Importing the necessary libraries

In [2]:
```
import sympy as sym
import numpy as np
import matplotlib.pyplot as plt
```

Problem Statement:

State-Space Definition of the given System:

$$\dot{x_1}(t) = x_2(t)$$

$$\dot{x_2}(t) = -2x_1(t) + 2x_2(t) + 2u(t)$$

Boundary Conditions:

$$t \in [0, 6]$$

$$x_1(0) = 1$$

$$x_2(0) = -2$$

Performance Index:

$$PI = \frac{1}{2}\left[x_1^2(6) + 2x_1(6)x_2(6) + 2x_2^2(6)\right]$$

$$+ \int_0^6 \left[2x_1^2(t) + 3x_1(t)x_2(t) + 2x_2^2(t) + \frac{1}{2}u^2(t)\right] dt$$

Find the optimal control input $u(t)$ for the given system to minimize the performance index.

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$t_f = 6$$

$$Q = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

$$R = 1$$

Eigen-Vales of A are:

$$\lambda_1 = 1 + j$$

$$\lambda_2 = 1 - j$$

Since the Real-Part of both the Eigen-Values is positive, the system is unstable.

Matrix Differential Riccati Equation:

$$\dot{P} = -(A^T P + PA - PBR^{-1}B^T P + Q)$$

$$\dot{P} = \begin{bmatrix} \dot{P}_{11}(t) & \dot{P}_{12}(t) \\ \dot{P}_{12}(t) & \dot{P}_{22}(t) \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix}$$
$$+ \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Now, lets use **SymPy** to simplify the above equation:

In [3]:
```
t = sym.symbols('t')
P11, P12, P22 = sym.symbols('P11 P12 P22')
P = sym.Matrix([[P11, P12], [P12, P22]])
A = sym.Matrix([[0, 1], [-2, 2]])
B = sym.Matrix([[0], [2]])
Q = sym.Matrix([[4, 3], [3, 4]])
R = 1

P_dot = -A.T*P - P*A + P*B*1*B.T*P - Q
P_dot
```

Out[3]:
$$\begin{bmatrix} 4P_{12}^2 + 4P_{12} - 4 & -P_{11} + 4P_{12}P_{22} - 2P_{12} + 2P_{22} - 3 \\ -P_{11} + 4P_{12}P_{22} - 2P_{12} + 2P_{22} - 3 & -2P_{12} + 4P_{22}^2 - 4P_{22} - 4 \end{bmatrix}$$

$$\begin{bmatrix} \dot{P}_{11}(t) & \dot{P}_{12}(t) \\ \dot{P}_{12}(t) & \dot{P}_{22}(t) \end{bmatrix}$$
$$= \begin{bmatrix} 4P_{12}^2 + 4P_{12} - 4 & -P_{11} + 4P_{12}P_{22} - 2P_{12} + 2P_{22} - 3 \\ -P_{11} + 4P_{12}P_{22} - 2P_{12} + 2P_{22} - 3 & -2P_{12}^2 + 4P_{22}^2 - 4P_{22} - 4 \end{bmatrix}$$

$$\dot{P}_{11}(t) = 4P_{12}^2(t) + 4P_{12}(t) - 4$$

$$\dot{P_{12}}(t) = -P_{11}(t) + 4P_{12}(t)P_{22}(t) - 2P_{12}(t) + 2P_{22}(t) - 3$$

$$\dot{P_{22}}(t) = -2P_{12}^2(t) + 4P_{22}^2(t) - 4P_{22}(t) - 4$$

$$P_{11}(6) = 1$$

$$P_{12}(6) = 1$$

$$P_{22}(6) = 2$$

Now, we will solve above equations using **Runge-Kutta Method** and **Euler's Method** with a small time-step.

Runge-Kutta Method:

Solving the above equations using Runge-Kutta Method:

$$\dot{P_{11}}(t) = 4P_{12}^2(t) + 4P_{12}(t) - 4$$

$$\dot{P_{12}}(t) = -P_{11}(t) + 4P_{12}(t)P_{22}(t) - 2P_{12}(t) + 2P_{22}(t) - 3$$

$$\dot{P_{22}}(t) = -2P_{12}^2(t) + 4P_{22}^2(t) - 4P_{22}(t) - 4$$

$$P_{11}(6) = 1$$

$$P_{12}(6) = 1$$

$$P_{22}(6) = 2$$

Let the time-step be

$$\Delta t = 0.001$$

Converting to vector form:

$$\vec{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} P_{11}(t) \\ P_{12}(t) \\ P_{22}(t) \end{bmatrix}$$

$$\dot{\vec{X}} = \begin{bmatrix} \dot{P_{11}} \\ \dot{P_{12}} \\ \dot{P_{22}} \end{bmatrix} = \begin{bmatrix} 4P_{12}^2 + 4P_{12} - 4 \\ -P_{11} + 4P_{12}P_{22} - 2P_{12} + 2P_{22} - 3 \\ -2P_{12}^2 + 4P_{22}^2 - 4P_{22} - 4 \end{bmatrix} = \begin{bmatrix} 4x_2^2 + 4x_2 - 4 \\ -x_1 + 4x_2x_3 - 2x_2 + 2x \\ -2x_2^2 + 4x_3^2 - 4x_3 - \end{bmatrix}$$

