Introduction to Navigation, Guidance and Control of Unmanned Systems: Modelling

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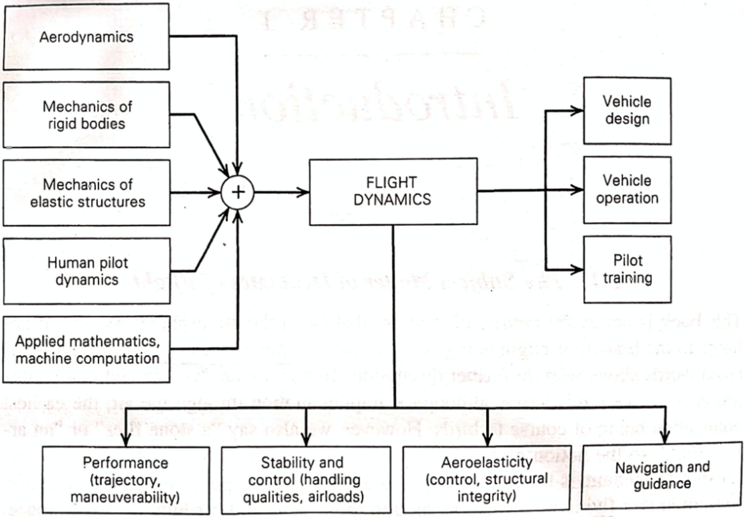
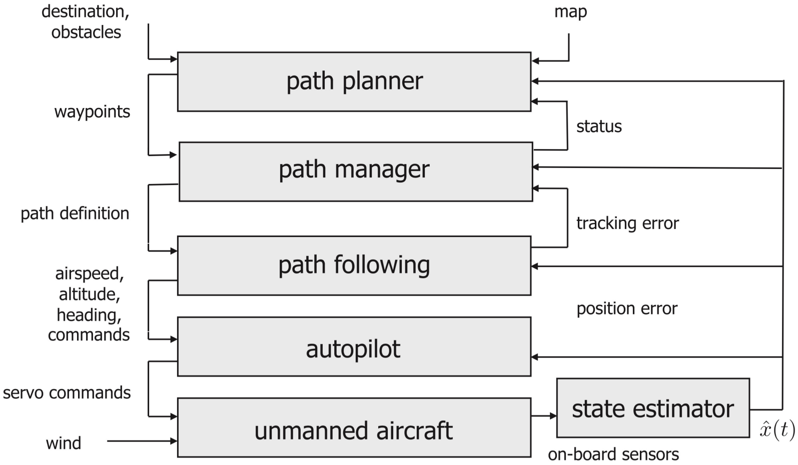
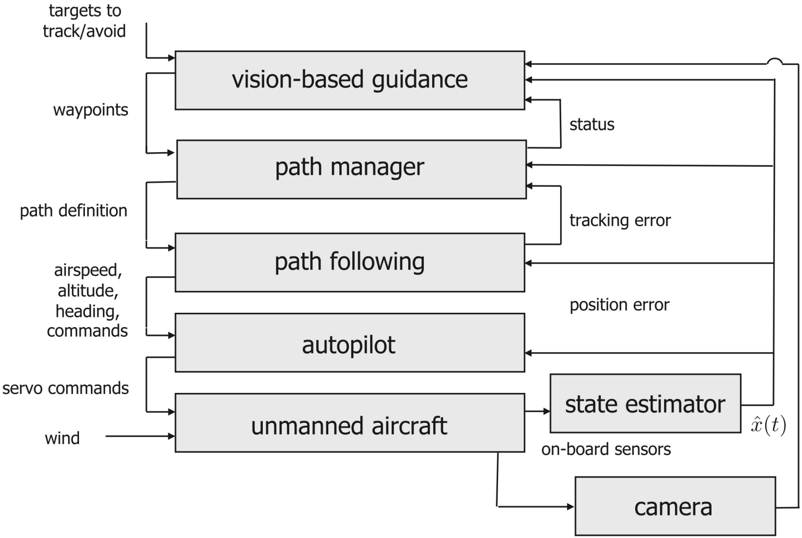

Aerodynamics
Mechanics of rigid bodies
Mechanics of elastic structures
Human pilot dynamics
Applied mathematics, machine computation
+
FLIGHT DYNAMICS
Vehicle design
Vehicle operation
Pilot training
Performance (trajectory, maneuverability)
Stability and control (handling qualities, airloads)
Aeroelasticity (control, structural integrity)
Navigation and guidance

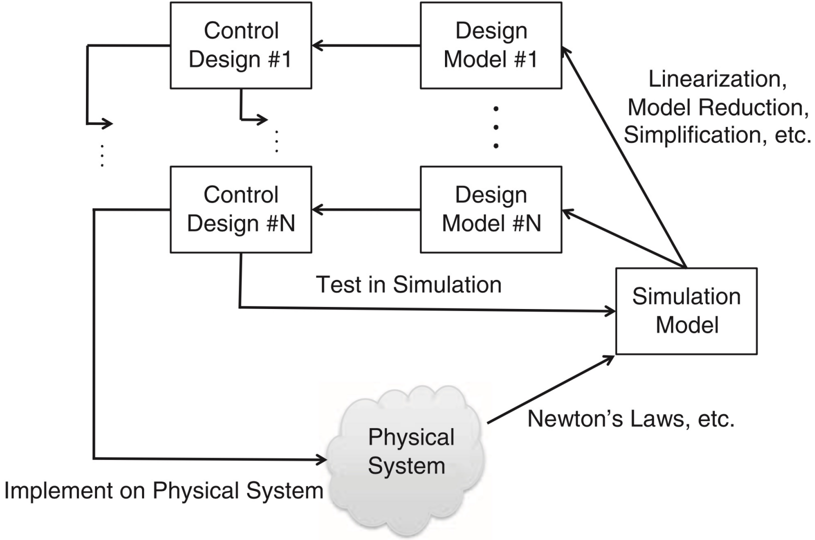
Unmanned Autonomous Vehicles: System Architecture



Unmanned Autonomous Vehicles: System Architecture



Unmanned Autonomous Vehicles: Design Models



- Physical System: UAV including actuators (control flaps and propeller) and sensors (IMU, GPS, camera, etc.)
- Design process: Model the physical system using nonlinear differential equations
  - ⇒ Appropriate approximations and simplifications
  - ⇒ Capture all of the important characteristics of physical system
- Simulation Model: High fidelity computer simulation of physical system
  - ⇒ Rigid body kinematics and dynamics, aerodynamic forces and moments, and the onboard sensors
  - ⇒ only an approximation of the physical system
- An effective design on simulation model may not function properly on the physical system.
- Simulation model is typically nonlinear and of high order, and is too mathematically complex to be useful for control design.
- For any physical system, there may be multiple design models capturing certain aspects of the design process.

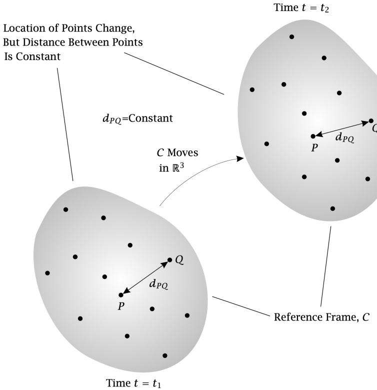
- Mechanics: Study of physical forces' effects on objects.
- Mechanics: Statics and dynamics
- Kinematics: Study of the geometry of motion without regard to the forces that cause the motion.
- Objectives of kinematics
  - ⇒ A set of reference frames in which to observe the motion of a system
  - ⇒ A set of coordinate systems fixed in the chosen reference frames
  - ⇒ Angular velocity and angular acceleration of each reference frame (and/or rigid body) resolved in the chosen coordinate systems
  - ⇒ Position, velocity, and acceleration of each particle in the system
- Any physical motion cannot occur without some kind of external inputs.
- Kinetics of the particle:
  - ⇒ To describe quantitatively the forces that act on a particle
  - ⇒ To determine the motion that results from the application of these forces using postulated laws of physics
  - ⇒ To analyze the motion

- Are reference frame and coordinate system the same? Not really
- **Reference frame:** A perspective from which observations are made regarding the motion of a system.
- $\mathcal{C}$: A collection of at least three noncolinear points that move in three-dimensional Euclidean space $\mathbb{R}^3$.

Rigidly connected or rigidly attached

Points P and Q in $\mathcal{C}$ are said to be rigidly connected or rigidly attached if the distance between P and Q, denoted d_{PQ}, is constant regardless of how P and Q move in $\mathbb{R}^3$.

A collection $\mathcal{C}$ is then said to be a reference frame if the distance between every pair of points in $\mathcal{C}$ is rigidly connected.



- Example of reference frames:
  - ⇒ Any three-dimensional rigid body (e.g., a cube, a sphere, or a cylinder)
  - ⇒ Any planar rigid object (e.g., a square, a circle, or a triangle)
- What about an isolated point in $\mathbb{R}^3$? No
- What about a line in $\mathbb{R}^3$? No, why?
- Points on a line are colinear.

- **Assumption of Newtonian mechanics**: Space and time are invariant with respect to changes in reference frames.
- For any two reference frames $\mathcal{A}$ and $\mathcal{B}$,

$$\boxed{\boldsymbol{b}^{\mathcal{A}} = \boldsymbol{b}^{\mathcal{B}} = \boldsymbol{b}; \quad t^{\mathcal{A}} = t^{\mathcal{B}} = t}$$

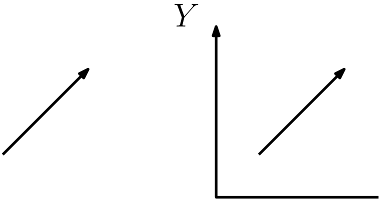
Nature of Reference frames

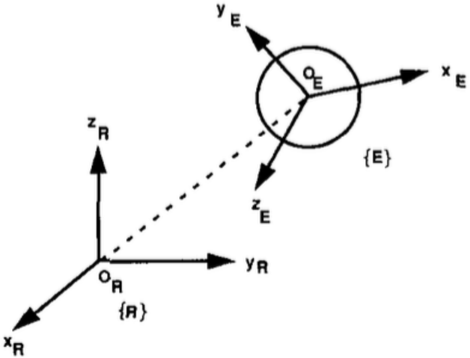
- Classification of reference frames: Inertial or noninertial.
- Inertial or Newtonian reference frame: one whose points are either absolutely fixed in space or at most translate relative to an absolutely fixed set of points with the same constant velocity.
- **Noninertial or non-Newtonian reference frame**: one whose points accelerate with time.
- **Axiom of Newtonian mechanics**:
  - $\Rightarrow$ The inertial reference frames exist and that the laws of mechanics are valid only in an inertial reference frame.
- Any possible difficulties with reference frame?
- Definition of a reference frame does not directly provide any means of measuring the observations that may be made of the motion by an observer fixed in the reference frame.
- Coordinate system: to quantify or realize the motion of the system

- **Coordinate system**
  - ⇒ What are the elements of a coordinate system?
  - ⇒ A point (the origin) fixed in reference frame $\mathcal{A}$, from which all distances are measured.
  - ⇒ A basis, which is a set of three linearly independent directions that are fixed in reference frame $\mathcal{A}$.
  - ⇒ A basis provides a way to resolve the vectors in $\mathbb{R}^3$.
- A coordinate system must be fixed in a unique reference frame.
- A coordinate system cannot be fixed simultaneously in two distinct reference frames.
- How to choose the basis of coordinate frame?
  - ⇒ Any set of three linearly independent vectors
  - ⇒ Is it convenient? May not be always convenient
  - ⇒ A set of mutually orthogonal unit vectors for better convenience.

- Reasons for requirements of different coordinate frames
- Newton's equations of motion are derived relative to a fixed, inertial reference frame. However, motion is most easily described in a body-fixed frame.
- Aerodynamic forces and torques act on the aircraft body and are most easily described in a body-fixed reference frame.
- On-board sensors like accelerometers and rate gyros measure information with respect to the body frame.
- GPS measures position, ground speed, and course angle with respect to the inertial frame.
- Most mission requirements, like loiter points and flight trajectories, as well as map information are specified in the inertial frame.
- **Transformation of one frame to another**: Translation and rotation

Coordinate Transformation





- Position of a rigid body: position vector $O_R O_E$ of origin
- Orientation of rigid body: 3×3 rotation matrix
- For simplification, we assume $O_E = O_R$

Coordinate Transformation

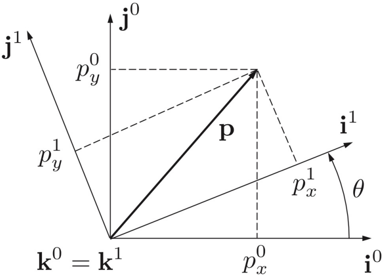
- Rotation matrix approach utilizes 9 parameters, which obey the orthogonality and unit length constraints, to describe the orientation of the rigid body.
- A rigid body possesses 3 rotational DOF, 3 independent parameters are sufficient to characterize completely and unambiguously its orientation.
- Three-parameter representations are popular in engineering because they minimize the dimensionality of rigid-body control problem
- Transformation of coordinate axes is an important necessity in resolving angular positions and rates from one coordinate system to another.
- Transformation matrix: Mapping of the components of a vector, resolved in one frame, into the same resolved into the another frame.
  - $\Rightarrow$ Direction cosine matrix (DCM)
  - $\Rightarrow$ Euler Angles
  - $\Rightarrow$ Quaternions

- Vector $\boldsymbol{p}$ in $\mathcal{F}^0$ frame

$$\boldsymbol{p} = p_x^0 \boldsymbol{i}^0 + p_y^0 \boldsymbol{j}^0 + p_z^0 \boldsymbol{k}^0$$

- Vector $\boldsymbol{p}$ in $\mathcal{F}^1$ frame

$$\boldsymbol{p} = p_x^1 \boldsymbol{i}^1 + p_y^1 \boldsymbol{j}^1 + p_z^1 \boldsymbol{k}^1$$



- Since the vector is same, we have

$$p_x^1 \boldsymbol{i}^1 + p_y^1 \boldsymbol{j}^1 + p_z^1 \boldsymbol{k}^1 = p_x^0 \boldsymbol{i}^0 + p_y^0 \boldsymbol{j}^0 + p_z^0 \boldsymbol{k}^0$$

- If we denote vector representations in $\mathcal{F}^0$ and $\mathcal{F}^1$ as $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ then

$$\boxed{\boldsymbol{p}^1 = \mathcal{R}_0^1 \boldsymbol{p}^0}$$

- Taking dot products on both sides with unit vectors $\boldsymbol{i}^1$, $\boldsymbol{j}^1$, and $\boldsymbol{k}^1$,

$$\begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{i}^1 \cdot \boldsymbol{i}^0 & \boldsymbol{i}^1 \cdot \boldsymbol{j}^0 & \boldsymbol{i}^1 \cdot \boldsymbol{k}^0 \\ \boldsymbol{j}^1 \cdot \boldsymbol{i}^0 & \boldsymbol{j}^1 \cdot \boldsymbol{j}^0 & \boldsymbol{j}^1 \cdot \boldsymbol{k}^0 \\ \boldsymbol{k}^1 \cdot \boldsymbol{i}^0 & \boldsymbol{k}^1 \cdot \boldsymbol{j}^0 & \boldsymbol{k}^1 \cdot \boldsymbol{k}^0 \end{pmatrix} \begin{pmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{pmatrix}$$

- General transformation matrix (also called DCM Matrix)

$$\mathcal{R}_0^1 = \begin{pmatrix} \boldsymbol{i}^1 \cdot \boldsymbol{i}^0 & \boldsymbol{i}^1 \cdot \boldsymbol{j}^0 & \boldsymbol{i}^1 \cdot \boldsymbol{k}^0 \\ \boldsymbol{j}^1 \cdot \boldsymbol{i}^0 & \boldsymbol{j}^1 \cdot \boldsymbol{j}^0 & \boldsymbol{j}^1 \cdot \boldsymbol{k}^0 \\ \boldsymbol{k}^1 \cdot \boldsymbol{i}^0 & \boldsymbol{k}^1 \cdot \boldsymbol{j}^0 & \boldsymbol{k}^1 \cdot \boldsymbol{k}^0 \end{pmatrix}$$

- For above figure, transformation matrix for rotation about $z-$axis

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Transformation matrix for rotation about $y-$axis

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

- Transformation matrix for rotation about $x-$axis

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

- Transformation matrix is an orthogonal rotation matrix.

$$\boxed{\left(\mathcal{R}_a^b\right)^{-1} = \left(\mathcal{R}_a^b\right)^\top = \mathcal{R}_b^a; \;\; \mathcal{R}_b^c\mathcal{R}_a^b = \mathcal{R}_a^c; \;\; \det\left(\mathcal{R}_a^b\right) = 1}$$

Coordinate Transformation

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If $\boldsymbol{i}' = \boldsymbol{j}$, $\boldsymbol{j}' = -\boldsymbol{i}$, and $\boldsymbol{k}' = \boldsymbol{k}$ then what would be the DCM matrix?

