APPENDIX A
Vector and Matrix Algebra

A.1

Evaluate the following determinant:

Solution:

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 1 \\ 1 & 2 & 1 \\ 0 & -5 & 2 \end{vmatrix} = (-1)^{1+2}(1)\begin{vmatrix} -3 & 1 \\ -5 & 2 \end{vmatrix} = (-1)[(-3\times 2) - (-5\times 1)] = 1$$

A.2

Evaluate the following determinant:

Solution:

$$\begin{vmatrix} 0 & 2 & 3 & 2 \\ 0 & 4 & 5 & 4 \\ 1 & -2 & -2 & 1 \\ 3 & -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \\ 1 & -3 & -2 & 1 \\ 3 & -2 & 2 & 1 \end{vmatrix} = (-1)^{2+3}(-1)\begin{vmatrix} 0 & 0 & 2 \\ 1 & -3 & 1 \\ 3 & -2 & 1 \end{vmatrix} = (-1)^{1+3}(2)\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} = 2(-2+9) = 14$$

A.3

Evaluate the following determinant:

Solution:

$$\begin{vmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 5 & 3 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & -3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 2 \text{ x } 1 \text{ x } 5 \text{ x } (-2) = -20$$

A.4

Calculate values of the scalar λ for which the determinant vanishes:

Solution:

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 3 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)[(2-\lambda)(3-\lambda) - 1] = 0;$$

or $(2-\lambda)(\lambda^2 - 5\lambda + 5) = 0$; ; $\lambda_1 = (5-\sqrt{5})/2$, $\lambda_2 = 2$, $\lambda_3 = (5+\sqrt{5})/2$

A.5

Calculate values of the scalar λ for which the determinant vanishes:

Solution:

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)[(2-\lambda)^2 - 2] = 0;$$

or $(2-\lambda)(\lambda^2 - 4\lambda + 2) = 0;\ \lambda_1 = 2-\sqrt{2},\ \lambda_2 = 2,\ \lambda_3 = 2+\sqrt{2}$

A.6

Determine rank of the following matrix:

Solution:

$$\begin{bmatrix} 3 & 0 & 1 & 3 \\ 2 & 0 & 3 & 2 \\ 0 & 2 & -8 & 1 \\ -2 & -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 3 & 2 \\ -2 & -1 & 2 & -1 \\ 3 & 0 & 1 & 3 \\ 0 & 2 & -8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/2 & 1 \\ 0 & -1 & 5 & 1 \\ 0 & 0 & -7/2 & 0 \\ 0 & 2 & -8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & -7/2 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ \text{Rank is 4.}$$

A.7

Determine rank of the following matrix:

Solution:

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 4 \\ 1 & 6 & 3 & 0 & 3 \\ 2 & 2 & 3 & 3 & 2 \\ 1 & 3 & 2 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & -2 & -1 \\ 0 & -2 & -1 & -1 & -6 \\ 0 & 1 & 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & -2 & -1 \\ 0 & -1 & -2 & -1 & -6 \\ 0 & 0 & 1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -3 & -7 \\ 0 & 0 & 1 & 3 & -3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 9/2 & -13/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix};\ \text{Rank is 4.}$$

A.8

Determine rank of the following matrix:

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 3 & 2 & 3 & 0 \\ 2 & 3 & 1 & 4 \\ 2 & 0 & 6 & 0 \\ 1 & 2 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -4 & -6 & -12 \\ 0 & -1 & -5 & -4 \\ 0 & -4 & 0 & -8 \\ 0 & 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -5 & -4 \\ 0 & -4 & -6 & -12 \\ 0 & -4 & 0 & -8 \\ 0 & 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 18 & 4 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

Rank is 4.

