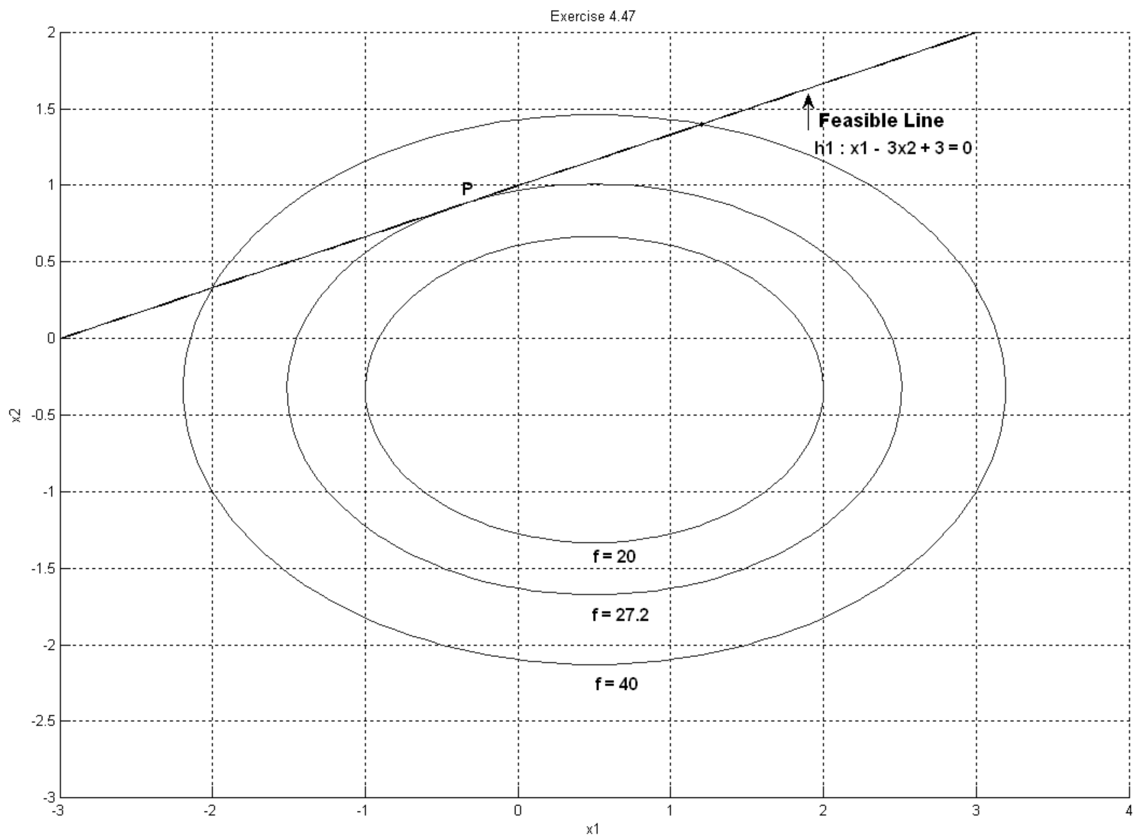
4.100

Exercise 4.46

Minimize $f(x_1, x_2) = 4x_1^2 + 9x_2^2 + 6x_2 - 4x_1 + 13$

subject to $x_1 - 3x_2 + 3 = 0$

Solution



Referring to Exercise 4.46, the point satisfying the KKT necessary conditions is
$x_1 = -0.4, \quad x_2 = 0.866667, \quad \nu = 7.2, \quad f = 27.2$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 that the point is an isolated global minimum.

The gradient of cost and constraint functions are

$\nabla f = \begin{bmatrix} 8x_1 - 4 \\ 18x_2 + 6 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

At optimum point P (-0.4, 0.866667)

$\nabla f(-0.4, 0.866667) = \begin{bmatrix} 8(-0.4) - 4 \\ 18(0.686667) + 6 \end{bmatrix} = \begin{bmatrix} -7.2 \\ 21.6 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

These vectors are shown at point P in above figure. Note that they are along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial b} = -\nu^* = -(7.2)$$

If we set b= 1, the new value of cost function will be approximately
$f^* = 27.2 — (7.2)(1) = 20$

MATLAB Code Exercise 4.100

```
clear all
axis equal
[x1,x2]=meshgrid(-3:0.1:4, -3:0.1:2);
f=4*x1.^2+9*x2.^2+6*x2-4*x1+13;
h1=x1-3*x2+3;

cla reset
axis equal
axis ([-3 4 -3 2])
xlabel('x1'),ylabel('x2')
title('Exercise 4.46')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

fv=[20 27.2 40];
fs=contour(x1,x2,f,fv,'b');

a=[  1.2];
b=[ 1.4];
plot(a,b,'.k');

grid
hold off
```

4.101

Exercise 4.47

Minimize $f(x) = (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 - 2)^2$

subject to $2x_1 + 3x_2 - 1 = 0$

$x_1 + x_2 + 2x_3 - 4 = 0$

Solution

No graphical solution. (3 design variables)

Referring to Exercise 4.47, the point satisfying the KKT necessary conditions is

$\mathrm{x}_1 = 1.71698,\ \mathrm{x}_2 = -0.81132,\ \mathrm{x}_3 = 1.547140,\ \nu_1 = -0.943396,\ \nu_2 = 0.4528299,$

$\mathrm{f} = 2.11318$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are

$$\nabla \mathrm{f} = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 + 2) \\ 2(x_3 - 2) \end{bmatrix}, \nabla \mathrm{h}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \text{ and } \nabla \mathrm{h}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

We need to check (4.52) from P.131

$$-\nabla \mathrm{f} = \nu_1 \nabla \mathrm{h}_1 + \nu_2 \nabla \mathrm{h}_2$$

$$-\begin{bmatrix} 2(1.71698 - 1) \\ 2(-0.81132 + 2) \\ 2(1.54717 - 2) \end{bmatrix} = -0.943396 * \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + 0.4528299 * \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathrm{LHS} = \mathrm{RHS} = \begin{bmatrix} -1.43396 \\ -2.37736 \\ 0.90566 \end{bmatrix}$$

which shows local minimum point.

By Theorem 4.7,

$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{b1}} = -\nu_1{}^* = -(-0.943396)$$

$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{b2}} = -\nu_2{}^* = -0.4528299$$

If we set b_1= 1 and b_2= 1, the new value of cost function will be approximately

$\mathrm{f}^* = 2.1318 — (-0.943396)(1) - (0.4528299)(1) = 2.62237$

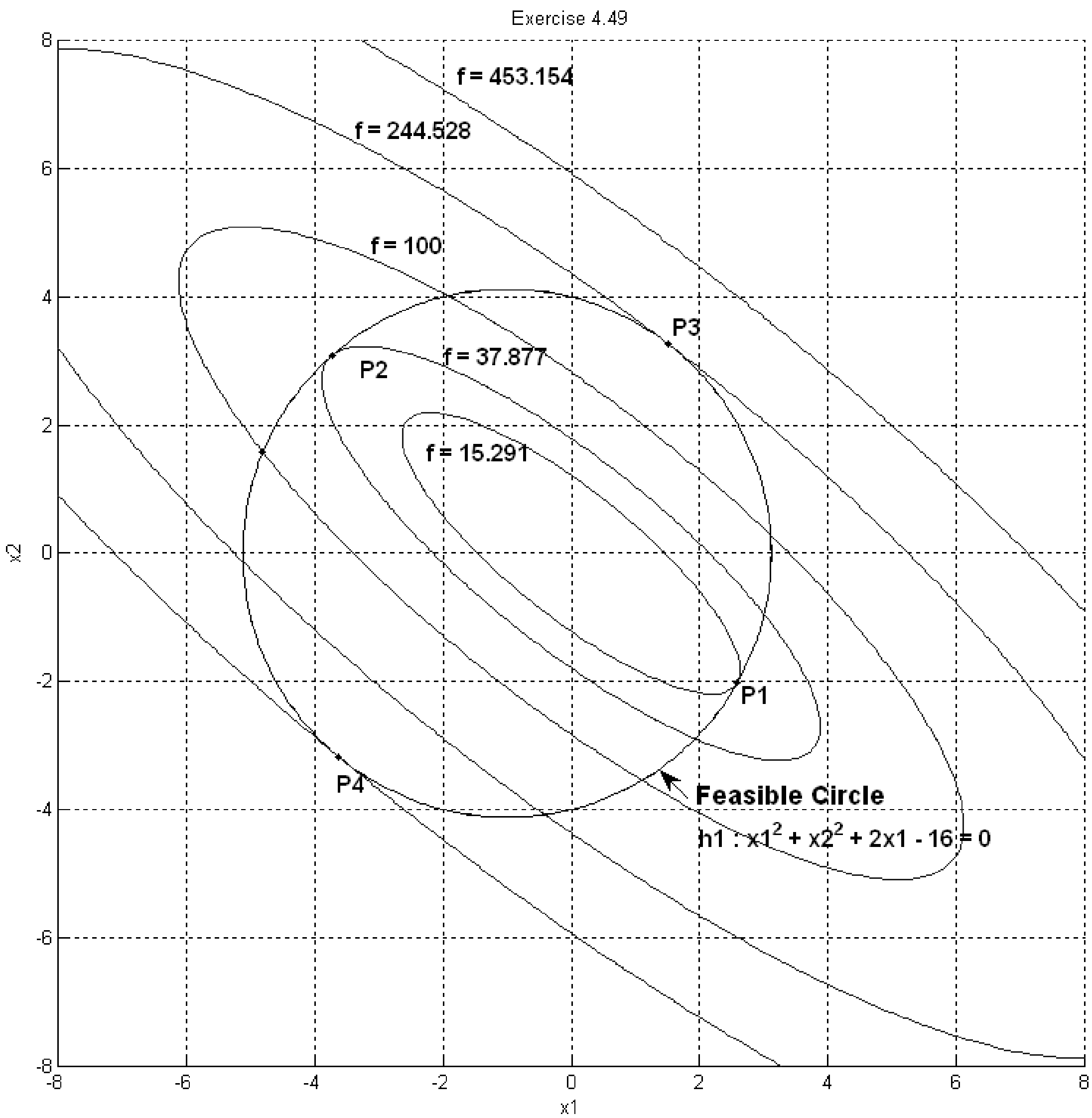
4.102

Exercise 4.48

Minimize $f(x_1, x_2) = 9x_1^2 + 18x_1x_2 + 13x_2^2 - 4$

subject to $x_1^2 + x_2^2 + 2x_1 = 16$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.48, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	ν_1^*
P1	2.5945	-2.0198	-1.4390
P2	-3.7322	3.0879	-2.1222
P3	1.5088	3.2720	-17.1503
P4	-3.630	-3.1754	-23.2885

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

Hessian of cost function, the gradient and Hessian of the constraint are

$$\nabla^2 f = \begin{bmatrix} 18 & 18 \\ 18 & 26 \end{bmatrix}; \nabla h = \begin{bmatrix} 2x_1 + 2 \\ 2x_2 \end{bmatrix}; \nabla^2 h = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

1. At point P1, $\nabla^2 L = \nabla^2 f + \nu \nabla^2 h = \begin{bmatrix} 15.122 & 18 \\ 18 & 23.122 \end{bmatrix}$

Since $M_1 = 15.122 > 0$, and $M_2 = 25.6509 > 0$, $\nabla^2 L$ is positive definite. Therefore, from *Theorem 5.3*, the point $x_1 = 2.5945, x_2 = -2.0198$ is an **isolated local minimum**.

2. At point P2, $\nabla^2 L = \nabla^2 f + \nu \nabla^2 h = \begin{bmatrix} 13.7556 & 18 \\ 18 & 21.7556 \end{bmatrix}$
Let $\mathbf{d} = d_1 \cdot d_2$. We need to find $\mathbf{d}$ such that $\nabla h \cdot \mathbf{d} = 0$. This gives $\mathbf{d} = c(1, 0.8848)$, where $c \neq 0$ is any constant. $Q = \mathbf{d}^T(\nabla^2 L)\mathbf{d} = 62.6402c^2 > 0$ for $c \neq 0$
The sufficient condition is satisfied. Thus, $x_1 = -3.7322, x_2 = 3.0879$ is an **isolated local minimum**.

3. At point P3,
$\nabla^2 L = \nabla^2 f + \nu \nabla^2 h = \begin{bmatrix} -16.3006 & 18 \\ 18 & -8.30006 \end{bmatrix}$
Let $\mathbf{d} = d_1 \cdot d_2$. We need to find $\mathbf{d}$ such that $\nabla h \cdot \mathbf{d} = 0$. This gives $\mathbf{d} = c(1, -0.7667)$, where $c \neq 0$ is any constant. $Q = \mathbf{d}^T(\nabla^2 L)\mathbf{d} = -48.7811c^2 < 0$ for $c \neq 0$
The sufficient condition is not satisfied, so $x_1 = 1.5088$, $x_2 = 3.2720$ is not an isolated local minimum. Since $Q < 0$, second order necessary condition is violated, so the point cannot be a minimum point.

4. At point P4, $\nabla^2 L = \nabla^2 f + \nu \nabla^2 h = \begin{bmatrix} -28.577 & 18 \\ 18 & -20.577 \end{bmatrix}$
Let $\mathbf{d} = d_1 \cdot d_2$. We need to find $\mathbf{d}$ such that $\nabla h \cdot \mathbf{d} = 0$. This gives $\mathbf{d} = c(1, -0.8282)$, where $c \neq 0$ is any constant. $Q = \mathbf{d}^T(\nabla^2 L)\mathbf{d} = -72.5063c^2 < 0$ for $c \neq 0$.
Both sufficient and second order necessary conditions are violated, so the point cannot be a minimum.

So only point P1 and P2 have isolated local minimum.

The gradient of cost and constraint functions are

$\nabla f = \begin{bmatrix} 18x_1 + 18x_2 \\ 18x_1 + 26x_2 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 2x_1 + 2 \\ 2x_2 \end{bmatrix}$

At point P1 (2.5945, -2.0198)

$\nabla f(2.5945, -2.0198) = \begin{bmatrix} 18(2.5945) + 18(-2.0198) \\ 18(2.5945) + 26(-2.0198) \end{bmatrix} = \begin{bmatrix} 10.3446 \\ -5.8138 \end{bmatrix} = -5.8138 \begin{bmatrix} -1.779 \\ 1 \end{bmatrix}$ and

$\nabla h = \begin{bmatrix} 7.189 \\ -4.0396 \end{bmatrix} = -4.0396 \begin{bmatrix} -1.779 \\ 1 \end{bmatrix}$

Note that they are along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial b} = -v^* = -(-1.4390)$$

If we set b_1= 1, the new value of cost function will be approximately

$f^* = 15.291 — (-1.4390)(1) = 16.73$

At point P2 (-3.7322, 3.0879)

$\nabla f(-3.7322, 3.0879) = \begin{bmatrix} 18(-3.7322) + 18(3.0879) \\ 18(-3.7322) + 26(3.0879) \end{bmatrix} = \begin{bmatrix} -11.5974 \\ 13.1058 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.13 \end{bmatrix}$ and

$\nabla h = \begin{bmatrix} 2(-3.7322) + 2 \\ 2(3.0879) \end{bmatrix} = \begin{bmatrix} -5.4644 \\ 6.1758 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.13 \end{bmatrix}$

Note that they are along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial b} = -v^* = -(-2.1222)$$

If we set b= 1, the new value of cost function will be approximately

$f^* = 37.877 — (-2.1222)(1) = 40$

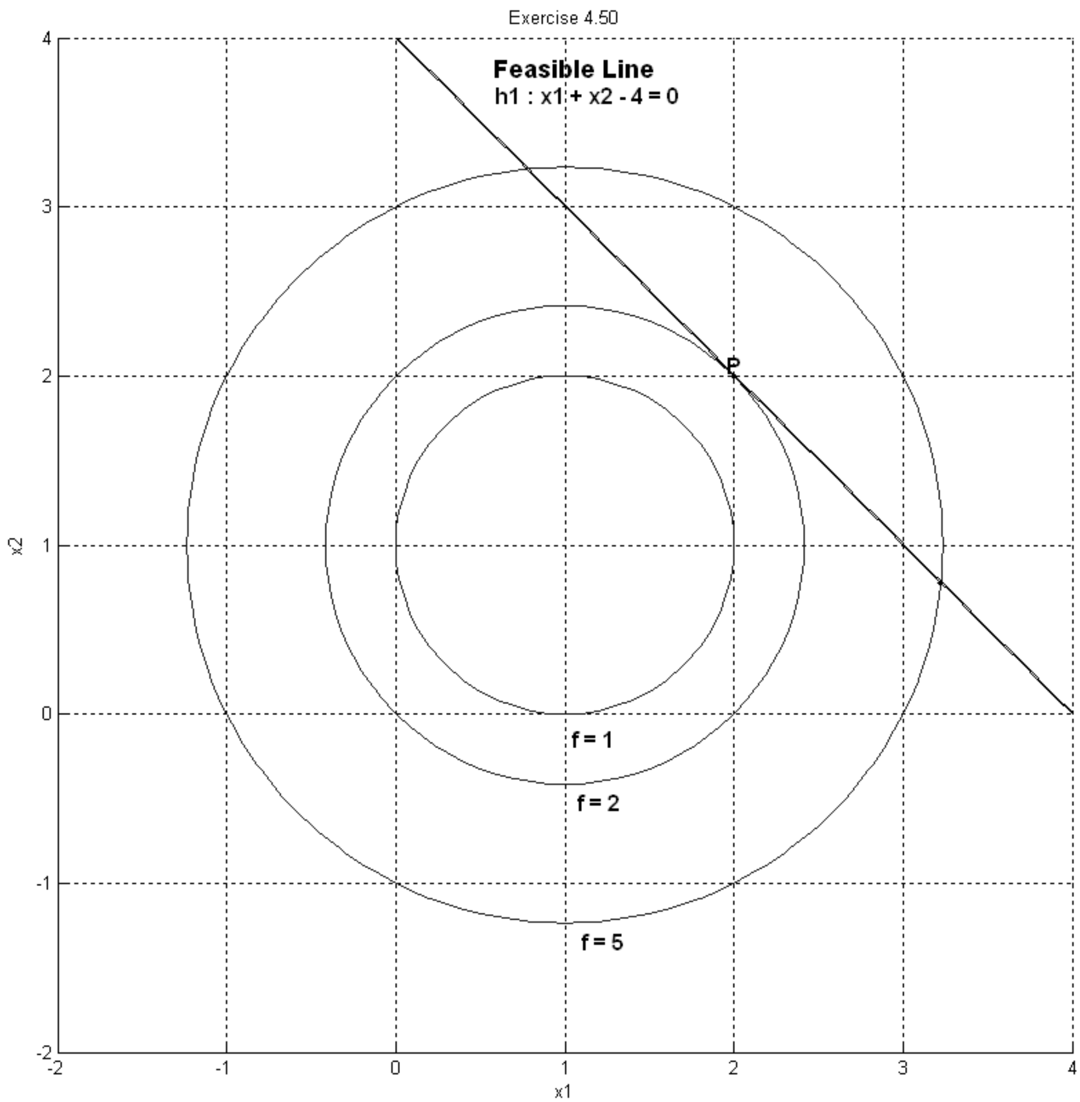
MATLAB Code for Exercise 4.102

```
clear all
axis equal
[x1,x2]=meshgrid(-8:0.1:8, -8:0.1:8);
f=9*x1.^2+18*x1.*x2+13*x2.^2-4;
h1=x1.^2+x2.^2+2*x1-16;

cla reset
axis equal
axis ([-8 8 -8 8])
xlabel('x1'),ylabel('x2')
title('Exercise 4.48')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

fv=[15.291 37.877 100 244.528 453.154];
fs=contour(x1,x2,f,fv,'b');

a=[1.5088 2.5945 -3.630 -3.7322 -4.80963 3.01213];
b=[3.2720 -2.0198 -3.1754 3.0879 1.57693 -0.950153];
plot(a,b,'.k');

grid
hold off
```

4.103

Exercise 4.49
Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 = 0$

Solution



Referring to Exercise 4.49, the point satisfying the KKT necessary conditions is
$x_1 = 2\ , x_2 = 2\ , \nu = -2, f = 2$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

At optimum point P (2, 2)
$\nabla f(2,2) = \begin{bmatrix} 2(2-1) \\ 2(2-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
These vectors can be shown at point P in above figure. Note that they will be along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial b} = -v^* = -(-2)$$

If we set b= 1, the new value of cost function will be approximately
$f^* = 2 — (-2)(1) = 4$

MATLAB Code for Exercise 4.103

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.1:4, -2:0.1:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1+x2-4;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.49')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

fv=[1 2 5];
fs=contour(x1,x2,f,fv,'b');

a=[2 3.22474];
b=[2 0.775255];
plot(a,b,'.k');

grid
hold off
```

4.104

Exercise 4.50

Consider the following problem with equality constraints:

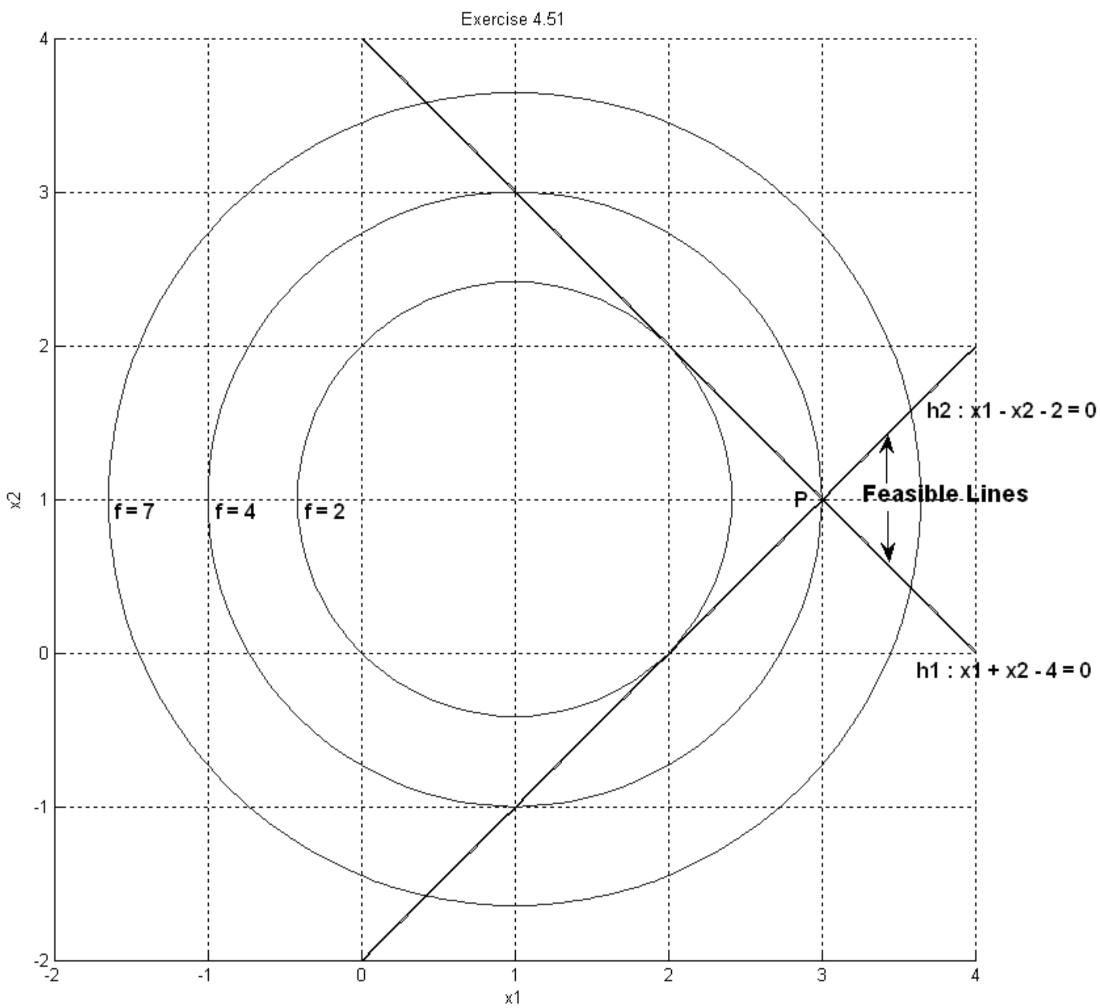
Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

subject to $x_1 + x_2 - 4 = 0$

$x_1 - x_2 - 2 = 0$

1. Is it a valid optimization problem? Explain.
2. Explain how you would solve the problem? Are necessary conditions needed to find the optimum solution?

Solution



Minimize $f = (x_1 - 1)^2 + (x_2 - 1)^2$; subject to $x_1 + x_2 - 4 = 0$, and $x_1 - x_2 - 2 = 0$ (Ref. Exercise 4.50)
1. It is not a valid optimization problem because there is only one feasible point of the constraint set; solution of the two linear equalities.
2. Solving the constraint equations, we get $x_1 = 3$, $x_2 = 1$, $f(3,1) = 4$.
Necessary conditions are not needed for this case since a unique solution has been found by solving the constraint equations. If Lagrange multipliers for the constraints are needed, then we need to write the necessary conditions and solve for them.

MATLAB Code for Exercise 4.104

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.1:4, -2:0.1:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1+x2-4;
h2=x1-x2-2;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.50')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

cv2=[0 0.01];
const2=contour(x1,x2,h2,cv2,'k');

fv=[2 4 7];
fs=contour(x1,x2,f,fv,'b');

a=[3];
b=[1];
plot(a,b,'.k');

grid
hold off
```