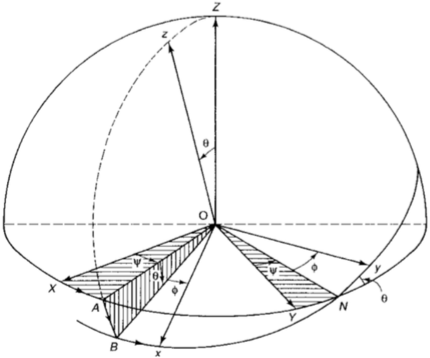
$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider two coordinate frames with their unit vectors as $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}')$, respectively. If $\boldsymbol{i}' = \boldsymbol{i}$, $\boldsymbol{j}' = -\boldsymbol{k}$, and $\boldsymbol{k}' = \boldsymbol{j}$ then what would be the DCM matrix?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

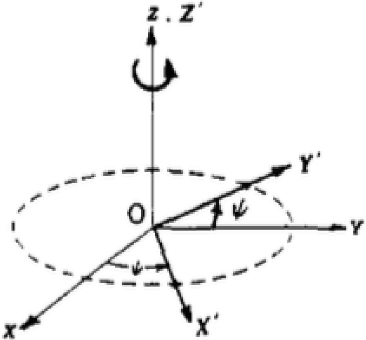
Coordinate Transformation

- Euler angles
  - $\Rightarrow$ Method to specify the angular orientation of one coordinate frame w.r.t. another frame
  - $\Rightarrow$ A series of three ordered right-handed rotations
  - $\Rightarrow$ Correspond to the conventional roll pitch yaw angles



- Nonunique Euler angles
  - Interchange in order of rotation $\Rightarrow$ different Euler angle representation.
  - Rotations are made about the Z, Y, X axes through an angle ψ, θ, ϕ angles.
  - These rotations are made in the positive (anticlockwise sense) when looking down the axis of rotation toward the origin.

- Euler angles: Three ordered elemental rotations
- Extrinsic rotations: Rotations about the axes xyz of the original coordinate system, which is assumed to remain motionless.
- Intrinsic rotations: Rotations about the axes of rotating coordinate system XYZ, which changes its orientation after each elemental rotation.
- Another classification
  - ⇒ Proper Euler angles
  - ⇒ Tait-Bryan angles
- Proper Euler angles: $(zxz, zyz, xyx, xzx, yzy, yxy)$
- Tait-Bryan angles: $(zyx, zxy, xyz, xzy, yzx, yxz)$
- What is the major difference between Proper Euler and Tait-Bryan angles?
- Tait-Bryan angles represent rotations about three distinct axes, while proper Euler angles use the same axis for both the first and third elemental rotations.



- Rotation about Z axis in anticlockwise direction by an angle ψ

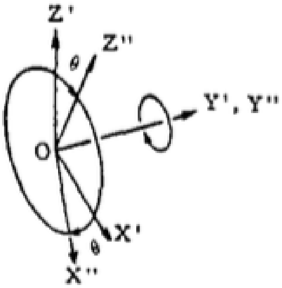
$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$= \mathcal{A} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where

$$\mathcal{A} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Coordinate Transformation



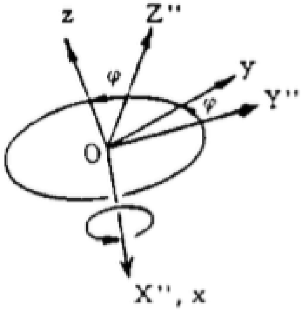
- Rotation about Y axis in anticlockwise direction by an angle θ

$$\begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

$$= \mathcal{B} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

where

$$\mathcal{B} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$



- Rotation about X axis in anticlockwise direction by an angle ϕ

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix}$$

$$= \mathcal{D} \begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix}$$

where

$$\mathcal{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

- If the consecutive rotations are performed in the order ψ, θ, ϕ i.e., (yaw, pitch, and roll) on reference frame XYZ, then we obtain other reference frame xyz.
- Rotation matrix for representing these three rotations

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathcal{D} \begin{bmatrix} X^{''} \\ Y^{''} \\ Z^{''} \end{bmatrix} = \mathcal{D}\mathcal{B} \begin{bmatrix} X^{'} \\ Y^{'} \\ Z^{'} \end{bmatrix} = \mathcal{D}\mathcal{B}\mathcal{A} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Equivalently,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\mathcal{D}\mathcal{B}\mathcal{A}}_{\mathcal{C}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

- Equivalent rotation matrix $\mathcal{C} = \mathcal{DBA}$ can be written as

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

- This rotation matrix is called Euler angle transformation matrix.
- Range of Euler angles:

$$-\pi \leq \psi \leq \pi, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad -\pi \leq \phi \leq \pi$$

- Is there any issue with $|\theta| > \pi/2$?

Simple Rotation

A motion of a rigid body or reference frame B relative to a rigid body or reference frame A is called a simple rotation of B in A if there exists a line L, called an axis of rotation, whose orientation relative to both A and B remains unaltered throughout the motion.

Euler's Theorem on Rotation

Every change in the relative orientation of two rigid bodies or reference frames A and B can be produced by means of a simple rotation of B in A.

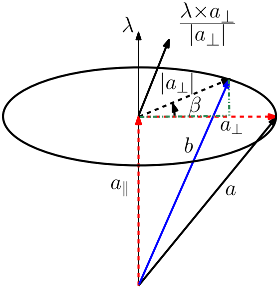
- Consider the rotation of vector $\boldsymbol{a}$ in fixed frame about an arbitrary axis to result in $\boldsymbol{b}$.
- Component of $\boldsymbol{a} \parallel$ axis of rotation $\boldsymbol{\lambda}$

$$\boldsymbol{a}_{\parallel} = (\boldsymbol{a} \cdot \boldsymbol{\lambda})\,\boldsymbol{\lambda}$$

- Component of $\boldsymbol{a} \perp$ to axis of rotation $\boldsymbol{\lambda}$

$$\boldsymbol{a}_{\perp} = \boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{\lambda})\,\boldsymbol{\lambda}$$

- Parallel component remain unaltered.
  - Rotated vector of $\boldsymbol{a}_{\perp}$



$\boldsymbol{\lambda}$: unit vector along axis of rotation

$$\boldsymbol{a}_{\perp}^{\text{rot}} = \boldsymbol{a}_{\perp}\cos\beta + |\boldsymbol{a}_{\perp}|\sin\beta\left(\frac{\boldsymbol{\lambda} \times \boldsymbol{a}_{\perp}}{|\boldsymbol{a}_{\perp}|}\right)$$

$$\boldsymbol{a}_{\perp}^{\text{rot}} = (\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{\lambda})\,\boldsymbol{\lambda})\cos\beta + \sin\beta\boldsymbol{\lambda} \times (\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{\lambda})\,\boldsymbol{\lambda})$$

- Overall rotated vector

$$\boldsymbol{b} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp}^{\text{rot}} = (\boldsymbol{a}\cdot\boldsymbol{\lambda})\,\hat{\boldsymbol{\lambda}} + (\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{\lambda})\,\hat{\boldsymbol{\lambda}})\cos\beta + \sin\beta\boldsymbol{\lambda}\times(\boldsymbol{a} - (\boldsymbol{a}\cdot\boldsymbol{\lambda})\,\boldsymbol{\lambda})$$

$$\boldsymbol{b} = \boldsymbol{a}\cos\beta + (1-\cos\beta)\,(\boldsymbol{a}\cdot\boldsymbol{\lambda})\,\boldsymbol{\lambda} + \sin\beta\boldsymbol{\lambda}\times\boldsymbol{a}$$

- In case of coordinate frame rotation, vector $\boldsymbol{b}$ in new frame is

$$\boldsymbol{b} = \cos\beta\boldsymbol{a} + (1-\cos\beta)\boldsymbol{\lambda}\boldsymbol{\lambda}^T\boldsymbol{a} - \sin\beta\boldsymbol{\lambda}\times\boldsymbol{a} = \mathcal{C}\boldsymbol{a}$$

- Transformation matrix (Rodrigue's Rotation Formula)

$$\boxed{\mathcal{C} = \cos\beta\boldsymbol{I} + (1-\cos\beta)\boldsymbol{\lambda}\boldsymbol{\lambda}^T - \sin\beta\boldsymbol{S}(\boldsymbol{\lambda})}$$

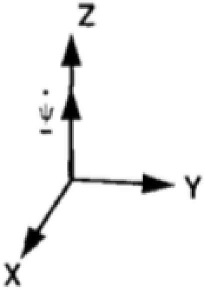
where the skew-symmetric matrix $\boldsymbol{S}(\boldsymbol{\lambda})$ is given by

$$\boldsymbol{S}(\boldsymbol{\lambda}) = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}$$

Coordinate Transformation

- Euler angles vary with time when an input angular velocity vector is applied between two reference frames.
- Angular velocity vector ω, in body-fixed coordinate system, has components p, q, and r in the x, y, and z directions, respectively.
- Consider each derivative of an Euler angle as magnitude of angular velocity vector in coordinate system in which angle is defined.
- $\dot{\psi}$ is the magnitude of $\dot{\boldsymbol{\psi}}$ that lies along the Z axis of Earth-fixed coordinate system.

$$\dot{\boldsymbol{\psi}} = \begin{bmatrix} \dot{\psi}_x \\ \dot{\psi}_y \\ \dot{\psi}_z \end{bmatrix} = \mathcal{C} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -\dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi \\ \dot{\psi}\cos\theta\cos\phi \end{bmatrix}$$

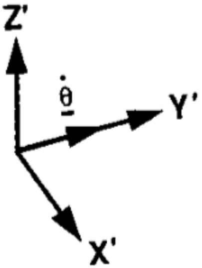


- Similarly, the components of $\dot{\boldsymbol{\theta}}$ in $X'Y'Z'$ are given by $(0, \dot{\theta}, 0)^T$.
- In body frame, it can be obtained as

$$\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \mathcal{DB} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

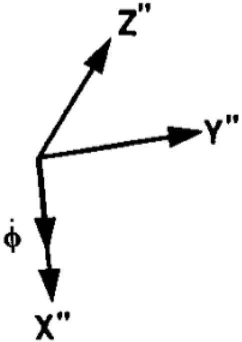
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \dot{\theta}\cos\phi \\ -\dot{\theta}\sin\phi \end{bmatrix}$$



- Similarly, the components of $\dot{\phi}$ in $X^{''}Y^{''}Z^{''}$ are given by $(\dot{\psi}, 0, 0)^T$.
- In body frame, it can be obtained as

$$
\begin{aligned}
\dot{\boldsymbol{\phi}} = \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} &= \mathcal{D} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$



- Components of $\boldsymbol{\omega}$ in body-fixed coordinate system is given by

$$\boldsymbol{\omega} = \dot{\boldsymbol{\psi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}}$$

- Now, we have body rates given by

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\psi}_x + \dot{\theta}_x + \dot{\phi}_x \\ \dot{\psi}_y + \dot{\theta}_y + \dot{\phi}_y \\ \dot{\psi}_z + \dot{\theta}_z + \dot{\phi}_z \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi \\ \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \end{bmatrix}$$

- Euler angle rates

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{q\sin\phi + r\cos\phi}{\cos\theta} \\ q\cos\phi - r\sin\phi \\ p + \tan\theta(q\sin\phi + r\cos\phi) \end{bmatrix}$$

- What happen when $\theta = \pm 90^\circ$? Gimbal lock problem
- How to avoid such difficulties? Nonsingular representation, e.g., quaternions

- Rate of change of DCM matrix

$$\dot{\mathcal{C}}_b^a(t) = \mathcal{C}_b^a(t)\boldsymbol{\Omega}_{ab}^b = \mathcal{C}_b^a(t)\begin{bmatrix} 0 & -\omega_Y & \omega_P \\ \omega_Y & 0 & -\omega_R \\ -\omega_P & \omega_R & 0 \end{bmatrix}$$

- Quaternion: $[Q] = q_0 + \boldsymbol{q} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\boldsymbol{q}}$
- Rotation matrix using quaternion

$$\begin{aligned} L_Q(\boldsymbol{v}) =& [Q]\boldsymbol{v}[Q]^\star \\ =& \cos\theta\boldsymbol{v} + (1-\cos\theta)(\hat{q}.\boldsymbol{v})\hat{\boldsymbol{q}} + \sin\theta(\hat{\boldsymbol{q}}\times\boldsymbol{v}) \\ =& \underbrace{\left[q_0^2 - \|\boldsymbol{q}\|^2)I_{3\times 3} + 2\boldsymbol{q}\boldsymbol{q}^T + 2q_0(\boldsymbol{q}\times\right]}_{\text{Rotation Matrix}}\boldsymbol{v} \end{aligned}$$

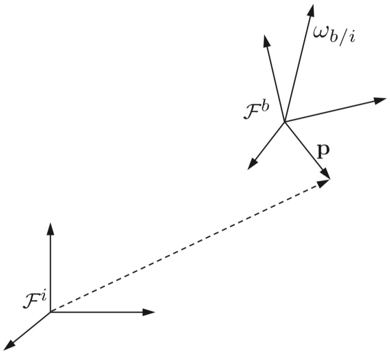
- Rate of change of quaternion

$$[\dot{Q}(t)] = \frac{1}{2}\boldsymbol{\omega}[Q(t)]$$

- Euler angle
  - Only 3 differential equations, No redundancy
  - Direct initialization from initial Euler angles
  - Nonlinear differential equations
  - Singularities and Gimbal lock problem
  - Transformation matrix needs to be computed
  - Order of rotation important
- Direction cosine matrix (DCM)
  - Linear differential equations, No singularity, Direct computation of DCM
  - Euler angles, required for initial calculation, are not directly available
  - Computational burden
- Quaternions
  - Only 4 linear coupled differential equations
  - No singularity thus avoids gimbal lock problem
  - Minimum redundancy to avoid singularity, Computationally simpler
  - If the coordinate systems do not coincides at $t = 0$ then Euler angle required for initial calculation
  - Transformation matrix needs to be computed
  - Euler angles are not directly available

- Consider two frames, inertial ($\mathcal{F}^i$) and body ($\mathcal{F}^b$) frames
- Frame $\mathcal{F}^b$ is rotating (but not translating) w.r.t. $\mathcal{F}^i$ with angular velocity of $\boldsymbol{\omega}_{b/i}$
- Vector $\boldsymbol{p}$ in body frame

$$\boldsymbol{p} = p_x \boldsymbol{i}^b + p_y \boldsymbol{j}^b + p_z \boldsymbol{k}^b$$



Time derivative of $\boldsymbol{p}$ w.r.t. $\mathcal{F}^i$

$$\frac{d}{dt_i}\boldsymbol{p} = \dot{p}_x \boldsymbol{i}^b + \dot{p}_y \boldsymbol{j}^b + \dot{p}_z \boldsymbol{k}^b + p_x \frac{d}{dt_i}\boldsymbol{i}^b + p_y \frac{d}{dt_i}\boldsymbol{j}^b + p_z \frac{d}{dt_i}\boldsymbol{k}^b$$

Differentiation of a Vector

- First three terms represent the change in $\boldsymbol{p}$ as viewed by an observer in the rotating frame.

$$\frac{d}{dt_b}\boldsymbol{p} = \dot{p}_x \boldsymbol{i}^b + \dot{p}_y \boldsymbol{j}^b + \dot{p}_z \boldsymbol{k}^b$$

- Last three terms represent the change in $\boldsymbol{p}$ due to the rotation of frame $\mathcal{F}^b$ relative to $\mathcal{F}^i$.
- We have the relations

$$\frac{d}{dt_i}\boldsymbol{i}^b = \boldsymbol{\omega}_{b/i} \times \boldsymbol{i}^b, \;\; \frac{d}{dt_i}\boldsymbol{j}^b = \boldsymbol{\omega}_{b/i} \times \boldsymbol{j}^b, \;\; \frac{d}{dt_i}\boldsymbol{k}^b = \boldsymbol{\omega}_{b/i} \times \boldsymbol{k}^b$$

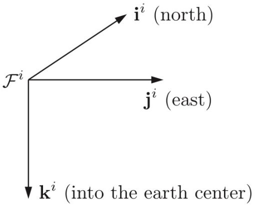
- Thus, we can write

$$\begin{aligned} p_x \frac{d}{dt_i}\boldsymbol{i}^b + p_y \frac{d}{dt_i}\boldsymbol{j}^b + p_z \frac{d}{dt_i}\boldsymbol{k}^b =& p_x \boldsymbol{\omega}_{b/i} \times \boldsymbol{i}^b + p_y \boldsymbol{\omega}_{b/i} \times \boldsymbol{j}^b + p_z \boldsymbol{\omega}_{b/i} \times \boldsymbol{k}^b \\ =& \boldsymbol{\omega}_{b/i} \times \left(p_x \boldsymbol{i}^b + p_y \boldsymbol{j}^b + p_z \boldsymbol{k}^b \right) = \boldsymbol{\omega}_{b/i} \times \boldsymbol{p} \end{aligned}$$

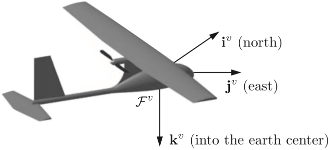
- Derivative of a vector

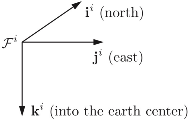
$$\frac{d}{dt_i}\boldsymbol{p} = \frac{d}{dt_b}\boldsymbol{p} + \boldsymbol{\omega}_{b/i} \times \boldsymbol{p}$$

Unmanned Autonomous Vehicles: Inertial and Vehicle Frames



- Earth-fixed coordinate system with its origin at the defined home location.
- Referred as a north-east-down (NED) reference frame.