A.9

Obtain solution of the following equations using the Gaussian elimination procedure:

Solution:

$2x_1+2x_2+x_3=5$

$x_1-2x_2+2x_3=1$

$x_2+2x_3=3$

$$\mathbf{A} = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -2 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 5/2 \\ 0 & -3 & 3/2 & -3/2 \\ 0 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 5/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 5/2 & 5/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 5/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right];$$

$$\begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \\ 1 \end{bmatrix};\ x_3 = 1;\ \ x_2 = (1/2) + (1/2)x_3 = 1;\ \ x_1 = (5/2) - x_2 - (1/2)x_3 = 1$$

A.10

Obtain solution of the following equations using the Gaussian elimination procedure:

Solution:

$x_2-x_3=0$

$x_1+x_2+x_3=3$

$x_1-3x_2=-2$

$$\mathbf{A} = \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 3 \\ 1 & -3 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & -3 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & -1 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & -5 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right];\text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix};\quad x_3 = 1;\quad x_2 = x_3 = 1;\quad x_1 = 3 - x_1 - x_2 = 1$$

A.11

Obtain solution of the following equations using the Gaussian elimination procedure:

$2x_1+x_2+x_3=7$

$4x_2-5x_3=-7$

$x_1-2x_2+4x_3=9$

Solution:

$$\mathbf{A}=\left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 4 & -5 & -7 \\ 1 & -2 & 4 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 4 & -5 & -7 \\ 0 & -5/2 & 7/2 & 11/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -5/4 & -7/4 \\ 0 & 0 & 3/8 & 9/8 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 7/2 \\ 0 & 1 & -5/4 & -7/4 \\ 0 & 0 & 1 & 3 \end{array}\right]; \quad \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -5/4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 7/2 \\ -7/4 \\ 3 \end{bmatrix};$$

$x_3 = 3;\quad x_2 = (-7/4) + (5/4)x_3 = 2;\quad x_1 = (7/2) - (1/2)x_2 - (1/2)x_3 = 1$

A.12

Obtain solution of the following equations using the Gaussian elimination procedure:

$2x_1+x_2-3x_3+x_4=1$

$x_1+2x_2+5x_3-x_4=7$

$-x_1+x_2+x_3+4x_4=5$

$2x_1-3x_2+2x_3-5x_4=-4$

Solution:

$$\mathbf{A}=\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 2 & 1 & -3 & 1 & 1 \\ 1 & 2 & 5 & -1 & 7 \\ -1 & 1 & 1 & 4 & 5 \\ 2 & -3 & 2 & -5 & -4 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 5 & -1 & 7 \\ -1 & 1 & 1 & 4 & 5 \\ 2 & 1 & -3 & 1 & 1 \\ 2 & -3 & 2 & -5 & -4 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 5 & -1 & 7 \\ 0 & 3 & 6 & 3 & 12 \\ 0 & -3 & -13 & 3 & -13 \\ 0 & -7 & -8 & -3 & -18 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 5 & -1 & 7 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & -7 & 6 & -1 \\ 0 & 0 & 6 & 4 & 10 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 5 & -1 & 7 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & -6/7 & 1/7 \\ 0 & 0 & 0 & 64/7 & 64/7 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 5 & -1 & 7 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & -6/7 & 1/7 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array};$$

$$\begin{bmatrix} 1 & 2 & 5 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -6/7 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 7 \\ 4 \\ 1/7 \\ 1 \end{bmatrix}$$

$x_4 = 1;\quad x_3 = (1/7) + (6/7)x_4 = 1;\quad x_2 = 4 - 2x_3 - x_4 = 1;\quad x_1 = 7 - 2x_2 - 5x_3 + x_4 = 1$

A.13

Obtain solution of the following equations using the Gaussian elimination procedure:

$x_1+x_2+x_3=8$

$2x_1-x_2-x_3=-3$

$x_1+2x_2-x_3=2$

Solution:

$$A=\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\ \left[\begin{array}{ccc|c} 3 & 1 & 1 & 8 \\ 2 & -1 & -1 & -3 \\ 1 & 2 & -1 & 2 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccc} x_3 & x_2 & x_1 & b \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 3 & 8 \\ -1 & -1 & 2 & -3 \\ -1 & 2 & 1 & 2 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccc} x_3 & x_2 & x_1 & b \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 3 & 8 \\ 0 & 0 & 5 & 5 \\ 0 & 3 & 4 & 10 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccc} x_3 & x_2 & x_1 & b \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 3 & 8 \\ 0 & 3 & 4 & 10 \\ 0 & 0 & 5 & 5 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccc} x_3 & x_2 & x_1 & b \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 3 & 8 \\ 0 & 1 & 4/3 & 10/3 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}; \quad \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 10/3 \\ 1 \end{bmatrix}$$

$x_1 = 1$; $x_2 = (10/3) - (4/3)x_1 = 2$; $x_3 = 8 - x_2 - 3x_1 = 3$

A.14

Obtain solution of the following equations using the Gaussian elimination procedure:

$x_1+x_2-x_3=2$

$2x_1-x_2+x_3=4$

$-x_1+2x_2+3x_3=3$

Solution:

$$\mathbf{A} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ -1 & 2 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right];$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}; \quad x_3 = 1; \quad x_2 = x_3 = 1; \quad x_1 = 2 - x_2 + x_3 = 2$$

A.15

Obtain solution of the following equations using the Gaussian elimination procedure:

$-x_1+x_2-x_3=-2$

$-2x_1+x_2+2x_3=6$

$x_1+x_2+x_3=6$

Solution:

$$\mathbf{A}=\left[\begin{array}{ccc|c}-1&1&-1&-2\\-2&1&2&6\\1&1&1&6\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-1&1&2\\0&-1&4&10\\0&2&0&4\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-1&1&2\\0&1&-4&-10\\0&0&8&24\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-1&1&2\\0&1&-4&-10\\0&0&1&3\end{array}\right];$$

$$\begin{bmatrix}1&-1&1\\0&1&-4\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-10\\3\end{bmatrix};\quad x_3=3;\quad x_2=-10+4x_3=2;\quad x_1=2+x_2-x_3=1$$

A.16

Obtain solution of the following equations using the Gaussian elimination procedure:

$-x_1+2x_2+3x_3=4$

$2x_1-x_2-2x_3=-1$

$x_1-3x_2+4x_3=2$

Solution:

$$\mathbf{A}=\left[\begin{array}{ccc|c}-1&2&3&4\\2&-1&-2&-1\\1&-3&4&2\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-2&-3&-4\\0&3&4&7\\0&-1&7&6\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-2&-3&-4\\0&-1&7&6\\0&3&4&7\end{array}\right]\sim\left[\begin{array}{ccc|c}1&-2&-3&-4\\0&1&-7&-6\\0&0&25&25\end{array}\right]\sim$$

$$\left[\begin{array}{ccc|c}1&-2&-3&-4\\0&1&-7&-6\\0&0&1&1\end{array}\right];\begin{bmatrix}1&-2&-3\\0&1&-7\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-4\\-6\\1\end{bmatrix};$$

$x_3= 1$; $x_2= -6 +7x_3= 1$; $x_1= -4+2x_2+ 3x_3= 1$

A.17

Obtain solution of the following equations using the Gaussian elimination procedure:

$x_1+x_2+x_3+x_4=2$

$2x_1+x_2-x_3+x_4=2$

$-x_1+2x_2+3x_3+x_4=1$

$3x_1+2x_2-2x_3-x_4=8$

Solution:

$$\mathbf{A} = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 1 & 2 \\ -1 & 2 & 3 & 1 & 1 \\ 3 & 2 & -2 & -1 & 8 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 & -2 \\ 0 & 3 & 4 & 2 & 3 \\ 0 & -1 & -5 & -4 & 2 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & -5 & -1 & -3 \\ 0 & 0 & -2 & -3 & 4 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1/5 & 3/5 \\ 0 & 0 & 0 & -13/5 & 26/5 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1/5 & 3/5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right];$$

$x_4 = -2$; $x_3 = (3/5) - (1/5)x_4 = 1$; $x_2 = 2 - 3x_3 - x_4 = 1$; $x_1 = 2 - x_2 - x_3 - x_4 = 2$

A.18

Obtain solution of the following equations using the Gaussian elimination procedure:

$x_1+x_2+x_3+x_4=-1$

$2x_1-x_2+x_3-2x_4=8$

$3x_1+2x_2+2x_3+2x_4=4$

$-x_1-x_2+2x_3-x_4=-2$

Solution:

$$\mathbf{A} = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 2 & -1 & 1 & -2 & 8 \\ 3 & 2 & 2 & 2 & 4 \\ -1 & -1 & 2 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 0 & -3 & -1 & -4 & 10 \\ 0 & -1 & -1 & -1 & 7 \\ 0 & 0 & 3 & 0 & -3 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 0 & -1 & -1 & -1 & 7 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & -3 & -1 & -4 & 10 \end{array}\right] \sim$$

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -7 \\ 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 2 & -1 & -11 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -9 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & -7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \\ -1 \\ 9 \end{bmatrix}$$