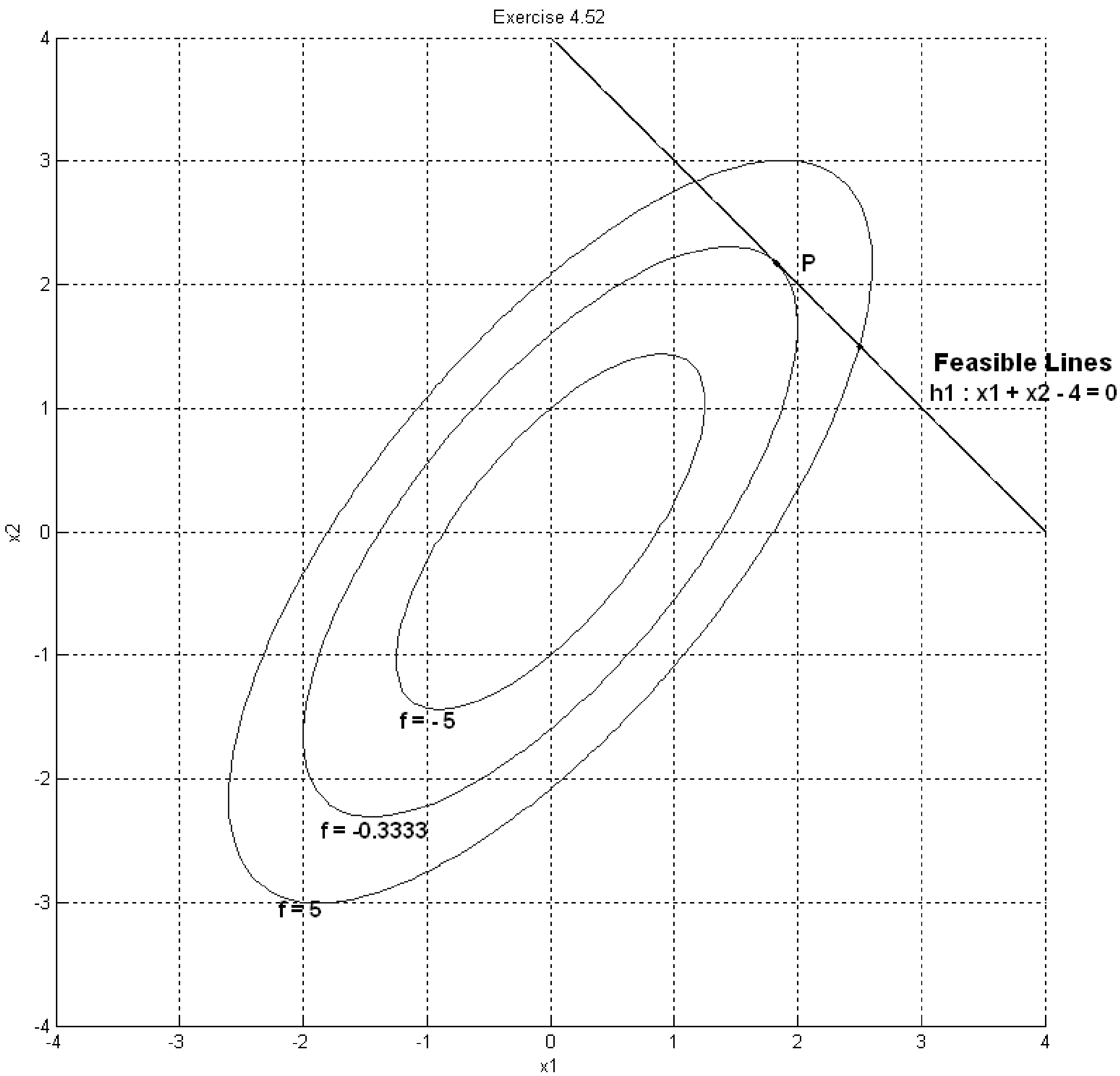
4.105

Exercise 4.51

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$

subject to $x_1 + x_2 = 4$

Solution



Referring to Exercise 4.51, the point satisfying the KKT necessary conditions is
$x_1 = 1.83333$, $x_2 = 2.16667$, $\nu = -3.83333$, $f = -0.3333$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 8x_1 - 5x_2 \\ 6x_2 - 5x_1 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

At optimum point P (1.83333, 2.16667)
$\nabla f(1.83333, 2.16667) = \begin{bmatrix} 8 * 1.83333 - 5 * 2.16667 \\ 6 * 2.16667 - 5 * 1.83333 \end{bmatrix} = \begin{bmatrix} 3.83333 \\ 3.83337 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
These vectors can be shown at point P in above figure. Note that they will be along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial b} = -\nu^* = -(-3.83333)$$

If we set b= 1, the new value of cost function will be approximately
$f^* = -0.33333 — (-3.83333)(1) = 3.5$

<u>MATLAB Code for Exercise 4.105</u>

```
clear all
axis equal
[x1,x2]=meshgrid(-4:0.1:4, -4:0.1:4);
f=4*x1.^2+3*x2.^2-5*x1.*x2-8;
h1=x1+x2-4;

cla reset
axis equal
axis ([-4 4 -4 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.51')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

fv=[-5 -0.3333 5];
fs=contour(x1,x2,f,fv,'b');

a=[1.83333 2.5];
b=[2.16667 1.5];
plot(a,b,'.k');

grid
hold off
```

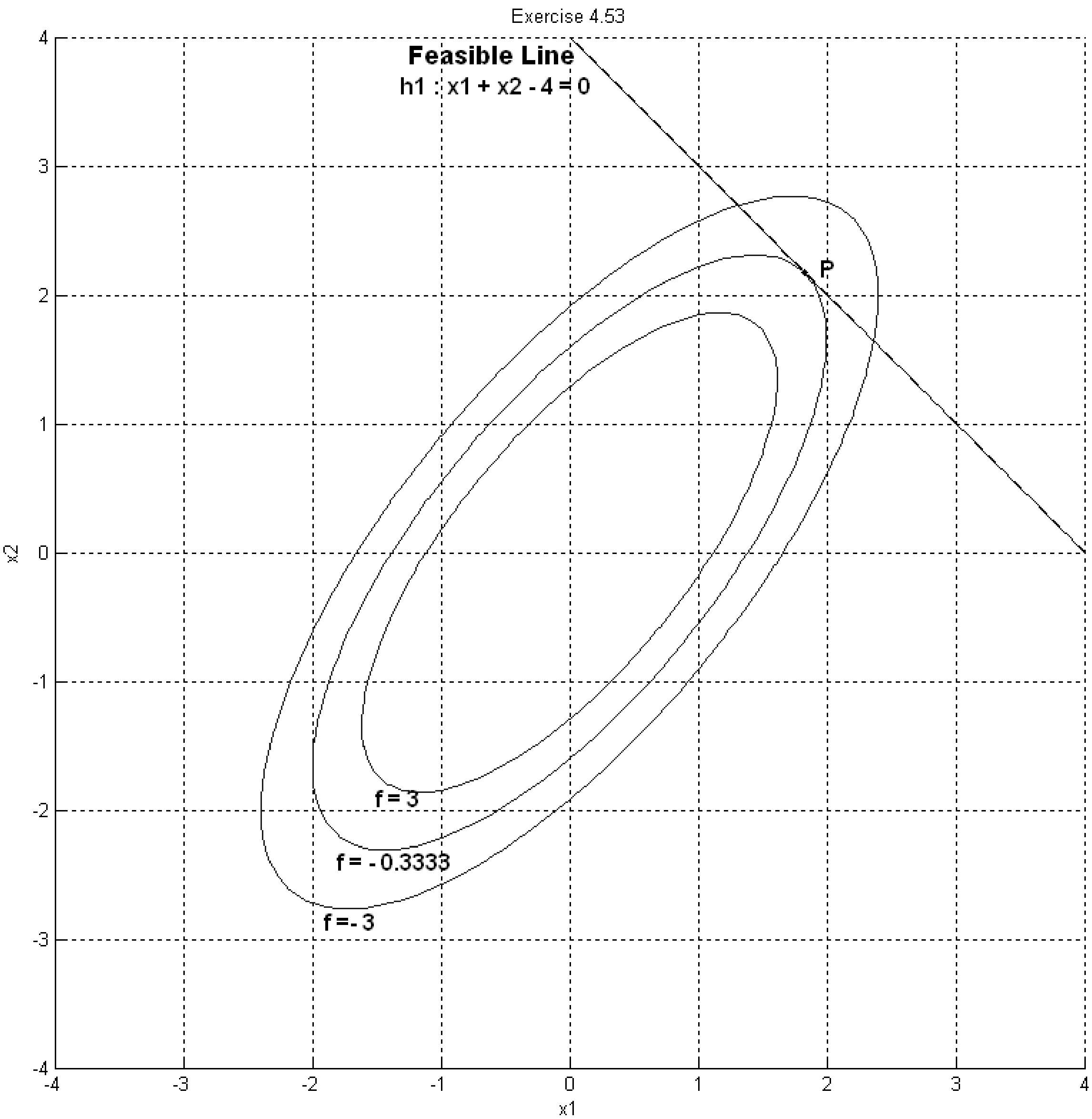
4.106

Exercise 4.52

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$

subject to $x_1 + x_2 = 4$

Solution



Referring to Exercise 4.52, the point satisfying the KKT necessary conditions is
$x_1 = 1.83333$, $x_2 = 2.16667$, $\nu = 3.83333$, $f = 0.3333$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$$\nabla L = \begin{bmatrix} -8x_1 + 5x_2 + \nu \\ -6x_2 + 5x_1 + \nu \end{bmatrix}$$

$$\nabla^2 L = \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix}$$

$M_1 = -8 < 0, M_2 = 48 - 25 = 23 > 0;$ Negative Definite

The Hessian of cost function is negative definite. So, this is not a convex problem.

$$\nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Sufficiency Check

$$\nabla h^T \cdot d = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 1 \cdot d_1 + 1 \cdot d_2 = 0$$

$d_1 = -d_2 = c$

$d = (c, -c)$ $\quad$ ($c \neq 0$ is an arbitry constant)

$$Q = d^T \cdot \nabla^2 L \cdot d = \begin{bmatrix} c & -c \end{bmatrix} \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} c \\ -c \end{bmatrix} = -24c^2 < 0 \ (c \neq 0)$$

The sufficient condition is NOT satisfied , so $x_1 = 1.83333$, $x_2 = 2.16667$ is NOT isolated minimum. Since $Q < 0$ second order necessary condition is violated ,so the point P cannot be a local minimum point.

MATLAB Code for Exercise 4.106

```
clear all
axis equal
[x1,x2]=meshgrid(-4:0.1:4, -4:0.1:4);
f=(-1)*(4*x1.^2+3*x2.^2-5*x1.*x2-8);
h1=x1+x2-4;


cla reset
axis equal
axis ([-4 4 -4 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.52')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

fv=[-3 0.3333 3];
fs=contour(x1,x2,f,fv,'b');

a=[1.83333];
b=[2.16667];
plot(a,b,'.k');

grid
hold off
```

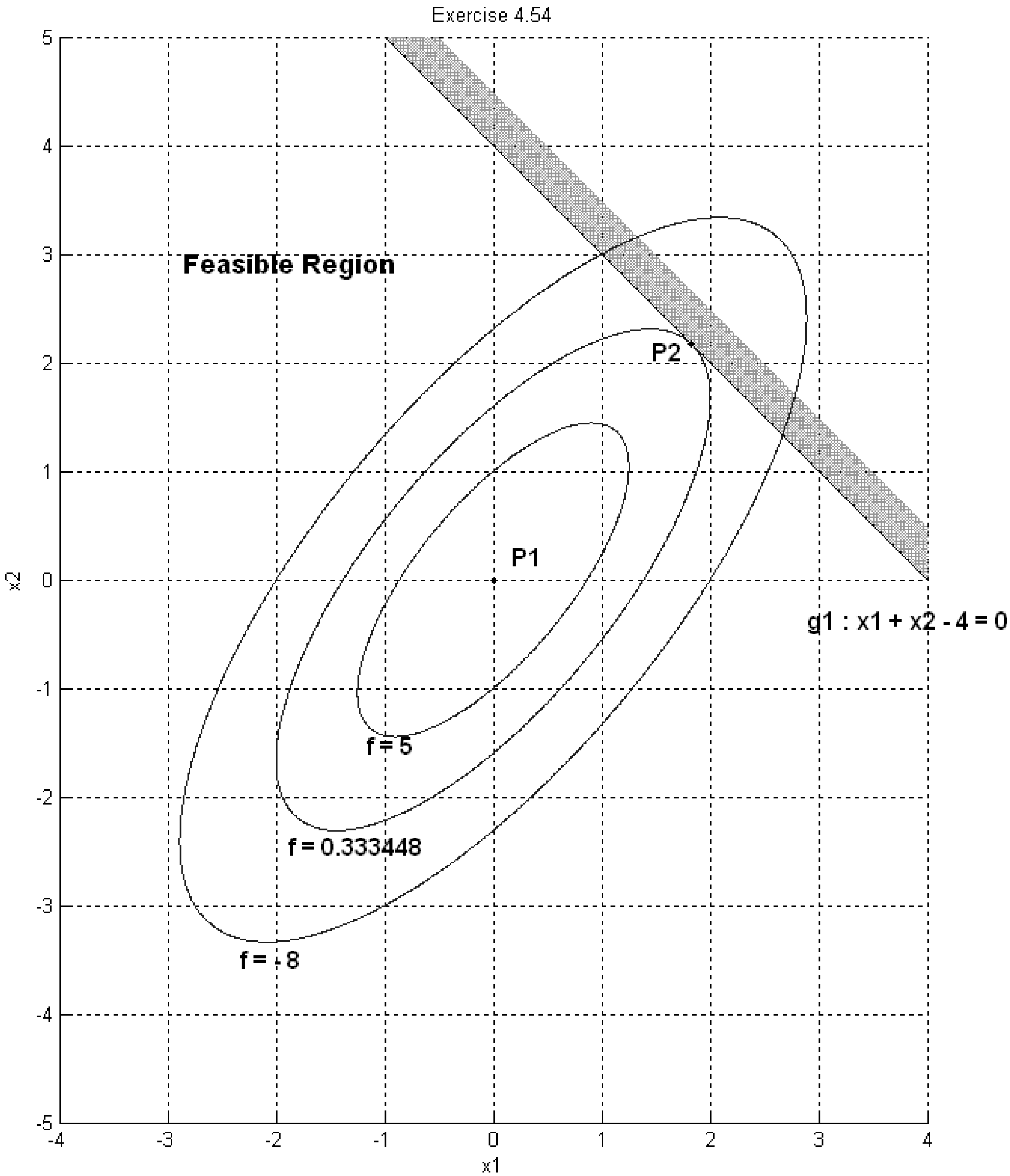
4.107

Exercise 4.54

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$

subject to $x_1 + x_2 \leq 4$

Solution



Referring to Exercise 4.54, the point satisfying the KKT necessary conditions is

	x_1^*	x_2^*	u^*
P1	0	0	0
P2	1.83333	2.16667	3.83329

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$$\nabla L = \begin{bmatrix} -8x_1 + 5x_2 + u \\ -6x_2 + 5x_1 + u \end{bmatrix}$$

$$\nabla^2 L = \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix}$$

$M_1 = -8 < 0, M_2 = 48 - 25 = 23 > 0;$ Negative definite

gradient of constraint

$$\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The Hessian of cost function is negative definite. So, this is not a convex problem.

1.At point P1, $x_1^* = 0\,, x_2^* = 0$
Since no constraint is active, Hessian of Lagrangian must be positive definite throughout to satisfy sufficient condition. But it is always negative definite so this is not an isolated minimum point because it violates the second order necessary condition.

2.At point P2, $x_1^* = 1.83333\,, x_2^* = 2.16667$

$$\nabla g^T \cdot d = [1 \quad 1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 1 \cdot d_1 + 1 \cdot d_2 = 0$$

$d_1 = -d_2 = c$

$d = (c, -c)$ ($c \neq 0$ is an arbitry constant)

$$Q = d^T \cdot \nabla^2 L \cdot d = [c \quad -c] \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} c \\ -c \end{bmatrix} = -24c^2 < 0 \ (c \neq 0)$$

Since $Q < 0$ second order necessary condition is violated ,so the point cannot be a local minimum point.

MATLAB Code for Exercise 107

```
clear all
axis equal
[x1,x2]=meshgrid(-4:0.01:4, -5:0.01:5);
f=(-1)*(4*x1.^2+3*x2.^2-5*x1.*x2-8);
g1=x1+x2-4;

cla reset
axis equal
axis ([-4 4 -5 5])
xlabel('x1'),ylabel('x2')
title('Exercise 4.54')
hold on

cv1=[0:0.03:0.5];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');

fv=[-8 0.333448 5];
fs=contour(x1,x2,f,fv,'b');

a=[0 1.83333];
b=[0 2.16667];
plot(a,b,'.k');

grid
hold off
```

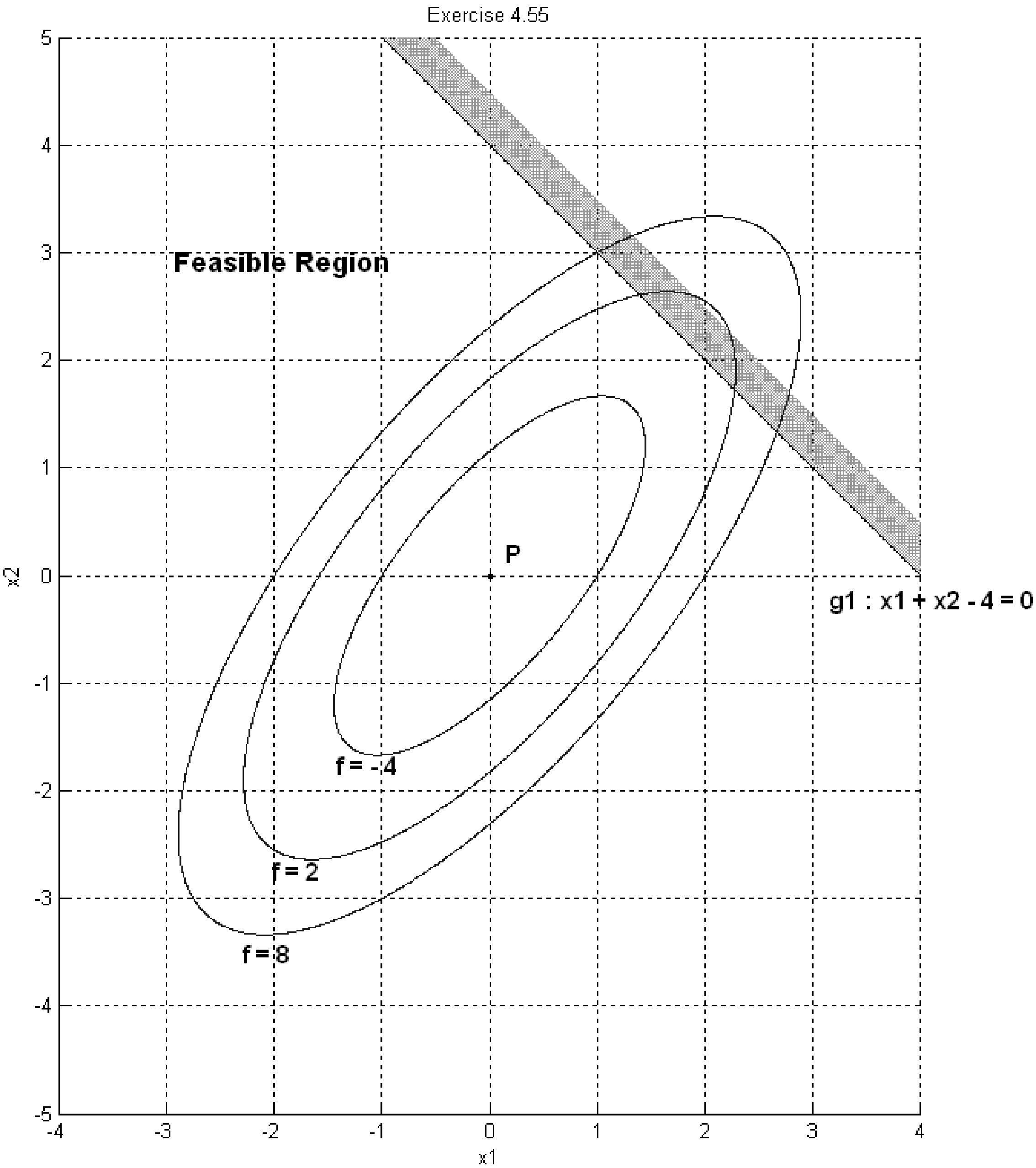
4.108

Exercise 4.55

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8$

subject to $x_1 + x_2 \leq 4$

Solution



Referring to Exercise 4.55, the point satisfying the KKT necessary conditions is
$x_1 = 0, x_2 = 0\ , u = 0, f = \ -8$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The gradient of cost function is
$\nabla f = \begin{bmatrix} 8x_1 - 5x_2 \\ 6x_2 - 5x_1 \end{bmatrix}$
Also,
$\nabla^2 f(0\ ,0) = \begin{bmatrix} 8 & -5 \\ -5 & 6 \end{bmatrix}$
$M_1 = 8 > 0, M_2 = 48 - 25 = 23 > 0;$ Positive definite

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.
Also, it is a local minimum with $f(x^*) = -8$.

<u>MATLAB Code for Exercise 108</u>

```
clear all
axis equal
[x1,x2]=meshgrid(-4:0.01:4, -5:0.01:5);
f=(4*x1.^2+3*x2.^2-5*x1.*x2-8);
g1=x1+x2-4;

cla reset
axis equal
axis ([-4 4 -5 5])
xlabel('x1'),ylabel('x2')
title('Exercise 4.55')
hold on

cv1=[0:0.03:0.5];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');

fv=[-8 -4 2 8];
fs=contour(x1,x2,f,fv,'b');

a=[0];
b=[0];
plot(a,b,'.k');

grid
hold off
```

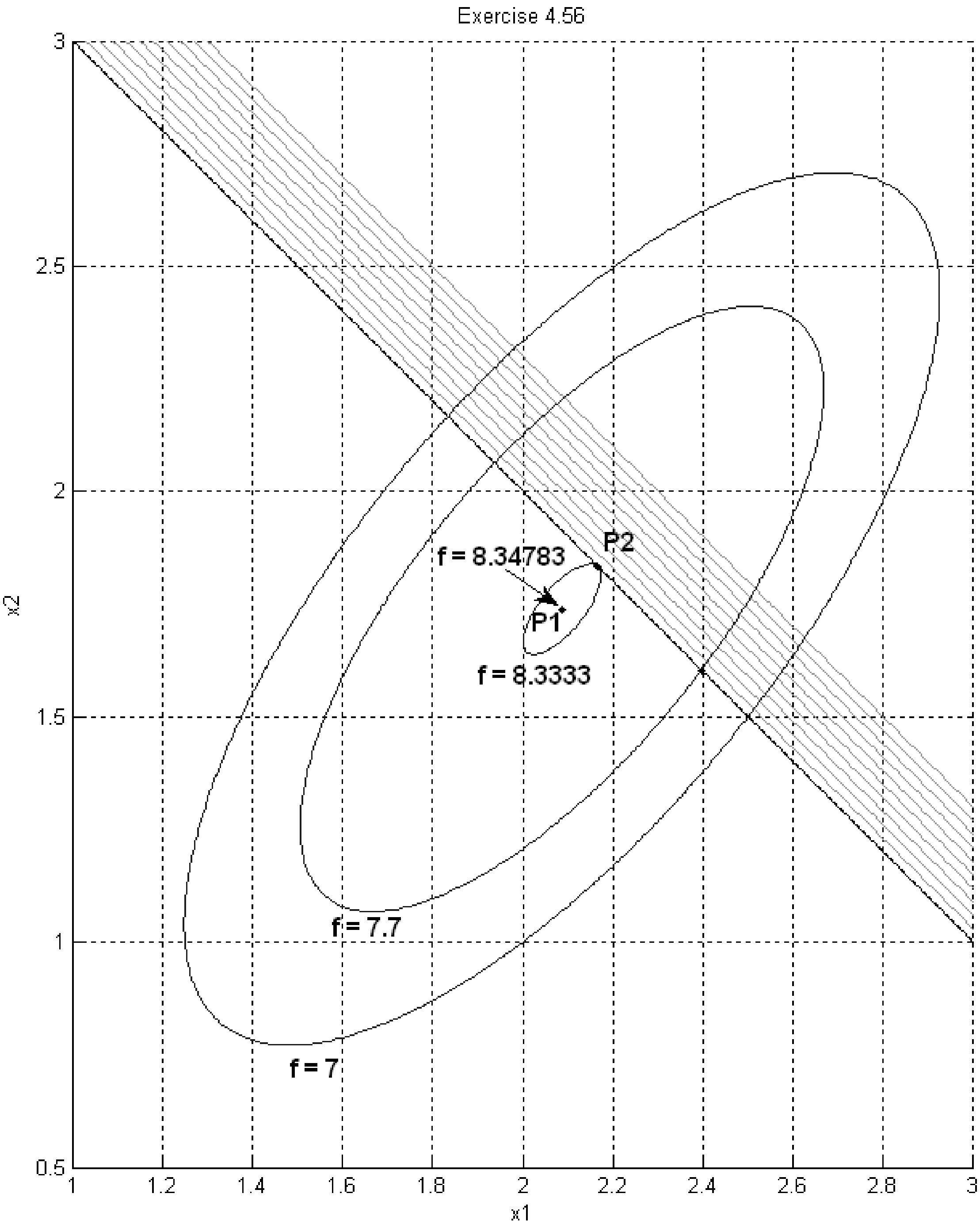
4.109

Exercise 4.56

Maximize $F(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$

subject to $x_1 + x_2 \leq 4$

Solution



Referring to Exercise 4.56, the point satisfying the KKT necessary conditions is

	x_1^*	x_2^*	u^*
P1	2.08696	1.73913	0
P2	2.16667	1.83333	0.16667

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$$\nabla L = \begin{bmatrix} -8x_1 + 5x_2 + 8 + u \\ -6x_2 + 5x_1 + u \end{bmatrix}$$

$$\nabla^2 L = \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix}$$

$M_1 = -8 < 0, M_2 = 48 - 25 = 23 > 0;$ Negative definite

gradient of constraint

$$\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The Hessian of cost function is negative definite. So, this is not a convex problem.

1.At point P1, $x_1^* = 2.08696$, $x_2^* = 1.73913$; $f^* = 8.348$

This is an unconstrained KKT point. The Hessain of the cost function is negative definite, so $x_1 = 2.08696$, $x_2 = 1.73913$ is NOT isolated minimum. Actually, the second order necessary condition is violated, so the point P1 cannot be a local minimum point.

2.At point P2, $x_1^* = 2.16667$, $x_2^* = 1.83333$; $f^* = 8.333$

$$\nabla g^T \cdot d = [1 \quad 1] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 1 \cdot d_1 + 1 \cdot d_2 = 0$$

$d_1 = -d_2 = c$

$d = (c, -c)$ ($c \neq 0$ is an arbitry constant)

$$Q = d^T \cdot \nabla^2 L \cdot d = [c \quad -c] \begin{bmatrix} -8 & 5 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} c \\ -c \end{bmatrix} = -24c^2 < 0 \ (c \neq 0)$$

The sufficient condition is NOT satisfied , so $x_1 = 2.16667$, $x_2 = 1.83333$ is NOT an isolated minimum. Since $Q < 0$ second order necessary condition is violated, so the point P2 cannot be a local minimum point of *f*.