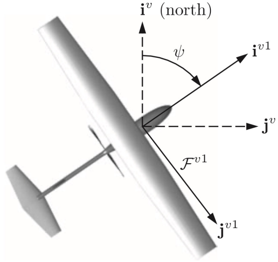




Origin of the vehicle frame is at the center of mass of the MAV, with the axes aligned with the inertial frame.

Inertial and vehicle frames are related by a translation, while the remaining frames are related by rotations.

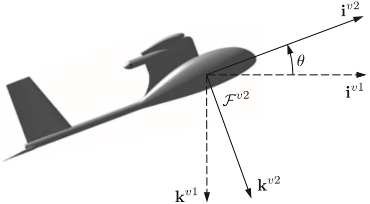
- Origin of the vehicle-1 frame is identical to the vehicle frame: the center of mass of the aircraft
- $\mathcal{F}^{v1}$ is rotated in positive right-handed direction about $\boldsymbol{k}_v$ by heading (or yaw) angle ψ.
- $\boldsymbol{i}_{v1}$ points out the nose of the airframe
- $\boldsymbol{j}_{v1}$ points out the right wing
- $\boldsymbol{k}_{v1}$ is aligned with $\boldsymbol{k}_v$ and points into the earth



$$\boldsymbol{p}^{v1} = \mathcal{R}_v^{v1}(\psi)\boldsymbol{p}^v$$

$$\mathcal{R}_v^{v1}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

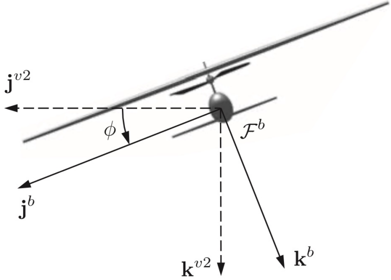
- Origin of the vehicle-2 frame is again the center of mass of the aircraft.
- Rotation of the vehicle-1 frame in a right-handed rotation about the $\boldsymbol{j}^{v1}$ axis by pitch angle θ.
- Unit vector $\boldsymbol{i}^{v2}$ points out the nose of the aircraft
- $\boldsymbol{j}^{v2}$ points out the right wing
- $\boldsymbol{k}^{v2}$ points out the belly



$$\boldsymbol{p}^{v2} = \mathcal{R}_{v1}^{v2}(\theta)\boldsymbol{p}^{v1}$$

$$\mathcal{R}_{v1}^{v2}(\psi) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

- Origin is the center of mass
- Body frame is obtained by rotating the vehicle-2 frame in a right-handed rotation about $\boldsymbol{i}^{v2}$ by roll angle ϕ.
- $\boldsymbol{i}^b$ points out the nose of the airframe
- $\boldsymbol{j}$ points out the right wing
- $\boldsymbol{k}^b$ points out the belly



$$\boldsymbol{p}^b = \mathcal{R}_{v2}^b(\phi)\boldsymbol{p}^{v2}$$

$$\mathcal{R}_{v2}^b(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

- Transformation from vehicle frame to body frame

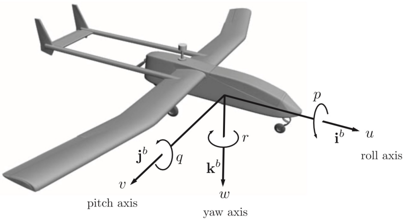
$$\mathcal{R}_v^b \triangleq \mathcal{R}_v^b(\phi,\theta,\psi) = \mathcal{R}_{v2}^b(\phi)\mathcal{R}_{v1}^{v2}(\theta)\mathcal{R}_v^{v1}(\psi)$$

$$\mathcal{R}_v^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Transformation matrix

$$\mathcal{R}_v^b(\phi,\theta,\psi) = \begin{pmatrix} \mathsf{c}_\theta\mathsf{c}_\psi & \mathsf{c}_\theta\mathsf{s}_\psi & -\mathsf{s}_\theta \\ \mathsf{s}_\phi s_\theta\mathsf{c}_\psi - \mathsf{c}_\phi\mathsf{s}_\psi & \mathsf{s}_\phi\mathsf{s}_\theta\mathsf{s}_\psi + \mathsf{c}_\phi\mathsf{c}_\psi & \mathsf{s}_\phi\mathsf{c}_\theta \\ \mathsf{c}_\phi\mathsf{s}_\theta\mathsf{c}_\psi + \mathsf{s}_\phi\mathsf{s}_\psi & \mathsf{c}_\phi\mathsf{s}_\theta\mathsf{s}_\psi - \mathsf{s}_\phi\mathsf{c}_\psi & \mathsf{c}_\phi\mathsf{c}_\theta \end{pmatrix}$$

- Euler angles: Singularity issues when $\theta = \pm\pi/2$ for $\psi - \theta - \phi$ sequence



Name	Description
p_n	Inertial north position of the MAV along $\mathbf{i}^i$ in $\mathcal{F}^i$
p_e	Inertial east position of the MAV along $\mathbf{j}^i$ in $\mathcal{F}^i$
p_d	Inertial down position (negative of altitude) of the MAV measured along $\mathbf{k}^i$ in $\mathcal{F}^i$
u	Body frame velocity measured along $\mathbf{i}^b$ in $\mathcal{F}^b$
v	Body frame velocity measured along $\mathbf{j}^b$ in $\mathcal{F}^b$
w	Body frame velocity measured along $\mathbf{k}^b$ in $\mathcal{F}^b$
ϕ	Roll angle defined with respect to $\mathcal{F}^{v2}$
θ	Pitch angle defined with respect to $\mathcal{F}^{v1}$
ψ	Heading (yaw) angle defined with respect to $\mathcal{F}^{v}$
p	Roll rate measured along $\mathbf{i}^b$ in $\mathcal{F}^b$
q	Pitch rate measured along $\mathbf{j}^b$ in $\mathcal{F}^b$
r	Yaw rate measured along $\mathbf{k}^b$ in $\mathcal{F}^b$

- Translational velocity of the vehicle
  - ⇒ Velocity components along each of the axes in a body-fixed coordinate frame
  - ⇒ Inertial velocity of the vehicle projected onto the $\boldsymbol{i}^b, \boldsymbol{j}^b$, and $\boldsymbol{k}^b$ axes
- Translational position of the vehicle: inertial reference frame
- How to relate translational position and velocity?
- Relation between translational position and velocity

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \mathcal{R}_b^v \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left(\mathcal{R}_v^b\right)^\top \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} \mathsf{c}_\theta \mathsf{c}_\psi & \mathsf{s}_\phi s_\theta \mathsf{c}_\psi - \mathsf{c}_\phi \mathsf{s}_\psi & \mathsf{c}_\phi \mathsf{s}_\theta \mathsf{c}_\psi + \mathsf{s}_\phi \mathsf{s}_\psi \\ \mathsf{c}_\theta \mathsf{s}_\psi & \mathsf{s}_\phi \mathsf{s}_\theta \mathsf{s}_\psi + \mathsf{c}_\phi \mathsf{c}_\psi & \mathsf{c}_\phi \mathsf{s}_\theta \mathsf{s}_\psi - \mathsf{s}_\phi \mathsf{c}_\psi \\ -\mathsf{s}_\theta & \mathsf{s}_\phi \mathsf{c}_\theta & \mathsf{c}_\phi \mathsf{c}_\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

- Dynamics using Newton's law
- Newton's laws hold in inertial reference frames
- Motion of the body of interest must be referenced to a fixed (i.e., inertial) frame of reference, (ground in our case).
- Flat earth model, appropriate for small and miniature air vehicles
- Newton's second law (translational motion)

$$\boxed{m\frac{d\boldsymbol{V}_g}{dt_i} = \boldsymbol{f}}$$

m: Mass of the vehicle
$\dfrac{d}{dt_i}$: Derivative w.r.t. inertial frame
and $\boldsymbol{f}$: Sum of all external forces acting on vehicle

- What are the possible external forces?
- **External forces**: gravity, aerodynamic and propulsion forces

- Using Transport Theorem between inertial and body frames

$$\boxed{\frac{d\boldsymbol{V}_g}{dt_i} = \frac{d\boldsymbol{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \boldsymbol{V}_g}$$

where $\boldsymbol{\omega}_{b/i}$ is the angular velocity of vehicle w.r.t. the inertial frame.

- Newton's second law in terms of derivative in body frame

$$\boxed{m\left(\frac{d\boldsymbol{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \boldsymbol{V}_g\right) = \boldsymbol{f}}$$

- Newton's second law in terms of forces and velocities in body frame

$$\boxed{m\left(\frac{d\boldsymbol{V}_g^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \boldsymbol{V}_g^b\right) = \boldsymbol{f}^b}$$

- What are these vectors $(\boldsymbol{V}_g^b, \boldsymbol{\omega}_{b/i}^b, \boldsymbol{f}^b)$?

$$\boldsymbol{V}_g^b = (u, v, w)^\top; \boldsymbol{\omega}_{b/i}^b = (p, q, r)^\top; \boldsymbol{f}^b = (f_x, f_y, f_z)^\top$$

- $\dfrac{d\boldsymbol{V}_g^b}{dt_b}$: Rate of change of velocity expressed in the body frame, as viewed by an observer on the moving body.
- As u, v, w are the instantaneous projections of $\boldsymbol{V}_g^b$ onto body axes,

$$\frac{d\boldsymbol{V}_g^b}{dt_b} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix}$$

- Translational motion equations

$$\boxed{m\left(\frac{d\boldsymbol{V}_g^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \boldsymbol{V}_g^b\right) = \boldsymbol{f}^b} \implies \frac{d\boldsymbol{V}_g^b}{dt_b} = \frac{\boldsymbol{f}^b}{m} - \boldsymbol{\omega}_{b/i}^b \times \boldsymbol{V}_g^b$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \frac{1}{m}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

- Newton's second law (rotational motion)

$$\boxed{\frac{d\boldsymbol{h}}{dt_i} = \boldsymbol{m}}$$

$\boldsymbol{h}$: Angular momentum in vector form

$\boldsymbol{m}$: Sum of all externally applied moments

- Is the above expression true for the moments about any point?
- No, true only when moments are summed about the center of mass of the vehicle.
- Derivative of angular momentum, taken in the inertial frame

$$\boxed{\frac{d\boldsymbol{h}}{dt_i} = \frac{d\boldsymbol{h}}{dt_b} + \boldsymbol{\omega}_{b/i} \times \boldsymbol{h} = \boldsymbol{m}}$$

- Newton's second law (rotational motion) in body frame

$$\boxed{\frac{d\boldsymbol{h}^b}{dt_b} + \boldsymbol{\omega}^b_{b/i} \times \boldsymbol{h}^b = \boldsymbol{m}^b}$$

- How to compute angular momentum of rigid body?
- **Angular momentum**: Product of the inertia matrix $\boldsymbol{J}$ and the angular velocity vector $\boldsymbol{\omega}^b_{b/i}$

$$\boldsymbol{h}^b = \boldsymbol{J}\boldsymbol{\omega}^b_{b/i}$$

$$\boldsymbol{J} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xydm & -\int xzdm \\ -\int xydm & \int (x^2 + z^2) dm & -\int yzdm \\ -\int xzdm & -\int yzdm & \int (x^2 + y^2) dm \end{pmatrix}$$

- Inertia matrix

$$\boldsymbol{J} \triangleq \begin{pmatrix} J_x & -J_{xy} & -J_{xz} \\ -J_{xy} & J_y & -J_{yz} \\ -J_{xz} & -J_{yz} & J_z \end{pmatrix}$$

- Diagonal and off-diagonal terms: Moments and Products of inertia, respectively
- Moment of inertia: Measure of the aircraft's tendency to oppose acceleration about a specific axis of rotation.
- **Interpretation of** J_x: Sum of the product of the mass of each element and the square of the distance of the mass element from the body x axis $(y^2 + z^2)$ and adding them up.
- Larger J_x, aircraft opposes angular acceleration about x-axis more.
- How to obtain inertia matrix?
- Numerically from mass properties using CAD models or measured experimentally.

- Is there any benefit of using body axes for the inertia matrix computation?
- Yes, it simplifies the equations of motion. How?
- Inertia matrix $\boldsymbol{J}$ is constant when viewed from the body frame, thus $\dot{\boldsymbol{J}} = 0$.
- As p, q, r are instantaneous components of angular velocity $\boldsymbol{\omega}_{b/i}^{b}$ along body axes,

$$\dot{\boldsymbol{\omega}}_{b/i} = \frac{d\boldsymbol{\omega}_{b/i}^{b}}{dt_b} = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix}$$

- Equations of rotational motion

$$\boldsymbol{J}\frac{d\boldsymbol{\omega}_{b/i}^{b}}{dt_b} + \boldsymbol{\omega}_{b/i}^{b} \times \left(\boldsymbol{J}\boldsymbol{\omega}_{b/i}^{b}\right) = \boldsymbol{m}^{b}$$

$$\Longrightarrow \boxed{\frac{d\boldsymbol{\omega}_{b/i}^{b}}{dt_b} = \boldsymbol{J}^{-1}\left\{-\boldsymbol{\omega}_{b/i}^{b} \times \left(\boldsymbol{J}\boldsymbol{\omega}_{b/i}^{b}\right) + \boldsymbol{m}^{b}\right\}}$$

- Assume externally applied moments along body axes

$$\boldsymbol{m}^b = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

- Equations of rotational motion

$$\boxed{\frac{d\boldsymbol{\omega}^b_{b/i}}{dt_b} = \boldsymbol{J}^{-1}\left\{-\boldsymbol{\omega}^b_{b/i} \times \left(\boldsymbol{J}\boldsymbol{\omega}^b_{b/i}\right) + \boldsymbol{m}^b\right\}}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} J_z/\Gamma & 0 & J_{xz}/\Gamma \\ 0 & 1/J_y & 0 \\ J_{xz}/\Gamma & 0 & J_x/\Gamma \end{pmatrix} \left[\begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix} \right]$$

$$\Gamma \triangleq J_x J_z - J_{xz}^2$$

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \frac{1}{m} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 & 0 & \Gamma_4 \\ 0 & 1/J_y & 0 \\ \Gamma_4 & 0 & \Gamma_8 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

- How are the kinematics and dynamics of different vehicles governed?
- Kinematics and dynamics equations remain the same for different vehicles.
- Forces and moments change depending on the types of vehicles.
  - $\Rightarrow$ Fixed-wing aircraft
  - $\Rightarrow$ Rotorcraft
  - $\Rightarrow$ Underwater vehicle
  - $\Rightarrow$ Surface vehicle
- Forces and moments acting on fixed-wing aerial vehicles
  - $\Rightarrow$ Gravity
  - $\Rightarrow$ Aerodynamics
  - $\Rightarrow$ Propulsion
- Total force and moment acting on the airframe

$$\boxed{\boldsymbol{f} = \boldsymbol{f}_g + \boldsymbol{f}_a + \boldsymbol{f}_p; \quad \boldsymbol{m} = \boldsymbol{m}_a + \boldsymbol{m}_p}$$

- Why is no moment here due to gravity?