$x_4 = 9$; $x_3 = -1$; $x_2 = -7 - x_3 - x_4 = -15$; $x_1 = -1 - x_2 - x_3 - x_4 = 6$

A.19

Check if the following system of equations is consistent. If it is, calculate its general solution:

$3x_1+x_2+5x_3+2x_4=2$

$2x_1-2x_2+4x_3=2$

$2x_1+2x_2+3x_3+2x_4=1$

$x_1+3x_2+x_3+2x_4=0$

Solution:

$$A = \begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array}\\ \left[\begin{array}{cccc|c} 3 & 1 & 5 & 2 & 2\\ 2 & -2 & 4 & 0 & 2\\ 2 & 2 & 3 & 2 & 1\\ 1 & 3 & 1 & 2 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array}\\ \left[\begin{array}{cccc|c} 1 & 1/3 & 5/3 & 2/3 & 2/3\\ 0 & -8/3 & 2/3 & -4/3 & 2/3\\ 0 & 4/3 & -1/3 & 2/3 & -1/3\\ 0 & 8/3 & -2/3 & 4/3 & -2/3 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array}\\ \left[\begin{array}{cc|cc|c} 1 & 1/3 & 5/3 & 2/3 & 2/3\\ 0 & 1 & -1/4 & 1/2 & -1/4\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & b \end{array}\\ \left[\begin{array}{cc|cc|c} 1 & 0 & 7/4 & 1/2 & 3/4\\ 0 & 1 & -1/4 & 1/2 & -1/4\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\end{array}$$

Since $\mathbf{P}_{(m-r\times 1)} = \mathbf{0}$, the system of equations is consistent.

General solution is $x_1 = (3/4) - (7/4)x_3 - 0.5x_4$; $x_2 = (-1/4) + x_3/4 - x_4/2$

A.20

Check if the following system of equations is consistent. If it is, calculate its general solution:

$x_1+x_2+x_3+x_4=10$

$-x_1+x_2-x_3+x_4=2$

$2x_1-3x_2+2x_3-2x_4=-6$

Solution:

$$\mathbf{A} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10\\ -1 & 1 & -1 & 1 & 2\\ 2 & -3 & 2 & -2 & -6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10\\ 0 & 2 & 0 & 2 & 12\\ 0 & -5 & 0 & -4 & -26 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10\\ 0 & 1 & 0 & 1 & 6\\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10\\ 0 & 1 & 1 & 0 & 6\\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 4\\ 0 & 1 & 0 & 0 & 2\\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$$; The system of equations is consistent.

General solution: $x_1 = 4 - x_3$; $x_2 = 2$; $x_4 = 4$.

A.21

Check if the following system of equations is consistent. If it is, calculate its general solution:

$x_2+2x_3+x_4=-2$

$x_1-2x_2-x_3-x_4=1$

$x_1-2x_2-3x_3+x_4=1$

Solution:

$$A=\left[\begin{array}{cccc|c}0&1&2&1&-2\\1&-2&-1&-1&1\\1&-2&-3&1&1\end{array}\right]\sim\left[\begin{array}{cccc|c}1&-2&-1&-1&1\\0&1&2&1&-2\\1&-2&-3&1&1\end{array}\right]\sim\left[\begin{array}{cccc|c}1&-2&-1&-1&1\\0&1&2&1&-2\\0&0&-2&2&0\end{array}\right]\sim$$

$$\left[\begin{array}{cccc|c}1&-2&-1&-1&1\\0&1&2&1&-2\\0&0&1&-1&0\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&0&4&-3\\0&1&0&3&-2\\0&0&1&-1&0\end{array}\right];$$

The system of equations is consistent.

General solution: $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-3\\-2\\0\end{bmatrix}-\begin{bmatrix}4\\3\\-1\end{bmatrix}x_4$

A.22

Check if the following system of equations is consistent. If it is, calculate its general solution:

$x_1+x_2+x_3+x_4=0$

$2x_1+x_2-2x_3-x_4=6$

$3x_1+2x_2+x_3+2x_4=2$

Solution:

$$\mathbf{A}=\left[\begin{array}{cccc|c}1&1&1&1&0\\2&1&-2&-1&6\\3&2&1&2&2\end{array}\right]\sim\left[\begin{array}{cccc|c}1&1&1&1&0\\0&-1&-4&-3&6\\0&-1&-2&-1&2\end{array}\right]\sim\left[\begin{array}{cccc|c}1&1&1&1&0\\0&1&4&3&-6\\0&0&2&2&-4\end{array}\right]\sim\left[\begin{array}{cccc|c}1&1&1&1&0\\0&1&4&3&-6\\0&0&1&1&-2\end{array}\right]\sim$$

$$\left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&-1&2\\0&0&1&1&-2\end{array}\right];$$ The system of equations is consistent.