<u>MATLAB Code for Exercise 109</u>

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1-x2-2;
g1=-x1-x2+4;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.56')
hold on

cv2=[0 0.01];
const2=contour(x1,x2,h1,cv2,'k');

cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');

fv=[2 4 6];
fs=contour(x1,x2,f,fv,'b');

a=[3];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.110

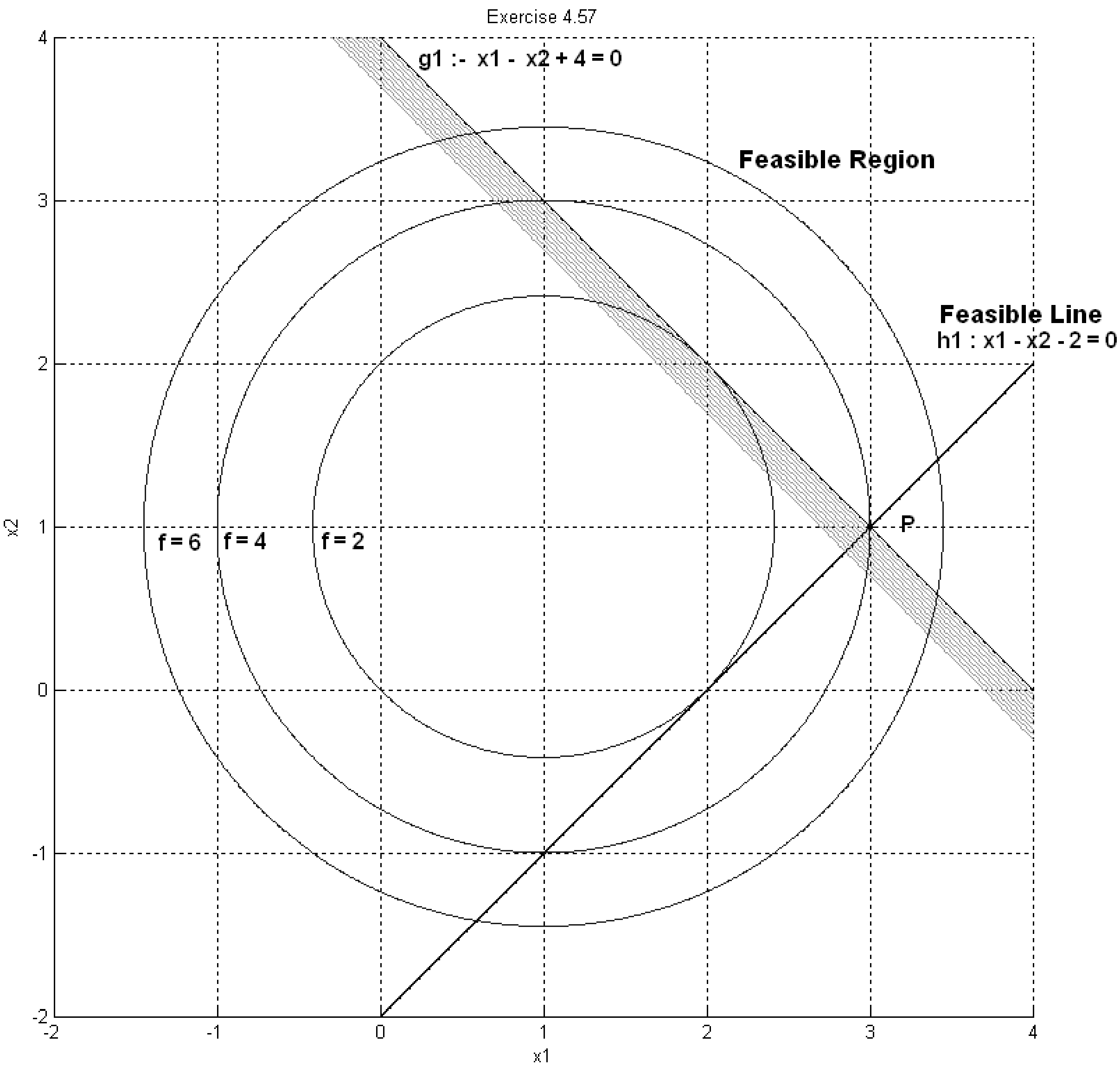
Exercise 4.57

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

subject to $x_1 + x_2 \geq 4$

$x_1 - x_2 - 2 = 0$

Solution



Referring to Exercise 4.57, the point satisfying the KKT necessary conditions is
$x_1 = 3, x_2 = 1, \nu = -2, u = 2, f = 4$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}, \nabla h = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

At optimum point P (3, 1)
$\nabla f(3,1) = \begin{bmatrix} 2(3-1) \\ 2(1-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \nabla h = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = \nu \nabla h + u \nabla g$

$$- \begin{bmatrix} 4 \\ 0 \end{bmatrix} = -2 * \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2 * \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\text{LHS} = \text{RHS} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

which shows local minimum point.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial b} = -\nu^* = -(-2)$$

$$\frac{\partial f(x^*)}{\partial e} = -u^* = -(2)$$

If we set b= 1 and e= 1, the new value of cost function will be approximately
$f^* = 4 - (-2)(1) - (2)(1) = 4$

MATLAB Code for Exercise 110

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1-x2-2;
g1=-x1-x2+4;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.57')
hold on

cv2=[0 0.01];
const2=contour(x1,x2,h1,cv2,'k');

cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');

fv=[2 4 6];
fs=contour(x1,x2,f,fv,'b');

a=[3];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.111

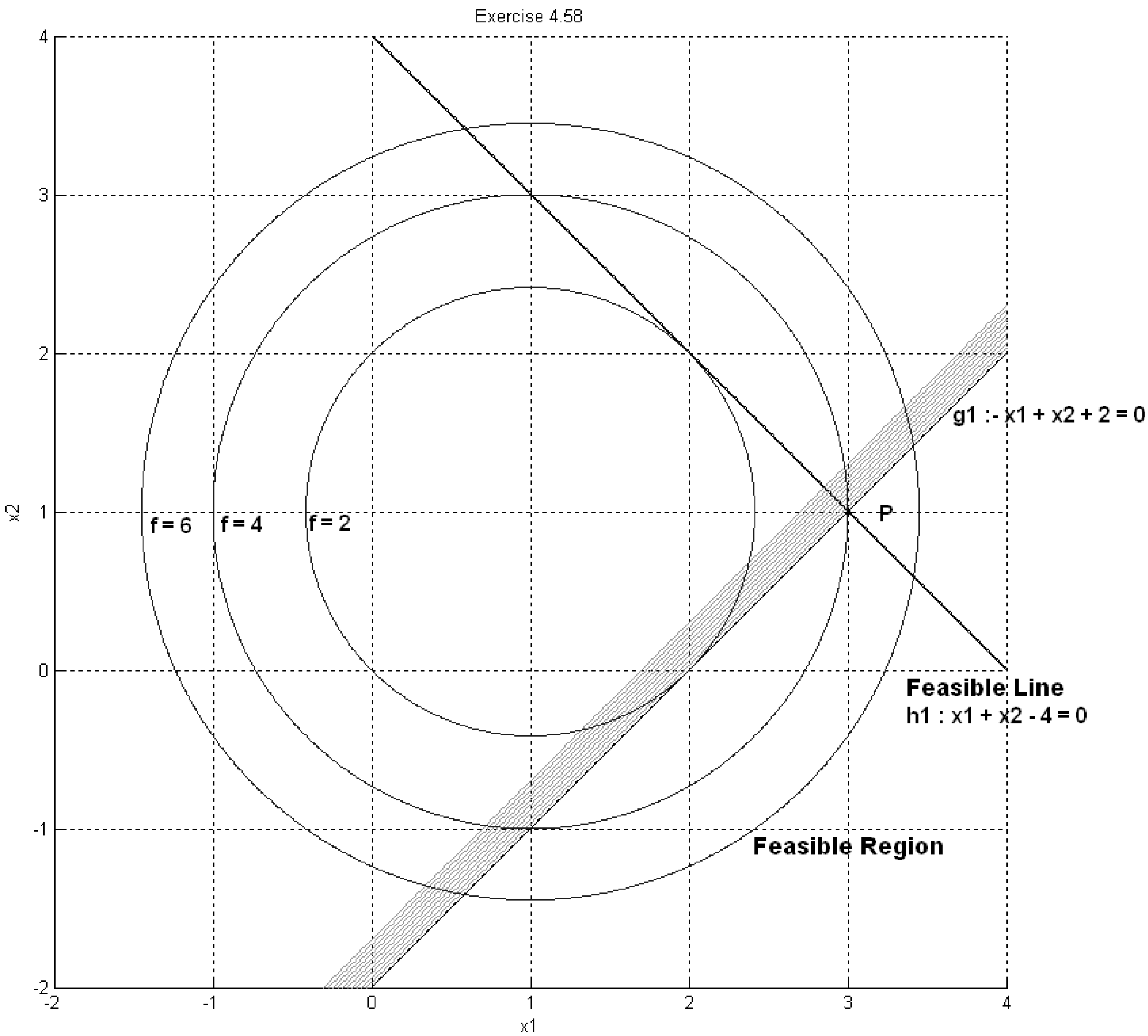
Exercise 4.58

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

subject to $x_1 + x_2 = 4$

$$x_1 - x_2 - 2 \geq 0$$

Solution



Referring to Exercise 4.58, the point satisfying the KKT necessary conditions is
$x_1 = 3, x_2 = 1, \nu = -2, u = 2, f = -8$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}, \nabla h = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = \nu \nabla h + u \nabla g$

$$- \begin{bmatrix} 2(3-1) \\ 2(1-1) \end{bmatrix} = -2 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 * \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\text{LHS} = \text{RHS} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

which shows local minimum point.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial b} = -\nu^* = -(-2)$$

$$\frac{\partial f(x^*)}{\partial e} = -u^* = -(2)$$

If we set b= 1 and e= 1, the new value of cost function will be approximately
$f^* = -8 — (-2)(1) - (2)(1) = -8$

MATLAB Code for Exercise 111

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
h1=x1+x2-4;
g1=-x1+x2+2;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.58')
hold on

cv2=[0 0.01];
const2=contour(x1,x2,h1,cv2,'k');

cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');

fv=[2 4 6];
fs=contour(x1,x2,f,fv,'b');

a=[3];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.112

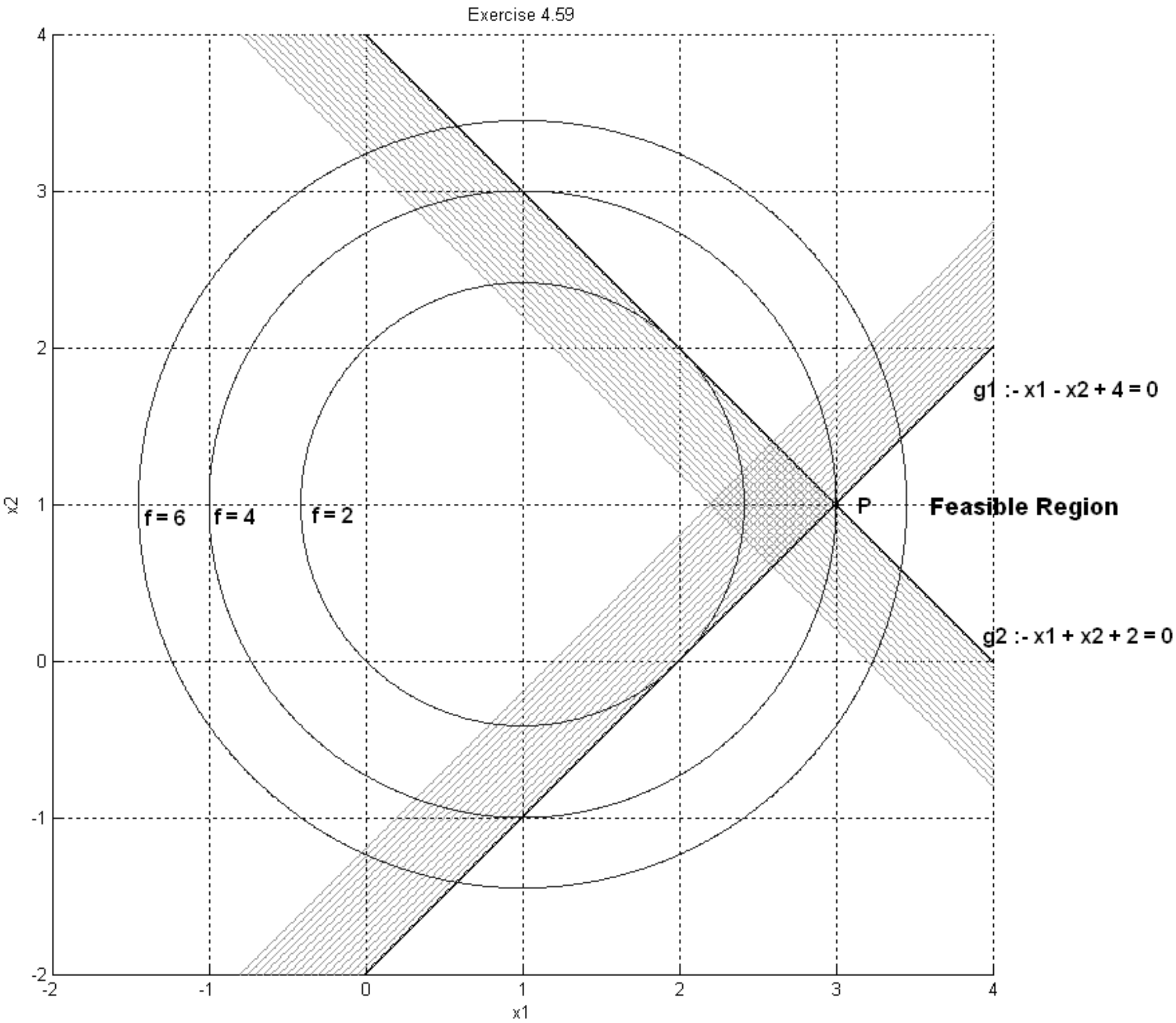
Exercise 4.59

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

subject to $x_1 + x_2 \geq 4$

$x_1 - x_2 \geq 2$

Solution



Referring to Exercise 4.59, the point satisfying the KKT necessary conditions is
$x_1 = 3, x_2 = 1, u_1 = 2, u_2 = 2, f = 4$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}, \nabla g_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\nabla g_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

At optimum point P (3, 1)
$\nabla f(3,1) = \begin{bmatrix} 2(3-1) \\ 2(1-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \nabla g_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\nabla g_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = u_1 \nabla g_1 + u_2 \nabla g_2$

$$-\begin{bmatrix} 4 \\ 0 \end{bmatrix} = -2 * \begin{bmatrix} -1 \\ -1 \end{bmatrix} + 2 * \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\text{LHS} = \text{RHS} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

which shows local minimum point.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(-2)$$

$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(2)$$

If we set $e_1 = 1$ and $e_2 = 1$, the new value of cost function will be approximately
$f^* = 4 - (-2)(1) - (2)(1) = 4$

MATLAB Code for Exercise 112

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;

g1=-x1-x2+4;
g2=-x1+x2+2;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.59')
hold on

cv1=[0:0.05:0.8];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x1,x2,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');

fv=[2];
fs=contour(x1,x2,f,fv,'b');

a=[3];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.113

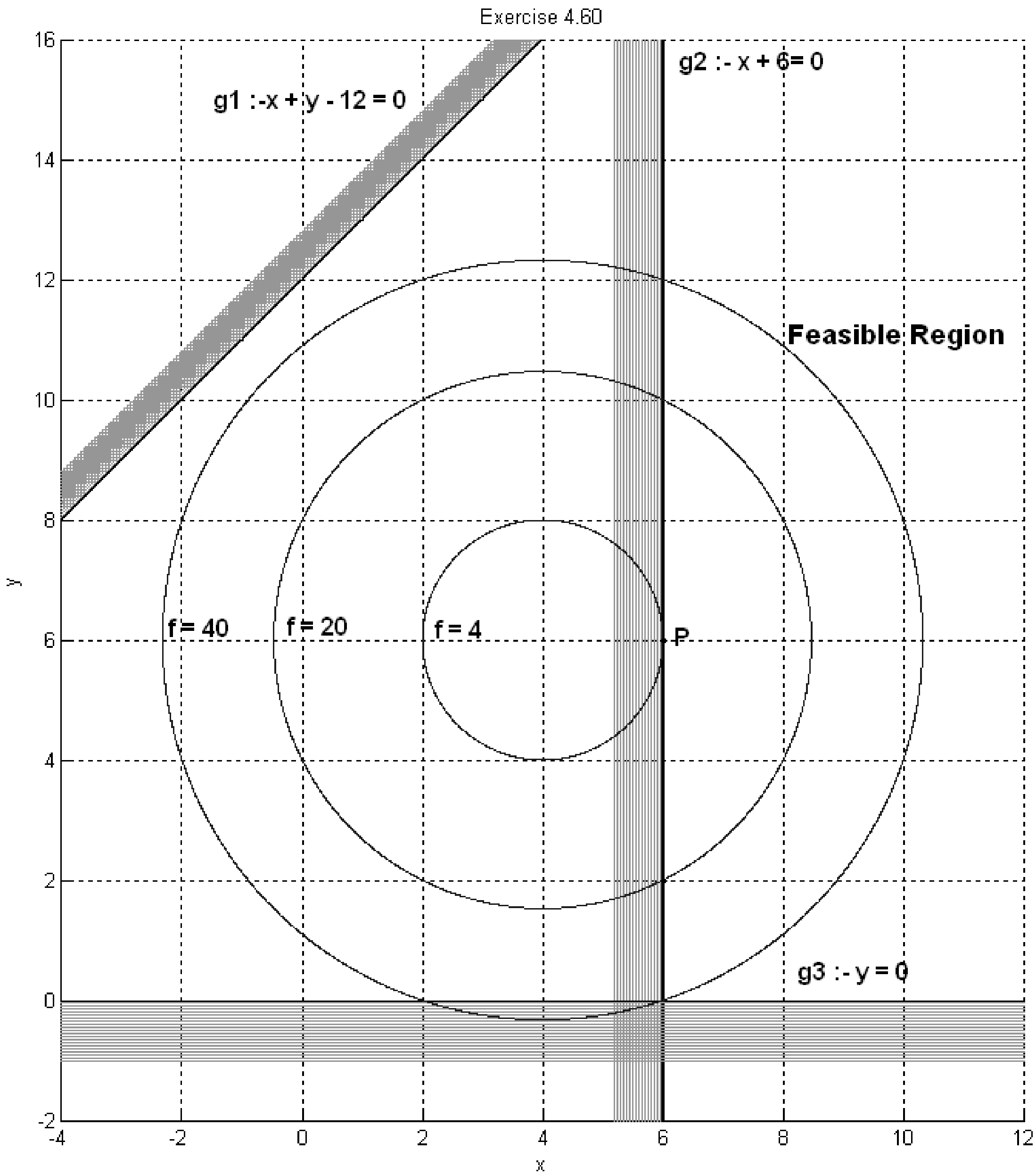
Exercise 4.60

Minimize $f(x, y) = (x-4)^2 + (y-6)^2$

subject to $12 \geq x + y$

$x \geq 6, y \geq 0$

Solution



Referring to Exercise 4.60, the points satisfying the KKT necessary conditions are
$x = y = 6, u_1 = 0, u_2 = 4, u_3 = 0$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x-4) \\ 2(y-6) \end{bmatrix}, \nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At optimum point P (6, 6)
$\nabla f(6,6) = \begin{bmatrix} 2(6-4) \\ 2(6-6) \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = u_1 \nabla g_1 + u_2 \nabla g_2 + u_3 \nabla g_3$

$$- \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 0 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4 * \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 * \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{LHS} = \text{RHS} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

which shows local minimum point.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(-0)$$

$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(4)$$

$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(-0)$$

If we set e_1=e_2= e_3=1, the new value of cost function will be approximately
$f^* = 4 — (4)(1) = 0$

MATLAB Code for Exercise 113

```
clear all
axis equal
[x,y]=meshgrid(-4:0.01:12, -2:0.01:16);
f=(x-4).^2+(y-6).^2;

g1=-x+y-12;
g2=-x+6;
g3=-y;

cla reset
axis equal
axis ([-4 12 -2 16])
xlabel('x'),ylabel('y')
title('Exercise 4.60')
hold on

cv1=[0:0.05:0.8];
const1=contour(x,y,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x,y,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x,y,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x,y,g2,cv2,'k');

cv3=[0:0.05:1];
const3=contour(x,y,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x,y,g3,cv3,'k');

fv=[4 20 40];
fs=contour(x,y,f,fv,'b');

a=[6 6];
b=[6 2];
plot(a,b,'.k');

grid
hold off
```

4.114

Exercise 4.61

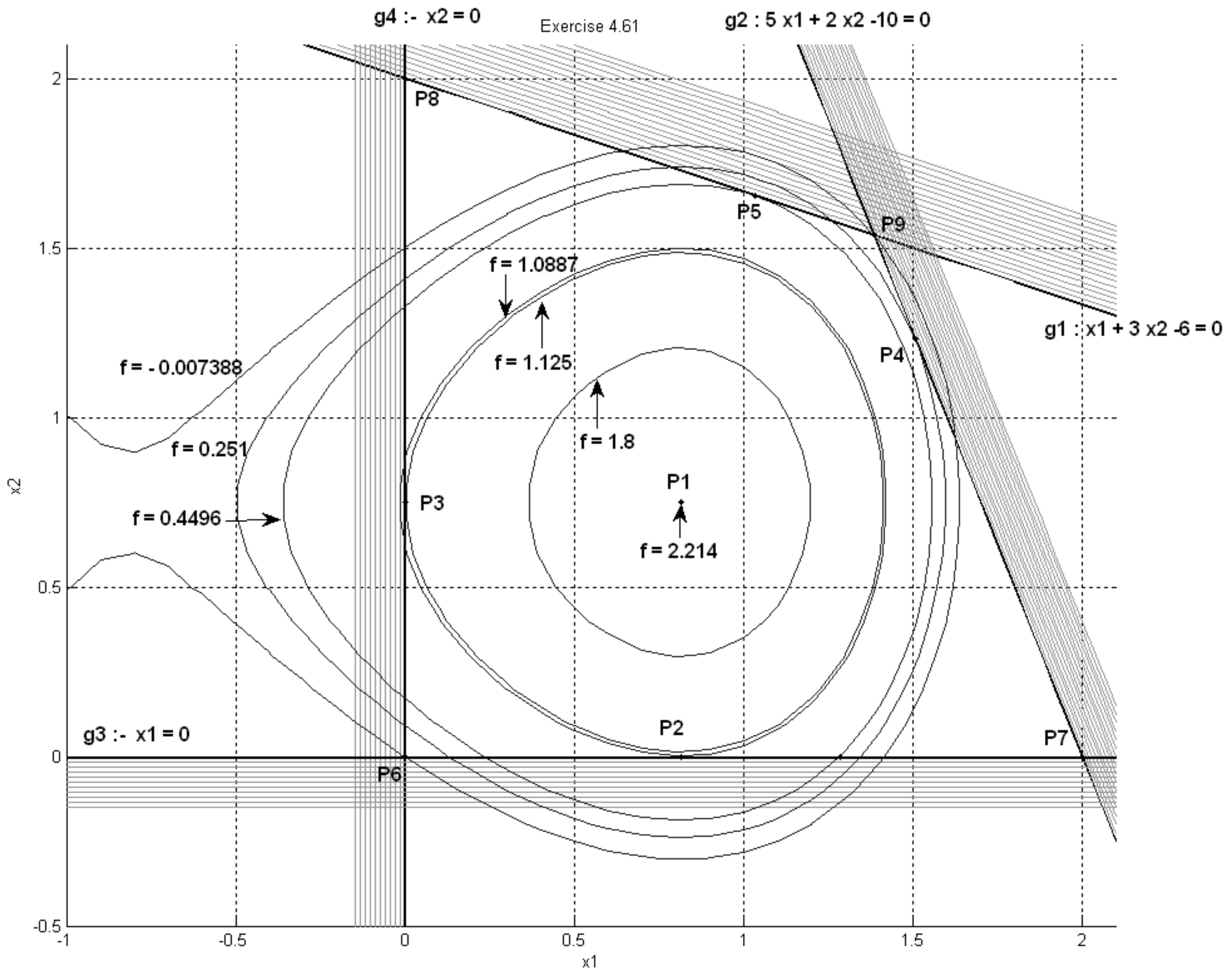
Minimize $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^3 - 2x_2^2$

subject to $x_1 + 3x_2 \leq 6$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.61, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	u_1^*	u_2^*	u_3^*	u_4^*
P1	0.816	0.75	0	0	0	0
P2	0.816	0	0	0	0	3
P3	0	0.75	0	0	2	0
P4	1.5073	1.2317	0	0.9632	0	0
P5	1.0339	1.655	1.2067	0	0	0
P6	0	0	0	0	2	3
P7	2	0	0	2	0	7
P8	0	2	1.667	0	3.667	0
P9	1.386	1.538	0.633	0.626	0	0

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of Lagrange function is $\nabla^2 L = \begin{bmatrix} -6x_1 & 0 \\ 0 & -4 \end{bmatrix}$ which is negative definite for all $x_1 > 0$.

The gradients of the constraints are as following

$$\nabla g_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

1. At point P1, $(0.813, 0.75)$, since no constraint is active, Hessian of Lagrangian must be positive definite throughout to satisfy sufficient condition. But it is always negative definite. So this is not an isolated minimum point (second order necessary condition is violated). Instead, it is a maximum point.
2. At point P2, $x_1 = 0.816,\ x_2 = 0,\ u_4 = 3,\ \nabla^2 L$ is negative definite. So, the point cannot be a minimum point because it violates second order necessary condition.
3. For points P3,P4 and P5, since the Hessian of the Lagrangian is negative definite, the three points cannot be local minima.
4. For points P6,P7,P8 and P9, the number of active constraints is equal to the number of design variables. There are no feasible directions in the neighborhood of the points that can reduce cost function any further. So, all the points are isolated local minima.