Gravitational Forces and Moments

- Effect of the earth's gravitational field: A force proportional to the mass acting at the center of mass in the $\boldsymbol{k}^i$ direction.
- Gravity force acting on the center of mass in inertial frame

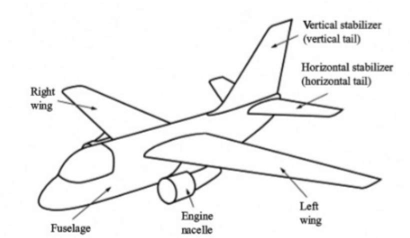
$$\boldsymbol{f}_g^v = \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}$$

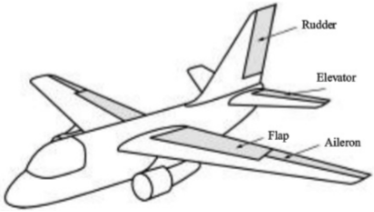
- In body frame, gravity force

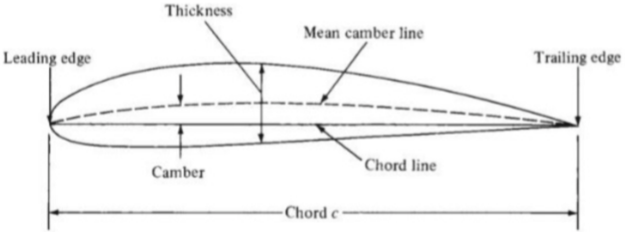
$$\boldsymbol{f}_g^b = \mathcal{R}_v^b \boldsymbol{f}_g^v = \begin{pmatrix} \mathsf{c}_\theta \mathsf{c}_\psi & \mathsf{c}_\theta \mathsf{s}_\psi & -\mathsf{s}_\theta \\ \mathsf{s}_\phi s_\theta \mathsf{c}_\psi - \mathsf{c}_\phi \mathsf{s}_\psi & \mathsf{s}_\phi \mathsf{s}_\theta \mathsf{s}_\psi + \mathsf{c}_\phi \mathsf{c}_\psi & \mathsf{s}_\phi \mathsf{c}_\theta \\ \mathsf{c}_\phi \mathsf{s}_\theta \mathsf{c}_\psi + \mathsf{s}_\phi \mathsf{s}_\psi & \mathsf{c}_\phi \mathsf{s}_\theta \mathsf{s}_\psi - \mathsf{s}_\phi \mathsf{c}_\psi & \mathsf{c}_\phi \mathsf{c}_\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}$$

$$= \begin{pmatrix} -mg \sin\theta \\ mg \cos\theta \sin\phi \\ mg \cos\theta \cos\phi \end{pmatrix}$$

- No gravitational moment as gravity acts through center of mass


Vertical stabilizer
(vertical tail)
Horizontal stabilizer
(horizontal tail)
Right
wing
Fuselage
Engine
nacelle
Left
wing


Rudder
Elevator
Flap
Aileron


Thickness
Mean camber line
Leading edge
Trailing edge
Camber
Chord line
Chord c

- Continuity equation

$$\boxed{\rho_1 A_1 V_1 = \rho_2 A_2 V_2}$$

- Euler equation

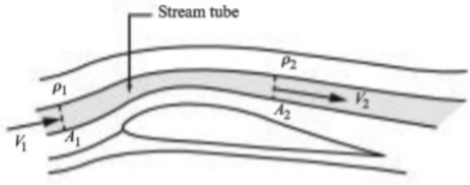
$$\boxed{dp = -\rho V dV}$$

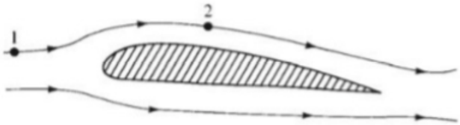
- Bernoulli equation, valid for incompressible flow only

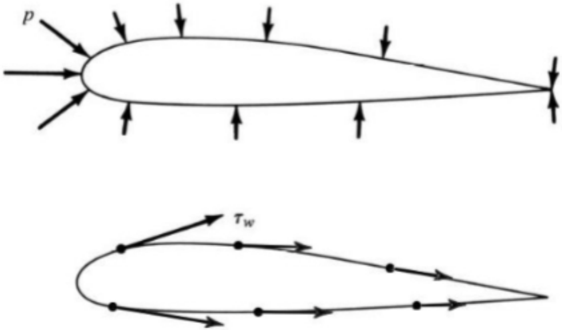
$$\boxed{p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2}$$

- Speed of sound

$$\boxed{a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}}$$

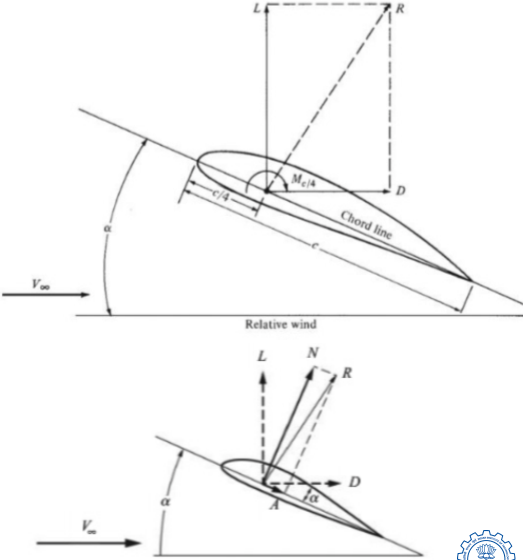


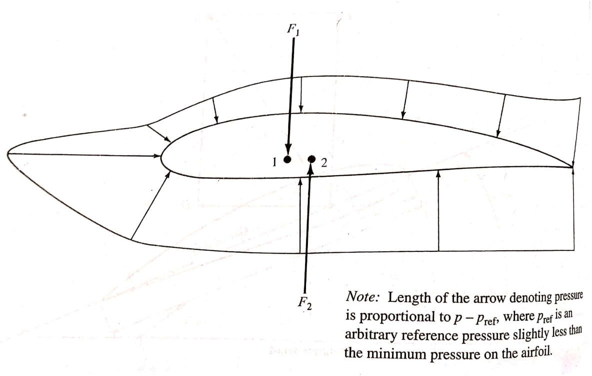




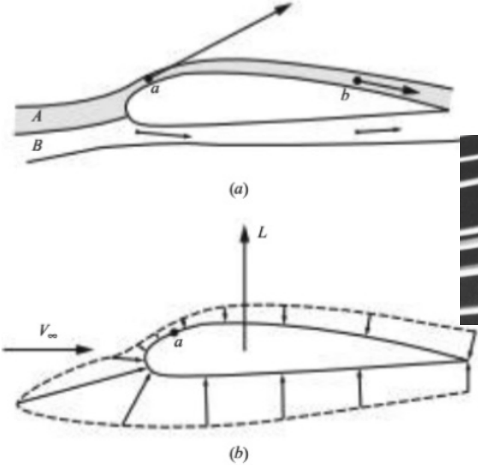
- **Pressure**: Force per unit area acting normal to the surface
- **Shear Stress**: Force per unit area tangential to the surface
- Unbalance of these pressure and shear stress create aerodynamic forces.

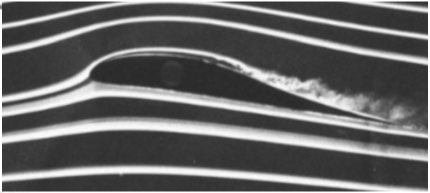
- **Relative wind (V_∞)** : Direction of free stream
- **Angle of attack (α)** : Angle between relative wind and chord line
- **Lift (L)**: Component of aerodynamic force $\perp V_\infty$
- **Drag (D)**: Component of aerodynamic force $\| V_\infty$
- **Normal and axial forces**: $\perp$ and $\|$ chord line





- **Quarter-chord point**: A point on the chord at a distance $c/4$ from the leading edge
- $M_{c/4}$: Moment about the quarter-chord point
- **Aerodynamic center**: Moments M_{ac} about this point do not vary with α.





- Why pressure is high on bottom and low on top?
- Mass continuity and Newton's second law

- **Lift**:

$$\boxed{\underbrace{L}_{\text{Lift}} = \underbrace{q_\infty}_{\text{Dynamic pressure}} \times \underbrace{S}_{\text{Wing area}} \times \underbrace{c_l}_{\text{Lift coefficient}}}$$

- **Drag**:

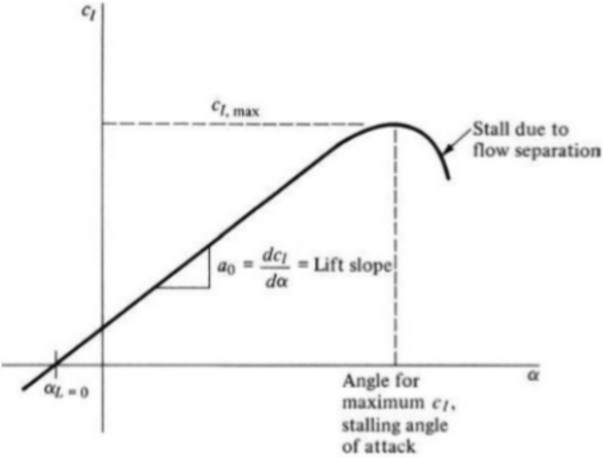
$$\boxed{\underbrace{D}_{\text{Drag}} = \underbrace{q_\infty}_{\text{Dynamic pressure}} \times \underbrace{S}_{\text{Wing area}} \times \underbrace{c_d}_{\text{Drag coefficient}}}$$

- **Moment**:

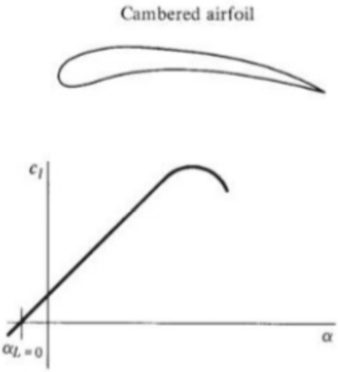
$$\boxed{\underbrace{M}_{\text{Moment}} = \underbrace{q_\infty}_{\text{Dynamic pressure}} \times \underbrace{S}_{\text{Wing area}} \times \underbrace{c_m}_{\text{Moment coefficient}} \times \underbrace{c}_{\text{Chord length}}}$$

- **Lift, drag, and moment coefficients**:

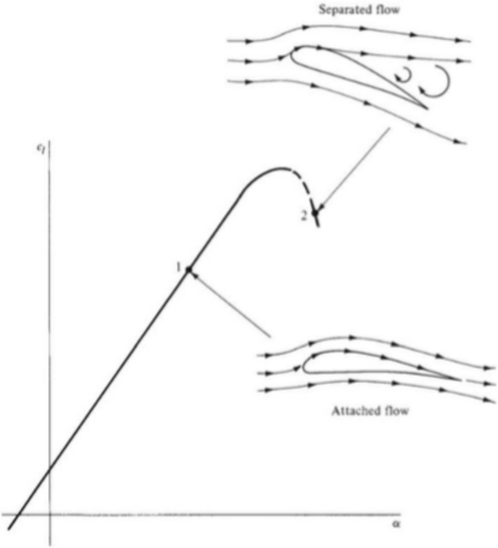
$$\boxed{c_l = \frac{L}{q_\infty S}, \quad c_d = \frac{D}{q_\infty S}, \quad c_m = \frac{M}{q_\infty S c}}$$



- c_l varies linearly with α, having lift slope a_0.
- Lift at $\alpha = 0$ is due to positive camber.
- Zero-lift angle of attack, $\alpha_{L=0}$.



- Linearity breaks down at some α, leading to maximum value of c_l, $c_{l,\max}$.
- Lift decreases significantly at high α, leading to stall of the airfoil.
- What is the cause of airfoil stall?

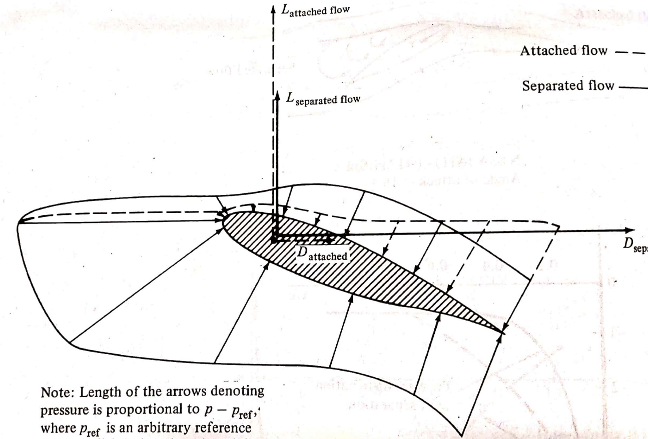


- Flow separation causes stalling
- With high α, adverse pressure gradient becomes stronger.
- Flow separation at stalling angle of attack

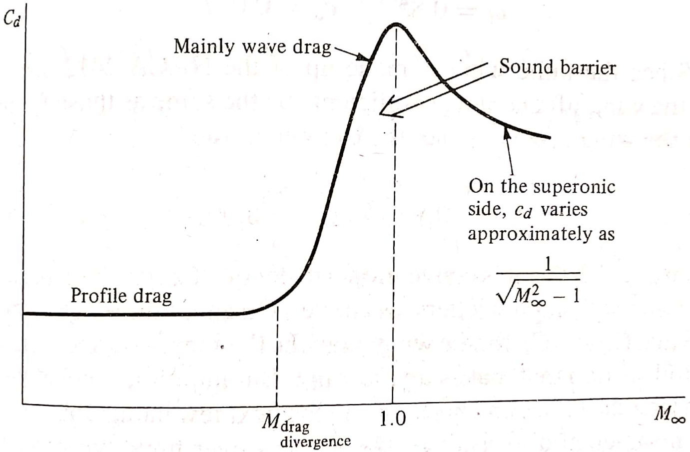
- **Drag**: Component of aerodynamic force parallel to relative wind
- **Sources of drag**:
  - ⇒ Skin friction drag (D_f): Shear stress at wall
  - ⇒ Pressure or form drag (D_p): Flow separation
  - ⇒ Wave drag (D_w): Shock wave at supersonic speed
- Total drag due to viscous effect

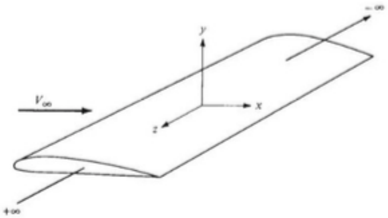
$$\boxed{D = \underbrace{D_f + D_p}_{\text{Profile drag}} + D_w}$$

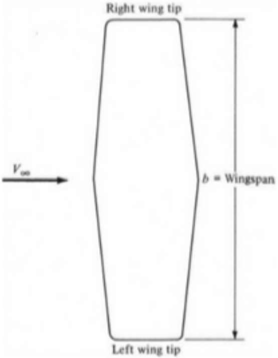
- Profile drag due to nature of source (shape and size or profile of body).
- Skin friction drag: More for turbulent and less for laminar
- Pressure drag: Less for turbulent and more for laminar why?