General solution: $x_1=-x_4$; $x_2=2+x_4$; $x_3=-2-x_4$

A.23

Check if the following system of equations is consistent. If it is, calculate its general solution:

$x_1+x_2+x_3+3x_4-x_5=5$

$2x_1-x_2+x_3-x_4+3x_5=4$

$-x_1+2x_2-x_3+3x_4-2x_5=1$

Solution:

Gauss-Jordan Elimination in Excel-Exercise A.23						
x_1	x_2	x_3	x_4	x_5	b	Remarks
1	1	1	3	-1	5	Pivot row; x1 is pivot column
2	-1	1	-1	3	4	
-1	2	-1	3	-2	1	
First iteration						
1	1	1	3	-1	5	Divide Row 3 with the pivot element
0	-3	-1	-7	5	-6	Multiply row 7 by a12 and subtract it from row 4
0	3	0	6	-3	6	Multiply row 7 by a13 and subtract it from row 5
Second iteration						
1	0	0.667	0.67	0.667	3	Multiply row 12 by a12 and subtract from row 7
0	1	0.333	2.33	-1.67	2	Divide row 8 (pivot row) by a22
0	0	-1	-1	2	0	Multiply row 12 by a32 and subtract from row 9
Third iteration						
1	0	0	0	2	3	Multiply row 17 by a13 and subtract from row 13
0	1	0	2	-1	2	
0	0	1	1	-2	0	Divide row 13 (pivot row) by a33

The system is consistent as the rank of the coefficient matrix and the augmented matrix is 3. In the third tableau the system of equations has been reduced to the canonical form as

$$x_1 + 2x_5 = 3$$
$$x_2 + 2x_4 - x_5 = 2$$
$$x_3 + x_4 - 2x_5 = 0$$

This gives the general solution as (x_4 and x_1 as independent variables, and x_1, x_2, x_3 as dependent variables

$$x_1 = 3 - 2x_5$$
$$x_2 = 2 - 2x_4 + x_5$$
$$x_3 = -x_4 + 2x_5$$

The basic solution: $x_1 = 3$, $x_2 = 2$, $x_3 = 0$; the nonbasic variables $x_4 = 0, x_5 = 0$.

A.27

Check if the following system of equations is consistent. If it is, calculate its general solution:

$x_2+2x_3+x_4+3x_5+2x_6=9$

$-x_1+5x_2+2x_3+x_4+2x_5+x_7=10$

$5x_1-3x_2+8x_3+6x_4+3x_5-2x_8=17$

$2x_1-x_2+x_4+5x_5-2x_8=5$

Solution:

$$A=\begin{array}{c}\begin{array}{cccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & b\end{array}\\ \left[\begin{array}{cccccccc|c} 0 & 1 & 2 & 1 & 3 & 2 & 0 & 0 & 9\\ -1 & 5 & 2 & 1 & 2 & 0 & 1 & 0 & 10\\ 5 & -3 & 8 & 6 & 3 & 0 & 0 & -2 & 17\\ 2 & -1 & 0 & 1 & 5 & 0 & 0 & -2 & 5\end{array}\right]\end{array} \rightarrow \begin{array}{c}\begin{array}{cccccccc|c} x_6 & x_7 & x_8 & x_1 & x_2 & x_3 & x_4 & x_5 & b\end{array}\\ \left[\begin{array}{cccccccc|c} 2 & 0 & 0 & 0 & 1 & 2 & 1 & 3 & 9\\ 0 & 1 & 0 & -1 & 5 & 2 & 1 & 2 & 10\\ 0 & 0 & -2 & 5 & -3 & 8 & 6 & 3 & 17\\ 0 & 0 & -2 & 2 & -1 & 0 & 1 & 5 & 5\end{array}\right]\end{array} \rightarrow$$