Therefore,

$$\dot{\vec{X}} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 4x_2^2 + 4x_2 - 4 \\ -x_1 + 4x_2x_3 - 2x_2 + 2x_3 - 3 \\ -2x_2^2 + 4x_3^2 - 4x_3 - 4 \end{bmatrix}$$

$$\vec{X}_{t_f=6} = \begin{bmatrix} x_1(6) \\ x_2(6) \\ x_3(6) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Runge-Kutta Method Implementation:

In [4]:
```
# Runge-Kutta 4th order
def rk4(f, x0, t0, tf, h):
    t = np.arange(t0, tf, h)
    x = np.zeros((len(t), len(x0)))
    x[0] = x0
    for i in range(len(t)-1):
        k1 = f(x[i], t[i])
        k2 = f(x[i] + h*k1/2, t[i] + h/2)
        k3 = f(x[i] + h*k2/2, t[i] + h/2)
        k4 = f(x[i] + h*k3, t[i] + h)
        x[i+1] = x[i] + h*(k1 + 2*k2 + 2*k3 + k4)/6
    return x

def f(x, t):
    return np.array([4*x[1]**2 + 4*x[1] - 4, -x[0] + 4*x[1]*x[2] - 2*x[1] +

x0 = np.array([1, 1, 2])

x = rk4(f, x0, 0, 6, 0.001)
```

In [5]:
```
x.shape
```

Out[5]:
```
(6000, 3)
```

Clearly, we have values for $\vec{X}$ from $t \in [0, 6]$ at a time-step of $\Delta t = 0.001$.

Now, let us plot $P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$ vs t

($P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$ are the values of $\vec{X}$ at each time-step)

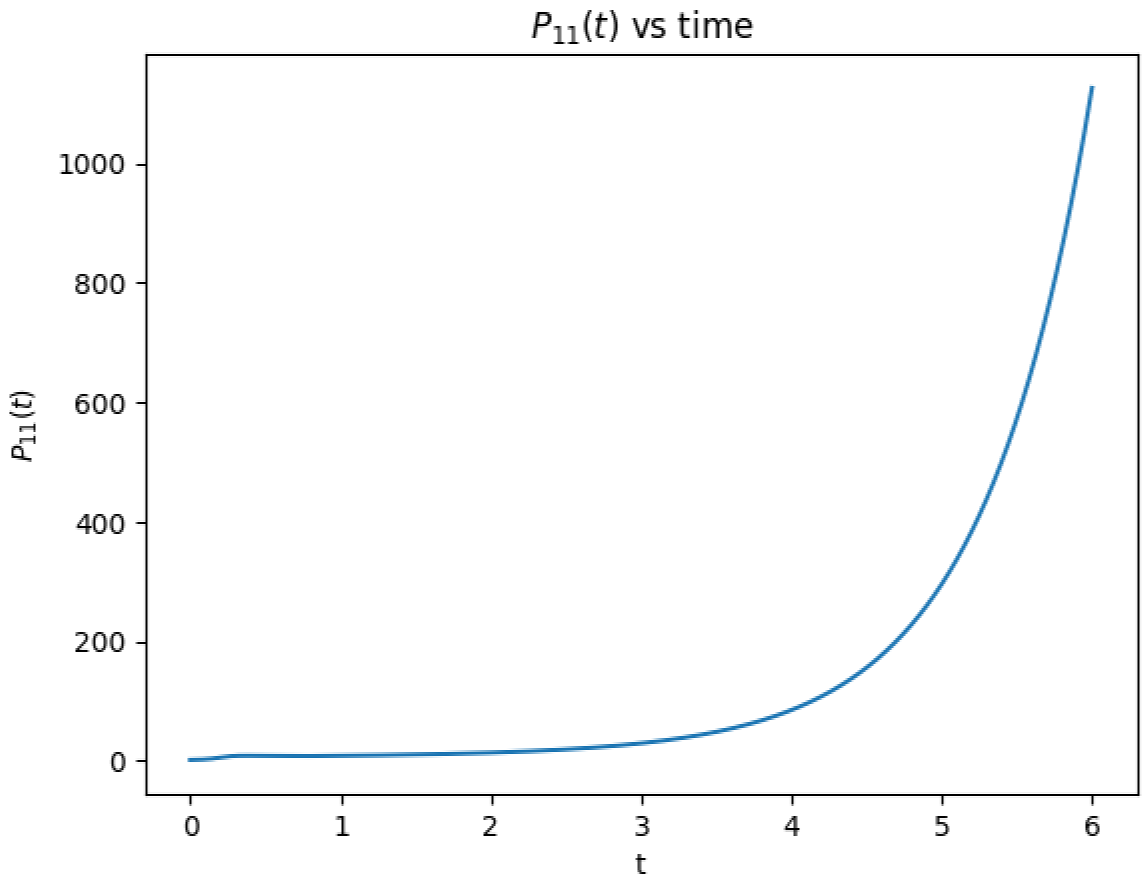
Plotting:

In [6]:
```
t = np.arange(0, 6, 0.001)
t.shape
```

Out[6]:
```
(6000,)
```

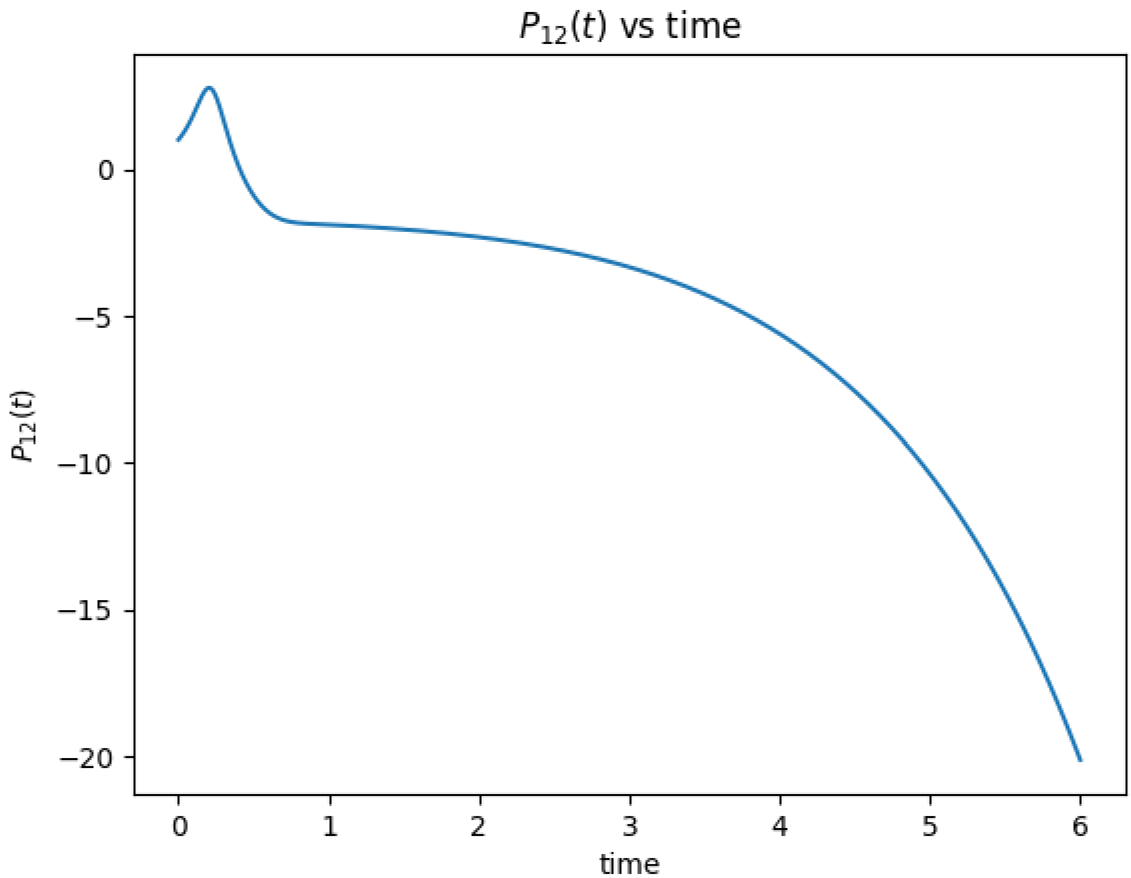
$P_{11}(t)$ vs t:

In [7]:
```
plt.plot(t, x[:,0])
plt.title(r'$P_{11}(t)$ vs time')
plt.xlabel('t')
plt.ylabel(r'$P_{11}(t)$')
plt.show()
```