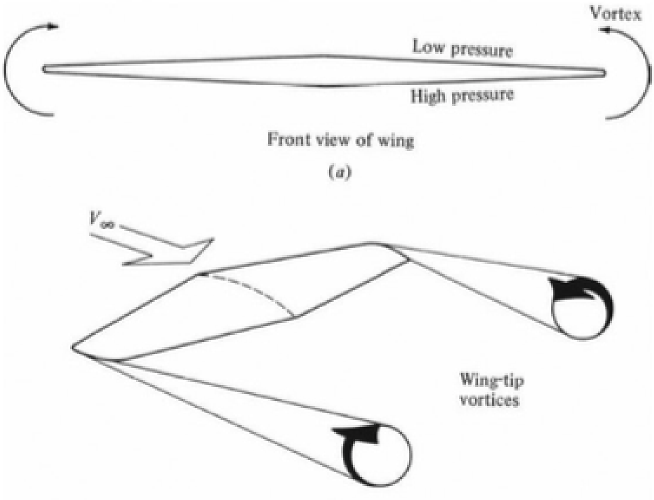
Note: Length of the arrows denoting pressure is proportional to $p - p_{\text{ref}}$, where p_{ref} is an arbitrary reference pressure slightly less than the minimum pressure on the airfoil

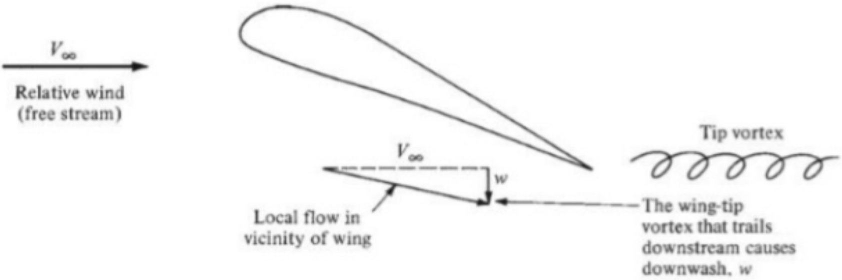

C_d
Mainly wave drag
Sound barrier
On the superonic side, c_d varies approximately as
$\frac{1}{\sqrt{M_\infty^2 - 1}}$
Profile drag
$M_{\text{drag divergence}}$
1.0
M_∞



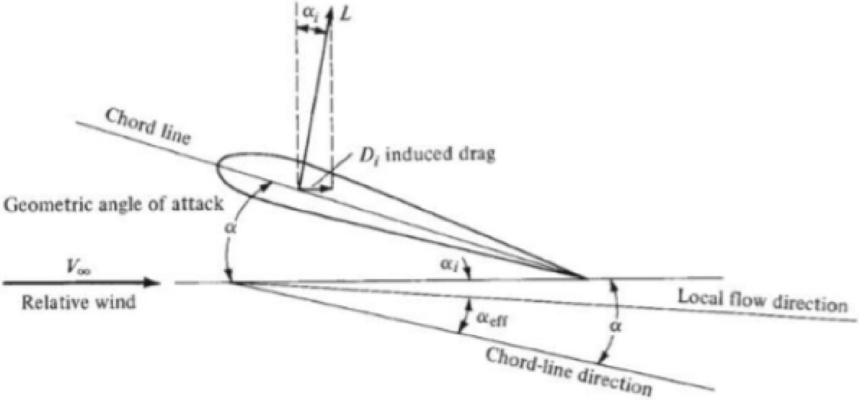


$$\text{Aspect ratio} = \text{AR} = \frac{b^2}{S}$$


Vortex
Low pressure
High pressure
Front view of wing
(a)
V_∞
Wing-tip
vortices



- **Vortex**: Trailing circulatory motion, as a result of pressure difference between upper and lower surfaces of wing.
- **Downwash**: Downward component of air velocity
- Local relative wind is changed.



- **Geometric angle of attack**: α between mean chord and V_∞
- **Induced angle of attack**: Difference between local and free stream flow directions
- **Effective angle of attack**: $\boxed{\alpha_{\text{eff}} = \alpha - \alpha_i}$

- Reduced angle of attack of the airfoil sections of the wing
- Increase in the drag, resulting in induced drag.
- What are the physical interpretations?

> Wing-tip vortices alter the flow field about wing to change the surface pressure distributions in the direction of increased drag.

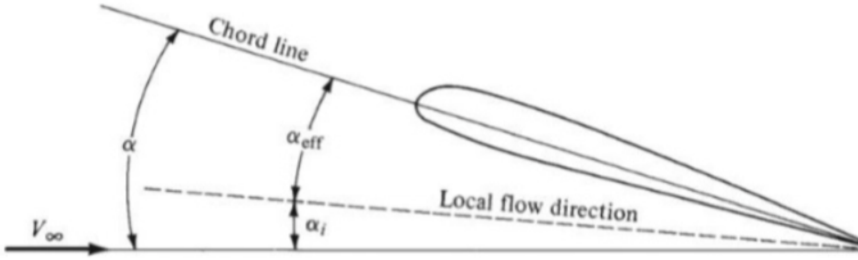
> Local relative wind is canted downward, L itself is "tilted back", contributing a certain component of force $\parallel V_\infty$ (a drag force).

> Wing-tip vortices contains a certain amount of kinetic energy, supplied by aircraft propulsion system. Extra power needs to be added to overcome this increment in drag due to induced drag.

- Decrease in lift coefficient and increase in drag coefficient

- Finite wing
  - ⇒ Induced drag
  - ⇒ Change in lift-slope
- Induced angle of attack

$$\alpha_i = \underbrace{\frac{C_L}{\pi e_1 \mathsf{AR}}}_{\text{Radians}} = \underbrace{\frac{57.3 C_L}{\pi e_1 \mathsf{AR}}}_{\text{Degrees}}$$



- Drag polar for complete airplane

$$C_D = \underbrace{C_{d,e}}_{\text{Parasite drag coefficient}} + \frac{C_L^2}{\pi e AR}$$

- Parasite drag include profile drag, and friction and pressure drag to other parts of airplane.
- Parasite drag coefficient

$$C_{D,e} = C_{D,0} + rC_L^2 \Rightarrow C_D = C_{d,e} + \left(r + \frac{1}{\pi e AR}\right) C_L^2$$

- Drag polar for complete airplane, with e as Oswald efficiency factor

$$\boxed{\mathsf{C}_D = \underbrace{C_{d,0}}_{\text{Zero}-\text{lift drag coeff.}} + \underbrace{C_{D,i}}_{\text{Drag coeff. due to lift}}, \; C_{D,i} = \frac{C_L^2}{\pi e AR}}$$

- **Stalling speed**: Slowest speed at which an airplane can fly in straight and level flight.

$$L = q_\infty S C_L \Rightarrow V_\infty = \sqrt{\frac{2L}{\rho_\infty S C_L}}$$

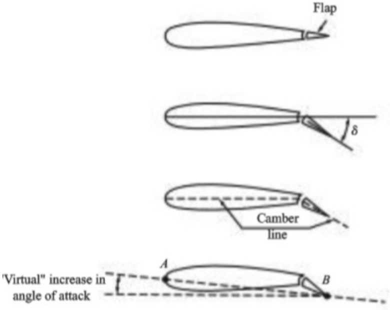
- For level and steady flight, $W = L$

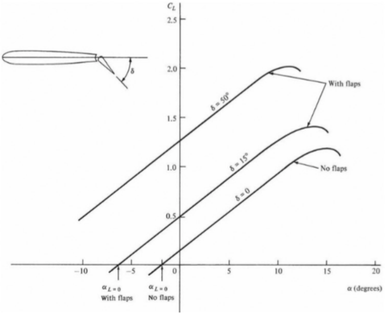
$$V_\infty = \sqrt{\frac{2W}{\rho_\infty S C_L}}$$

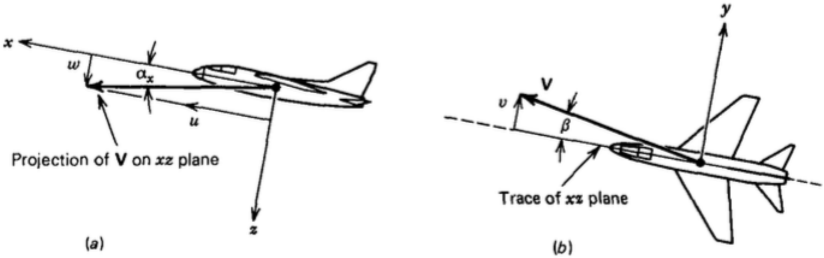
- Stalling speed correspond to α resulting in $C_{L,\max}$.

$$V_\infty = \sqrt{\frac{2W}{\rho_\infty S C_{L,\max}}}$$

- How to achieve enhanced lifting property for a given airfoil? One way is to use flap.


Flap
δ
Camber
line
A
B
"Virtual" increase in
angle of attack

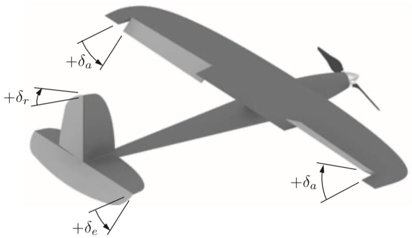

C_L
2.5
2.0
1.5
1.0
0.5
δ = 50°
δ = 15°
δ = 0
With flaps
No flaps
δ
−10
−5
0
5
10
15
20
$\alpha_{L=0}$
With flaps
$\alpha_{L=0}$
No flaps
α (degrees)



- Components of V: u, v, w
- Angle of attack and sideslip angle

$$\alpha_x = \tan^{-1}\left[\frac{w}{u}\right], \quad \beta = \sin^{-1}\left[\frac{v}{V}\right]$$

- Control surfaces
  - ⇒ Elevator, aileron, rudder
  - ⇒ Spoiler, flap, canard
- Positive direction of a control surface deflection: Right-hand rule to the hinge axis of control surface
- $\delta_e > 0 \rightarrow$ down deflection
- $\delta_r > 0 \rightarrow$ left deflection
- Positive aileron $\rightarrow$ trailing edge down

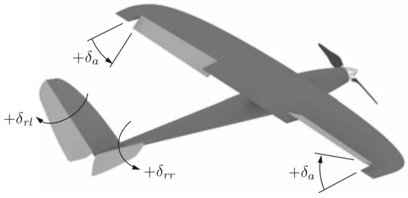


Aileron deflection

$$\delta_a = \frac{1}{2}\left(\delta_{a-\mathrm{left}} - \delta_{a-\mathrm{right}}\right)$$

$\delta_a > 0$ if left aileron down, right aileron up

- Small vehicles:
  - ⇒ V-tail configuration
  - ⇒ Flying wing configuration
- V-tail configuration:
  - ⇒ left and right ruddervators $(\delta_{rr}, \delta_{rl})$
  - ⇒ Differential ruddervators → rudder
  - ⇒ Altogether ruddervators → elevator



Mathematical model in standard rudder-elevator notation

Mathematical model for V-tail

$$\begin{pmatrix} \delta_e \\ \delta_r \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_{rr} \\ \delta_{rl} \end{pmatrix}$$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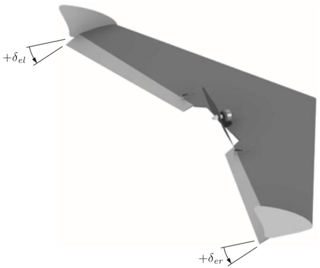
- Control surfaces for a flying wing: Elevons
- Elevons replaces aileron and elevator
  - $\Rightarrow$ left and right elevons $(\delta_{er}, \delta_{el})$
  - $\Rightarrow$ Differential elevons $\rightarrow$ ailerons
  - $\Rightarrow$ Altogether ruddervators $\rightarrow$ elevator



Mathematical model for flying wing

$$\begin{pmatrix} \delta_e \\ \delta_a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta_{er} \\ \delta_{el} \end{pmatrix}$$

Mathematical model in standard aileron-elevator notation

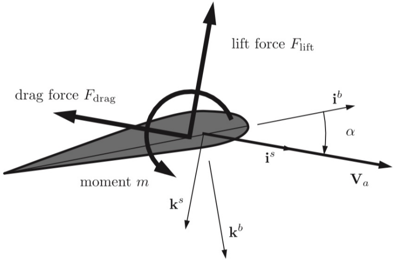
- Pitch plane: motion in $\boldsymbol{i}^b - \boldsymbol{k}^b$ plane
- Lift and drag forces along stability frame axes

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S C_L(\alpha, q, \delta_e)$$
$$D_{\text{drag}} = \frac{1}{2}\rho V_a^2 S C_D(\alpha, q, \delta_e)$$

- Pitching moment along $\boldsymbol{j}^s$

$$m = \frac{1}{2}\rho V_a^2 S c C_m(\alpha, q, \delta_e)$$

- Nonlinear equations



For small α, flow is laminar and linear approximation valid

- Lift force using Taylor series expansion

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S\left[C_{L0} + \underbrace{\frac{\partial C_L}{\partial \alpha}}_{C_{L_\alpha}}\alpha + \underbrace{\frac{\partial C_L}{\partial q}}_{C_{L_q}} q + \underbrace{\frac{\partial C_L}{\partial \delta_e}}_{C_{L_{\delta_e}}}\delta_e\right]$$

- In nondimensional form

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S\left[C_{L0} + C_{L_\alpha}\alpha + C_{L_q}\frac{c}{2V_a}q + C_{L_{\delta e}}\delta_e\right]$$

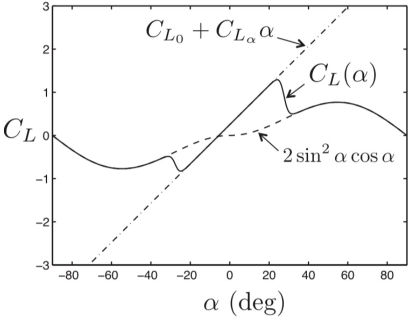
- Stability derivatives (C_{L_α}, C_{L_q}), Control derivative $(C_{L_{\delta_e}})$
- Drag and Moment using Taylor series expansion

$$F_{\text{drag}} = \frac{1}{2}\rho V_a^2 S\left[C_{D0} + C_{D_\alpha}\alpha + C_{D_q}\frac{c}{2V_a}q + C_{D_{\delta e}}\delta_e\right]$$

$$m = \frac{1}{2}\rho V_a^2 S\left[C_{m0} + C_{m_\alpha}\alpha + C_{m_q}\frac{c}{2V_a}q + C_{m_{\delta e}}\delta_e\right]$$

- For high α, the wing acts roughly like a flat plate.
- Lift coefficient at high α

$$C_{L,\text{flatplate}} = 2\sin^2\alpha\cos\alpha\,\mathsf{sign}(\alpha)$$



- Lift model accounting for stall

$$C_{L_\alpha} = (1-\sigma(\alpha))\left[C_{L_0} + C_{L_\alpha}\alpha\right] + \sigma(\alpha)\left[2\mathsf{sign}(\alpha)\sin^2\alpha\cos\alpha\right]$$

- Sigmoid function (blending function) with cutoff at $\pm\alpha_0$ and transition rate M

$$\sigma(\alpha) = \frac{1 + e^{-M(\alpha-\alpha_0)} + e^{M(\alpha-\alpha_0)}}{(1+e^{-M(\alpha-\alpha_0)})(1+e^{M(\alpha-\alpha_0)})},\ M,\ \alpha_0 > 0$$

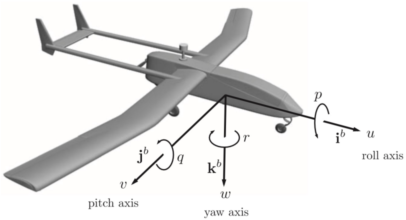
- Lift and drag forces expressed in stability frame
- Lift and drag forces in body frame

$$\begin{pmatrix} f_x \\ f_z \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -F_{\text{drag}} \\ -F_{\text{lift}} \end{pmatrix}$$

$$= \frac{1}{2}\rho V_a^2 S \begin{pmatrix} -C_D(\alpha)\cos\alpha + C_L(\alpha)\sin\alpha \\ + \left[-C_{D_q}\cos\alpha + C_{L_q}\sin\alpha\right] \frac{c}{2V_a} q \\ + \left[-C_{D_{\delta_e}}\cos\alpha + C_{L_{\delta_e}}\sin\alpha\right] \delta_e \\ \\ -C_D(\alpha)\sin\alpha + C_L(\alpha)\cos\alpha \\ + \left[-C_{D_q}\sin\alpha + C_{L_q}\cos\alpha\right] \frac{c}{2V_a} q \\ + \left[-C_{D_{\delta_e}}\sin\alpha + C_{L_{\delta_e}}\cos\alpha\right] \delta_e \end{pmatrix}$$