MATLAB Code for Exercise 114

```
clear all
axis equal
[x1,x2]=meshgrid(-1:0.1:2.1, -0.5:0.1:2.1);
f=2*x1+3*x2-x1.^3-2*x2.^2;

g1=x1+3*x2-6;
g2=5*x1+2*x2-10;
g3=-x1;
g4=-x2;

cla reset
axis equal
axis ([-1 2.1 -0.5 2.1])
xlabel('x1'),ylabel('x2')
title('Exercise 4.61')
hold on

cv1=[0:0.05:0.8];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x1,x2,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');

cv3=[0:0.015:0.15];
const3=contour(x1,x2,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x1,x2,g3,cv3,'k');

cv4=[0:0.015:0.15];
const4=contour(x1,x2,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x1,x2,g4,cv4,'k');

fv=[-0.007388 0.251 0.4496 1.0887 1.125 1.8 2.214];
fs=contour(x1,x2,f,fv,'b');

a=[0.816 0.816 0 1.5073 1.0339 0 2 0 1.386 1.28452];
b=[0.75 0 0.75 1.2317 1.655 0 0 2 1.538 0];
plot(a,b,'.k');

grid
hold off
```

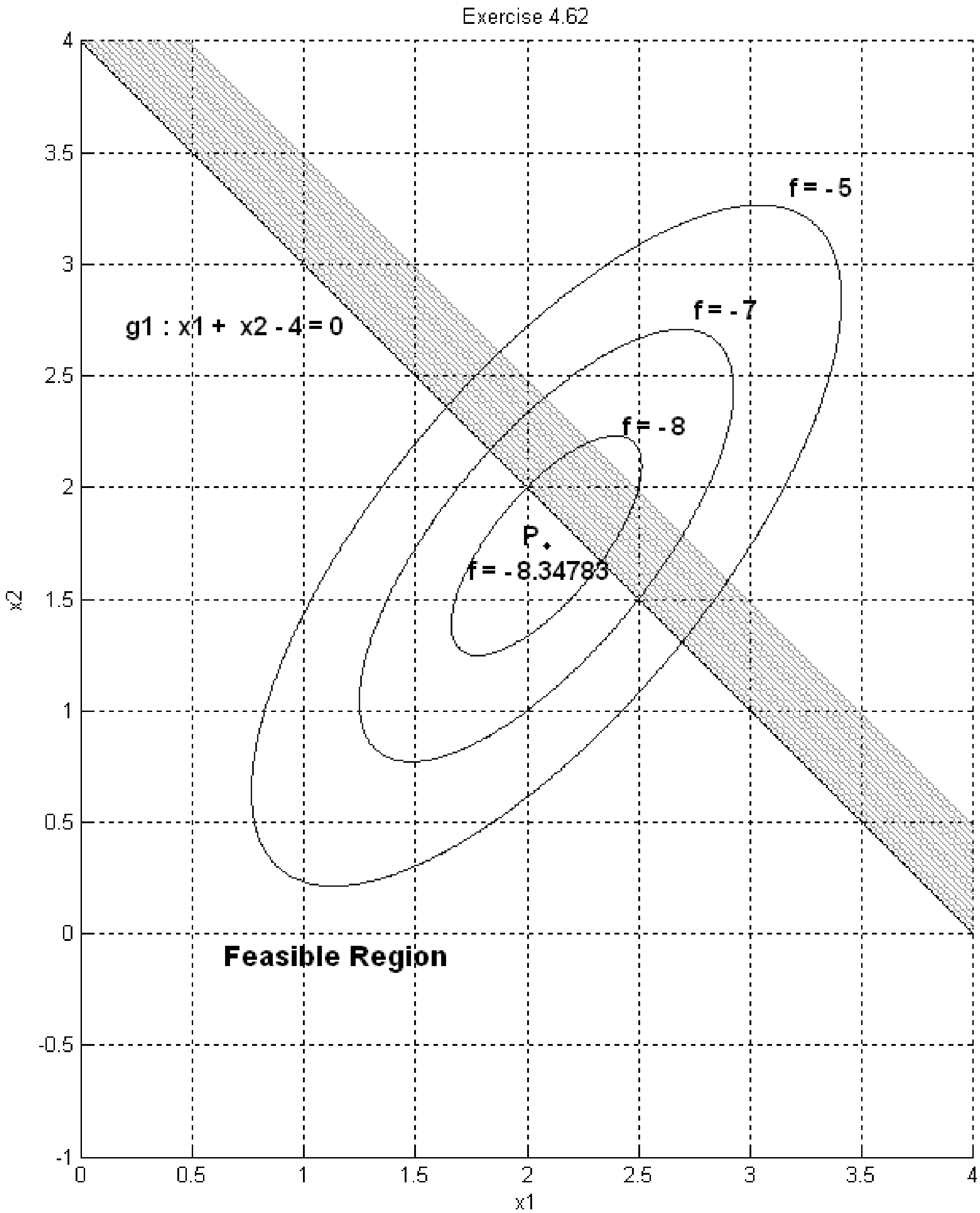
4.115

Exercise 4.62

Minimize $f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$

subject to $x_1 + x_2 \le 4$

Solution



Referring to Exercise 4.62, the point satisfying the KKT necessary conditions is
$$\mathrm{x}_1 = \frac{48}{23} = 2.0870, \mathrm{x}_2 = \frac{40}{23} = 1.7391 , \mathrm{u} = 0, \mathrm{f} = -\frac{192}{23} = 8.3478$$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$
The gradient of cost function is
$$\nabla \mathrm{f} = \begin{bmatrix} 8\mathrm{x}_1 - 5\mathrm{x}_2 - 8 \\ 6\mathrm{x}_2 - 5\mathrm{x}_1 \end{bmatrix}$$
Also,
$$\nabla^2 \mathrm{f} = \begin{bmatrix} 8 & -5 \\ -5 & 6 \end{bmatrix}$$
$\mathrm{M}_1 = 8 > 0, \mathrm{M}_2 = 48 - 25 = 23 > 0;$ Positive definite

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.
Also, it is a local minimum with $\mathrm{f}(\mathrm{x}^*) = -\frac{192}{23} = -8.3478$

<u>MATLAB Code for Exercise 115</u>

```
clear all
axis equal

[x1,x2]=meshgrid(0:0.01:4, -1:0.01:4);
f=(4*x1.^2+3*x2.^2-5*x1.*x2-8*x1);
g1=x1+x2-4;

cla reset
axis equal

axis ([0 4 -1 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.62')
hold on

cv1=[0:0.03:0.5];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[-8.34783 -8 -7 -5];
fs=contour(x1,x2,f,fv,'b');

a=[2.08696];
b=[1.73913];
plot(a,b,'.k');

grid
hold off
```

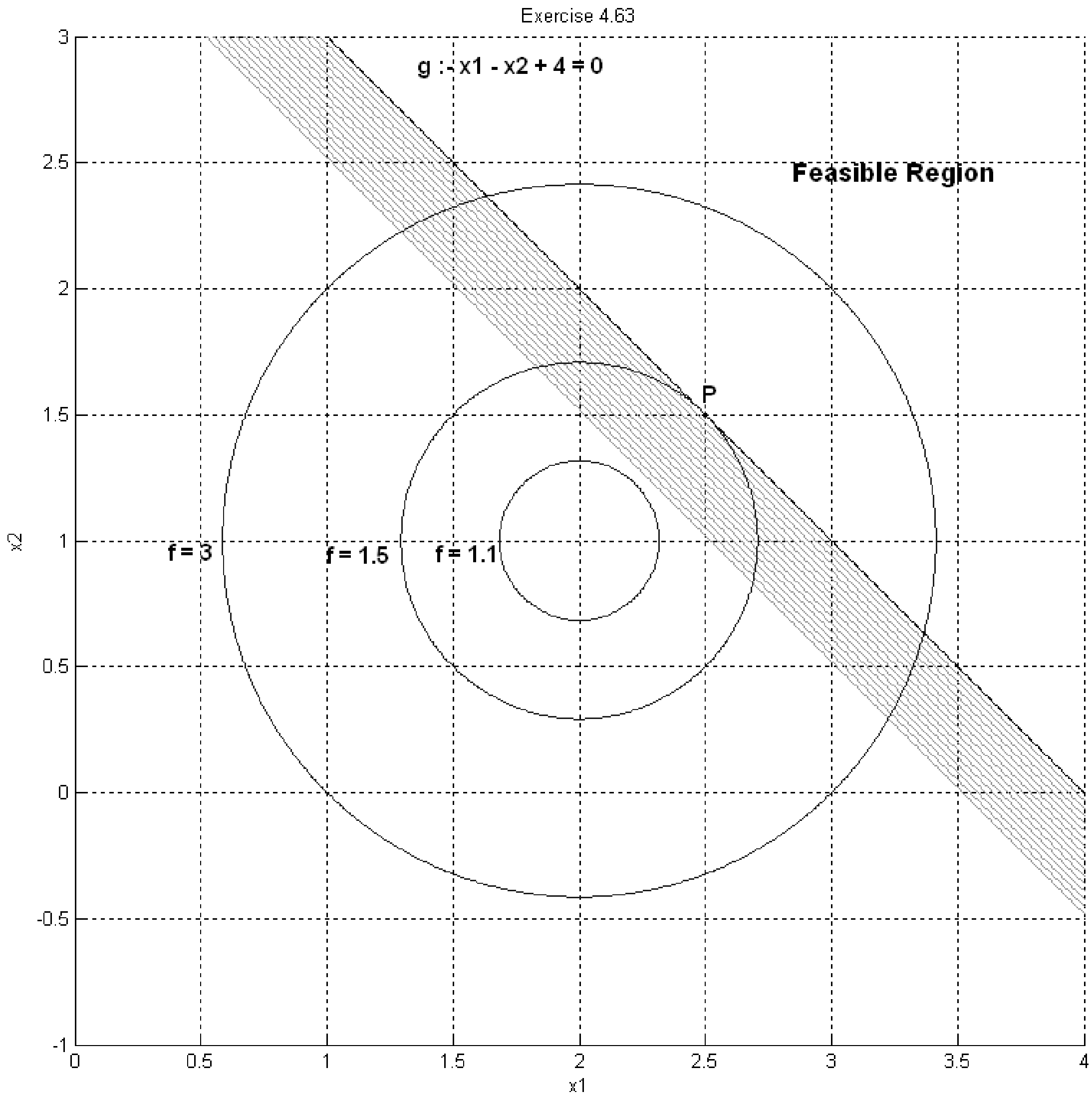
4.116

Exercise 4.63

Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$

subject to $x_1 + x_2 \geq 4$

Solution



Referring to Exercise 4.63, the point satisfying the KKT necessary conditions is
$x_1 = 2.5, x_2 = 1.5$, $u = 1, f = 1.5$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

$f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$
$g = -x_1 - x_2 + 4 \le 0$

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2x_1 - 4 \\ 2x_2 - 2 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

At optimum point P (2.5, 1.5)
$\nabla f(2.5, 1.5) = \begin{bmatrix} 2(2.5) - 4 \\ 2(1.5) - 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla g = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = u\nabla g$
$-\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 * -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\text{LHS} = \text{RHS} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
which shows local minimum point.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e} = -u^* = -(1)$$

If we set e= 1, the new value of cost function will be approximately
$f^* = 1.5 - (1)(1) = 0.5$

MATLAB Code for Exercise 116

```
clear all
axis equal

[x1,x2]=meshgrid(0:0.01:4, -1:0.01:3);
f=(x1.^2+x2.^2-4*x1-2*x2+6);
g1=-x1-x2+4;

cla reset
axis equal
axis ([0 4 -1 3])
xlabel('x1'),ylabel('x2')
title('Exercise 4.63')
hold on

cv1=[0:0.03:0.5];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[1.1 1.5 3];
fs=contour(x1,x2,f,fv,'b');

a=[2.5];
b=[1.5];
plot(a,b,'.k');

grid
hold off
```

4.117

Exercise 4.64

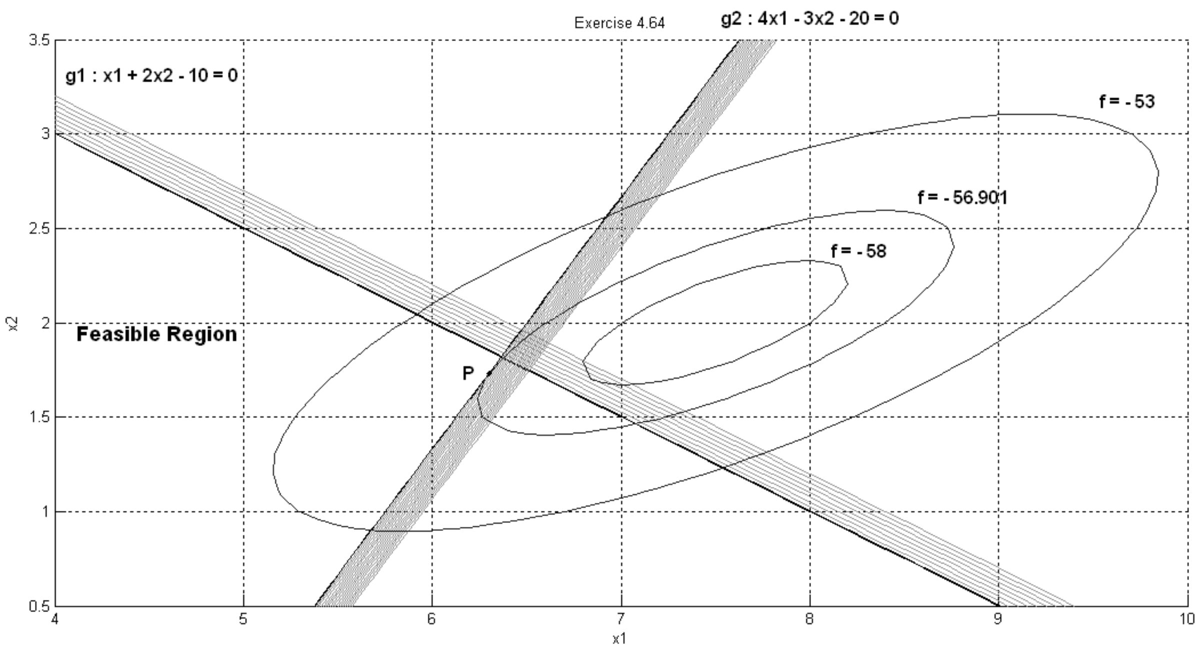
Minimize $f(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$

subject to $x_1 + 2x_2 \leq 10$

$4x_1 - 3x_2 \leq 20$

$x_i \geq 0; i = 1,2$

Solution



Referring to Exercise 4.64, the point satisfying the KKT necessary conditions is
$x_1 = 6.3, x_2 = 1.733, u_2 = 1, f = -56.901$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

$f(x_1, x_2) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$
$g_1 = x_1 + 2x_2 - 10 \leq 0$
$g_2 = 4x_1 - 3x_2 - 20 \leq 0$
$g_3 = -x_1 \leq 0$
$g_4 = -x_2 \leq 0$

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 4x_1 - 6x_2 - 18 \\ -6x_1 + 18x_2 + 9 \end{bmatrix}, \nabla g_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At optimum point P (6.3,1.733)
$\nabla f(6.3, 1.733) = \begin{bmatrix} 4x_1 - 6x_2 - 18 \\ -6x_1 + 18x_2 + 9 \end{bmatrix} = \begin{bmatrix} 4(6.3) - 6(1.733) - 18 \\ -6(6.3) + 18(1.733) + 9 \end{bmatrix} = \begin{bmatrix} -3.198 \\ 2.394 \end{bmatrix} = 2.394 \begin{bmatrix} -1.33 \\ 1 \end{bmatrix}$
$\nabla g_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 4 \\ -3 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = u_i \nabla g_i$
$-\begin{bmatrix} -3.198 \\ 2.394 \end{bmatrix} = 1 * \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$\text{LHS} = \text{RHS} \approx -\begin{bmatrix} 1.333 \\ 1 \end{bmatrix}$
which shows local minimum point.

By Theorem 4.7,
$\dfrac{\partial f(x^*)}{\partial e} = -u_2{}^* = -(1)$
If we set e= 1, the new value of cost function will be approximately
$f^* = -56.901 - (1)(1) = -57.901$

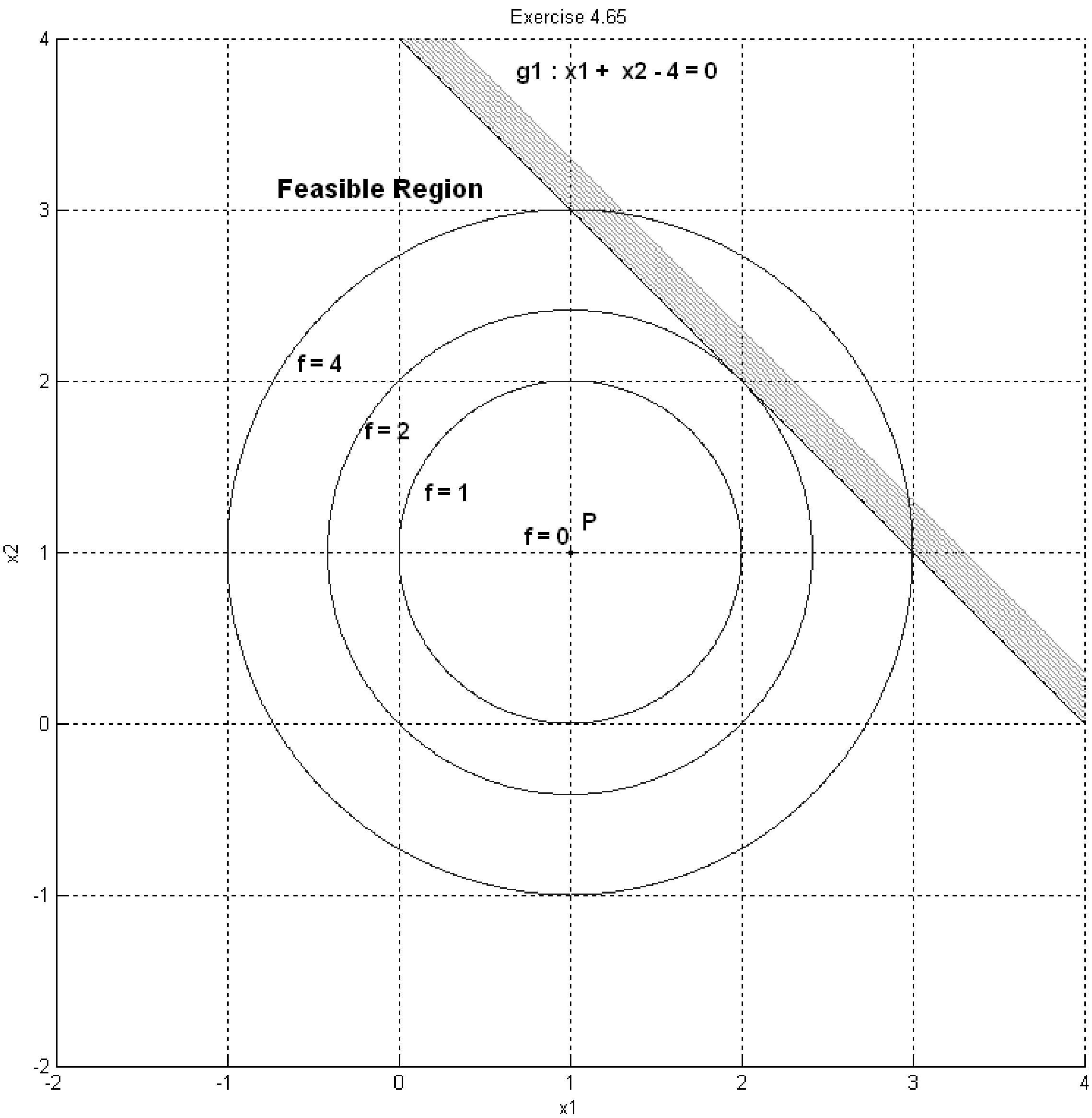
MATLAB Code for Exercise 117

```
clear all
axis equal
[x1,x2]=meshgrid(4:0.1:10, 0.5:0.1:3.5);

f=2*x1.^2-6*x1.*x2+9*x2.^2-18*x1+9*x2;
g1=x1+2*x2-10;
g2=4*x1-3*x2-20;
g3=-x1;
g4=-x2;

cla reset
axis equal
axis ([4 10 0.5 3.5])
xlabel('x1'),ylabel('x2')
title('Exercise 4.64')
hold on

cv1=[0:0.05:0.4];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x1,x2,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(x1,x2,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x1,x2,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(x1,x2,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x1,x2,g4,cv4,'k');

fv=[-58 -56.901 -53];
fs=contour(x1,x2,f,fv,'b');

a=[6.3];
b=[1.733];
plot(a,b,'.k');

grid
hold off
```

4.118

Exercise 4.65 Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
subject to $x_1 + x_2 - 4 \leq 0$

Solution



Referring to Exercise 4.65, the point satisfying the KKT necessary conditions is
$x_1 = 1, x_2 = 1, u = 0, f = 0$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
The gradient of cost function is
$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}$
Also,
$\nabla^2 f(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
$M_1 = 2 > 0, M_2 = 4 > 0;$ Positive definite

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.
Also, it is a local minimum with $f(x^*) = 0$