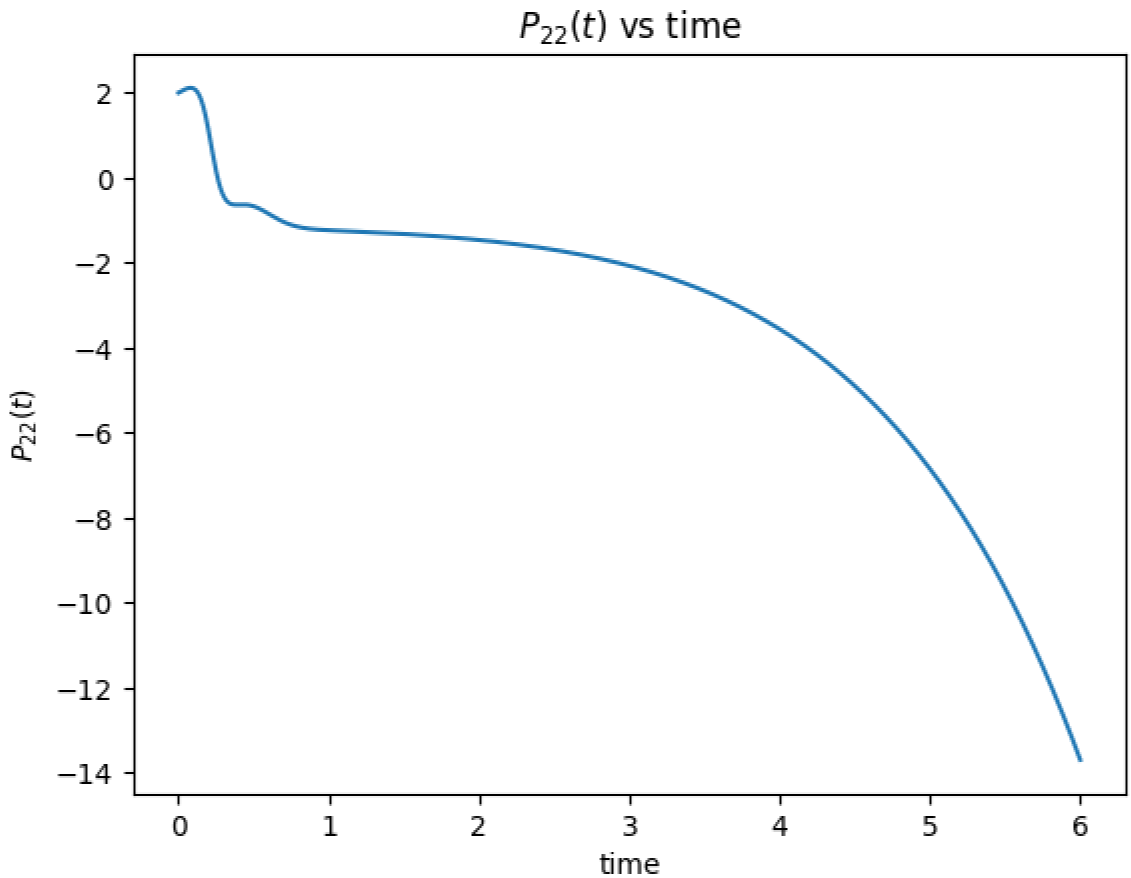
$P_{12}(t)$ vs t

In [8]:
```
plt.plot(t, x[:,1])
plt.title(r'$P_{12}(t)$ vs time')
plt.xlabel('time')
plt.ylabel(r'$P_{12}(t)$')
plt.show()
```



$P_{22}(t)$ vs t

In [9]:
```
plt.plot(t, x[:,2])
plt.title(r'$P_{22}(t)$ vs time')
plt.xlabel('time')
plt.ylabel(r'$P_{22}(t)$')
plt.show()
```



