Importing the necessary libraries

In [1]:
```
import sympy as sym
import numpy as np
import matplotlib.pyplot as plt
```

Problem Statement:

State-Space Definition of the given System:

$$\dot{x_1}(t) = x_2(t)$$

$$\dot{x_2}(t) = -2x_1(t) + 2x_2(t) + 2u(t)$$

Boundary Conditions:

$$t \in [0, 6]$$

$$x_1(0) = 1$$

$$x_2(0) = -2$$

Performance Index:

$$PI = \frac{1}{2}\left[x_1^2(6) + 2x_1(6)x_2(6) + 2x_2^2(6)\right] + \int_0^6 \left[2x_1^2(t) + 3x_1(t)x_2(t) + 2x_2^2(t) + \frac{1}{2}u^2$$

Find the optimal control input $u(t)$ for the given system to minimize the performance index.

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$t_f = 6$$

$$Q = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

$$R = 1$$

Eigen-Vales of A are:

$$\lambda_1 = 1 + j$$

$$\lambda_2 = 1 - j$$

Since the Real-Part of both the Eigen-Values is positive, the system is unstable.

Matrix Differential Riccati Equation:

$$\dot{P} = -(A^T P + PA - PBR^{-1}B^T P + Q)$$

Non-Linear Algebraic Riccati Equation:

$$\dot{P} = 0$$

P is a Constant Matrix

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

$$-\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} = 0$$

Now, lets use **SymPy** to simplify the above equation:

In [2]:
```
t = sym.symbols('t')
P11, P12, P22 = sym.symbols('P11 P12 P22')
P = sym.Matrix([[P11, P12], [P12, P22]])
A = sym.Matrix([[0, 1], [-2, 2]])
B = sym.Matrix([[0], [2]])
Q = sym.Matrix([[4, 3], [3, 4]])
R = 1

# Solve for P
P = sym.solve(P * A + A.T * P + Q - P * B * B.T * P, P)

# Print P beutifully
sym.pprint(P)
```

```
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√5
⎢⎜-2 + ╲╱ 20 - 5⋅√5 , - ── - ─, ─ - ──────────⎟, ⎜-2 + ╲╱ 5⋅√5 + 20 , - ─ +
──
⎣⎝                        2    2  2      2     ⎠  ⎝                       2
2

        ________⎞  ⎛                                    ________⎞  ⎛
  1   ╲╱ √5 + 4 ⎟  ⎜      ___________     √5   1  1  ╲╱ 4 - √5 ⎟  ⎜    ___
___
, ─ + ──────────⎟, ⎜- ╲╱ 20 - 5⋅√5  - 2, - ── - ─, ─ + ──────────⎟, ⎜- ╲╱ 5
⋅√5
  2       2     ⎠  ⎝                      2    2  2       2     ⎠  ⎝

                                 ________⎞⎤
             1   √5  1   ╲╱ √5 + 4 ⎟⎥
______
 + 20  - 2, - ─ + ──, ─ - ──────────⎟⎥
             2   2   2       2     ⎠⎦
```

We have **4 distinct solutions for P**, they are

$$P_1 = \begin{bmatrix} -2+\sqrt{20-5\sqrt{5}} & \frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & \frac{1-\sqrt{4-\sqrt{5}}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0.969791257395215 & -1.61803398874989 \\ -1.61803398874989 & -0.164065513052028 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -2+\sqrt{20+5\sqrt{5}} & \frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & \frac{1+\sqrt{4+\sqrt{5}}}{2} \end{bmatrix} = \begin{bmatrix} 3.58393587781047 & 0.618033988 \\ 0.618033988749895 & 1.74860602 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} -2-\sqrt{20-5\sqrt{5}} & \frac{-\sqrt{5}-1}{2} \\ \frac{-\sqrt{5}-1}{2} & \frac{1+\sqrt{4-\sqrt{5}}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} -4.96979125739522 & -1.61803398874989 \\ -1.61803398874989 & 1.16406551305203 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} -2 - \sqrt{20 + 5\sqrt{5}} & \frac{\sqrt{5} - 1}{2} \\ \frac{\sqrt{5} - 1}{2} & \frac{1 - \sqrt{4 + \sqrt{5}}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} -7.58393587781047 & 0.618033988749895 \\ 0.618033988749895 & -0.748606020478416 \end{bmatrix}$$