MATLAB Code for Exercise 118

```
clear all
axis equal

[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
g1=x1+x2-4;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.65')
hold on

cv1=[0:0.03:0.3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[0 1 2 4];
fs=contour(x1,x2,f,fv,'b');

a=[1];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.119

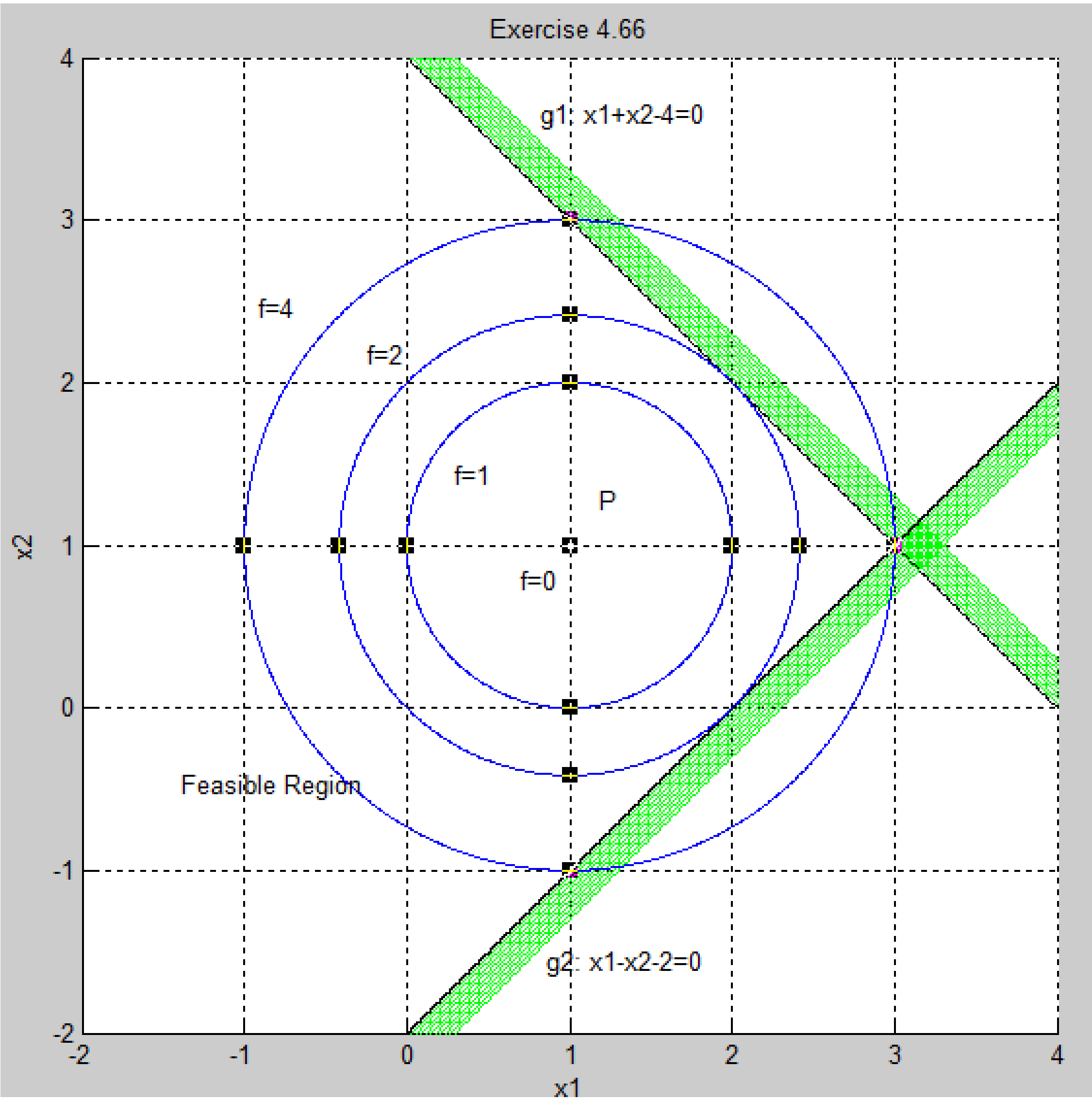
Exercise 4.66

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

subject to $x_1 + x_2 - 4 \leq 0$

$x_1 - x_2 - 2 \leq 0$

Solution



Referring to Exercise 4.66, the point satisfying the KKT necessary conditions is $x_1 = 1, x_2 = 1, u_1 = 0, u_2 = 0, f = 0$

$f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The gradient of cost function is

$$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}$$

Also,

$$\nabla^2 f(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$M_1 = 2 > 0, M_2 = 4 > 0;$ Positive definite

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.
Also, it is a local minimum with $f(x^*) = 0$

MATLAB Code for Exercise 119

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;
g1=x1+x2-4;
g2=x1-x2-2;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.66')
hold on

cv1=[0:0.03:0.3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');

cv2=[0:0.03:0.3];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');

fv=[0 1 2 4];
fs=contour(x1,x2,f,fv,'b');

a=[1];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.120

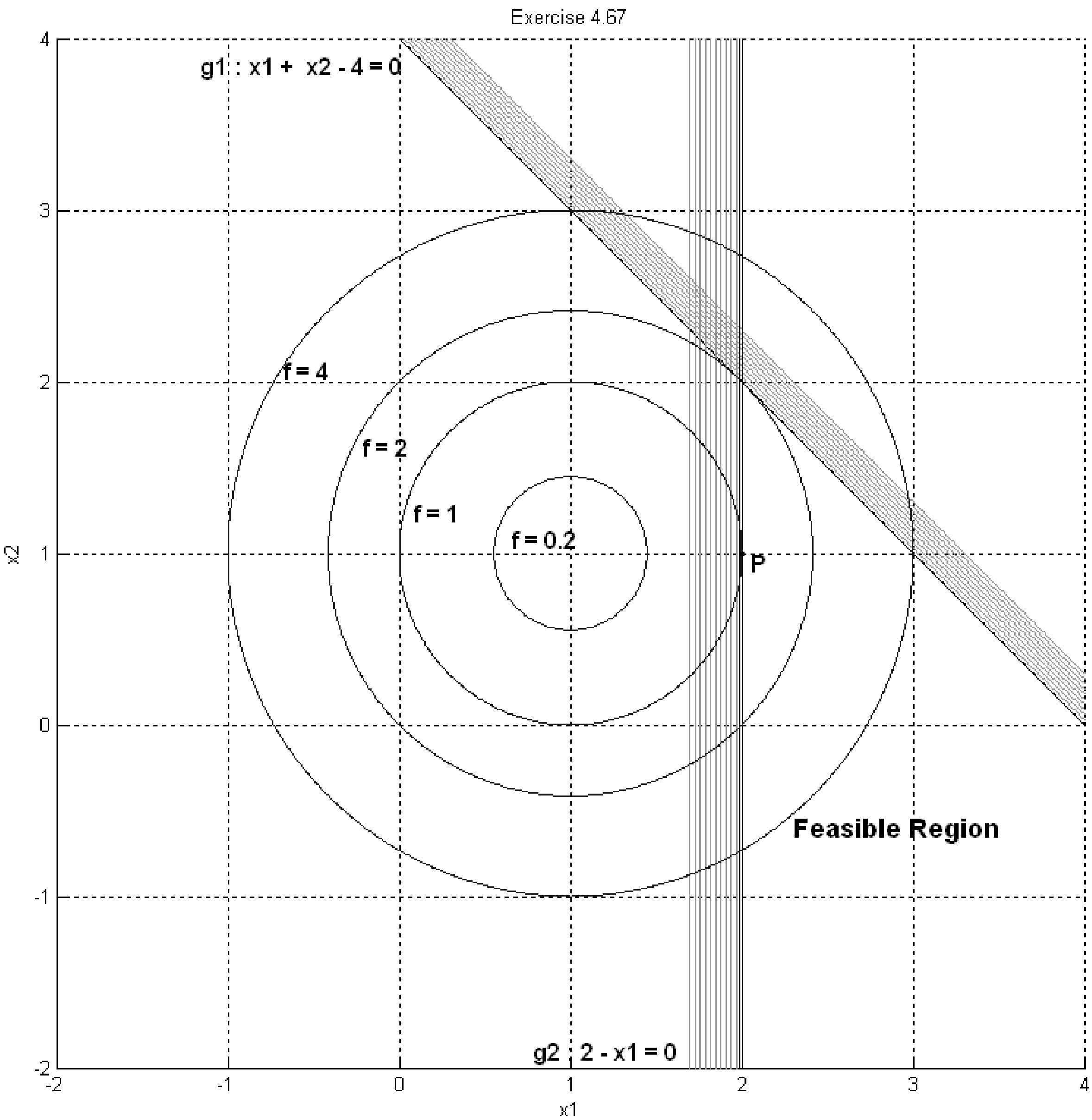
Exercise 4.67

Minimize $f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$

subject to $x_1 + x_2 - 4 \le 0$

$2 - x_1 \le 0$

Solution



Referring to Exercise 4.67, the point satisfying the KKT necessary conditions is
$x_1 = 2, x_2 = 1, u_2 = 2, f = 1$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

$f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 1)^2$
$g_1 = x_1 + x_2 - 4 \leq 0$
$g_2 = 2 - x_1 \leq 0$

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{bmatrix}, \nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

At optimum point P (2,1)
$\nabla f(2,1) = \begin{bmatrix} 2(2 - 1) \\ 2(1 - 1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\nabla g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = u_i \nabla g_i$
$-\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 * \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
$\text{LHS} = \text{RHS} = -\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
which shows local minimum point.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e} = -u^* = -(2)$$
If we set e= 1, the new value of cost function will be approximately
$f^* = 1 - (2)(1) = -1.5$

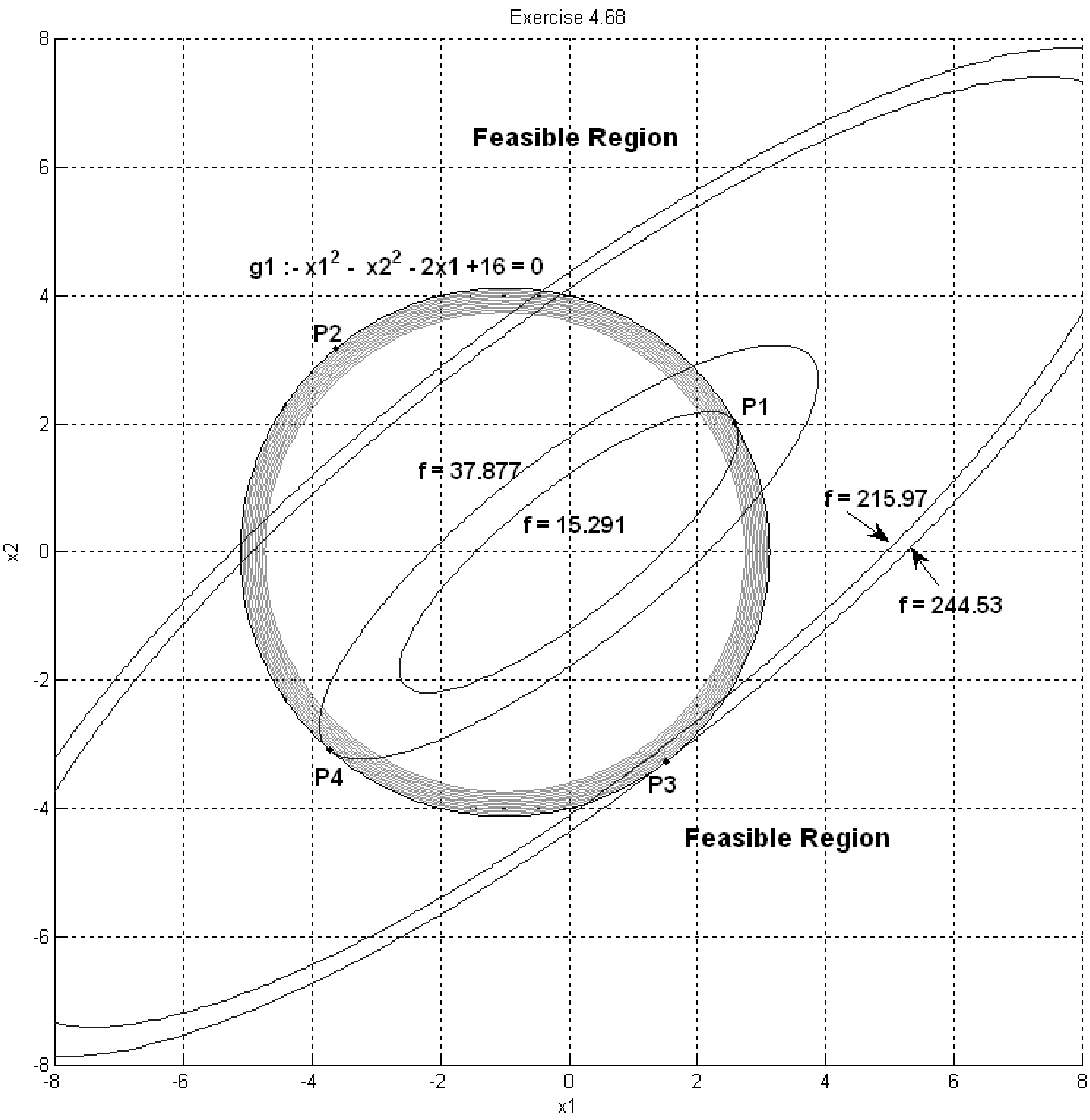
MATLAB Code for Exercise 120

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:4, -2:0.01:4);
f=(x1-1).^2+(x2-1).^2;

g1=x1+x2-4;
g2=2-x1;

cla reset
axis equal
axis ([-2 4 -2 4])
xlabel('x1'),ylabel('x2')
title('Exercise 4.67')
hold on

cv1=[0:0.03:0.3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');

cv2=[0:0.03:0.3];
const2=contour(x1,x2,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x1,x2,g2,cv2,'k');

fv=[0.2 1 2 4];
fs=contour(x1,x2,f,fv,'b');

a=[2];
b=[1];
plot(a,b,'.k');

grid
hold off
```

4.121

Exercise 4.68
Minimize $f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$
subject to $x_1^2 + x_2^2 + 2x_1 \geq 16$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.68, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	u_1^*
P1	2.5945	2.0198	1.4390
P2	-3.630	3.1754	23.2885
P3	1.5088	-3.2720	17.1503
P4	-3.7322	-3.0879	2.1222

$f(x_1, x_2) = 9x_1^2 - 18x_1x_2 + 13x_2^2 - 4$
$\mathrm{g} = -x_1^2 - x_2^2 - 2x_1 + 16 \leq 0$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

Hessian of cost function, the gradient and Hessian of the constraint are

$\nabla^2\mathrm{f} = \begin{bmatrix} 18 & -18 \\ -18 & 26 \end{bmatrix}$; $\nabla\mathrm{g} = \begin{bmatrix} -2\mathrm{x}_1 - 2 \\ -2\mathrm{x}_2 \end{bmatrix}$; $\nabla^2\mathrm{g} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

1. At point P1,

$\nabla^2\mathrm{L} = \nabla^2\mathrm{f} + \mathrm{u}\nabla^2\mathrm{g} = \begin{bmatrix} 15.122 & -18 \\ -18 & 23.122 \end{bmatrix}$

Since M_1 = 15.122 > 0, and M_2 = 25.6509 > 0, $\nabla^2\mathrm{L}$ is positive definite. Therefore, from *Theorem 5.3*, the point $\mathrm{x}_1 = 2.5945, \mathrm{x}_2 = 2.0198$ is an **isolated local minimum**.

2. At point P2,

$\nabla^2\mathrm{L} = \nabla^2\mathrm{f} + \mathrm{u}\nabla^2\mathrm{g} = \begin{bmatrix} -28.577 & -18 \\ -18 & -20.577 \end{bmatrix}$

$\nabla^2\mathrm{L}$ is not positive definite.
Let **d** $= \mathrm{d}_1 \cdot \mathrm{d}_2$. We need to find **d** such that $\nabla\mathrm{g} \cdot \mathbf{d} = 0$. This gives **d** $= \mathrm{c}(1, 0.82824)$, where $\mathrm{c} \neq 0$ is any constant. $Q = \mathbf{d}^\mathrm{T}(\nabla^2\mathrm{L})\mathbf{d} = -72.5091\mathrm{c}^2 < 0$ for $\mathrm{c} \neq 0$
The sufficient condition is not satisfied, so x_1 = - 3.630, x_2 = 3.1754 is not an isolated local minimum. Since Q < 0, second order necessary condition is violated, so the point cannot be a local minimum point.

3. At point P3,

$\nabla^2\mathrm{L} = \nabla^2\mathrm{f} + \mathrm{u}\nabla^2\mathrm{g} = \begin{bmatrix} -16.3006 & -18 \\ -18 & -8.3006 \end{bmatrix}$

$\nabla^2\mathrm{L}$ is not positive definite.
Let **d** $==[\mathrm{d}_1, \mathrm{d}_2]$. We need to find **d** such that $\nabla\mathrm{g} \cdot \mathbf{d} = 0$. This gives **d** $= \mathrm{c}(1, 0.7667)$, where $\mathrm{c} \neq 0$ is any constant. $Q = \mathbf{d}^\mathrm{T}(\nabla^2\mathrm{L})\mathbf{d} = -48.7835\mathrm{c}^2 < 0$ for $\mathrm{c} \neq 0$
The sufficient condition is not satisfied, so x_1 = 1.5088, x_2 = - 3.2720 is not an isolated local minimum. Since Q < 0, second order necessary condition is violated, so the point cannot be a local minimum point.

4. At point P4,

$\nabla^2\mathrm{L} = \nabla^2\mathrm{f} + \mathrm{u}\nabla^2\mathrm{g} = \begin{bmatrix} 13.7556 & -18 \\ -18 & 21.7556 \end{bmatrix}$

$\nabla^2 L$ is not positive definite
Let **d** ==$[d_1, d_2]$. We need to find **d** such that $\nabla g \cdot$ **d** = 0. This gives **d** =$c(1, -0.87509)$, where $c \neq 0$ is any constant. $Q = \mathbf{d}^T(\nabla^2 L)\mathbf{d} = 61.9189c^2 > 0$ for $c \neq 0$.
Therefore, from *Theorem 5.1*, the point $x_1 = -3.7322, x_2 = -3.0879$ is an **local minimum**.
So only point P1 and P4 have local minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 18x_1 - 18x_2 \\ -18x_1 + 26x_2 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} -2x_1 - 2 \\ -2x_2 \end{bmatrix}$

At point P1 (2.5945, 2.0198)
$\nabla f(2.5945, 2.0198) = \begin{bmatrix} 18(2.5945) - 18(2.0198) \\ -18(2.5945) + 26(2.0198) \end{bmatrix} = \begin{bmatrix} 10.3446 \\ 5.8138 \end{bmatrix} = 5.8138 \begin{bmatrix} 1.779 \\ 1 \end{bmatrix}$ and
$\nabla g = \begin{bmatrix} -7.189 \\ -4.0396 \end{bmatrix} = -4.0396 \begin{bmatrix} 1.779 \\ 1 \end{bmatrix}$
Note that they are along the same line.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e} = -u^* = -(1.4390)$$

If we set e = 1, the new value of cost function will be approximately
$f^* = 15.291 — (1.4390)(1) = 13.852$

At point P4 (-3.7322, - 3.0879)
$\nabla f(-3.7322, -3.0879) = \begin{bmatrix} 18(-3.7322) - 18(-3.0879) \\ -18(-3.7322) + 26(-3.0879) \end{bmatrix} = \begin{bmatrix} -11.5974 \\ -13.1058 \end{bmatrix} = -11.594 \begin{bmatrix} 1 \\ 1.13 \end{bmatrix}$ and
$\nabla g = \begin{bmatrix} -2(-3.7322) - 2 \\ -2(-3.0879) \end{bmatrix} = \begin{bmatrix} -5.4644 \\ -6.1758 \end{bmatrix} = -5.4644 \begin{bmatrix} 1 \\ 1.13 \end{bmatrix}$
Note that they are along the same line.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e} = -u^* = -(2.1222)$$

If we set e= 1, the new value of cost function will be approximately
$f^* = 37.877 — (2.1222)(1) = 35.7548$

MATLAB Code for Exercise 121

```
clear all
axis equal
[x1,x2]=meshgrid(-8:0.1:8, -8:0.1:8);
f=9*x1.^2-18*x1.*x2+13*x2.^2-4;
g1=-x1.^2-x2.^2-2*x1+16;

cla reset
axis equal
axis ([-8 8 -8 8])
xlabel('x1'),ylabel('x2')
title('Exercise 4.68')
hold on

cv1=[0:0.3:3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');

fv=[15.291 37.877 215.97 244.53 ];
fs=contour(x1,x2,f,fv,'b');

a=[2.5945 -3.630 1.5088 -3.7322];
b=[2.0198 3.1754 -3.2720 -3.0879];
plot(a,b,'.k');

grid
hold off
```

4.122

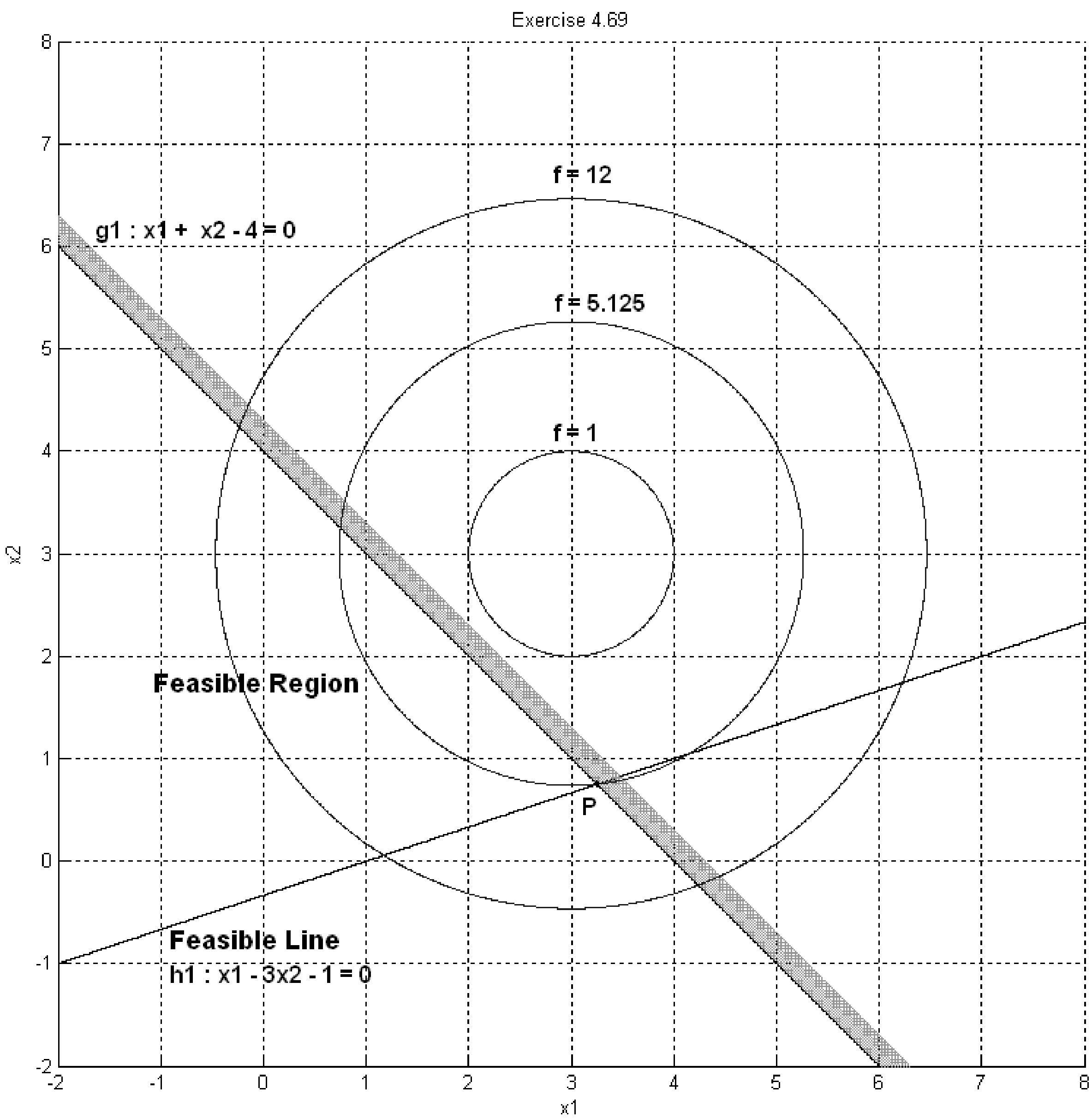
Exercise 4.69

Minimize $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$

subject to $x_1 + x_2 \le 4$

$$x_1 - 3x_2 = 1$$

Solution



Referring to Exercise 4.69, the point satisfying the KKT necessary conditions is
$x_1 = 3.25, x_2 = 0.75$, $\nu = -1.25, u = 0.75, f = 5.125$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

$f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 3)^2$
$g = x_1 + x_2 - 4 \leq 0$
$h = x_1 - 3x_2 - 1 = 0$

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x_1 - 3) \\ 2(x_2 - 3) \end{bmatrix}$, $\nabla h = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We need to check (4.52) from P.131
$- \nabla f = \nu \nabla h + u \nabla g$
$- \begin{bmatrix} 2(3.25 - 3) \\ 2(0.75 - 3) \end{bmatrix} = -1.25 * \begin{bmatrix} 1 \\ -3 \end{bmatrix} + 0.75 * \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\text{LHS} = \text{RHS} = \begin{bmatrix} -0.5 \\ 4 \end{bmatrix}$

which shows local minimum point.

By Theorem 4.7,
$\frac{\partial f(x^*)}{\partial b} = -\nu^* = -(-1.25)$
$\frac{\partial f(x^*)}{\partial e} = -u^* = -(0.75)$

If we set b= 1 and e= 1, the new value of cost function will be approximately
$f^* = 5.125 — (-1.25)(1) - (0.75)(1) = 5.625$

MATLAB Code for Exercise 122

```
clear all
axis equal
[x1,x2]=meshgrid(-2:0.01:8, -2:0.01:8);
f=(x1-3).^2+(x2-3).^2;
h1=x1-3*x2-1;
g1=x1+x2-4;

cla reset
axis equal
axis ([-2 8 -2 8])
xlabel('x1'),ylabel('x2')
title('Exercise 4.69')
hold on

cv1=[0 0.01];
const1=contour(x1,x2,h1,cv1,'k');

cv2=[0:0.03:0.3];
const2=contour(x1,x2,g1,cv2,'g');
cv2=[0 0.001];
const2=contour(x1,x2,g1,cv2,'k');
fv=[1 5.125 12];
fs=contour(x1,x2,f,fv,'b');

a=[3.25];
b=[0.75];
plot(a,b,'.k');

grid
hold off
```

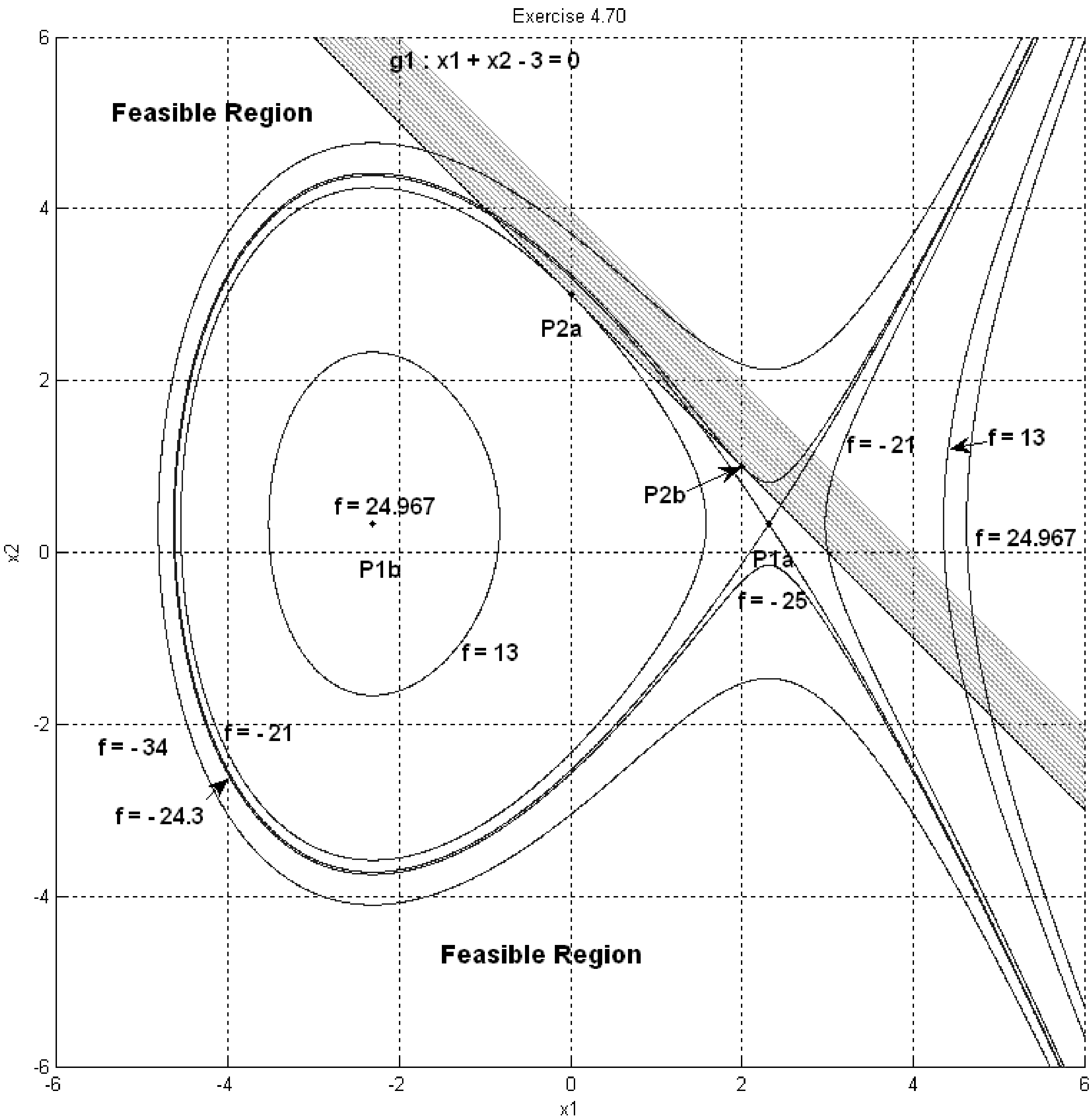
4.123

Exercise 4.70

Minimize $f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$

subject to $x_1 + x_2 \leq 3$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.70, the points satisfying the KKT necessary conditions are

	x_1^*	x_2^*	u_1^*
P1a	$\frac{4}{\sqrt{3}}$	$\frac{1}{3}$	0
P1b	$-\frac{4}{\sqrt{3}}$	$\frac{1}{3}$	0
P2a	0	3	16
P2b	2	1	4

$f(x_1, x_2) = x_1^3 - 16x_1 + 2x_2 - 3x_2^2$
g = $x_1 + x_2 - 3 \leq 0$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of Lagrange function is $\nabla^2 \mathrm{L} = \begin{bmatrix} 6\mathrm{x}_1 & 0 \\ 0 & -6 \end{bmatrix}$which is negative definite for all $x_1 < 0$.
The gradients of the constraints are as following
$\nabla \mathrm{g} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1. At point P1a $\left(\frac{4}{\sqrt{3}}, \frac{1}{3}\right)$,
Since no constraint is active, Hessian of Lagrangian must be positive definite throughout to satisfy sufficient condition. But it is indefinite. So this is not an isolated minimum point (second order necessary condition is violated).
At point P1b $\left(-\frac{4}{\sqrt{3}}, \frac{1}{3}\right)$,
Since no constraint is active, Hessian of Lagrangian must be positive definite throughout to satisfy sufficient condition. But it is always negative definite. So this is not an isolated minimum point (second order necessary condition is violated). Instead, it is a maximum point.