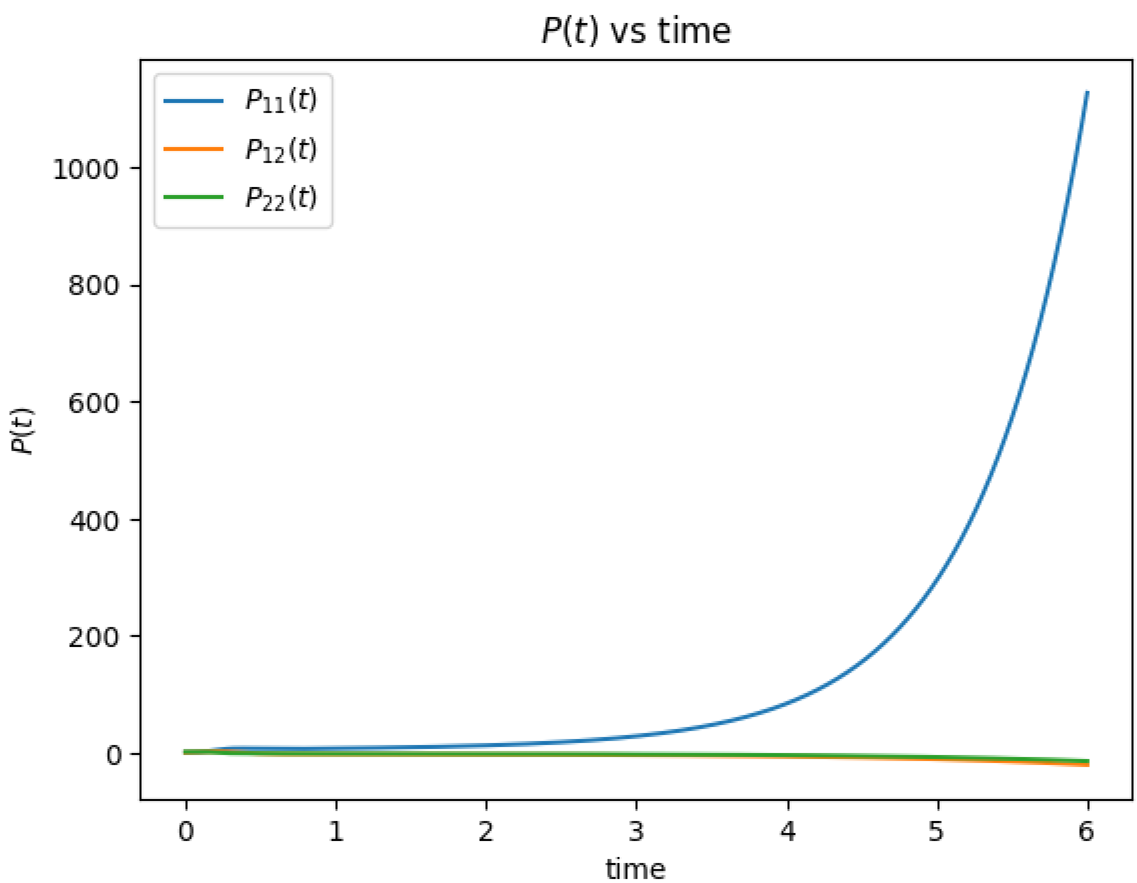
Plotting them together:

In [10]:
```
plt.plot(t, x[:,0], label=r'$P_{11}(t)$')
plt.plot(t, x[:,1], label=r'$P_{12}(t)$')
plt.plot(t, x[:,2], label=r'$P_{22}(t)$')

plt.title(r'$P(t)$ vs time')

plt.xlabel('time')
plt.ylabel(r'$P(t)$')
plt.legend()

plt.show()
```



In [11]:
```
x_rk4 = x.copy()
```

Euler's Method:

$$\dot{P}_{11}(t) = 4P_{12}^2(t) + 4P_{12}(t) - 4$$

$$\dot{P}_{12}(t) = -P_{11}(t) + 4P_{12}(t)P_{22}(t) - 2P_{12}(t) + 2P_{22}(t) - 3$$

$$\dot{P}_{22}(t) = -2P_{12}^2(t) + 4P_{22}^2(t) - 4P_{22}(t) - 4$$

$$P_{11}(6) = 1$$

$$P_{12}(6) = 1$$

$$P_{22}(6) = 2$$

Solving the above equations using Euler's Method:

Let the time-step be

$$\Delta t = 0.001$$

$$\dot{\vec{X}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 4x_2^2 + 4x_2 - 4 \\ -x_1 + 4x_2x_3 - 2x_2 + 2x_3 - 3 \\ -2x_2^2 + 4x_3^2 - 4x_3 - 4 \end{bmatrix}$$

$$\vec{X}_{t_f=6} = \begin{bmatrix} x_1(6) \\ x_2(6) \\ x_3(6) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Euler's Method Implementation:

In [12]:
```
# Solve using Euler's method
def euler(f, x0, t0, tf, h):
    t = np.arange(t0, tf, h)
    x = np.zeros((len(t), len(x0)))
    x[0] = x0
    for i in range(len(t)-1):
        x[i+1] = x[i] + h*f(x[i], t[i])
    return x

def f(x, t):
    return np.array([4*x[1]**2 + 4*x[1] - 4, -x[0] + 4*x[1]*x[2] - 2*x[1] +


x0 = np.array([1, 1, 2])

x = euler(f, x0, 0, 6, 0.001)
```

In [13]:
```
x.shape
```

Out[13]:
```
(6000, 3)
```

Clearly, we have values for $\vec{X}$ from $t \in [0, 6]$ at a time-step of $\Delta t = 0.001$.