- Simpler model for $C_L(\alpha), C_D(\alpha), C_m(\alpha)$

$$C_L(\alpha) = C_{L_0} + C_{L_\alpha}\alpha, \; C_D(\alpha) = C_{D_0} + C_{D_\alpha}\alpha, C_m(\alpha) = C_{m_0} + C_{m_\alpha}\alpha$$



- Lateral aerodynamic force and moments
  - ⇒ Translational motion in the lateral direction along the $\boldsymbol{j}^b$ axis
  - ⇒ Rotational motions in roll and yaw
  - ⇒ Directional changes in flight path of the vehicle
- Important variables:
  - ⇒ Sideslip slip angle (β)
  - ⇒ Roll rate (p)
  - ⇒ Yaw rate (r)
  - ⇒ Aileron and rudder deflections (δ_a, δ_r)

- Lateral force and moments

$$f_y = \frac{1}{2}\rho V_a^2 S C_Y(\beta, p, r, \delta_a, \delta_r), \quad l = \frac{1}{2}{}_a^2 SbC_l(\beta, p, r, \delta_a, \delta_r)$$
$$n = \frac{1}{2}{}_a^2 SbC_n(\beta, p, r, \delta_a, \delta_r)$$

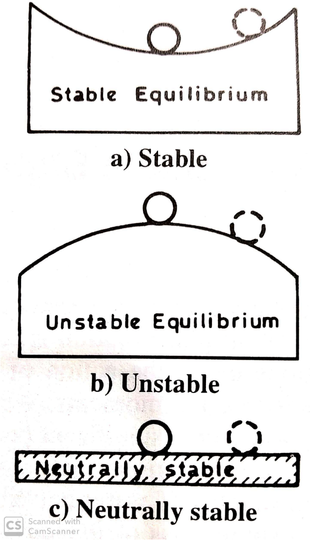
- Nonlinear nature of force and moments coefficients in general
- Linear approximations for insights

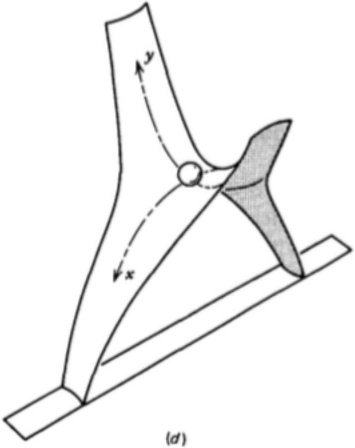
$$f_y = \frac{1}{2}\rho V_a^2 S\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r}\frac{b}{2V_a}r + C_{Y\delta_a}\delta_a + C_{Y_{\delta_r}}\delta_r\right]$$
$$l = \frac{1}{2}\rho b V_a^2 S\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r\right]$$
$$n = \frac{1}{2}\rho b V_a^2 S\left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r\right]$$

- For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

- Notions of stability: Static or Dynamic
- **Static stability**: If the forces and moments on the body caused by a disturbance tend initially to return the body toward its equilibrium position, the body is statically stable.
  - ⇒ No requirement of actual return of vehicle to equilibrium.
  - ⇒ If the forces and moments are such that the body continues to move away from its equilibrium position after being disturbed, the body is statically unstable.
- **Dynamic stability**: A body is dynamically stable if, of its own accord, it eventually returns to and remains at its equilibrium position over time.
- Static stability is related to initial tendency while dynamic stability focus on final state.

Static Stability

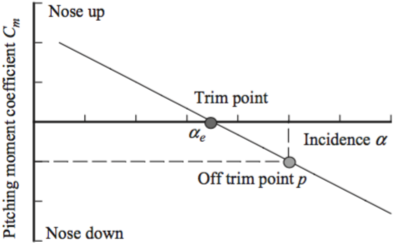


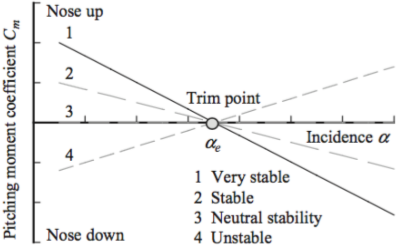


(d)

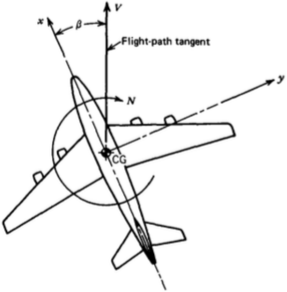
Balance or Equilibrium

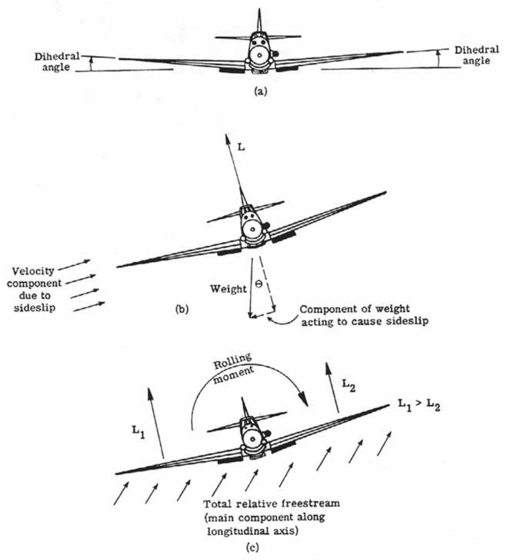
- An airplane can continue in steady unaccelerated flight only when the resultant external force and moment about the CG both vanish.
- Longitudinal balance: Zero pitching moment
- Nonzero pitching moment $\Longrightarrow$ Rotation in direction of unbalanced moment





- **Stability derivatives**: $C_{m_\alpha}, C_{l_\beta}, C_{n_\beta}, C_{m_q}, C_{l_p}$, and C_{n_r}
- Important insights about static and dynamic stability of the vehicle
  - $\Rightarrow$ C_{m_α} (< 0): Longitudinal static stability derivative
    - $\rightarrow$ Positive α causes negative down pitching moment
  - $\Rightarrow$ C_{l_β} (< 0): Roll static stability derivative
    - $\rightarrow$ For positive β, it generates restoring roll moment
  - $\Rightarrow$ C_{n_β} (> 0): Weathercock (yaw) static stability derivative
    - $\rightarrow$ Statically stable aircraft in yaw, naturally orients aircraft in wind direction.
    - $\rightarrow$ Positive β causes positive down yawing moment





- Dihedral angle: Angle between plane of wing and horizontal plane.
- Positive if wing tip is above wing root.
- Dihedral effect: generation of rolling moment due to sideslip
- In dihedral wing, angle of attack during sideslip is different on both sides.
- Different lift forces on both sides.
- Restoring moment

Total forces on vehicle

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} -mg\sin\theta \\ mg\cos\theta\sin\phi \\ mg\cos\theta\cos\phi \end{pmatrix} + \frac{1}{2}\rho S_{\text{prop}} C_{\text{prop}} \begin{pmatrix} (K_{\text{motor}}\delta_t)^2 - \rho V_a^2 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \frac{\rho V_a^2 S}{2} \begin{bmatrix} C_X(\alpha) + C_{X_q}(\alpha)\frac{cq}{2V_a} + C_{X_{\delta_e}}(\alpha)\delta_e \\ C_{Y_0} + C_{Y_\beta}(\beta) + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r \\ C_Z(\alpha) + C_{Z_q}\frac{cq}{2V_a} + C_{Y_{\delta_e}}\delta_e \end{bmatrix}$$

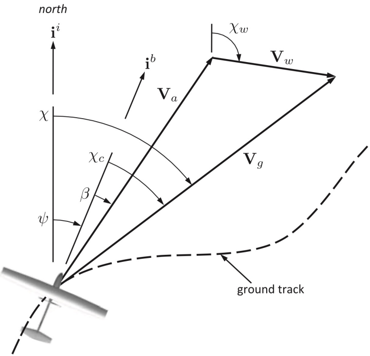
Total moments on vehicle

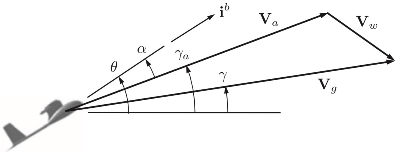
$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \frac{1}{2}\rho V_a^2 S \begin{bmatrix} b\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r\right] \\ c\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{b}{2V_a}q + C_{m_{\delta_e}}\delta_e\right] \\ b\left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r\right] \end{bmatrix}$$

$$+ \begin{pmatrix} -K_{T_p}(k_\Omega\delta_t)^2 \\ 0 \\ 0 \end{pmatrix}$$

- **Equations of motion**: A set of 12 nonlinear, coupled, first-order, ordinary differential equations
- Difficult to design controller based on complete nonlinear vehicle model
- Reduced order model for control design using decoupling and linearization around equilibrium point
- Equations of motion
  - $\Rightarrow$ Longitudinal motion: airspeed, pitch angle, and altitude
  - $\Rightarrow$ Lateral motion: roll and heading angles
- For most of vehicles, weak dynamic coupling such that their unwanted effect can be handled by controller
- Trim: Force and moment equilibrium point
- Quasi-linear aerodynamic (nonlinear in α) and propulsion (nonlinear in throttle) models

UAV: Various Angles Definitions

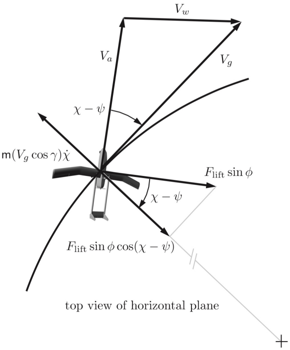


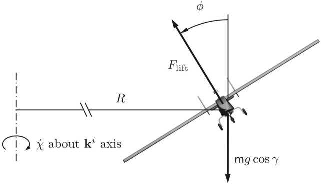


- Wind triangle:

$$\boldsymbol{V}_g = \boldsymbol{V}_a + \boldsymbol{V}_w$$

- Wind is significant (around 40-50%) for UAV applications as compared to large aircraft.





- Equation for a coordinated turn

$$\boxed{\dot{\chi} = \frac{g}{V_g} \tan\phi \cos(\chi - \psi)}$$

- In absence of wind, $V_a = V_g, \psi = \chi$,

$$\boxed{\dot{\chi} = \frac{g}{V_g} \tan\phi = \dot{\psi} = \frac{g}{V_a} \tan\phi}$$

- Given a nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}); \boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

- When do we say that a system is in its equilibrium?
- System is in equilibrium at state $\boldsymbol{x}^\star$ and $\boldsymbol{u}^\star$ if $\boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star) = \boldsymbol{0}$
- Trim condition: Aircraft in equilibrium
- When vehicle is in constant-altitude, wings-level steady flight, a subset of its states are in equilibrium.
- $h = -p_d$, constant body frame velocities, constant Euler angle and body rates
- In general, trim conditions may include non-constant states.
- Examples:
  - ⇒ Steady climb, wing level flight
  - ⇒ Constant turn

- General trim condition

$$\boxed{\dot{\boldsymbol{x}}^\star = \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}$$

- For trim calculations, treat wind as an unknown disturbance.
- **Objectives**: To compute trim states and inputs when vehicle simultaneously satisfies following conditions:
  - $\Rightarrow$ traveling at a constant speed $V_a^\star$,
  - $\Rightarrow$ climbing at a constant flight path angle of $\gamma^\star$
  - $\Rightarrow$ a constant orbit of radius $R^\star$ with $R^\star > R_{\text{min}}$
- What about other known flight conditions, such as constant altitude flight?
- For constant altitude, wing level flight,

$$\gamma^\star = 0, R^\star = \infty$$

- For constant altitude orbit with radius $R^\star$,

$$\gamma^\star = 0$$

- For fixed wing aircraft, system state vector and input vector are

$$\boldsymbol{x} = (p_n, p_e, p_d, u, v, w, \phi, \theta, \psi, p, q, r)^\top, \quad \boldsymbol{u} = (\delta_e, \delta_t, \delta_a, \delta_r)^\top$$

- Trimmed flight is independent of position as $\boldsymbol{f}(.)$ does not depend on position.
- As only $\dot{p}_e, \dot{p}_n$ depend on ψ, trimmed flight is independent of ψ.
- For constant climb orbit, no speed change and constant roll and pitch angles

$$\boxed{\dot{u}^\star = \dot{v}^\star = \dot{w}^\star = 0, \quad \dot{\theta}^\star = \dot{\phi}^\star = \dot{p}^\star = \dot{q}^\star = 0}$$

- Turn and climb rates

$$\boxed{\dot{\psi}^\star = \frac{V_a^\star}{R^\star}\cos\gamma^\star \implies \dot{r}^\star = 0; \quad \dot{h}^\star = V_a^\star \sin\gamma^\star}$$

Given $V_a^\star, \gamma^\star, R^\star$

$$\dot{\boldsymbol{x}}^\star = \begin{pmatrix} \dot{p}_n^\star \\ \dot{p}_e^\star \\ \dot{h}^\star \\ \dot{u}^\star \\ \dot{v}^\star \\ \dot{w}^\star \\ \dot{\phi}^\star \\ \dot{\theta}^\star \\ \dot{\psi}^\star \\ \dot{p}^\star \\ \dot{q}^\star \\ \dot{r}^\star \end{pmatrix} = \begin{pmatrix} \text{Don't care} \\ \text{Don't care} \\ V_a^\star \sin\gamma^\star \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{V_a^\star \cos\gamma^\star}{R^\star} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Computation of $\boldsymbol{x}^\star$: Solving nonlinear algebraic equations

- General nonlinear system of equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u});\ \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{u} \in \mathbb{R}^m$$

- Trim values: $\boldsymbol{x}^\star, \boldsymbol{u}^\star$ s.t. $\boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star) = \boldsymbol{0}$
- Define deviations in state as $\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}^\star$

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}} =& \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star) = \boldsymbol{f}(\boldsymbol{x}^\star + \boldsymbol{x} - \boldsymbol{x}^\star, \boldsymbol{u}^\star + \boldsymbol{u} - \boldsymbol{u}^\star) - \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star) \\ =& \boldsymbol{f}(\bar{\boldsymbol{x}} + \boldsymbol{x}^\star, \bar{\boldsymbol{u}} + \boldsymbol{u}^\star) - \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)\end{aligned}$$

- Using Taylor's series

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}} =& \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star) + \frac{\partial \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}{\partial \boldsymbol{x}} \bar{\boldsymbol{x}} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}{\partial \boldsymbol{u}} \bar{\boldsymbol{u}} + HOT - \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star) \\ \approx& \frac{\partial \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}{\partial \boldsymbol{x}} \bar{\boldsymbol{x}} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}{\partial \boldsymbol{u}} \bar{\boldsymbol{u}}\end{aligned}$$

- States vector

$$\boldsymbol{x}_{\text{lat}} \triangleq \begin{pmatrix} v \\ p \\ r \\ \phi \\ \psi \end{pmatrix}$$

- Input vector

$$\boldsymbol{u}_{\text{lat}} \triangleq \begin{pmatrix} \delta_a \\ \delta_r \end{pmatrix}$$

- Sideslip angle

$$\beta = \tan^{-1}\left(\frac{v}{u^2 + w^2}\right)$$

- Airspeed

$$V_a = \sqrt{u^2 + v^2 + w^2}$$

Dynamics in lateral direction in terms of lateral states and inputs

$$
\begin{aligned}
\dot{v} =& pw - ru + g\cos\theta\sin\phi + \frac{\rho b\sqrt{u^2+v^2+w^2}}{4m}\left(C_{Y_p}p + C_{Y_r}r\right) \\
&+ \frac{\rho S\left(u^2+v^2+w^2\right)}{2m}\left[C_{Y_0} + C_{Y_\beta}\tan^{-1}\left[\frac{v}{\sqrt{u^2+w^2}}\right] + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right] \\
\dot{p} =& \Gamma_1 pq - \Gamma_2 qr + \frac{\rho b^2\sqrt{u^2+v^2+w^2}}{4}\left(C_{p_p}p + C_{p_r}r\right) \\
&+ \frac{\rho Sb\left(u^2+v^2+w^2\right)}{2}\left[C_{p_0} + C_{p_\beta}\tan^{-1}\left[\frac{v}{\sqrt{u^2+w^2}}\right] + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r\right] \\
\dot{r} =& \Gamma_7 pq - \Gamma_1 qr + \frac{\rho b^2\sqrt{u^2+v^2+w^2}}{4}\left(C_{r_p}p + C_{r_r}r\right) \\
&+ \frac{\rho Sb\left(u^2+v^2+w^2\right)}{2}\left[C_{r_0} + C_{r_\beta}\tan^{-1}\left[\frac{v}{\sqrt{u^2+w^2}}\right] + C_{r_{\delta_a}}\delta_a + C_{r_{\delta_r}}\delta_r\right] \\
\dot{\phi} =& p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\psi} =& q\sin\phi\sec\theta + r\cos\phi\sec\theta
\end{aligned}
$$