2. At point P2a$(0, 3)$,
$\nabla^2 \mathrm{L} = \begin{bmatrix} 0 & 0 \\ 0 & -6 \end{bmatrix}$
Hessian of Lagrangian is negative semidefinite.
Let **d** ==$[\mathrm{d}_1, \mathrm{d}_2]$. . We need to find $\mathbf{d}$ such that $\nabla \mathrm{g} \cdot \mathbf{d}$ = 0. This gives **d** =$\mathrm{c}(1, -1)$, where $\mathrm{c} \neq 0$ is any constant. $Q = \mathbf{d}^{\mathrm{T}}(\nabla^2 \mathrm{L})\mathbf{d} = -6\mathrm{c}^2 < 0$ for $\mathrm{c} \neq 0$
The sufficient condition is not satisfied, so $x_1 = 0$, $x_2 = 3$ not an isolated local minimum. Since $Q < 0$, second order necessary condition is violated, so the point cannot be a local minimum point.

At point P2b, $(2, 1)$,
$\nabla^2 \mathrm{L} = \begin{bmatrix} 12 & 0 \\ 0 & -6 \end{bmatrix}$

Hessian of Lagrangian is indefinite.
Let $\mathbf{d}$ =$[\mathrm{d}_1, \mathrm{d}_2]$. We need to find $\mathbf{d}$ such that $\nabla g \cdot \mathbf{d} = 0$. This gives $\mathbf{d}$ =$\mathrm{c}(1, -1)$, where $\mathrm{c} \neq 0$ is any constant. $Q = \mathbf{d}^{\mathrm{T}}(\nabla^2 \mathrm{L})\mathbf{d} = 6\mathrm{c}^2 > 0$ for $\mathrm{c} \neq 0$
The sufficient condition is satisfied, so $x_1 = 0$, $x_2 = 3$ an isolated local minimum.

MATLAB Code for Exercise 123

```
clear all
axis equal
[x1,x2]=meshgrid(-6:0.01:6, -6:0.01:6);
f=x1.^3-16*x1+2*x2-3*x2.^2;
g1=x1+x2-3;

cla reset
axis equal
axis ([-6 6 -6 6])
xlabel('x1'),ylabel('x2')
title('Exercise 4.70')
hold on

cv1=[0:0.07:1];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');

fv=[-34 -24.3 -25 -21 13 24.967];
fs=contour(x1,x2,f,fv,'b');

a=[4/sqrt(3) -4/sqrt(3) 0 2];
b=[1/3 1/3 3 1];
plot(a,b,'.k');

grid
hold off
```

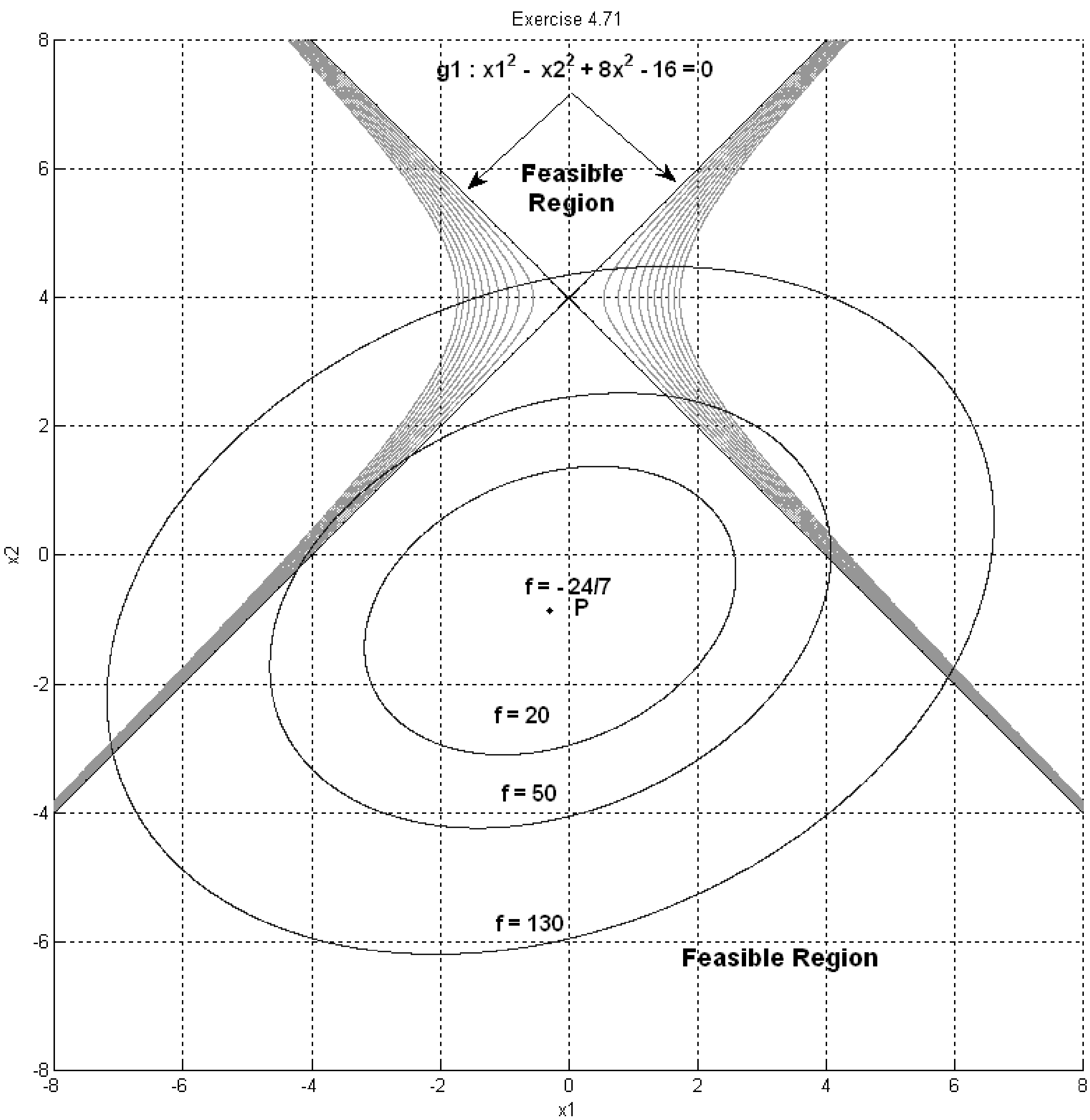
4.124

Exercise 4.71

Minimize $f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$

subject to $x_1^2 - x_2^2 + 8x_2 \le 16$

Solution



Referring to Exercise 4.71, the point satisfying the KKT necessary conditions is
$x_1 = -\frac{2}{7}, x_2 = -\frac{6}{7}, u = 0, f = \frac{-24}{7}$

$f(x_1, x_2) = 3x_1^2 - 2x_1x_2 + 5x_2^2 + 8x_2$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The gradient of cost function is
$\nabla f = \begin{bmatrix} 6x_1 - 2x_2 \\ -2x_1 + 10x_2 + 8 \end{bmatrix}$
Also,
$\nabla^2 f(-\frac{2}{7}, -\frac{6}{7}) = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}$
$M_1 = 6 > 0, M_2 = 60 - 4 = 56 > 0;$ Positive definite

The Hessian of cost function is positive definite; therefore, Hessian of the Lagrangian is positive definite. Second order sufficiency condition is satisfied; the point is a local minimum with $f(x^*) = \frac{-24}{7}$.

MATLAB Code for Exercise 124

```
clear all
axis equal
[x1,x2]=meshgrid(-8:0.01:8, -8:0.01:8);
f=3*x1.^2-2*x1.*x2+5*x2.^2+8*x2;
g1=x1.^2-x2.^2+8*x2-16;

cla reset
axis equal
axis ([-8 8 -8 8])
xlabel('x1'),ylabel('x2')
title('Exercise 4.71')
hold on

cv1=[0:0.3:3];
const1=contour(x1,x2,g1,cv1,'g');
cv1=[0 0.001];
const1=contour(x1,x2,g1,cv1,'k');
fv=[-24/7 20 50 130];
fs=contour(x1,x2,f,fv,'b');

a=[-2/7];
b=[-6/7];
plot(a,b,'.k');

grid
hold off
```

4.125

Exercise 4.72

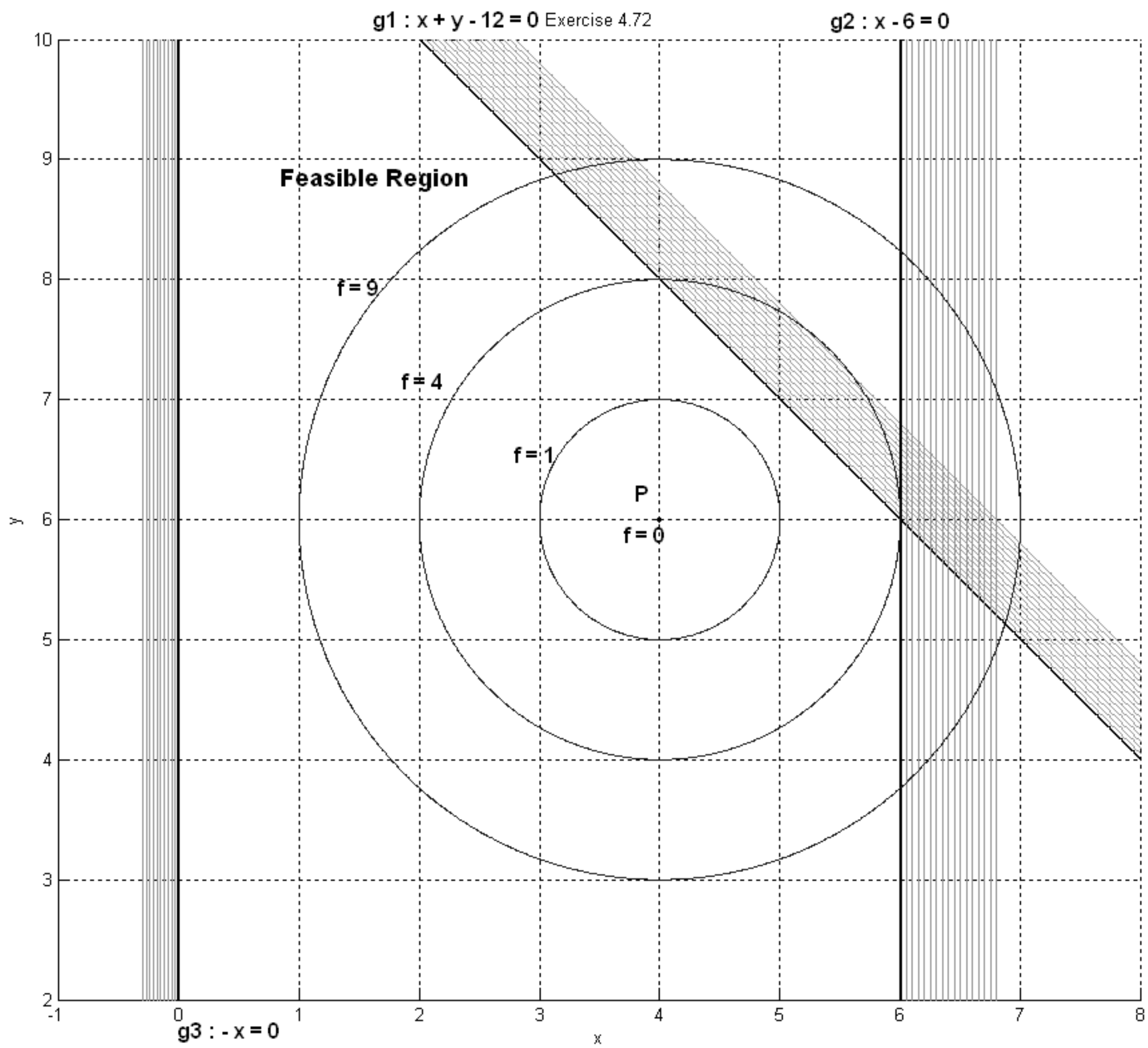
Minimize $f(x, y) = (x-4)^2 + (y-6)^2$

subject to $x + y \leq 12$

$x \leq 6$

$x, y \geq 0$

Solution



Referring to Exercise 4.72, the point satisfying the KKT necessary conditions is
$x = 4, y = 6, u = 0, f = 0$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(x, y) = (x - 4)^2 + (y - 6)^2$
The gradient of cost function is
$\nabla f = \begin{bmatrix} 2(x-4) \\ 2(y-6) \end{bmatrix}$
Also,
$\nabla^2 f(4,6) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
$M_1 = 2 > 0, M_2 = 4 > 0$; Positive definite

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.
Also, it is a local minimum with $f(x^*) = 0$

MATLAB Code for Exercise 125

```
clear all
axis equal
[x,y]=meshgrid(-1:0.01:8, 2:0.01:10);
f=(x-4).^2+(y-6).^2;

g1=x+y-12;
g2=x-6;
g3=-x;
g4=-y;

cla reset
axis equal
axis ([-1 8 2 10])
xlabel('x'),ylabel('y')
title('Exercise 4.72')
hold on

cv1=[0:0.05:0.8];
const1=contour(x,y,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x,y,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x,y,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x,y,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(x,y,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x,y,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(x,y,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x,y,g4,cv4,'k');

fv=[0 1 4 9];
fs=contour(x,y,f,fv,'b');

a=[4];
b=[6];
plot(a,b,'.k');

grid
hold off
```

4.126

Exercise 4.73

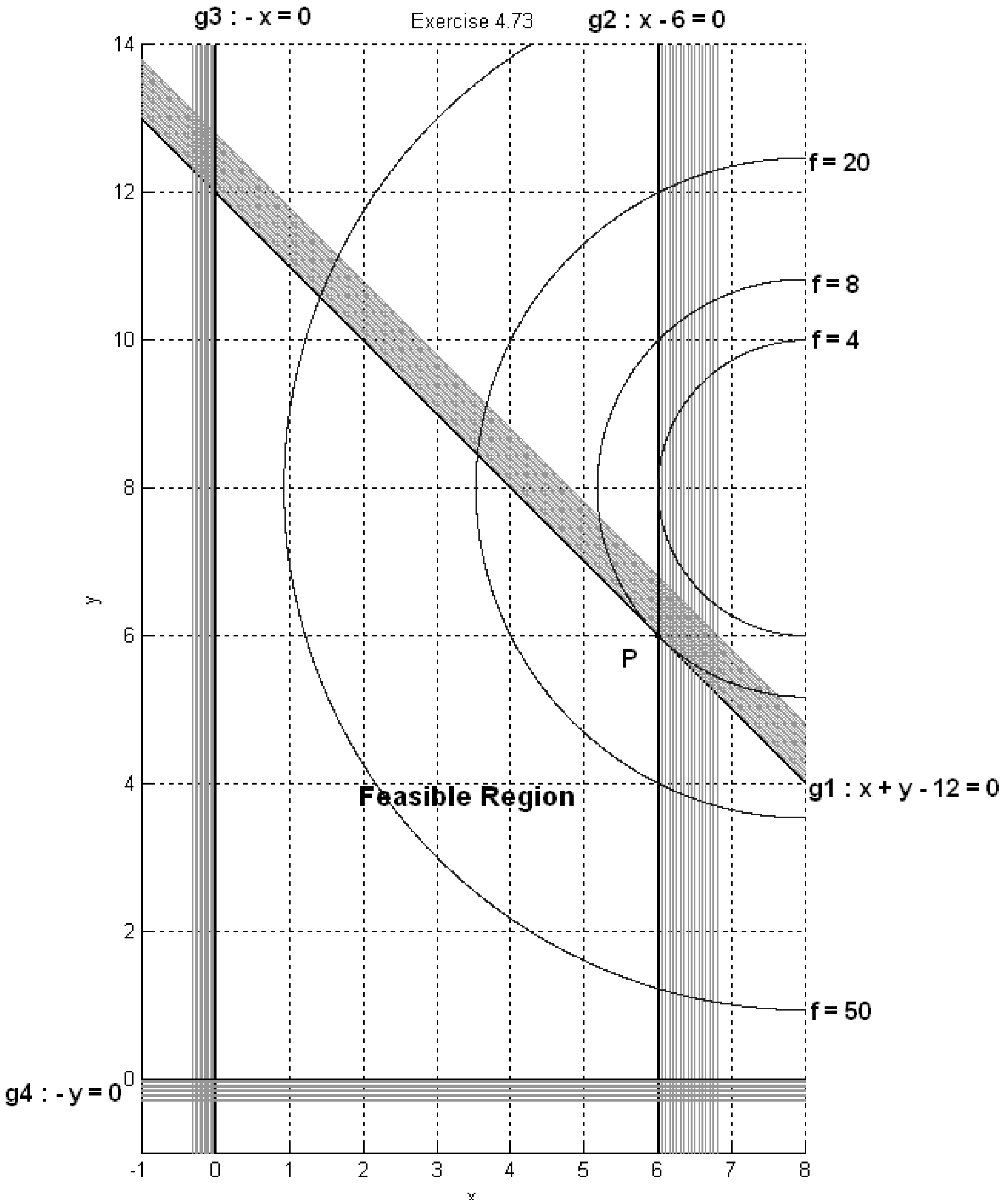
Minimize $f(x, y) = (x - 8)^2 + (y - 8)^2$

subject to $x + y \leq 12$

$x \leq 6$

$x, y \geq 0$

Solution



Referring to Exercise 4.73, the points satisfying the KKT necessary conditions are
$x = y = 6, u_1 = 4, u_2 = 0, u_3 = 0, u_4 = 0$ and $f = 8$

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

The Hessian of cost function is positive definite, and the constraint function is linear. So, this is a convex problem. It follows from Theorem 4.11 the point is an isolated global minimum.

$f(x, y) = (x - 8)^2 + (y - 8)^2$
$g_1 = x + y - 12 \leq 0$
$g_2 = x - 6 \leq 0$
$g_3 = -x \leq 0$
$g_4 = -y \leq 0$
The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} 2(x-8) \\ 2(y-8) \end{bmatrix}, \nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At optimum point P (6, 6)
$\nabla f(6,6) = \begin{bmatrix} 2(6-8) \\ 2(6-8) \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

We need to check (4.52) from P.131
$-\nabla f = u_1 \nabla g_1 + u_2 \nabla g_2 + u_3 \nabla g_3 + u_4 \nabla g_4$

$- \begin{bmatrix} -4 \\ -4 \end{bmatrix} = 4 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 * \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 * \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 0 * \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\text{LHS} = \text{RHS} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$
which shows local minimum point.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(4)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(0)$$

If we set e_1=1, the new value of cost function will be approximately
$f^* = 8 — (4)(1) = 4$

MATLAB Code for Exercise 126

```
clear all
axis equal
[x,y]=meshgrid(-1:0.01:8, -1:0.01:14);
f=(x-8).^2+(y-8).^2;

g1=x+y-12;
g2=x-6;
g3=-x;
g4=-y;

cla reset
axis equal
axis ([-1 8 -1 14])
xlabel('x'),ylabel('y')
title('Exercise 4.73')
hold on

cv1=[0:0.05:0.8];
const1=contour(x,y,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x,y,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x,y,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x,y,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(x,y,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x,y,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(x,y,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x,y,g4,cv4,'k');

fv=[4 8 20 50];
fs=contour(x,y,f,fv,'b');

a=[6];
b=[6];
plot(a,b,'.k');

grid
hold off
```

4.127

Exercise 4.74

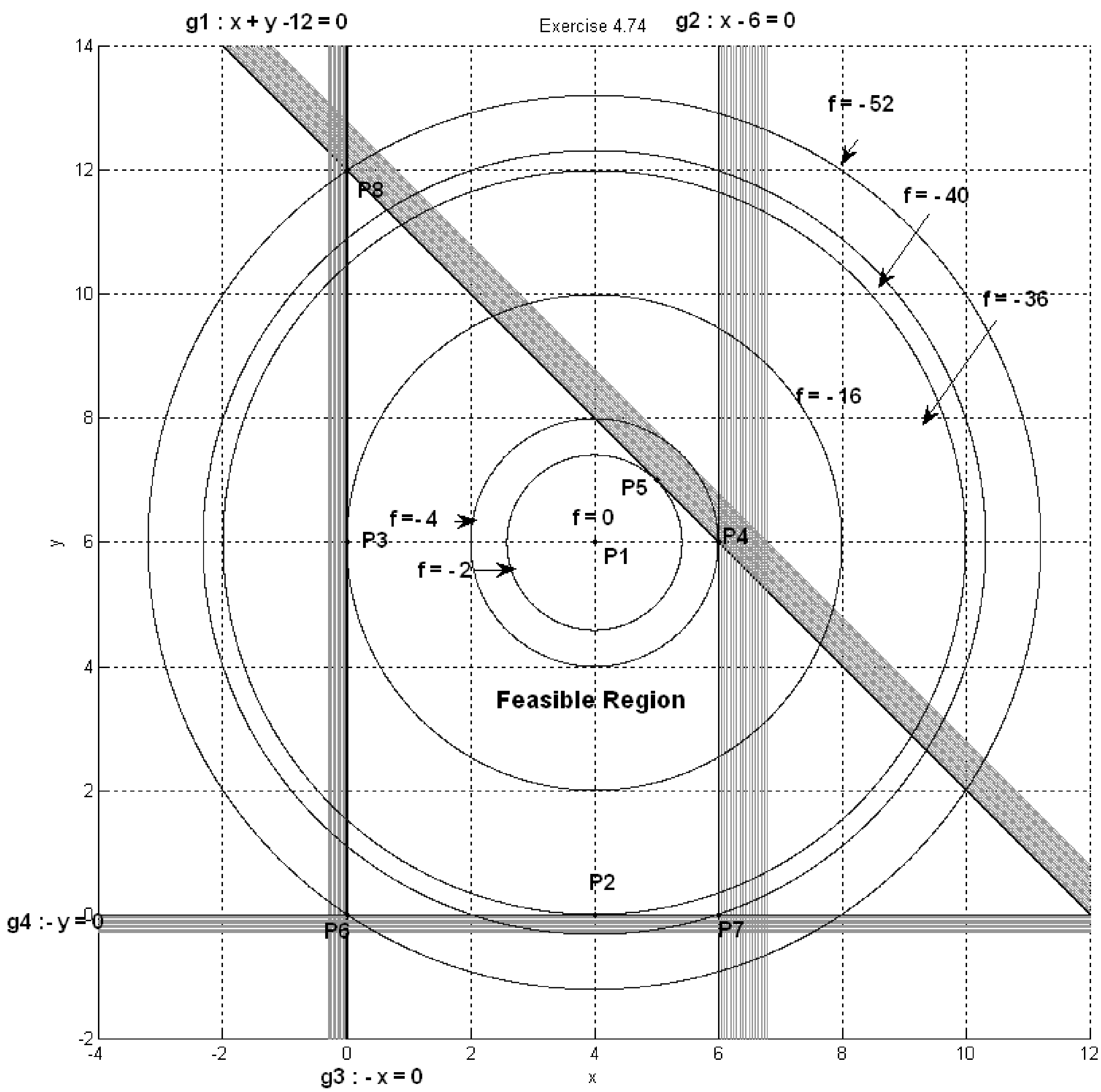
Maximize $F(x, y) = (x - 4)^2 + (y - 6)^2$

subject to $x + y \leq 12$

$6 \geq x$

$x, y \geq 0$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.74, the points satisfying the KKT necessary conditions are

	x	y	u_1	u_2	u_3	u_4
P1	4	6	0	0	0	0
P2	4	0	0	0	0	12
P3	0	6	0	0	8	0
P4	6	6	0	4	0	0
P5	5	7	2	0	0	0
P6	0	0	0	0	8	12
P7	6	0	0	4	0	12
P8	0	12	12	0	20	0

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(x,y) = -(x-4)^2 - (y-6)^2$
$g_1 = x + y - 12 \leq 0$
$g_2 = x - 6 \leq 0$
$g_3 = -x \leq 0$
$g_4 = -y \leq 0$

$\nabla^2 L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$; $M_1 = -2 < 0, M_2 = 4 > 0$; Negative definite
Gradient of constraints are
$\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

1. At point P1 (4, 6)
Since no constraint is active, Hessian of Lagrangian must be positive definite throughout to satisfy sufficient condition. But it is always negative definite. So this is not an isolated minimum point for *f* . (the second order necessary condition is violated). Instead, it is a maximum point for *f*.

2. At points P2 (4, 0), P3 (0,6) , P4 (6,6)and P5 (5,7)
Since the Hessian of the Lagrangian is negative definite, the four points cannot be local minima for *f*. Or, local maxima for *F*. The second order necessary condition is violated.

3. At points P6 (0, 0) and P7 (6, 0)
The number of active constraints is equal to the number of design variables. There are no feasible directions in the neighborhood of the points that can reduce cost function any further. So, these points are local minima for *f*.

So only point P6, P7 and P8 have local minimum for *f*.