Now, let us plot $P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$ vs t

($P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$ are the values of $\vec{X}$ at each time-step)

Plotting:

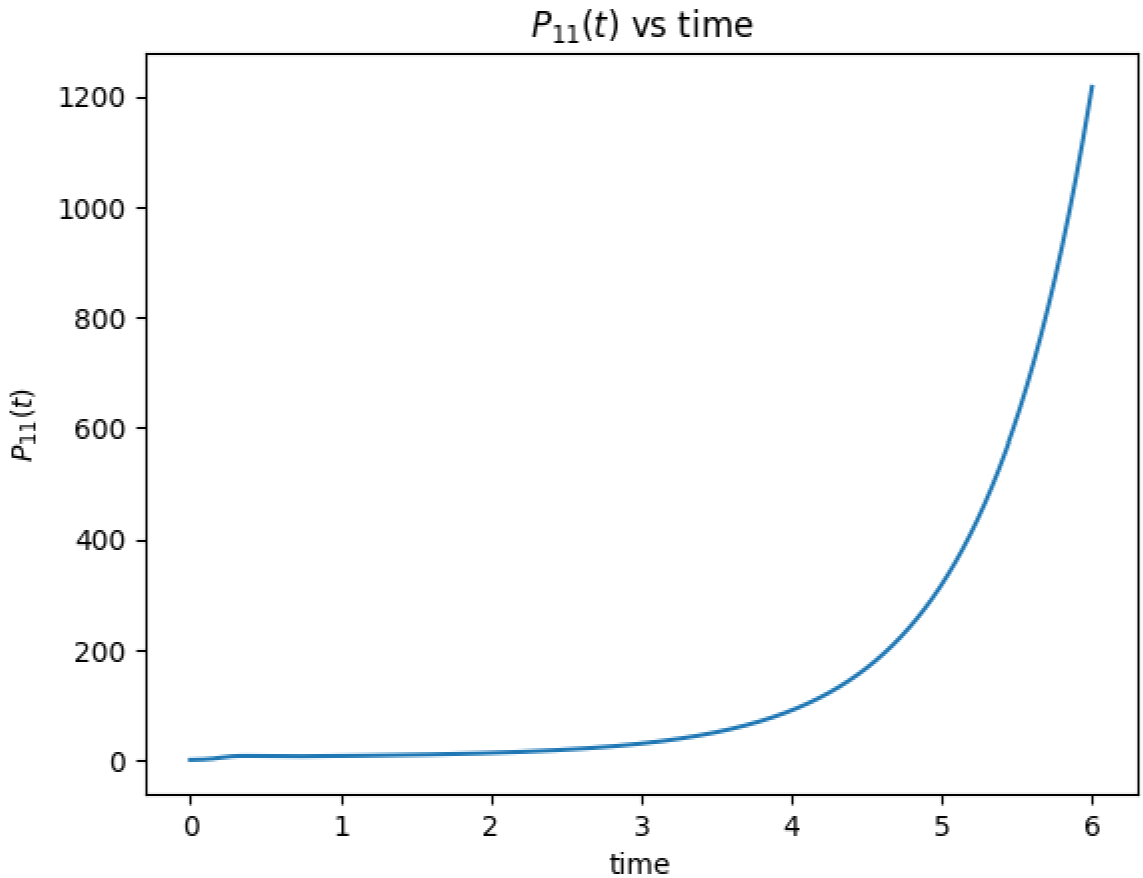
In [14]:
```
t = np.arange(0, 6, 0.001)
t.shape
```