- Linearized equations

$$\boxed{\dot{\boldsymbol{x}} \approx \frac{\partial \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}{\partial \boldsymbol{x}} \bar{\boldsymbol{x}} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{u}^\star)}{\partial \boldsymbol{u}} \bar{\boldsymbol{u}}}$$

- Jacobians of equations

$$\frac{\partial \boldsymbol{f}_{\text{lat}}}{\partial \boldsymbol{x}_{\text{lat}}} = \begin{pmatrix} \frac{\partial \dot{v}}{\partial v} & \frac{\partial \dot{v}}{\partial p} & \frac{\partial \dot{v}}{\partial r} & \frac{\partial \dot{v}}{\partial \phi} & \frac{\partial \dot{v}}{\partial \psi} \\ \frac{\partial \dot{p}}{\partial v} & \frac{\partial \dot{p}}{\partial p} & \frac{\partial \dot{p}}{\partial r} & \frac{\partial \dot{p}}{\partial \phi} & \frac{\partial \dot{p}}{\partial \psi} \\ \frac{\partial \dot{r}}{\partial v} & \frac{\partial \dot{r}}{\partial p} & \frac{\partial \dot{r}}{\partial r} & \frac{\partial \dot{r}}{\partial \phi} & \frac{\partial \dot{r}}{\partial \psi} \\ \frac{\partial \dot{\phi}}{\partial v} & \frac{\partial \dot{\phi}}{\partial p} & \frac{\partial \dot{\phi}}{\partial r} & \frac{\partial \dot{\phi}}{\partial \phi} & \frac{\partial \dot{\psi}}{\partial \psi} \\ \frac{\partial \dot{\psi}}{\partial v} & \frac{\partial \dot{\phi}}{\partial p} & \frac{\partial \dot{\phi}}{\partial r} & \frac{\partial \dot{\psi}}{\partial \psi} & \frac{\partial \dot{\psi}}{\partial \psi} \end{pmatrix}, \quad \frac{\partial \boldsymbol{f}_{\text{lat}}}{\partial \boldsymbol{u}_{\text{lat}}} = \begin{pmatrix} \frac{\partial \dot{v}}{\partial \delta_a} & \frac{\partial \dot{v}}{\partial \delta_r} \\ \frac{\partial \dot{p}}{\partial v} & \frac{\partial \dot{p}}{\partial \delta_r} \\ \frac{\partial \dot{r}}{\partial \delta_a} & \frac{\partial \dot{r}}{\partial \delta_r} \\ \frac{\partial \dot{\phi}}{\partial \delta_a} & \frac{\partial \dot{\phi}}{\partial \delta_r} \\ \frac{\partial \dot{\psi}}{\partial \delta_a} & \frac{\partial \dot{\phi}}{\partial \delta_r} \end{pmatrix}$$

- Stability and control derivatives

Linearized equations in lateral direction

$$\begin{pmatrix} \dot{\bar{v}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} Y_v & Y_p & Y_r & g\cos\theta^\star\cos\phi^\star & 0 \\ L_v & L_p & Y_r & 0 & 0 \\ N_v & N_p & Y_r & 0 & 0 \\ 0 & 1 & \cos\phi^\star\tan\theta^\star & q^\star\cos\phi^\star\tan\theta^\star - r^\star\sin\phi^\star\tan\theta^\star & 0 \\ 0 & 0 & \cos\phi^\star\sec\theta^\star & p^\star\cos\phi^\star\sec\theta^\star - r^\star\sin\phi^\star\sec\theta^\star & 0 \end{pmatrix} \begin{pmatrix} \bar{v} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix}$$

$$+ \begin{pmatrix} Y_{\delta_a} & Y_{\delta_r} \\ Y_{\delta_a} & Y_{\delta_r} \\ Y_{\delta_a} & Y_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_a \\ \bar{\delta}_r \end{pmatrix}$$

- Can we use the sideslip angle in place of lateral speed?
- Yes, one can use sidelslip angle β in place of $\bar{v}$. How?

$$v = V_a \sin\beta \implies \bar{v} = V_a^\star \cos\beta^\star \bar{\beta} \implies \boxed{\dot{\bar{\beta}} = \left(\frac{1}{V_a^\star \cos\beta^\star}\right)\dot{\bar{v}}}$$

Linearized equations in lateral direction

$$
\begin{pmatrix} \dot{\bar{\beta}} \\ \dot{\bar{p}} \\ \dot{\bar{r}} \\ \dot{\bar{\phi}} \\ \dot{\bar{\psi}} \end{pmatrix} = \begin{pmatrix} Y_v & \frac{Y_p}{V_a^\star \cos\beta^\star} & \frac{Y_r}{V_a^\star \cos\beta^\star} & \frac{g\cos\theta^\star \cos\phi^\star}{V_a^\star \cos\beta^\star} & 0 \\ L_v V_a^\star \cos\beta^\star & L_p & Y_r & 0 & 0 \\ N_v V_a^\star \cos\beta^\star & N_p & Y_r & 0 & 0 \\ 0 & 1 & \cos\phi^\star \tan\theta^\star & q^\star \cos\phi^\star \tan\theta^\star - r^\star \sin\phi^\star \tan\theta^\star & 0 \\ 0 & 0 & \cos\phi^\star \sec\theta^\star & p^\star \cos\phi^\star \sec\theta^\star - r^\star \sin\phi^\star \sec\theta^\star & 0 \end{pmatrix}
$$

$$
\times \begin{pmatrix} \bar{\beta} \\ \bar{p} \\ \bar{r} \\ \bar{\phi} \\ \bar{\psi} \end{pmatrix} + \begin{pmatrix} \frac{Y_{\delta_a}}{V_a^\star \cos\beta^\star} & \frac{Y_{\delta_r}}{V_a^\star \cos\beta^\star} \\ Y_{\delta_a} & Y_{\delta_r} \\ Y_{\delta_a} & Y_{\delta_r} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_a \\ \bar{\delta}_r \end{pmatrix}
$$

Linearized equations in longitudinal direction

$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{w}} \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \\ \dot{\bar{h}} \end{pmatrix} = \begin{pmatrix} X_u & X_w & X_q & -g\cos\theta^\star & 0 \\ Z_u & Z_w & Z_q & -g\sin\theta^\star & 0 \\ M_u & M_w & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\theta^\star & -\cos\theta^\star & 0 & u^\star\cos\theta^\star + w^\star\sin\theta^\star & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{q} \\ \bar{\theta} \\ \bar{h} \end{pmatrix}$$

$$+ \begin{pmatrix} X_{\delta_e} & X_{\delta_t} \\ Z_{\delta_e} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_e \\ \bar{\delta}_t \end{pmatrix}$$

- Can we use angle of attack to express longitudinal dynamics? Yes, How?
- Use of angle of attack α in place of $\bar{w}$, with $\beta = 0$,

$$w = V_a \sin\alpha\cos\beta = V_a\sin\alpha \implies \bar{w} = V_a\cos\alpha^\star\bar{\alpha}$$

$$\implies \boxed{\dot{\bar{\alpha}} = \left(\frac{1}{V_a^\star\cos\alpha^\star}\right)\dot{\bar{w}}}$$

Linearized equations in longitudinal direction

$$\begin{pmatrix} \dot{\bar{u}} \\ \dot{\bar{\alpha}} \\ \dot{\bar{q}} \\ \dot{\bar{\theta}} \\ \dot{\bar{h}} \end{pmatrix} = \begin{pmatrix} X_u & X_w V_a^\star \cos\alpha^\star & X_q & -g\cos\theta^\star & 0 \\ \frac{Z_u}{V_a^\star \cos\alpha^\star} & Z_w & \frac{Z_q}{V_a^\star \cos\alpha^\star} & -\frac{g\sin\theta^\star}{V_a^\star \cos\alpha^\star} & 0 \\ M_u & M_w V_a^\star \cos\alpha^\star & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sin\theta^\star & -V_a^\star \cos\alpha^\star \cos\theta^\star & 0 & u^\star\cos\theta^\star + w^\star \sin\theta^\star & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{q} \\ \bar{\theta} \\ \bar{h} \end{pmatrix} + \begin{pmatrix} X_{\delta_e} & X_{\delta_t} \\ \frac{Z_{\delta_e}}{V_a^\star \cos\alpha^\star} & 0 \\ M_{\delta_e} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\delta}_e \\ \bar{\delta}_t \end{pmatrix}$$

Reference

1. Randal Beard and Timothy W. McLain, "Small Unmanned Aircraft: Theory and Practice", Princeton University Press, 2012.

- Variables of interest: ϕ, p, ψ, r
- Control inputs: δ_a, δ_r
- **Roll angle dynamics**:

$$\dot{\phi} = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta$$

- In most of flight, $\theta \approx 0$, thus

$$\dot{\phi} = p + d_{\phi_1}; \quad d_{\phi_1} \triangleq q \sin\phi \tan\theta + r \cos\phi \tan\theta$$

- On differentiating $\dot{\phi}$, we get

$$\begin{aligned}\ddot{\phi} &= \dot{p} + \dot{d}_{\phi_1} \\ &= \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb \Big[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta_a}}\delta_a \\ &\quad + C_{p_{\delta_r}}\delta_r \Big] + \dot{d}_{\phi_1}\end{aligned}$$

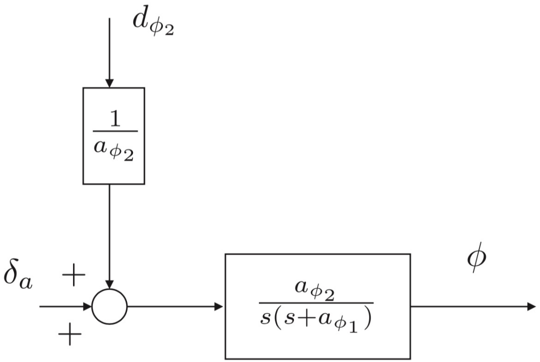
- $p = \dot{\phi} - d_{\phi_1}$
- On substituting for p, we get

$$\ddot{\phi} = \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb \left[C_{p0} + C_{p_\beta}\beta + C_{p_p}\frac{b(\dot{\phi} - d_{\phi_1})}{2V_a} + C_{p_r}\frac{br}{2V_a} \right.$$
$$\left. + C_{p_{\delta_a}}\delta_a + C_{p_{\delta_r}}\delta_r \right] + \dot{d}_{\phi_1}$$
$$= \underbrace{\frac{1}{2}\rho V_a^2 Sb C_{p_p}\frac{b}{2V_a}}_{-a_{\phi_1}}\dot{\phi} + \underbrace{\frac{1}{2}\rho V_a^2 Sb C_{p_{\delta_a}}}_{a_{\phi_2}}\delta_a$$
$$+ \left\{ \Gamma_1 pq - \Gamma_2 qr + \frac{1}{2}\rho V_a^2 Sb \left[C_{p0} + C_{p_\beta}\beta - C_{p_p}\frac{bd_{\phi_1}}{2V_a} + C_{p_r}\frac{br}{2V_a} \right. \right.$$
$$\left. \left. + C_{p_{\delta_r}}\delta_r \right] + \dot{d}_{\phi_1} \right\}$$
$$= -a_{\phi_1}\dot{\phi} + a_{\phi_2}\delta_a + d_{\phi_2}$$

- Can we write transfer function for this differential equation now?

- In Laplace domain,

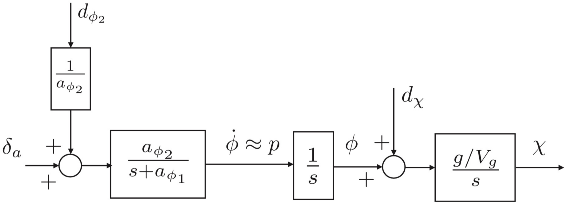
$$\phi(s) = \frac{a_{\phi_2}}{s(s+a_{\phi_1})}\left(\delta_a(s) + \frac{1}{a_{\phi_2}} d_{\phi_2}(s)\right)$$



UAV: Lateral Transfer Functions

- **Course and Heading angle dynamics**: In coordinated turn without wind, course angle dynamics
$$\dot{\chi} = \frac{g}{V_g}\tan\phi \implies \dot{\chi} = \frac{g}{V_g}\phi + \frac{g}{V_g}\underbrace{(\tan\phi - \phi)}_{d_\chi}$$
- In Laplace domain, $\chi(s) = \dfrac{g}{V_g s}\left(\phi(s) + d_\chi(s)\right)$
- With zero wind and $V_g \to V_a$, we have

$$\psi(s) = \frac{g}{V_g s}\left(\phi(s) + d_\chi(s)\right)$$



- **Sideslip angle**: In absence of wind, $v = V_a \sin\beta$
- For constant airspeed,

$$\dot{v} = V_a \cos\beta\dot{\beta}$$

$$\begin{aligned}V_a \cos\beta\dot{\beta} =& pw - ru + g\cos\theta\sin\phi + \frac{\rho V_a^2 S}{2m}\Big(C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V_a} \\ &+ C_{Y_r} r + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\Big)\end{aligned}$$

- With $\beta \approx 0$, $\cos\beta \approx 1$,

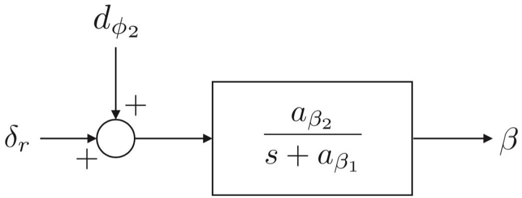
$$\dot{\beta} = -a_{\beta_1}\beta + a_{\beta_2}\delta_r + d_\beta$$

$$\begin{aligned}d_\beta =& \frac{pw - ru + g\cos\theta\sin\phi}{V_a} + \frac{\rho V_a S}{2m}\left(C_{Y_0} + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r} r + C_{Y_{\delta_a}}\delta_a\right) \\ a_{\beta_1} =& \frac{\rho V_a^2 S}{2m} C_{Y_\beta}, \quad a_{\beta_2} = \frac{\rho V_a^2 S}{2m} C_{Y_{\delta_r}}\end{aligned}$$

UAV:Lateral Transfer Functions

- In Laplace domain,

$$\beta(s) = \frac{a_{\beta_2}}{s + a_{\beta_1}} \left(\delta_r(s) + d_\beta(s)\right)$$



- Variables of interest: $\theta, V_a, h = -p_d, q,$
- Control inputs: Throttle and elevator inputs δ_t, δ_e
- **Pitch angle dynamics**:

$$\dot{\theta} = q\cos\phi - r\sin\phi = q + \underbrace{q(\cos\phi - 1) - r\sin\phi}_{d_{\theta_1}} \triangleq q + d_{\theta_1}$$

- Small d_{θ_1} for small ϕ
- On differentiating $\dot{\theta}$,

$$\ddot{\theta} = \dot{q} + \dot{d}_{\theta_1}$$

- Using $\theta = \alpha + \gamma_a, \gamma_a = \gamma$

$$\ddot{\theta} = \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{b}{2V_a}q + C_{m_{\delta_e}}\delta_e\right] + \dot{d}_{\theta_1}$$

- Dynamics of pitch angle

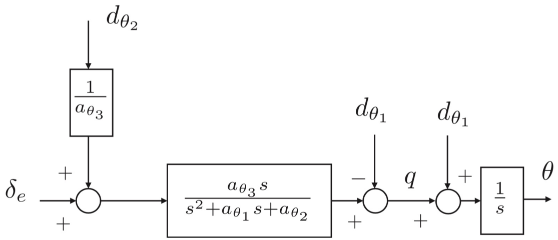
$$
\begin{aligned}
\ddot{\theta} =& \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \frac{\rho V_a^2 Sc}{2J_y}\Big[C_{m_0} + C_{m_\alpha}(\theta - \gamma) \\
&+ C_{m_q}\frac{b}{2V_a}(\dot{\theta} - d_{\theta_1}) + C_{m_{\delta_e}}\delta_e\Big] + \dot{d}_{\theta_1} \\
=& \underbrace{\frac{\rho V_a^2 Sc}{2J_y} C_{m_q}\frac{b}{2V_a}}_{-a_{\theta_1}}\dot{\theta} + \underbrace{\frac{\rho V_a^2 Sc}{2J_y} C_{m_\alpha}}_{-a_{\theta_2}}\theta + \underbrace{\frac{\rho V_a^2 Sc}{2J_y} C_{m_{\delta_e}}}_{a_{\theta_3}}\delta_e \\
&+ \Gamma_5 pr + \Gamma_6(r^2 - p^2) + \frac{\rho V_a^2 Sc}{2J_y}\Big[C_{m_0} + C_{m_\alpha}(-\gamma) \\
&+ C_{m_q}\frac{b}{2V_a}(-d_{\theta_1})\Big] + \dot{d}_{\theta_1} \\
=& -a_{\theta_1}\dot{\theta} - a_{\theta_2}\theta + a_{\theta_3}\delta_e + d_{\theta_2}
\end{aligned}
$$

- How to write this in Laplace domain?