The gradient of cost and constraint functions are

$\nabla f = \begin{bmatrix} -2(x-4) \\ -2(y-6) \end{bmatrix}$ and $\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\nabla g_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\nabla g_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At point P6 (0, 0)
By (4.52),
$-\nabla f(0,0) = -\begin{bmatrix} -2(0-4) \\ -2(0-6) \end{bmatrix} = -8\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$ and
$u_3\nabla g_3 + u_4\nabla g_4 = 8\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 12\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix} = -8\begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$
Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(8)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(12)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately
$f^* = -52 — (8)(1) — (12)(1) = -72$

At point P7 (6, 0)
By (4.52),
$-\nabla f(6,0) = -\begin{bmatrix} -2(6-4) \\ -2(0-6) \end{bmatrix} = -4\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and
$u_2\nabla g_2 + u_4\nabla g_4 = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 12\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -3 \end{bmatrix}$
Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(4)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(12)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately
$f^* = -40 — (4)(1) — (12)(1) = -56$

MATLAB Code for Exercise 127

```
clear all
axis equal
[x,y]=meshgrid(-4:0.01:12, -2:0.01:14);
f=(-1)*((x-4).^2+(y-6).^2);

g1=x+y-12;
g2=x-6;
g3=-x;
g4=-y;

cla reset
axis equal
axis ([-4 12 -2 14])
xlabel('x'),ylabel('y')
title('Exercise 4.74')
hold on

cv1=[0:0.05:0.8];
const1=contour(x,y,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(x,y,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(x,y,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(x,y,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(x,y,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(x,y,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(x,y,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(x,y,g4,cv4,'k');

fv=[-52 -40 -36 -16 -4 -2 0];
fs=contour(x,y,f,fv,'b');

a=[4 6 0 6 5 0 6 0 6 4];
b=[6 0 6 6 7 0 0 12 6 0];
plot(a,b,'.k');

grid
hold off
```

4.128

Exercise 4.75

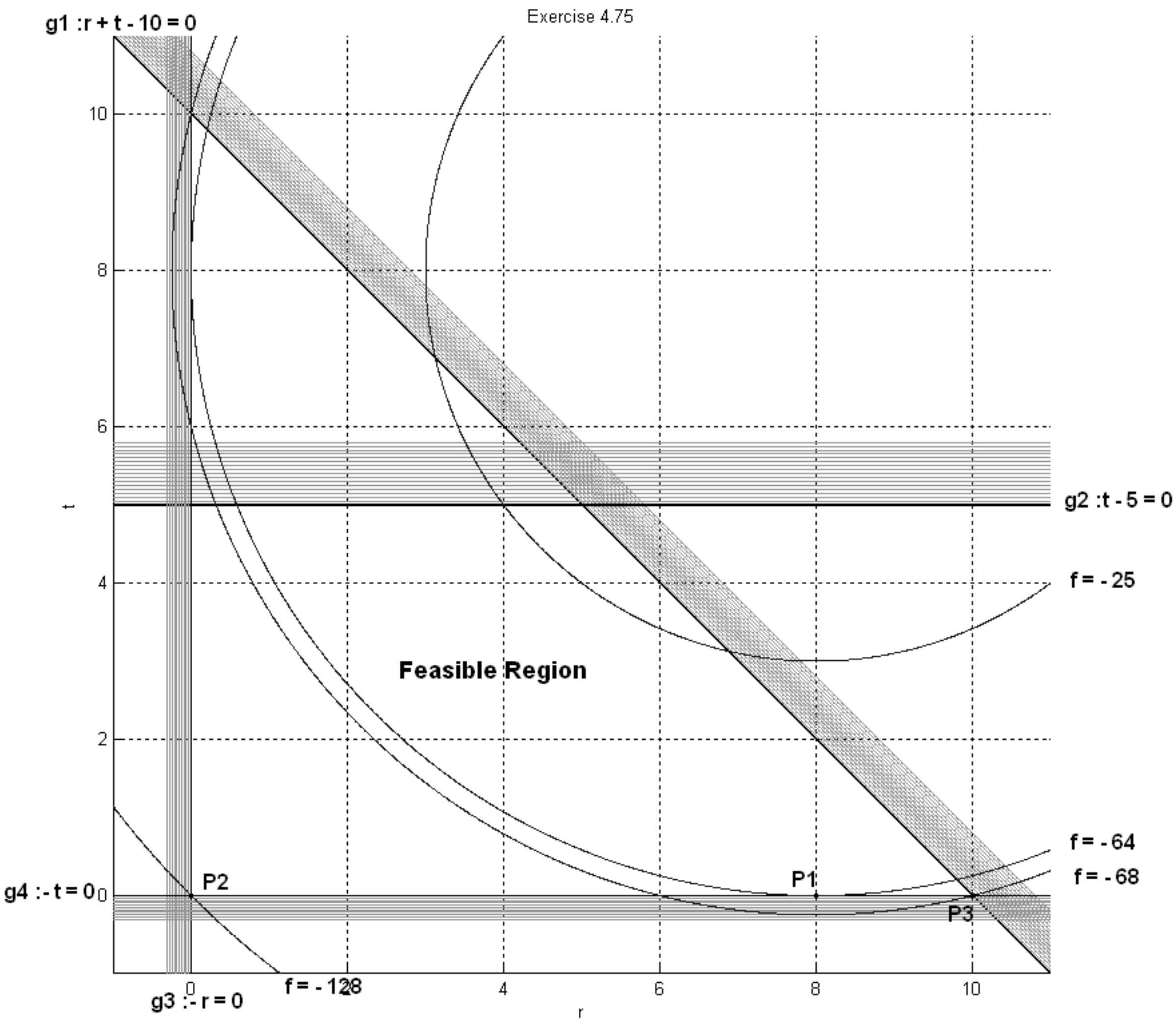
Maximize $F(r,t) = (r-8)^2 + (t-8)^2$

subject to $10 \geq r + t$

$t \leq 5$

$r, t \geq 0$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.75, the points satisfying the KKT necessary conditions are

	r	t	u_1	u_2	u_3	u_4
P1	8	0	0	0	0	4
P2	0	0	0	0	16	16
P3	10	0	4	0	0	20

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(r,t) = -(r-8)^2 - (t-8)^2$
$g_1 = r + t - 10 \le 0$
$g_2 = t - 5 \le 0$
$g_3 = -r \le 0$
$g_4 = -t \le 0$

$\nabla^2 L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$; $M_1 = -2 < 0, M_2 = 4 > 0$; Negative definite

Gradient of constraints are
$\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

1. At points P1 (8, 0)
Since the Hessian of the Lagrangian is negative definite, the point cannot be local minima.

2. At point P2 (0,0) and P3 (10, 0)
The number of active constraints is equal to the number of design variables. There are no feasible directions in the neighborhood of the points that can reduce cost function any further. So, these points are isolated local minima.

So only point P2 and P3 has local minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} -2(r-8) \\ -2(t-8) \end{bmatrix}$ and $\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At point P2 (0, 0)
By (4.52),
$-\nabla f(0,0) = -\begin{bmatrix} -2(0-8) \\ -2(0-8) \end{bmatrix} = -16\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and
$u_3 \nabla g_3 + u_4 \nabla g_4 = 16\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 16\begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\begin{bmatrix} 16 \\ 16 \end{bmatrix}$
Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(0)$$

$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(0)$$

$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(16)$$

$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(16)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately

$$f^* = -128 - (16)(1) - (16)(1) = -160$$

At point P3 (10, 0)

By (4.52),

$$-\nabla f(10,0) = -\begin{bmatrix}-2(10-8)\\-2(0-8)\end{bmatrix} = -\begin{bmatrix}-4\\16\end{bmatrix} = 4\begin{bmatrix}1\\-4\end{bmatrix} \text{ and}$$

$$u_1\nabla g_1 + u_4\nabla g_4 = 4\begin{bmatrix}1\\1\end{bmatrix} + 20\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}4\\-16\end{bmatrix} = 4\begin{bmatrix}1\\-4\end{bmatrix}$$

Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(4)$$

$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(0)$$

$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(0)$$

$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(20)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately

$$f^* = -68 - (4)(1) - (20)(1) = -92$$

MATLAB Code for Exercise 128

```
clear all
axis equal
[r,t]=meshgrid(-1:0.01:11, -1:0.01:11);
f=(-1)*((r-8).^2+(t-8).^2);

g1=r+t-10;
g2=t-5;
g3=-r;
g4=-t;

cla reset
axis equal
axis ([-1 11 -1 11])
xlabel('r'),ylabel('t')
title('Exercise 4.75')
hold on

cv1=[0:0.05:0.8];
const1=contour(r,t,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(r,t,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(r,t,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(r,t,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(r,t,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(r,t,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(r,t,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(r,t,g4,cv4,'k');

fv=[-25 -64 -68 -128];
fs=contour(r,t,f,fv,'b');

a=[8 0 10];
b=[0 0 0];
plot(a,b,'.k');

grid
hold off
```

4.129

Exercise 4.76

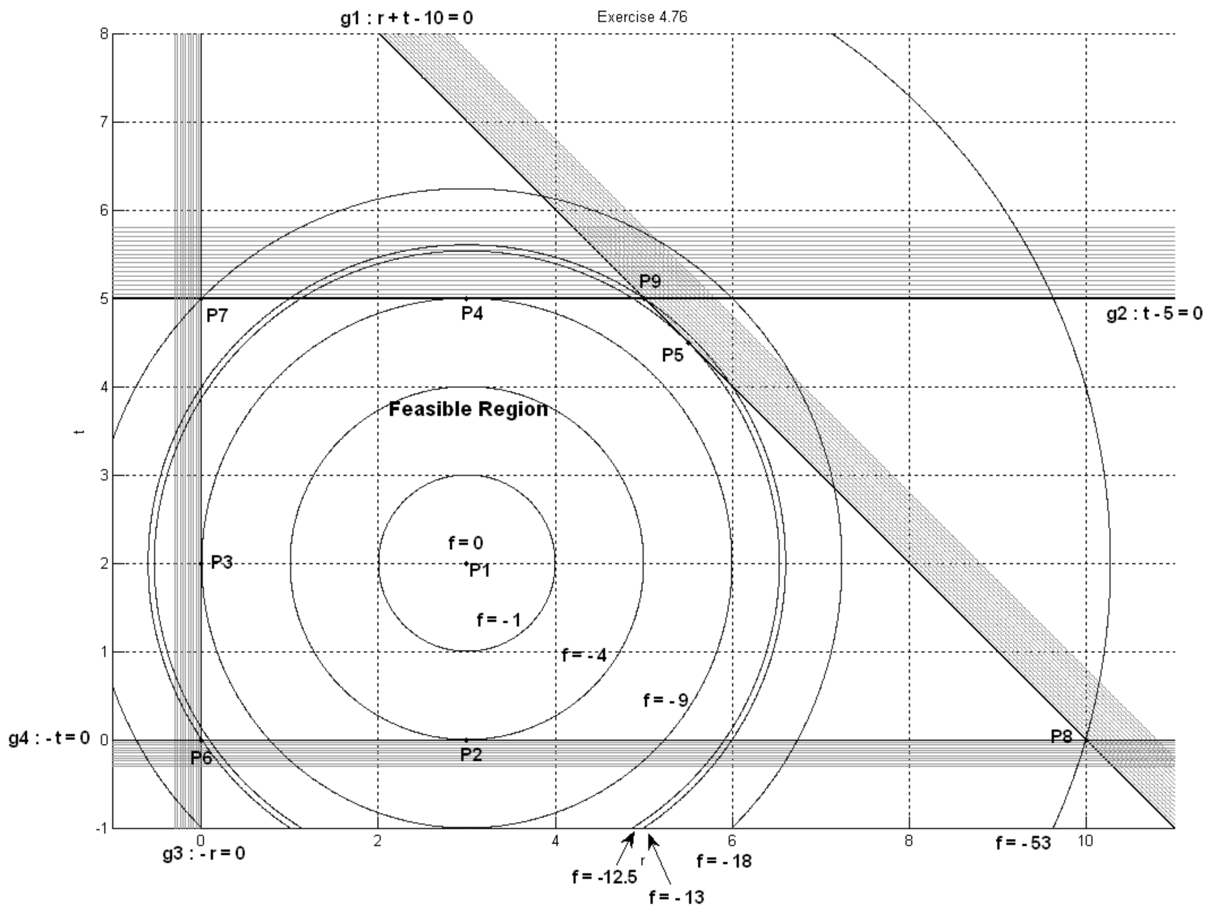
Maximize $F(r,t) = (r-3)^2 + (t-2)^2$

subject to $10 \geq r + t$

$t \leq 5$

$r, t \geq 0$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.76, the points satisfying the KKT necessary conditions are

	r	t	u_1	u_2	u_3	u_4
P1	3	2	0	0	0	0
P2	3	0	0	0	0	4
P3	0	2	0	0	6	0
P4	3	5	0	6	0	0
P5	5.5	4.5	5	0	0	0
P6	0	0	0	0	6	4
P7	0	5	0	6	6	0
P8	10	0	14	0	0	18
P9	5	5	4	2	0	0

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(r,t) = -(r-3)^2 - (t-2)^2$
$g_1 = r + t - 10 \le 0$
$g_2 = t - 5 \le 0$
$g_3 = -r \le 0$
$g_4 = -t \le 0$

$\nabla^2 L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$; $M_1 = -2 < 0, M_2 = 4 > 0$; Negative definite

Gradient of constraints are
$\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

1. At point P1 (3, 2)
Since no constraint is active, Hessian of Lagrangian must be positive definite throughout to satisfy sufficient condition. But it is always negative definite. So this is not an isolated minimum point for *f* (second order necessary condition is violated); or an isolated maximum point for *F*. Instead, it is a maximum point for *f*; or a minimum point for *F*.

2. At points P2 (3, 0), P3 (0, 2) , P4 (3, 5)and P5 (5.5, 4.5)
Since the Hessian of the Lagrangian is negative definite, these four points cannot be local minima for *f*; or local maxima for *F*.

3. At points P6 (0, 0) , P7 (0, 5), P8 (10, 0) and P9 (5, 5)
The number of active constraints is equal to the number of design variables. There are no feasible directions in the neighborhood of the points that can reduce cost function *f* any further. So, these points are local minima for *f*; or local maxima for *F*.

So only point P6, P7, P8 and P9 have local minimum for *f*; or local maxima for *F*.

The gradient of cost and constraint functions are

$\nabla \mathrm{f} = \begin{bmatrix} -2(\mathrm{r}-3) \\ -2(\mathrm{t}-2) \end{bmatrix}$ and $\nabla \mathrm{g}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\nabla \mathrm{g}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\nabla \mathrm{g}_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla \mathrm{g}_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At point P6 (0, 0)
By (4.52),

$-\nabla \mathrm{f}(0,0) = -\begin{bmatrix} -2(0-3) \\ -2(0-2) \end{bmatrix} = -\begin{bmatrix} 6 \\ 4 \end{bmatrix} = -4\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$ and

$\mathrm{u}_3\nabla \mathrm{g}_3 + \mathrm{u}_4\nabla \mathrm{g}_4 = 6\begin{bmatrix} -1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$

Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,

$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_1} = -\mathrm{u}_1{}^* = -(0)$$
$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_2} = -\mathrm{u}_2{}^* = -(0)$$
$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_3} = -\mathrm{u}_3{}^* = -(6)$$
$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_4} = -\mathrm{u}_4{}^* = -(4)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately

$\mathrm{f}^* = -13 — (6)(1) — (4)(1) = -23$

At point P7 (0, 5)
By (4.52),

$-\nabla \mathrm{f}(0,5) = -\begin{bmatrix} -2(0-3) \\ -2(5-2) \end{bmatrix} = -4\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and

$\mathrm{u}_2\nabla \mathrm{g}_2 + \mathrm{u}_4\nabla \mathrm{g}_4 = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 12\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,

$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_1} = -\mathrm{u}_1{}^* = -(0)$$
$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_2} = -\mathrm{u}_2{}^* = -(4)$$
$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_3} = -\mathrm{u}_3{}^* = -(0)$$
$$\frac{\partial \mathrm{f}(\mathrm{x}^*)}{\partial \mathrm{e}_4} = -\mathrm{u}_4{}^* = -(12)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately

$\mathrm{f}^* = -18 — (4)(1) — (12)(1) = -34$

At point P8 (10, 0)
By (4.52),
$-\nabla f(10, 0) = -\begin{bmatrix} -2(10-3) \\ -2(0-2) \end{bmatrix} = -4\begin{bmatrix} 3.5 \\ -1 \end{bmatrix}$ and
$u_1 \nabla g_1 + u_4 \nabla g_4 = 14\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 18\begin{bmatrix} 0 \\ -1 \end{bmatrix} = -4\begin{bmatrix} 3.5 \\ -1 \end{bmatrix}$
Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(14)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(18)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately
$f^* = -53 — (14)(1) — (18)(1) = -85$

At point P9 (5, 5)
By (4.52),
$-\nabla f(5, 5) = -\begin{bmatrix} -2(5-3) \\ -2(5-2) \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$ and
$u_1 \nabla g_1 + u_2 \nabla g_2 = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$
Note that the two vectors are along the same line, verifying the KKT necessary conditions.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(4)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(2)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(0)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately
$f^* = -1 — (4)(1) — (2)(1) = -7$

MATLAB Code for Exercise 129

```
clear all
axis equal
[r,t]=meshgrid(-1:0.01:11, -1:0.01:8);
f=(-1)*((r-3).^2+(t-2).^2);

g1=r+t-10;
g2=t-5;
g3=-r;
g4=-t;

cla reset
axis equal
axis ([-1 11 -1 8])
xlabel('r'),ylabel('t')
title('Exercise 4.76')
hold on

cv1=[0:0.05:0.8];
const1=contour(r,t,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(r,t,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(r,t,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(r,t,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(r,t,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(r,t,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(r,t,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(r,t,g4,cv4,'k');

fv=[0 -1 -4 -9 -12.5 -13 -18 -53];
fs=contour(r,t,f,fv,'b');

a=[3 3 0 3 5.5 0 0 10 5];
b=[2 0 2  5 4.5 0 5 0 5];
plot(a,b,'.k');

grid
hold off
```

4.130

Exercise 4.77

Maximize $F(r,t) = (r-8)^2 + (t-8)^2$

subject to $r + t \leq 10$

$t \geq 0$

$r \leq 0$

Solution

Referring to Exercise 4.77: Minimize $f(r,t) = -(r-8)^2 - (t-8)^2$;

subject to $g_1 = r + t - 10 \leq 0$; $g_2 = -r \leq 0$; $g_3 = -t \leq 0$;

No KKT solution. No candidate minimum.

4.131

Exercise 4.78

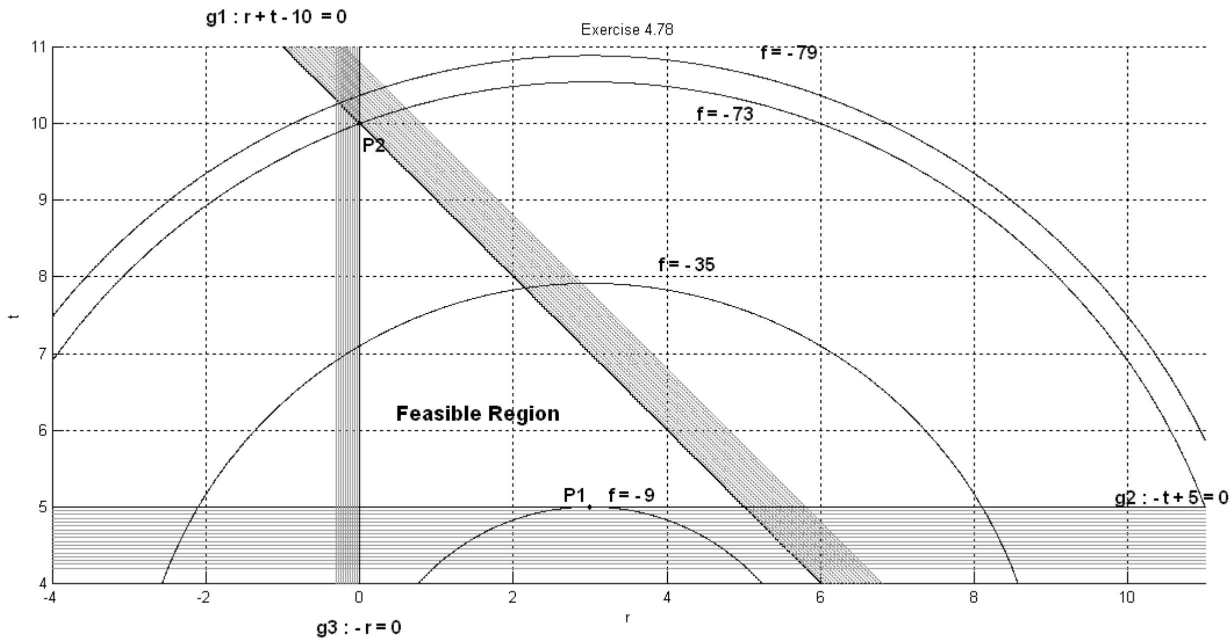
Maximize $F(r,t) = (r-3)^2 + (t-2)^2$

subject to $10 \geq r + t$

$t \geq 5$

$r, t \geq 0$

Solution



We need to find isolated or local minimum point(s) which satisfy both KKT necessary conditions and sufficient or the second order necessary conditions.

Referring to Exercise 4.78, the points satisfying the KKT necessary conditions are

	r	t	u_1	u_2	u_3	u_4
P1	3	5	0	0	6	0
P2	0	10	16	0	22	0

SECOND ORDER CONDITIONS ARE DISCUSSED IN CHAPTER 5

$f(r,t) = -(r-3)^2 - (t-8)^2$
$g_1 = r + t - 10 \le 0$
$g_2 = t - 5 \le 0$
$g_3 = -r \le 0$
$g_4 = -t \le 0$
$\nabla^2 L = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$; $M_1 = -2 < 0, M_2 = 4 > 0$; Negative definite

Gradient of constraints are
$\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

1. At point P1 (3, 5)
Since the Hessian of the Lagrangian is negative definite, the four points cannot be local minima.

2. At points P2 (0, 10)
The number of active constraints is equal to the number of design variables. There are no feasible directions in the neighborhood of the points that can reduce cost function any further. So, these points are isolated local minima.

So only point P2 has local minimum.

The gradient of cost and constraint functions are
$\nabla f = \begin{bmatrix} -2(r-3) \\ -2(t-2) \end{bmatrix}$ and $\nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \nabla g_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \nabla g_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\nabla g_4 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

At point P2 (0, 10)
By (4.52),
$-\nabla f(0,10) = -\begin{bmatrix} -2(0-3) \\ -2(10-2) \end{bmatrix} = -\begin{bmatrix} 6 \\ -16 \end{bmatrix} = -6\begin{bmatrix} 1 \\ -8/3 \end{bmatrix}$ and
$u_1 \nabla g_1 + u_3 \nabla g_3 = 16\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 22\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 16 \end{bmatrix} = -6\begin{bmatrix} 1 \\ -8/3 \end{bmatrix}$
Note that they are along the same line.

By Theorem 4.7,

$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(16)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(22)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(0)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately
$f^* = -73 — (16)(1) — (22)(1) = -111$

At point P7 (0, 5)
By (4.52),
$-\nabla f(0,5) = -\begin{bmatrix} -2(0-3) \\ -2(5-2) \end{bmatrix} = -4\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and
$u_2\nabla g_2 + u_4\nabla g_4 = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 12\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -3 \end{bmatrix}$
Note that they are along the same line.

By Theorem 4.7,
$$\frac{\partial f(x^*)}{\partial e_1} = -u_1{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_2} = -u_2{}^* = -(4)$$
$$\frac{\partial f(x^*)}{\partial e_3} = -u_3{}^* = -(0)$$
$$\frac{\partial f(x^*)}{\partial e_4} = -u_4{}^* = -(12)$$

If we set e_1=e_2= e_3= e_4=1, the new value of cost function will be approximately
$f^* = -18 — (4)(1) — (12)(1) = -34$

MATLAB Code for Exercise 131

```
clear all
axis equal

[r,t]=meshgrid(-4:0.01:11, 4:0.01:11);
f=(-1)*((r-3).^2+(t-2).^2);

g1=r+t-10;
g2=-t+5;
g3=-r;
g4=-t;

cla reset
axis equal
axis ([-4 11 4 11])
xlabel('r'),ylabel('t')
title('Exercise 4.78')
hold on

cv1=[0:0.05:0.8];
const1=contour(r,t,g1,cv1,'g');
cv1=[0 0.01];
const1=contour(r,t,g1,cv1,'k');

cv2=[0:0.05:0.8];
const2=contour(r,t,g2,cv2,'g');
cv2=[0 0.01];
const2=contour(r,t,g2,cv2,'k');

cv3=[0:0.03:0.3];
const3=contour(r,t,g3,cv3,'g');
cv3=[0 0.005];
const3=contour(r,t,g3,cv3,'k');

cv4=[0:0.03:0.3];
const4=contour(r,t,g4,cv4,'g');
cv4=[0 0.005];
const4=contour(r,t,g4,cv4,'k');

fv=[-9 -35 -73 -79];
fs=contour(r,t,f,fv,'b');

a=[3 0];
b=[5 10];
plot(a,b,'.k');

grid
hold off
```

Section 4.8 Global Optimality

4.132

Answer True or False

1. A linear inequality constraint always defines a convex feasible region.
   True
2. A linear equality constraint always defines a convex feasible region.
   True
3. A nonlinear equality constraint cannot give a convex feasible region.
   True
4. A function is convex if and only if its Hessian is positive definite everywhere.
   False
5. An optimum design problem is convex if all constraints are linear and cost function is convex.
   True
6. A convex programming problem always has an optimum solution.
   False
7. An optimum solution for a convex programming problem is always unique.
   False
8. A nonconvex programming problem cannot have global optimum solution.
   False
9. For a convex design problem, the Hessian of the cost function must be positive semidefinite everywhere.
   False
10. Checking for the convexity of a function can actually identify a domain over which the function may be convex.
    True

4.133

Using the definition of a line segment given in Eq. (4.71), show that the following set is convex

$$S = \{x | x_1^2 + x_2^2 - 1.0 \le 0\}$$

Solution

Assume $\mathbf{x} \varepsilon S \to x_1^2 + x_2^2 - 1 \le 0$; $\mathbf{y} \varepsilon S \to y_1^2 + y_2^2 - 1 \le 0$;

$$\alpha \mathbf{x} + (1-\alpha)\mathbf{y} = \begin{bmatrix} \alpha x_1 + (1-\alpha) y_1 \\ \alpha x_2 + (1-\alpha) y_2 \end{bmatrix};$$

If we can show that $[\alpha x_1 + (1-\alpha) y_1]^2 + [\alpha x_2 + (1-\alpha) y_2]^2 - 1 \le 0$, then S is a convex set.

$$[\alpha x_1 + (1-\alpha) y_1]^2 + [\alpha x^2 + (1-\alpha) y_2]^2$$

$$= \alpha^2 x_1^2 + 2\alpha(1-\alpha) x_1 y_1 + (1-\alpha)^2 y_1^2 + \alpha^2 x_2^2 + 2\alpha(1-\alpha) x_2 y_2 + (1-\alpha)^2 y_2^2$$

$$\alpha^2 (x_1^2 + x_2^2) + (1-\alpha)^2 (y_1^2 + y_2^2) + 2\alpha(1-\alpha)(x_1 y_1 + x_2 y_2)$$

$$\le \alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha) = \alpha^2 + 1 - 2\alpha + \alpha^2 + 2\alpha - 2\alpha^2 = 1$$

where $x_1^2 + x_2^2 \le 1, y_1^2 + y_2^2 \le 1$ and $x_1 y_1 + x_2 y_2 \le 1$ are used.

The first two inequalities are derived by definition. The last inequality is derived as follows:

$x_1 y_1 + x_2 y_2 = \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \; \|\mathbf{y}\| \cos(\mathbf{x}, \mathbf{y})$.

Since $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2} \le 1$; $\|\mathbf{y}\| = \sqrt{y_1^2 + y_2^2} \le 1$; $\cos(\mathbf{x}, \mathbf{y}) \le 1$, it follows that $x_1 y_1 + x_2 y_2 \le 1$.

4.134

Find the domain for which the following functions are convex: (i) *sin x* , (ii) *cos x*.