Out[14]:
```
(6000,)
```

$P_{11}(t)$ vs t

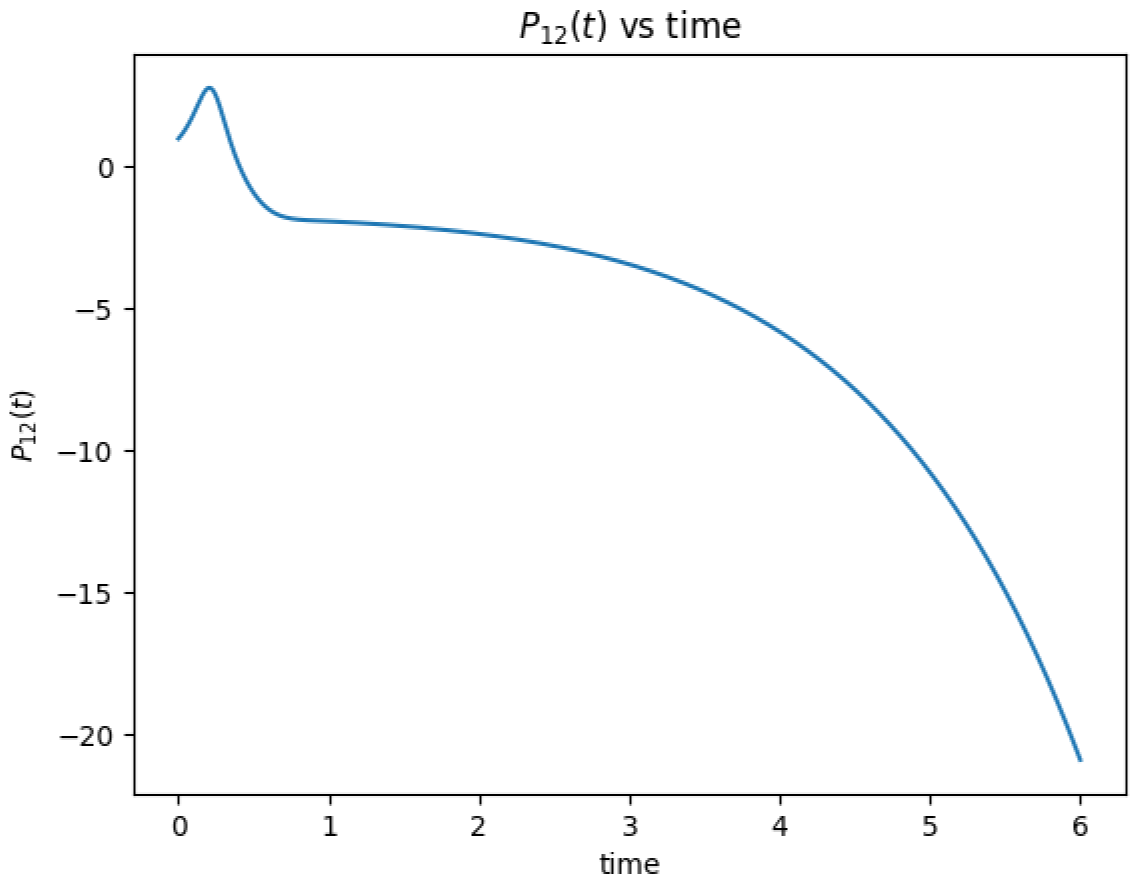
In [15]:
```
plt.plot(t, x[:,0])
plt.title(r'$P_{11}(t)$ vs time')
plt.xlabel('time')
```

```
plt.ylabel(r'$P_{11}(t)$')
plt.show()
```



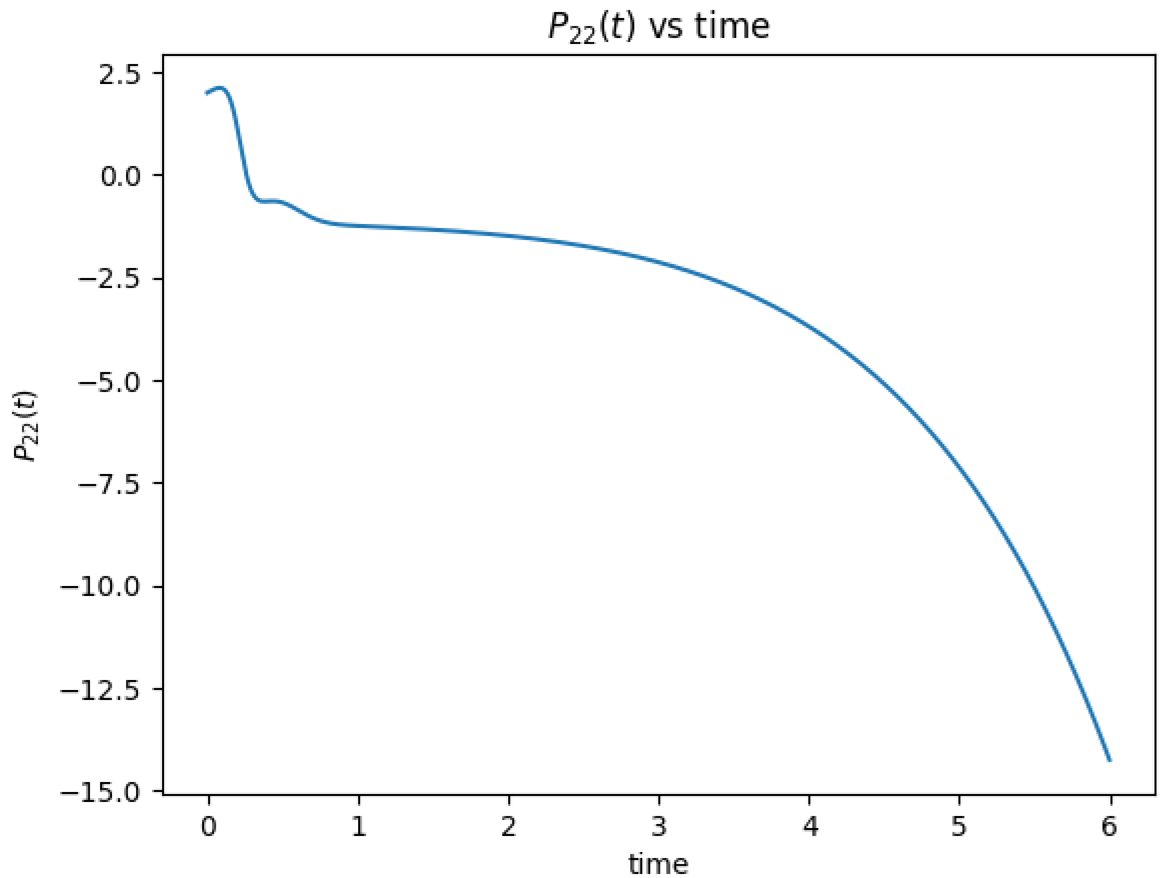
$P_{12}(t)$ vs t

In [16]:
```
plt.plot(t, x[:,1])
plt.title(r'$P_{12}(t)$ vs time')
plt.xlabel('time')
plt.ylabel(r'$P_{12}(t)$')
plt.show()
```