$$\begin{array}{c}\begin{array}{cccccccc|c} x_6 & x_7 & x_8 & x_1 & x_2 & x_3 & x_4 & x_5 & b\end{array}\\ \left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 1/2 & 1 & 1/2 & 3/2 & 9\\ 0 & 1 & 0 & -1 & 5 & 2 & 1 & 2 & 10\\ 0 & 0 & 1 & -5/2 & 3/2 & -4 & -3 & -3/2 & 17\\ 0 & 0 & 0 & -3 & 2 & -8 & -5 & 2 & 5\end{array}\right]\end{array} \rightarrow$$

$$\begin{array}{c}\begin{array}{cccccccc|c} x_6 & x_7 & x_8 & x_1 & x_2 & x_3 & x_4 & x_5 & b\end{array}\\ \left[\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 1/2 & 1 & 1/2 & 3/2 & 9\\ 0 & 1 & 0 & 0 & 13/3 & 14/3 & 8/3 & 4/3 & 10\\ 0 & 0 & 1 & 0 & -1/6 & 8/3 & 7/6 & -19/6 & 17\\ 0 & 0 & 0 & 1 & -2/3 & 8/3 & 5/3 & -2/3 & 5\end{array}\right]\end{array}$$

The system of equations is consistent. General solution:

$$\begin{bmatrix} x_6\\ x_7\\ x_8\\ x_1\end{bmatrix}=\begin{bmatrix} 9/2\\ 14\\ 3/2\\ 4\end{bmatrix}-\begin{bmatrix} 1/2 & 1 & 1/2 & 3/2\\ 13/3 & 14/3 & 8/3 & 4/3\\ -1/6 & 8/3 & 7/6 & -19/6\\ -2/3 & 8/3 & 5/3 & -2/3\end{bmatrix}\begin{bmatrix} x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}$$

A.28

Check the linear independence of the following set of vectors.

$$\mathbf{a}^{(1)} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{a}^{(2)} = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{a}^{(3)} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{a}^{(4)} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

The number of vectors is 4 (k) and each vector has 3 components (n). Since k > n, the set of vectors is linearly dependent.

A.29

Check the linear independence of the following set of vectors.

$$\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mathbf{a}^{(2)} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{a}^{(3)} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

Solution:

Assume x_1, x_2 and x_3 are constants. Then

$$\mathbf{a}^{(1)}x_1 + \mathbf{a}^{(2)}x_2 + \mathbf{a}^{(3)}x_3 = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & 0 \\ 3 & 0 & -3 \\ 4 & 1 & 2 \\ 5 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Only solution for the system is $x_1 = x_2 = x_3 = 0$. Therefore the set of vectors is linearly independent.

A.30

Find eigenvalues for the following matrix.

$$\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Solution:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 4 = \lambda^2 - 6\lambda + 1 = 0; \quad \lambda_1 = 3 - 2\sqrt{2}, \quad \lambda_2 = 3 + 2\sqrt{2}$$

A.31

Find eigenvalues for the following matrix:

$$\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

Solution:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 4 = \lambda^2 - 6\lambda + 4 = 0; \quad \lambda_1 = 3 - \sqrt{5}, \quad \lambda_2 = 3 + \sqrt{5}$$

A.32

Find eigenvalues for the following matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (5-\lambda)\,[(1-\lambda)(4-\lambda) - 1] = (5-\lambda)(\lambda^2 - 5\lambda + 3) = 0;$$

$$\lambda_1 = (5 - \sqrt{13})/2, \quad \lambda_2 = (5 + \sqrt{13})/2, \quad \lambda_3 = 5$$

A.33

Find eigenvalues for the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)[-\lambda\ (2-\lambda) - 1] = (1-\lambda)(\lambda^2 - 2\lambda - 1) = 0;$$

$\lambda_1 = 1 - \sqrt{2}$, $\quad \lambda_2 = 1$, $\quad \lambda_3 = 1 + \sqrt{2}$

A.34

Find eigenvalues for the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Solution:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 5-\lambda \end{vmatrix} = -\lambda\ \left[(1-\lambda)(5-\lambda) - 1\right] = -(\lambda^2 - 6\lambda + 4) = 0;$$

$\lambda_1 = 0$, $\lambda_2 = 3 - \sqrt{5}$, $\lambda_3 = 3 + \sqrt{5}$