Solution

1. $f = \sin x$; $0 \le x \le 2\pi$; $f' = \cos x$, $f'' = -\sin x$

For a single variable function to be convex, its second derivative must be nonnegative, i.e., $f'' = -\sin x \ge 0$, or $\pi \le x \le 2\pi$

2. $f = \cos x$; $0 \le x \le 2\pi$; $f' = -\sin x$, $f'' = -\cos x \ge 0$, so $\pi/2 \le x \le 3\pi/2$

4.135

Check for convexity of the following function. If the function is not convex everywhere, than determine the domain (feasible set S)over which the function is convex.

$$f(x_1, x_2) = 3x_1^2 + 2x_1 x_2 + 2x_2^2 + 7$$

Solution

$$f(x_1, x_2) = 3x_1^2 + 2x_1 x_2 + 2x_2^2 + 7$$

$$\tilde{\mathbf{N}} f = \begin{bmatrix} 6x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}; \quad M_1 = 6 > 0; \quad M_2 = 20 > 0.$$

Function f is convex everywhere since its Hessian is positive definite.

4.136

Check for convexity of the following function. If the function is not convex everywhere, than determine the domain (feasible set S)over which the function is convex.

$$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3$$

Solution

$$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3$$

$$\nabla f = \begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}; \quad M_1 = 2 > 0; \quad M_2 = -12 > 0.$$

Function f is not convex since the Hessian is indefinite. We cannot find a domain over which function f is convex because the Hessian is always indefinite.

4.137

Check for convexity of the following function. If the function is not convex everywhere, than determine the domain (feasible set S)over which the function is convex.

$$f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$$

Solution

$$f(x_1, x_2) = x_1^3 + 12x_1x_2^2 + 2x_2^2 + 5x_1^2 + 3x_2$$

$$\partial f/\partial x_1 = 3x_1^2 + 12x_2^2 + 10x_1; \quad \partial f/\partial x_2 = 24x_1x_2 + 4x_2 + 3; \quad \mathbf{H} = \begin{bmatrix} 6x_1 + 10 & 24x_2 \\ 24x_2 & 24x_1 + 4 \end{bmatrix}$$

Since Hessian is not always positive semidefinite, the function is not convex everywhere. To find the domain over which the function is convex, we need to impose the following conditions:

$$M_1 = 6x_1 + 10 \geq 0 \qquad (1)$$

$$M_2 = (6x_1 + 10)(24x_1 + 4) - (24x_2)^2 \geq 0 \qquad (2)$$

From (1), $x_1 \geq -5/3$.

From (2), $144x_1^2 + 264x_1 + 40 - 576x_2^2 \geq 0$, or $x_2^2 \leq (1/576)\left[144\left(x_1^2 + 264x_1/144 + 40/144\right)\right]$,

or $x_2^2 \leq (1/4)\left[(x_1 + 11/12)^2 - 9/16\right]$, or $(x_1 + 11/12)^2 - 4x_2^2 - 9/16 \geq 0.$

4.138

Check for convexity of the following function. If the function is not convex everywhere, than determine the domain (feasible set S)over which the function is convex.

$$f(x_1, x_2) = 5x_1 - (1/16)x_1^2 x_2^2 + x_2^2/4x_1$$

Solution

$f(x_1, x_2) = 5x_1 - (1/16)x_1^2 x_2^2 + x_2^2/4x_1$,

$$\tilde{\mathbf{N}}f = \begin{bmatrix} 5 - x_1 x_2^2/8 - x_2^2/4x_1^2 \\ -x_1^2 x_2/8 + x_2/2x_1 \end{bmatrix}; \quad \mathbf{H} = \begin{bmatrix} -x_2^2/8 + x_2^2/2x_1^3 & -x_1x_2/4 - x_2/2x_1^2 \\ -x_1x_2/4 - x_2/2x_1^2 & -x_1^2/8 + 1/2x_1 \end{bmatrix}$$

The Hessian of this function is not always positive semidefinite; so, this function is not convex everywhere. To find the domain over which the function is convex, we need to impose the following conditions:

$$M_1 = -x_2^2/8 + x_2^2/2x_1^3 \geq 0 \qquad (1)$$

$$M_2 = \left(-x_2^2/8 + x_2^2/2x_1^3\right)\left(-x_1^2/8 + 1/2x_1\right) - \left(-x_1x_2/4 - x_2/2x_1^2\right)^2 \geq 0 \qquad (2)$$

From (1), $x_2^2\left(-1/8 + 1/2x_1^3\right) \geq 0$ $\left(\text{since } x_2^2 \geq 0\right)$; or $-1/8 + 1/2x_1^3 \geq 0$, or

$1/2\,x_1^3 \geq 1/8$, or $0 < x_1^3 \leq 4$, or $0 < x_1 \leq (2)^{2/3}$

From (2), $x_1^2x_2^2/64 - x_2^2/16x_1 - x_2^2/16x_1 + x_2^2/4x_1^4 - x_1^2x_2^2/16 - x_2^2/4x_1 - x_2^2/4x_1^4 \geq 0$

or, $-3x_1^2x_2^2/64 - 3x_2^2/8x_1 \geq 0$; $\left(3x_2^2/64x_1\right)\left(-x_1^3 - 8\right) \geq 0$ $\left(\text{since } 3x_2^2/64x_1 \geq 0\right)$;

$-x_1^3 \geq 8$, or $x_1^3 \leq -8$; or $x_1 \leq -2$

This contradicts the condition derived from (1) which requires $x_1 > 0$. So, the function is not convex.

4.139

Check for convexity of the following function. If the function is not convex everywhere, than determine the domain (feasible set S)over which the function is convex.

$$f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$$

Solution

Since $M_1 > 0$ and $M_2 > 0$, the Hessian is positive definite. Consequently, the function is convex everywhere.

4.140

Check for convexity of the following function. If the function is not convex everywhere, than determine the domain (feasible set S)over which the function is convex.

$$U(V,C) = \frac{21.9\times10^7}{V^2C} + 3.9\times10^6C + 1000V$$

Solution

$$U(V,\,C) = \frac{21.9\times10^7}{V^2C} + 3.9\times10^6C + 1000V;\quad \mathbf{\tilde{N}}U = \begin{bmatrix} -43.8\times10^7/V^3C + 1000 \\ -21.9\times10^7/V^2C^2 + 3.9\times10^6 \end{bmatrix};$$

$$\mathbf{H} = \begin{bmatrix} 131.4\times10^7/V^4C & 43.8\times10^7/V^3C^2 \\ 43.8\times10^7/V^3C^2 & 43.8\times10^7/V^2C^3 \end{bmatrix} = \frac{43.8\times10^7}{V^4C^4}\begin{bmatrix} 3C^3 & VC^2 \\ VC^2 & V^2C \end{bmatrix}$$

Neglecting the positive coefficient of the Hessian, $M_1 = 3C^3 \geq 0$ if $C \geq 0$;

$M_2 = 3C^3\left(V^2C\right) - \left(VC^2\right)^2 = 2V^2C^4 \geq 0$.

The Hessian is positive semidefinite if $C \geq 0$. So, the function is convex if C is nonnegative.

4.141

Consider the problem of designing the “can” formulated in Section 2.2. Check convexity of the problem. Solve the problem graphically and check the KKT conditions at the solution point.

Solution

Minimize $f(D,H) = \pi DH + \pi D^2/2$ subject to $\pi D^2H/4 \geq 400$, or $g_1 = 400 - \pi D^2H/4 \leq 0$

$3.5 \leq D \leq 8.0;\quad 8.0 \leq H \leq 18.0$

Hessian of g_1 is $\begin{bmatrix} -\pi H/2 & -\pi D/2 \\ -\pi D/2 & 0 \end{bmatrix}$; $M_1 = -\pi H/2 < 0$; $M_2 = -\pi^2D^2/4 < 0$;

Since Hessian is not positive semidefinite, the first constraint function is not convex. The problem is not a convex programming problem.

4.142

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.1

Solution

Referring to Exercise 4.83, the problem is written in the standard form as

Minimize $f = 0.6h + 0.001A$ subject to $g_1 = 20000 - hA/3.5 \le 0$;

$g_2 = A(h+14)/14 - 10000 \le 0$; $g_3 = 3.5 - h \le 0$; $g_4 = h - 21 \le 0$; $g_5 = -A \le 0$

The convexity of each nonlinear equation has to be checked:

$$\nabla g_1 = \begin{bmatrix} \partial g_1/\partial A \\ \partial g_1/\partial h \end{bmatrix} = \begin{bmatrix} -h/3.5 \\ -A/3.5 \end{bmatrix}; \quad \mathbf{H}g_1 = \begin{bmatrix} 0 & -1/3.5 \\ -1/3.5 & 0 \end{bmatrix}$$

Hessian of g_1 is not positive semidefinite, so the function is not convex. So the problem is not a convex programming problem.

4.143

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.3.

Solution

Referring to Exercise 4.85, the problem is written in the standard form as

Minimize $f = -\pi R^2 H$,

subject to
$$g_1 = 2\pi RH - 900 \le 0$$
$$g_2 = 5 - R \le 0;$$
$$g_3 = R - 20 \le 0;$$
$$g_4 = -H \le 0; \quad g_5 = H - 20 \le 0$$

$$\mathbf{H}g_1 = \begin{bmatrix} 0 & 2\pi \\ 2\pi & 0 \end{bmatrix}; \text{This is indefinite.}$$

$$\nabla f = \begin{bmatrix} \partial f/\partial R \\ \partial f/\partial H \end{bmatrix} = \begin{bmatrix} -2\pi RH \\ -\pi R^2 \end{bmatrix}$$

$$\mathbf{H}_f = \begin{bmatrix} -2\pi H & -2\pi R \\ -2\pi R & 0 \end{bmatrix}; \quad M_1 = -2\pi H < 0; \quad M_2 = -4\pi^2 R^2 < 0$$

Hessian of f is indefinite. Therefore the cost function is also not convex. The problem is not a convex programming problem.

4.144

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.4

Solution

Referring to Exercise 4.86, the problem is written in the standard form as

Minimize $f = -2\pi LNR$, subject to $g_1 = 0.5 - R \le 0$; $g_2 = \pi NR - 2000 \le 0$; $g_3 = -N \le 0$.

$\mathrm{H}g_2 = (2\pi)\begin{bmatrix} N & R \\ R & 0 \end{bmatrix}$; This is indefinite, so the constraint function is nonconvex.

Hessian of the cost function is indefinite, so it is also not a convex function. The problem is not a convex programming problem.

4.145

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.5

Solution

Referring to Exercise 4.87, the problem is written in the standard form as

Minimize $f = 200W + 100D$, subject to $g_1 = W - 100 \le 0$; $g_2 = D - 200 \le 0$;

$g_3 = 10000 - WD \le 0$;

$g_4 = D - 2W \le 0$; $g_5 = W - 2D \le 0$; $g_6 = -W \le 0$; $g_7 = -D \le 0$

$\tilde{\mathrm{N}}g_3 = \begin{bmatrix} \partial g_3/\partial W \\ \partial g_3/\partial D \end{bmatrix} = \begin{bmatrix} -D \\ -W \end{bmatrix}$; $\mathbf{H}g_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Hessian is indefinite. So the constraint function g_3 is not convex. Therefore, this is not a convex programming problem.

4.146

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.9

Solution

Referring to Exercise 4.91, the problem is written in the standard form as

Minimize $f = \pi r^2 + 2\pi rh$, subject to $h_1 = \pi r^2 h - 600 = 0$; $g_1 = 1 - h/2r \le 0$

$g_2 = h/2r - 1.5 \le 0$; $g_3 = h - 20 \le 0$; $g_4 = -h \le 0$; $g_5 = -r \le 0$

Since the equality constraint is not linear, the feasible region is not convex.

4.147

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.10

Solution

Referring to Exercise 4.92, the problem is written in the standard form as

Minimize $f = (32/15)(1/h + 2/b)$,

subject to $g_1 = b - 10 \le 0$; $g_2 = h - 18 \le 0$; $g_3 = -b \le 0$; $g_4 = -h \le 0$

$$\nabla f = \begin{bmatrix} \partial f/\partial b \\ \partial f/\partial h \end{bmatrix} = \begin{bmatrix} -64/(15b^2) \\ -32/(15h^2) \end{bmatrix}; \quad \mathbf{H}_f = \begin{bmatrix} 128/(15b^3) & 0 \\ 0 & 64/(15h^3) \end{bmatrix}$$

Hessian of cost function is positive definite if both *b* and *h* are greater than zero. So, cost function is convex. All constraints are linear, so they define a convex set. Therefore, the problem is convex.

4.148

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.12

Solution

Referring to Exercise 4.94, we have

Minimize $f = 400(\pi D^2/2 + DH)$, subject to $h_1 = \pi D^2 H/4 - 150 = 0$; $g_1 = H + D/2 - 10 \le 0$;

$g_2 = -H \le 0$; $g_3 = -D \le 0$

Since the equality constraint is not linear, the feasible region is not convex.

4.149

Formulate and check convexity of the following problem; solve the problems graphically and verify the KKT conditions at the solution point.

Exercise 2.14

Solution

Referring to Exercise 4.96, the problem is written in the standard form as

Minimize $f = (1 - P_1 + P_1^2) + (1 + 0.6P_2 + P_2^2)$, subject to $g_1 = 60 - P_1 - P_2 \le 0$;

$g_2 = -P_1 \le 0$; $g_3 = -P_2 \le 0$

$$\nabla f = \begin{bmatrix} -1 + 2P_1 \\ 0.6 + 2P_2 \end{bmatrix}; \quad \mathbf{H}_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Hessian of cost function is positive definite and all constraints are linear, therefore this is a convex programming problem.

Section 4.9 Engineering Design Examples

4.150

The problem of minimum weight design of the symmetric three-bar truss of Fig. 2-6 is formulated as follows:

Minimize $f(x_1, x_2) = 2\sqrt{2}x_1 + x_2$

Subject to the constraints

$$g_1 = \frac{1}{\sqrt{2}}\left[\frac{P_u}{x_1} + \frac{P_v}{(x_1+\sqrt{2}x_2)}\right] - 20{,}000 \le 0$$

$$g_2 = \frac{\sqrt{2}P_v}{(x_1+\sqrt{2}x_2)} - 20{,}000 \le 0$$

$$g_3 = -x_1 \le 0$$

$$g_4 = -x_2 \le 0$$

Solution

Minimize $f = 2\sqrt{2}\,x_1 + x_2$, subject to $g_1 = \left(1/\sqrt{2}\right)\left[P_u / x_1 + P_v\big/\left(x_1 + \sqrt{2}x_2\right)\right] - 20{,}000 \le 0$

$g_2 = \sqrt{2}P_v\big/\left(x_1 + \sqrt{2}x_2\right) - 20{,}000 \le 0;\quad g_3 = -x_1 \le 0;\quad g_4 = -x_2 \le 0$

where $P_u = P\cos\theta$, $P_v = P\sin\theta$, $P > 0$ and $\theta = 60°$

$$\tilde{\mathbf{N}}g_1 = \begin{bmatrix} -P_u\big/\sqrt{2}x_1^2 - P_v\big/\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2 \\ -P_v\big/\left(x_1+\sqrt{2}x_2\right)^2 \end{bmatrix};$$

$$\mathbf{H}g_1 = \begin{bmatrix} \sqrt{2}\,P_u\big/x_1^3 + \sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 & 2P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 \\ 2P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 & 2\sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 \end{bmatrix}$$

$$M_1 = \sqrt{2}\,P_u\big/x_1^3 + \sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 \ge 0;\quad M_2 = 4\,P_u P_v\big/x_1^3\left(x_1+\sqrt{2}x_2\right)^3 \ge 0$$

The Hessian of g_1 is positive semidefinite, so g_1 is a convex function.

$$\tilde{\mathbf{N}}g_2 = \begin{bmatrix} -\sqrt{2}P_v\big/\left(x_1+\sqrt{2}x_2\right)^2 \\ -2\,P_v\big/\left(x_1+\sqrt{2x_2}\right)^2 \end{bmatrix};\quad \mathbf{H}g_2 = \begin{bmatrix} 2\sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 & 4P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 \\ 4P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 & 4\sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 \end{bmatrix}$$

$$M_1 = 2\sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right)^3 \ge 0;\quad M_2 = 0$$

The Hessian of g_2 is positive semidefinite, so g_2 is a convex function. The other two constraints (g_3 and g_4) are linear, so the constraint set is convex. Since cost function is also linear, the problem is convex.

4.151

For the three-bar truss problem of Exercise 4.150, consider the case of KKT conditions with g_1 as the only active constraint. Solve the conditions for optimum solution and determine the range for the load angle θ for which the solution is valid.

Solution

Referring to Exercise 4.150, the Lagrange function is

$$L = 2\sqrt{2}x_1 + x_2 + u_1\left[\left(1/\sqrt{2}\right)\left(P_u/x_1 + P_v/\left(x_1+\sqrt{2}x_2\right)\right) - 20{,}000 + s_1^2\right]$$

$$+u_2\left(\sqrt{2}\,P_v/\left(x_1+\sqrt{2}x_2\right) - 20{,}000 + s_2^2\right) + u_3\left(-x_1+s_3^2\right) + u_4\left(-x_2+s_4^2\right)$$

Assuming that only g_1 is active, i.e., $u_2 = u_3 = u_4 = 0$ and $g_1 = 0\,(s_1 = 0)$, the KKT necessary conditions give

$$\partial L/\partial x_1 = 2\sqrt{2} + u_1\left[-P_u/\sqrt{2}x_1^2 - P_v/\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2\right] = 0 \quad (1)$$

$$\partial L/\partial x_2 = 1 + u_1\left[-P_v/\left(x_1+\sqrt{2}x_2\right)^2\right] = 0; \quad \text{or} \quad u_1 = \left(x_1+\sqrt{2}x_2\right)^2/P_v \quad (2)$$

$$g_1 = \left(1/\sqrt{2}\right)\left(P_u/x_1 + P_v/\left(x_1+\sqrt{2}x_2\right)\right) - 20{,}000 = 0 \quad (3)$$

Substituting (2) into (1),

$$2\sqrt{2} + \left[\left(x_1+\sqrt{2}x_2\right)^2/P_v\right]\left[-P_u/\sqrt{2}\,x_1^2 - P_v/\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2\right] = 0$$

$$2\sqrt{2} - \left(P_u/P_v\right)\left(x_1+\sqrt{2}x_2\right)^2/\sqrt{2}x_1^2 - 1/\sqrt{2} = 0; \text{or } x_1+\sqrt{2}x_2 = x_1\left(3P_v/P_u\right)^{1/2} \quad (4)$$

$$\mathrm{x}_2 = \left(\mathrm{x}_1/\sqrt{2}\right)\left[\left(3\mathrm{P_u}/\mathrm{P_u}\right)^{1/2} - 1\right] \quad (5)$$

Substituting (4) into (3),

$$\left(1/\sqrt{2}\right)\left[\mathrm{P}_u/\mathrm{x}_1 + \mathrm{P}_v/\mathrm{x}_1\left(3\mathrm{P}_v/\mathrm{P}_u\right)^{1/2}\right] - 20{,}000 = 0$$

$$x_1 = \left[P_u/\sqrt{2} + \left(P_uP_v/6\right)^{1/2}\right]/20{,}000 \quad (6)$$

Substituting (6) into (5), $x_2 = \left[P_u + \left(P_uP_v/3\right)^{1/2}\right]\left[\left(3P_v/P_u\right)^{1/2} - 1\right]/40{,}000$ (7)

Note that $x_2 \le 0$ requires that $3P_v/P_u$, which is equivalent to $3\tan\theta \ge 1$, or $\theta \ge$ 18.43°.

We still need to check the feasibility of constraint g_2, i.e.,

$\sqrt{2}P_v/\left(x_1+\sqrt{2}x_2\right) - 20{,}000 \le 0$. Substituting x_1 and x_2 from Eqs. (6) and (7), we get

$$\frac{\sqrt{2}P_v}{\left[P_u + \left(P_uP_v/3\right)^{1/2}\right]/\left(20{,}000\sqrt{2}\right) + \sqrt{2}\left[P_u + \left(P_uP_v/3\right)^{1/2}\right]\left[\left(3P_v/P_u\right)^{1/2} - 1\right]/40{,}000} - 20{,}000 \le 0$$

$$\frac{\sqrt{2}P_v}{\left[P_u + \left(P_uP_v/3\right)^{1/2}\right]\left(3P_v/P_u\right)^{1/2}/\left(20{,}000\sqrt{2}\right)} - 20{,}000 \le 0,$$

$\dfrac{2P_v}{(3P_uP_v)^{1/2}+P_v}-1\le 0$, or $P_v \le 3P_u$; This is equivalent to $\tan\theta \le 3$, or $\theta \le 71.57°$.

Therefore, this case yields an optimum solution only when 18.43º ≤ θ ≤ 71.57º

4.152

For the three-bar truss problem of Exercise 4.150, consider the case of KKT conditions with g_1 and g_2 as active constraints. Solve the conditions for optimum solution and determine the range for the load angle θ for which the solution is valid.

Solution

Referring to Exercise 4.150, we write the KKT conditions for the case

$g_1 = g_2 = 0\ (s_1 = s_2 = 0)$ and $u_3 - u_4 = 0$:

$$\partial L/\partial x_1 = 2\sqrt{2} + u_1\left[-P_u\Big/\sqrt{2}x_1^2 - P_v\Big/\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2\right] + u_2\left[-\sqrt{2}\,P_v\Big/\left(x_1+\sqrt{2}x_2\right)^2\right] = 0 \quad (1)$$

$$\partial L/\partial x_2 = 1 + u_1\left[-P_v\Big/\left(x_1+\sqrt{2}x_2\right)^2\right] + u_2\left[-2\,P_v\Big/\left(x_1+\sqrt{2}x_2\right)^2\right] = 0 \quad (2)$$

$$g_1 = \left(1\big/\sqrt{2}\right)\left(P_u\big/x_1 + P_v\big/\left(x_1+\sqrt{2}x_2\right)\right) - 20{,}000 = 0 \quad (3)$$

$$g_2 = \sqrt{2}\,P_v\big/\left(x_1+\sqrt{2}x_2\right) - 20{,}000 = 0;\quad u_1, u_2 \ge 0,\quad x_1, x_2 \ge 0 \quad (4)$$

From (4), $x_1 + \sqrt{2}x_2 = \sqrt{2}\,P_v/20{,}000$ (5)

Substituting (5) into (3), $\left(1\big/\sqrt{2}\right)\left[P_u\big/x_1 + P_v\big/\left(\sqrt{2}\,P_v/20{,}000\right)\right] - 20{,}000 = 0$

$P_u\big/\sqrt{2}\,x_1 + 10{,}000 - 20{,}000 = 0$, or $x_1 = P_u\big/\left(10{,}000\sqrt{2}\right)$ (6)

From (5) and (6), $x_2 = \left(P_v - P_u\right)/20{,}000$ (7)

Note that $x_2 > 0$ requires that $P_v - P_u \ge 0$, which is equivalent to $\tan\theta \ge 1$, or $\theta \ge 45°$. Substituting x_1 and x_2 from Eqs. (6) and (7) into (1) and (2), solving these equations for u_1 and u_2, we get $u_1 = 1.5\times10^{-7}P_u$, $u_2 = 2.5\times10^{-9}\left(P_v - 3P_u\right)$. Thus, for $u_2 \ge 0, P_v - 3P_u \ge 0$, which is equivalent to $\tan\theta \ge 3$, or $\theta \ge 71.57°$. Therefore, this case gives an optimal solution only when $\theta \ge 71.57°$.

4.153

For the three-bar truss problem of Exercise 4.150, consider the case of KKT conditions with g_2 as the only active constraint. Solve the conditions for optimum solution and determine the range for the load angle θ for which the solution is valid.

Solution

Ref. to Exercise 4.150, the KKT conditions for the case $g_2 = 0 \; (s_2 = 0)$ and $u_1 = u_3 = u_4 = 0$, are

$$\partial L/\partial x_1 = 2\sqrt{2} + u_2\left[-\sqrt{2}\,P_v\Big/\left(x_1+\sqrt{2}x_2\right)^2\right] = 0 \qquad (1)$$

$$\partial L/\partial x_2 = 1 + u_2\left[-2\,P_v\Big/\left(x_1+\sqrt{2}x_2\right)^2\right] = 0 \qquad (2)$$

$$g_2 = \sqrt{2}\,P_v\Big/\left(x_1+\sqrt{2}x_2\right) - 20{,}000 = 0, \quad u_2 \ge 0, \quad x_1, x_2 \ge 0 \qquad (3)$$

From (1), $u_2 = 2\left(x_1+\sqrt{2}x_2\right)^2\Big/P_v$; From (2), $u_2 = \left(x_1+\sqrt{2}x_2\right)^2\Big/2P_v$. These two equations are inconsistent, so there is no solution in this case.

4.154

For the three-bar truss problem of Exercise 4.150, consider the case of KKT conditions with g_1 and g_4 as active constraints. Solve the conditions for optimum solution and determine the range for the load angle θ for which the solution is valid.

Solution

Referring to Exercise 4.150, we write the KKT conditions for this case,

$g_1 = g_4 = 0 \; (s_1 = s_4 = 0)$ and $u_2 = u_3 = 0$, as

$$\partial L/\partial x_1 = 2\sqrt{2} + u_1\left[-P_u\Big/\sqrt{2}x_1^2 - P_v\Big/\sqrt{2}\left(x_1+\sqrt{2}x_2\right)^2\right] = 0 \qquad (1)$$

$$\partial L/\partial x_2 = 1 + u_1\left[-P_v\Big/\left(x_1+\sqrt{2}x_2\right)^2\right] - u_4 = 0 \qquad (2)$$

$$g_1 = \left(1/\sqrt{2}\right)\left[\left(P_u/x_1 + \left(P_v/x_1+\sqrt{2}\,x_2\right)\right)\right] - 20{,}000 = 0 \quad (3)$$

$$g_4 = x_2 = 0; \quad u_1, u_2 \ge 0, \quad x_1 \ge 0 \qquad (4)$$

Substituting $x_2 = 0$ into (1), (2) and (3) respectively, we get

$$2\sqrt{2} + u_1\left[\left(-P_u - P_v\right)\Big/\sqrt{2}x_1^2\right] = 0; \quad 1 - u_1\left(P_v/x_1^2\right) - u_4 = 0$$

$$\left(1/\sqrt{2}\right)\left(\left(P_u + P_v\right)/x_1\right) - 20{,}000 = 0$$

From the last equation, we get $x_1 = \left(P_u + P_v\right)\Big/\left(20{,}000\sqrt{2}\right)$.

Substituting x_1 into the previous two equations and solving for u_1 and u_4, we obtain

$$u_1 = \left(P_u + P_v\right)\Big/\left(2\times 10^8\right), \quad u_4 = \left(P_u - 3P_v\right)\Big/\left(P_u + P_v\right).$$

Now $u_4 \ge 0$ requires that $P_u - 3P_v \ge 0$ which is equivalent to $\tan\theta \le 1/3$, or $\theta \le 18.43°$.

Thus there is an optimum solution only when $\theta \le 18.43°$.