$P_{22}(t)$ vs t

In [17]:
```
plt.plot(t, x[:,2])
plt.title(r'$P_{22}(t)$ vs time')
plt.xlabel('time')
plt.ylabel(r'$P_{22}(t)$')
plt.show()
```



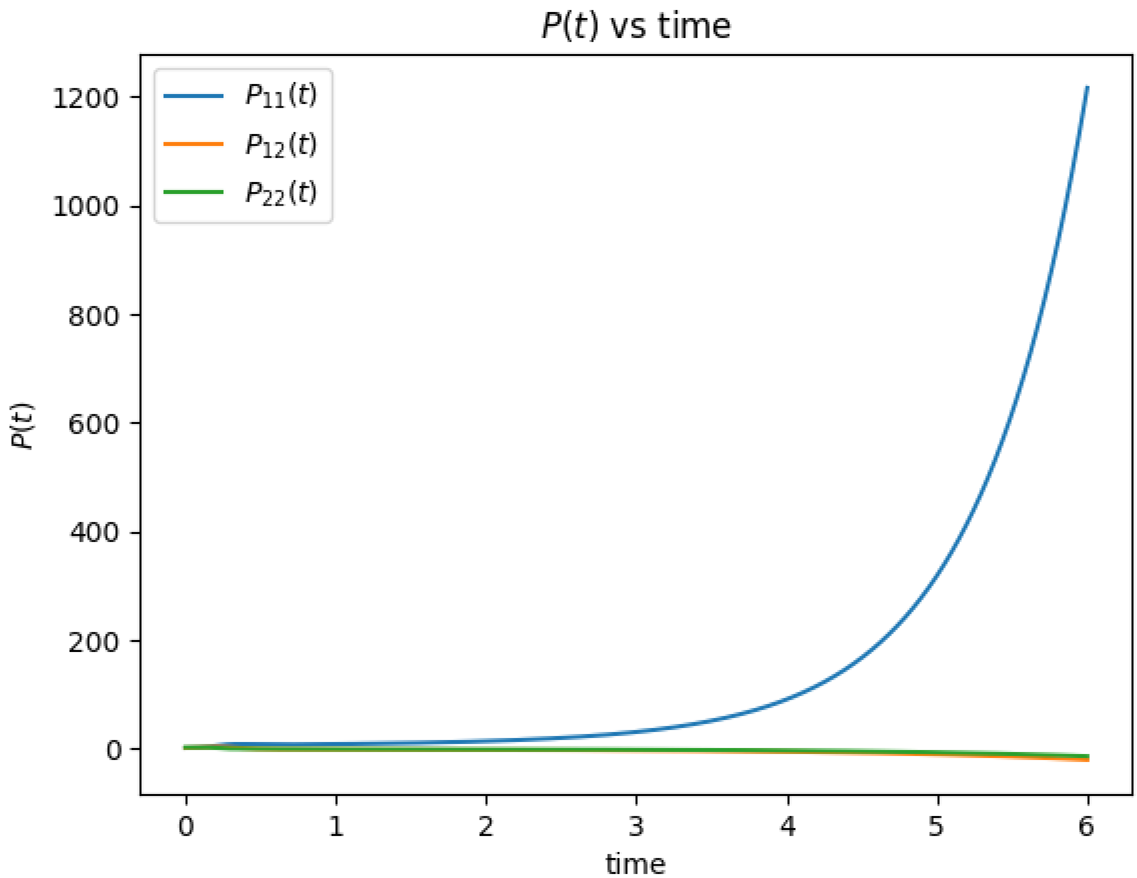
Plotting them together:

In [18]:
```
plt.plot(t, x[:,0], label=r'$P_{11}(t)$')
plt.plot(t, x[:,1], label=r'$P_{12}(t)$')
plt.plot(t, x[:,2], label=r'$P_{22}(t)$')

plt.title(r'$P(t)$ vs time')

plt.xlabel('time')
plt.ylabel(r'$P(t)$')
plt.legend()

plt.show()
```



$P(t)$ vs time
$P_{11}(t)$
$P_{12}(t)$
$P_{22}(t)$
1200
1000
800
600
400
200
0
$P(t)$
0
1
2
3
4
5
6
time