Calculating

$$K = -R^{-1}B^TP$$
$$u(t) = -Kx(t)$$

In [3]:
```
P1 = sym.Matrix([[P[0][0], P[0][1]], [P[0][1], P[0][2]]])
K1 = -B.T * P1
sym.pprint(K1)

P2 = sym.Matrix([[P[1][0], P[1][1]], [P[1][1], P[1][2]]])
K2 = -B.T * P2
sym.pprint(K2)

P3 = sym.Matrix([[P[2][0], P[2][1]], [P[2][1], P[2][2]]])
K3 = -B.T * P3
sym.pprint(K3)

P4 = sym.Matrix([[P[3][0], P[3][1]], [P[3][1], P[3][2]]])
K4 = -B.T * P4
sym.pprint(K4)
```

```
[                  ____________]
[1 + √5   -1 + ╲╱ 4 - √5 ]
[             ____________    ]
[1 - √5   - ╲╱ √5 + 4   - 1]
[             ____________    ]
[1 + √5   - ╲╱ 4 - √5   - 1]
[                  ____________]
[1 - √5   -1 + ╲╱ √5 + 4 ]
```

For P_1:

$$K_1 = \begin{bmatrix} 1 + \sqrt{5} & -1 + \sqrt{4 - \sqrt{5}} \end{bmatrix} = \begin{bmatrix} 3.23606797749979 & 0.328131026104055 \end{bmatrix}$$

For P_2:

$$K_2 = \begin{bmatrix} 1 - \sqrt{5} & -1 - \sqrt{4 + \sqrt{5}} \end{bmatrix} = \begin{bmatrix} -1.23606797749979 & -3.49721204095683 \end{bmatrix}$$

For P_3:

$$K_3 = \begin{bmatrix} 1 + \sqrt{5} & -1 - \sqrt{4 + \sqrt{5}} \end{bmatrix} = \begin{bmatrix} 3.23606797749979 & -2.32813102610406 \end{bmatrix}$$

For P_4:

$$K_4 = \begin{bmatrix} 1-\sqrt{5} & -1+\sqrt{4+\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -1.23606797749979 & 1.49721204095683 \end{bmatrix}$$

Optimal Cost

$$C = \frac{1}{2} x^T(t_0) P x(t_0)$$

$$t_0 = 0$$

$$x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x^T(t_0) = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

In [5]:
```
x = sym.Matrix([[1], [-2]])
C1 = (x.T * P1 * x) / 2
C2 = (x.T * P2 * x) / 2
C3 = (x.T * P3 * x) / 2
C4 = (x.T * P4 * x) / 2

sym.pprint(C1)
sym.pprint(C2)
sym.pprint(C3)
sym.pprint(C4)
```

```
⎡                            _____________     ⎤
⎢      __________           ╲╱ 20 - 5⋅√5       ⎥
⎢- ╲╱  4 - √5    + 1 + ─────────────────── + √5⎥
⎣                              2               ⎦
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⎢                    ________   ╲╱ 5⋅√5 + 20   ⎥
⎢-√5 + 1 + ╲╱ √5 + 4    +   ─────────────────  ⎥
⎣                                  2           ⎦
⎡     _____________                            ⎤
⎢   ╲╱ 20 - 5⋅√5             __________        ⎥
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⎣         2                                    ⎦
⎡     _____________                            ⎤
⎢   ╲╱ 5⋅√5 + 20         ________              ⎥
⎢- ────────────────── - ╲╱ √5 + 4    - √5 + 1  ⎥
⎣         2                                    ⎦
```

For P_1:

$$C_1 = -\sqrt{4-\sqrt{5}} + 1 + \frac{\sqrt{20-5\sqrt{5}}}{2} + \sqrt{5} = 3.39283258009334$$

For P_2:

$$C_2 = -\sqrt{5} + 1 + \sqrt{\sqrt{5}+4} + \frac{\sqrt{20+5\sqrt{5}}}{2} = 4.05311200236228$$

For P_3:

$$C_3 = -\frac{\sqrt{20 - 5\sqrt{5}}}{2} + 1 + \sqrt{4 - \sqrt{5}} + \sqrt{5} = 3.07930337490624$$

For P_4:

$$C_4 = -\frac{\sqrt{20 + 5\sqrt{5}}}{2} - \sqrt{4 + \sqrt{5}} - \sqrt{5} + 1 = -6.52524795736186$$

Since $C_4 = -6.52524795736186$ is the minimum cost, we can say that P_4 is the optimal solution.