^{(0)} = -3.3E+27$; $\boldsymbol{x}^*$= (75, 0), ($f^* = 67.9$)

$\boldsymbol{x}^{(0)} = (100, 10000)$, $f^{(0)} = 4.02E+217$; $\boldsymbol{x}^*$= (75, 65), ($f^* = 42.2$)

$\boldsymbol{x}^{(0)} = (0.000001, 100000000)$, $f^{(0)} = -1.35E+31$; $\boldsymbol{x}^*$= (10.845, 64.544), ($f^* = 4.62$)

$\boldsymbol{x}^{(0)} = (10000000, 0.00000001)$, $f^{(0)} = -1.03E+23$; $\boldsymbol{x}^*$= (75, 12.8), ($f^* = 65.2$)

$\boldsymbol{x}^{(0)} = (90, 10)$, $f^{(0)} = 86.9$; $\boldsymbol{x}^*$= (13.549, 51.66), ($f_G^* = -7.8$)

Global minimum: $\boldsymbol{x}^*$= (13.549, 51.66), $f_G^* = -7.80$

The following Excel sheet gives a snapshot for the global solution:

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.046 Seconds.

Iterations: 12 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$F$9	Objective function: Min Sum of LHS	8.69E+01	-7.80E+00

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	90	13.54999288	Contin
$E$8	Variable value x2	10	51.66057566	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$12	Constraint 3 Sum of LHS	210.0075471	$F$12>=$G$12	Not Binding	210.0075471
$F$11	Constraint 2 Sum of LHS	50.19175721	$F$11>=$G$11	Not Binding	50.19175721
$F$10	Constraint 1 Sum of LHS	700.0004322	$F$10>=$G$10	Not Binding	0.000432214
$E$8	Variable value x2	51.66057566	$E$8>=0	Not Binding	51.66057566
$D$8	Variable value x1	13.54999288	$D$8>=0	Not Binding	13.54999288
$D$8	Variable value x1	13.54999288	$D$8<=75	Not Binding	61.45000712
$E$8	Variable value x2	51.66057566	$E$8<=65	Not Binding	13.33942434

16.9

Calculate a global minimum point for the problem (See Problem 71 in Hock and Schittkowski, 1981)

Minimize

$f(\mathbf{x}) = x_1x_4(x_1 + x_2 + x_3) + x_3$

subject to

$x_1x_2x_3x_4 - 25 \geq 0$

$x_1^2 + x_2^2 + x_3^2 + x_4^2 - 40 = 0$

$1 \leq x_i \leq 5; \quad i = 1 \text{ to } 4$

Solution:

Four local minima:

$\boldsymbol{x}^{(0)} = (4000, 555, 55, 555)$, $f^{(0)} = -3$; $\boldsymbol{x}^*$= (1, 4.74, 3.82, 1.37), ($f^* = 17.014$)
$\boldsymbol{x}^{(0)} = (-1, -1, -1, -10)$, $f^{(0)} = -31$; $\boldsymbol{x}^*$= (1, 5, 1.44, 3.44), ($f^* = 27.146$)
$\boldsymbol{x}^{(0)} = (-1, 6, -1, 0.5)$, $f^{(0)} = 10234200055$; $\boldsymbol{x}^*$= (1, 5, 4.61, 1.18), ($f^* = 17.221$)
$\boldsymbol{x}^{(0)} = (100, 100, 10, 100)$, $f^{(0)} = 2100010$; $\boldsymbol{x}^*$= (1, 4.74, 3.82, 1.37), ($f_G^* = 16.994$)
Global minimum: (1, 4.74, 3.82, 1.37), ($f_G^* = 16.994$).
The following Excel sheet gives a snapshot for the global solution:

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.047 Seconds.

Iterations: 11 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$H$9	Objective function: Min Sum of LHS	2100010	16.9948638

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	100	1	Contin
$E$8	Variable value x2	100	4.74282494	Contin
$F$8	Variable value x3	10	3.82645698	Contin
$G$8	Variable value x4	100	1.37611233	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$10	Constraint 1 Sum of LHS	24.97398328	$H$10>=I1C	Binding	0
$H$11	Constraint 2 Sum of LHS	40.0298466	$H$11=$I$11	Binding	0
$D$8	Variable value x1	1	$D$8<=5	Not Binding	4
$D$8	Variable value x1	1	$D$8>=1	Binding	0
$E$8	Variable value x2	4.742824943	$E$8<=5	Not Binding	0.25717506
$E$8	Variable value x2	4.742824943	$E$8>=1	Not Binding	3.74282494
$F$8	Variable value x3	3.826456979	$F$8<=5	Not Binding	1.17354302
$F$8	Variable value x3	3.826456979	$F$8>=1	Not Binding	2.82645698
$G$8	Variable value x4	1.376112331	$G$8<=5	Not Binding	3.62388767
$G$8	Variable value x4	1.376112331	$G$8>=1	Not Binding	0.37611233

16.10

Calculate a global minimum point for the problem (See Problem 118 in Hock and Schittkowski, 1981)

Minimize

$$f(\boldsymbol{x}) = \sum_{k=0}^{4}(2.3x_{3k+1} + (1.0E-4)x_{3k+1}^2 + 1.7x_{3k+2} + (1.0E-4)x_{3k+2}^2 + 2.2x_{3k+3} + (1.5E-4)x_{3k+3}^2)$$

subject to

$$0 \le x_{3j+1} - x_{3j-2} + 7 \le 13 \quad ; \; j = 1 \text{ to } 4$$
$$0 \le x_{3j+2} - x_{3j-1} + 7 \le 14 \quad ; \; j = 1 \text{ to } 4$$
$$0 \le x_{3j+3} - x_{3j} + 7 \le 13 \quad ; \; j = 1 \text{ to } 4$$
$$x_1 + x_2 + x_3 - 60 \ge 0$$
$$x_4 + x_5 + x_6 - 50 \ge 0$$
$$x_7 + x_8 + x_9 - 70 \ge 0$$
$$x_{10} + x_{11} + x_{12} - 85 \ge 0$$
$$x_{13} + x_{14} + x_{15} - 100 \ge 0$$

and the bounds are (k=1 to 4):

$8.0 \le x_1 \le 21.0$

$43.0 \le x_2 \le 57.0$

$3.0 \le x_3 \le 16.0$

$0.0 \le x_{3k+1} \le 90.0$

$0.0 \le x_{3k+2} \le 120.0$

$0.0 \le x_{3k+3} \le 60.0$

Solution:

Two local minima:

$\boldsymbol{x}^{(0)} = (20, 55, 15, 20, 60, 20, 20, 60, 20, 20, 60, 20, 20, 60, 20)$, $f^{(0)} = 942.625$;

$\boldsymbol{x}^*$= (8, 49, 3, 1, 56, 0, 1, 63, 6, 3, 70, 12, 5, 77, 18), ($f_G^* = 664.794$)

$\boldsymbol{x}^{(0)} = (-10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000, -10000)$, $f^{(0)} = -160000$;

$\boldsymbol{x}^*$= (8, 49, 3, 1, 56, 0, 1, 63, 6, 3, 70, 12, 9, 73, 18), ($f^* = 667.14$).

Global minimum: $\boldsymbol{x}^*$= (8, 49, 3, 1, 56, 0, 1, 63, 6, 3, 70, 12, 5, 77, 18), ($f_G^* = 664.794$)

The following shows Excel sheet snapshot for the global solution:

Solver Engine

Engine: GRG Nonlinear

Solution Time: 0.125 Seconds.

Iterations: 19 Subproblems: 0

Solver Options

Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling

Convergence 0.0001, Population Size 100, Random Seed 0, Derivatives Forward, Require Bounds

Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$S$9	Objective function: Min RHS Limit	942.625	664.794803

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$8	Variable value x1	20	8	Contin
$E$8	Variable value x2	55	49	Contin
$F$8	Variable value x3	15	3	Contin
$G$8	Variable value x4	20	1	Contin
$H$8	Variable value x5	60	56	Contin
$I$8	Variable value x6	20	0	Contin
$J$8	Variable value x7	20	1.0000006	Contin
$K$8	Variable value x8	60	63	Contin
$L$8	Variable value x9	20	6	Contin
$M$8	Variable value x10	20	3.00000058	Contin
$N$8	Variable value x11	60	70	Contin
$O$8	Variable value x12	20	12	Contin
$P$8	Variable value x13	20	5	Contin
$Q$8	Variable value x14	60	77	Contin
$R$8	Variable value x15	20	18	Contin