- In Laplace domain,

$$\theta(s) = \frac{a_{\theta_3}}{s^2 + a_{\theta_1} s + a_{\theta_2}} \left(\delta_e(s) + \frac{1}{a_{\theta_3}} d_{\theta_2}(s) \right)$$

- For straight and level flight, $r = p = \phi = \gamma = 0$
- For $C_{m_0} = 0$, we have $d_{\theta_2} = 0$.
- Also, $\dot{\theta} = q + d_{\theta_1}$



- **Altitude**: For a constant airspeed, pitch angle directly influences climb rate of the aircraft.
- Dynamics of altitude

$$
\begin{aligned}
\dot{h} =& u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\
=& V_a\theta + \underbrace{(u\sin\theta - V_a\theta) - v\sin\phi\cos\theta - w\cos\phi\cos\theta}_{d_h} \\
=& V_a\theta + d_h
\end{aligned}
$$

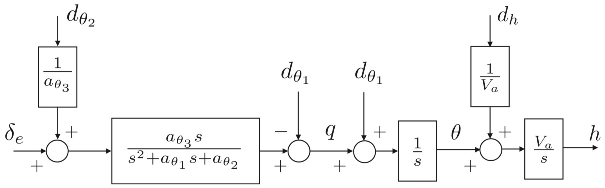
- In straight and level flight,

$$v \approx 0, w \approx 0, u \approx V_a, \phi \approx 0$$

and θ small, and thus $d_h \approx 0$.

- For constant V_a and θ as input, in Laplace domain,

$$h(s) = \frac{V_a}{s}\left(\theta + \frac{1}{V_a} d_h\right)$$



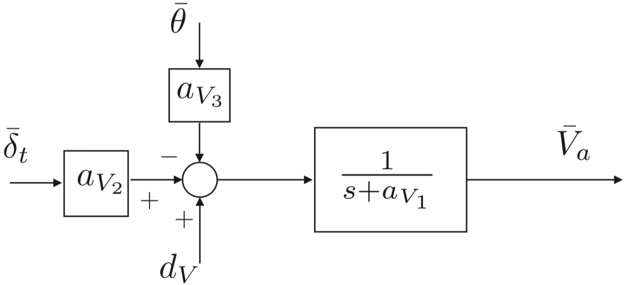
- For constant θ and V_a as input, in Laplace domain,

$$h(s) = \frac{\theta}{s}\left(V_a + \frac{1}{\theta} d_h\right)$$

- Either or both of these approaches can be used.

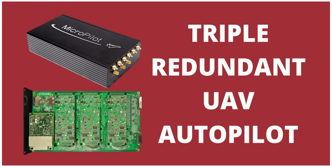
- How to obtain the relation between airspeed and throttle and pitch angle?
- In Laplace domain,

$$\bar{V}_a(s) = \frac{1}{s + a_{V_1}} \left(a_{V_2} \bar{\delta}_t(s) - a_{V_3} \bar{\theta}(s) + d_V(s) \right)$$

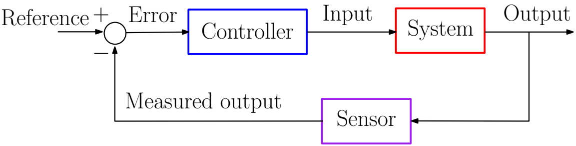


Unmanned Autonomous Vehicles: Autopilot

- Autopilot: A system used to guide an aircraft without the assistance of a pilot.
- Manned aircraft
  - ⇒ A single-axis wing-leveling autopilot
  - ⇒ A full flight control system to control position and attitude during various phases of flight
- Unmanned vehicle
  - ⇒ Complete control of aircraft during all phases of flight
- **Autopilot**: Suitable for available sensor and computational resources



PC: http://uavpropulsiontech.com



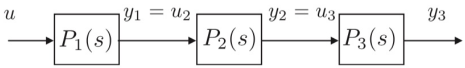
Primary goal of autopilot

To control the inertial position and attitude of the vehicle

- How to design autopilot for the UAV?
- Autopilot design: Decoupled dynamics (longitudinal and lateral)
- Simplified problem with reduced order dynamics
- Successive loop closure approach

Successive loop closure

To close several simple feedback loops in succession around the open-loop plant dynamics rather than designing a single control system.



- Open loop dynamics

$$P(s) = P_1(s)P_2(s)P_3(s)$$

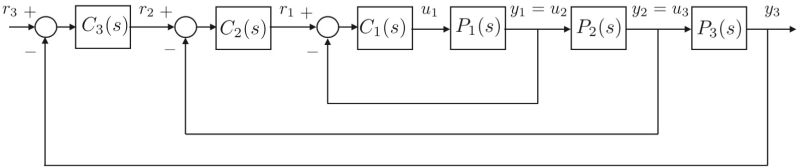
- $P_i(s)$: lower (first or second) order transfer functions

- Objective: To control y_3

- Closed-loop control: Feedback through output y_3

- Closure of feedback loops around all y_1, y_2, y_3 instead of only y_3

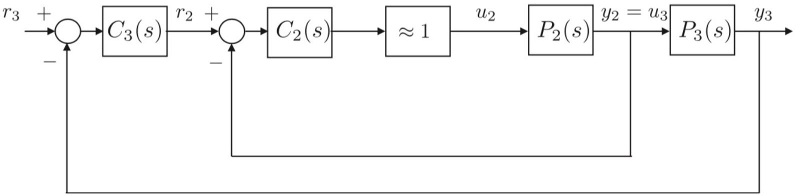
- Necessary condition: Inner loop should have highest bandwidth, with the bandwidth of each successive loop smaller, by a factor of 5-10 in frequency



- **Compensators**: $C_i(s)$
- Goal of compensator $C_1(s)$: To design a closed-loop system from r_1 to y_1 having a bandwidth ω_{BW1}.
- For frequencies well below ω_{BW1}, closed-loop transfer function

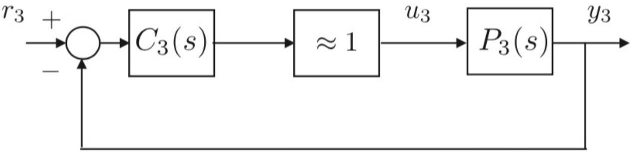
$$\frac{y_1(s)}{r_1(s)} \approx 1$$

- What would be the simplified model for second loop?



- Critical step: To design next loop bandwidth a factor of S_1 (5 to 10) smaller than the preceding loop.
- **Goal of compensator** $C_2(s)$: To design a closed-loop system from r_2 to y_2 having a bandwidth ω_{BW2}, with $\omega_{BW2} < \omega_{BW1}/S_1$.
- For frequencies well below ω_{BW2}, closed-loop transfer function

$$\frac{y_2(s)}{r_2(s)} \approx 1$$



- Goal of compensator $C_3(s)$: To design a closed-loop system from r_3 to y_3 having a bandwidth ω_{BW3}, with $\omega_{BW3} < \omega_{BW2}/S_2$.
- Nature of plant models $P_1(s), P_2(s)$, and $P_3(s)$: first or second order
- Conventional PID or lead-lag compensators
- Transfer function-based design methods, such as root-locus approach

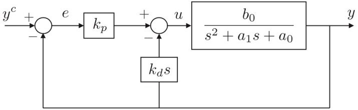
UAV Autopilot: Saturation Constraints and Performance

- Successive loop closure: Performance limitation of the system due to the inner-most loop

- What are other reasons for performance degradation?
  - $\Rightarrow$ Saturation constraints, such as bounded angular deflections
  - $\Rightarrow$ Physical bounds on ailerons restrict roll rate
  - $\Rightarrow$ Bound on total lateral acceleration due to aircraft stall at higher angle of attack

- Goal of autopilot design: To design bandwidth of the inner loop to be as large as possible, without violating the saturation constraints

- It allows outer loops to ensure sufficient bandwidth separation of the successive loops



- Consider second order system

$$P(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

- Closed-loop transfer function%pause

$$\frac{y(s)}{y^c(s)} = \frac{b_0 k_p}{s^2 + (a_1 + b_0 k_d)s + (a_0 + b_0 k_p)}$$

- **Closed-loop poles**: depends on gains
- **Actuator effort**: $u = k_p e - k_d \dot{y}$
- What terms affect the input demand or actuator effort?
- For $\dot{y} \approx 0$, size of error e and k_p decide about input u; $u \approx k_p e$
- For stable system, largest control effort occurs immediately after step input
- Determination of proportional gain

$$\boxed{u^{\max} = k_p e^{\max} \implies k_p = \frac{u^{\max}}{e^{\max}}}$$

- Can we achieve any bandwidth we choose? No
- How to compute the achievable bandwidth of a closed-loop system?
- Canonical form of second order system, with no zeros

$$\frac{y(s)}{y^c(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

  where ζ is damping ratio and ω_n is natural frequency.
- Under-damped system

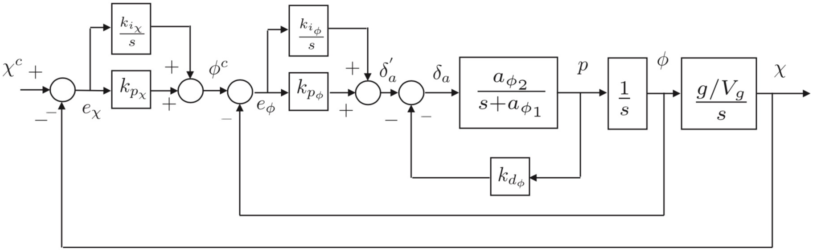
$$0 \le \zeta < 1; \;\; \lambda = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

- Natural frequency of closed-loop system under saturation constraints

$$\omega_n = \sqrt{a_0 + b_0 k_p} = \sqrt{a_0 + b_0 \frac{u^{\max}}{e^{\max}}}$$

- Upper limit on the bandwidth to avoid possible saturation on input

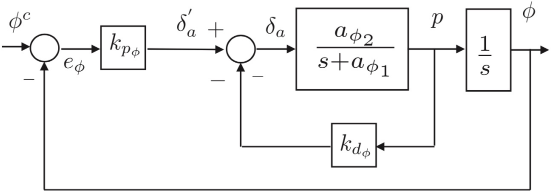
- How to design lateral autopilot?



- Roll rate damping: k_{d_ϕ}
- Roll attitude: k_{p_ϕ} and k_{i_ϕ}
- Course angle: k_{p_χ} and k_{i_χ}
- How to select these gains?
- Successive selection of gains beginning with the inner loop and working outward.

- Linear design model:

$$\phi(s) = \frac{a_{\phi_2}}{s(s + a_{\phi_1})}\left(\delta_a(s) + \frac{1}{a_{\phi_2}} d_{\phi_2}(s)\right)$$



- **Goal of inner loop**: To control roll angle and its rate
- **Gain design**: Based on desired response of closed-loop system
- Closed-loop transfer function

$$H_{\phi/\phi^c}(s) = \frac{k_{p_\phi} a_{\phi_2}}{s^2 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi})s + k_{p_\phi} a_{\phi_2}}$$

- Desired second order closed-loop transfer function

$$\frac{\phi^c(s)}{\phi(s)} = \frac{\omega_{n_\phi}^2}{s^2 + 2\zeta_\phi \omega_{n_\phi} s + \omega_{n_\phi}^2}$$

- How to find this desired closed-loop transfer function?
- Equating coefficients results in

$$\omega_{n_\phi}^2 = k_{p_\phi} a_{\phi_2}, \quad 2\zeta_\phi \omega_{n_\phi} = a_{\phi_1} + a_{\phi_2} k_{d_\phi}$$

- Gain k_{p_ϕ} is chosen as per saturation bound on aileron deflections

$$\boxed{k_{p_\phi} = \frac{\delta_a^{\max}}{e_\phi^{\max}} \mathsf{sign}(a_{\phi_2}) \implies \omega_{n_\phi} = \sqrt{|a_{\phi_2}| \frac{\delta_a^{\max}}{e_\phi^{\max}}}}$$

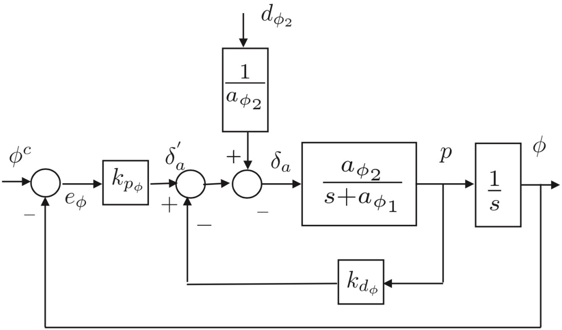
- Gain k_{d_ϕ} based on gain k_{p_ϕ}, for chosen ζ_ϕ to satisfy other specifications

$$\boxed{k_{d_\phi} = \frac{2\zeta_\phi \omega_{n_\phi} - a_{\phi_1}}{a_{\phi_2}}}$$

UAV Lateral Autopilot: Roll Attitude Loop Design Issues

- Zero steady-state tracking error in roll without an integrator (Type-1 open loop TF)
- Recall final value theorem and system type
- Presence of disturbance at the summing junction
- Neglected dynamics, wind gust, or turbulence

$$\phi(s) = \frac{a_{\phi_2}}{s(s + a_{\phi_1})}\left(\delta_a(s) + \frac{1}{a_{\phi_2}} d_{\phi_2}(s)\right)$$



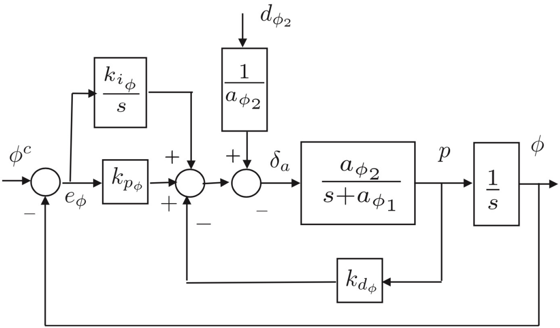
UAV Lateral Autopilot: Roll Attitude Loop Design Issues

- On solving for $\phi(s)$

$$\phi(s) = \left(\frac{1}{s^2 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi})s + k_{p_\phi} a_{\phi_2}}\right) d_{\phi_2} + \left(\frac{a_{\phi_2} k_{d_\phi}}{s^2 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi})s + k_{p_\phi} a_{\phi_2}}\right) \phi^c$$

- What would be the steady-state error for constant step disturbance $d_{\phi_2} = A/s$?
- Using final-value theorem, steady-state error is $A/(a_{\phi_2} k_{p_\phi})$, a nonzero value.
- For constant orbit, p, q, r and thus d_{ϕ_2} are constant.
- How to remove this steady-state error?
- To eliminate steady-state error, an integrator is required.
- How to see if an integrator rejects disturbances?

UAV Lateral Autopilot: Roll Attitude Loop Design using Integrator



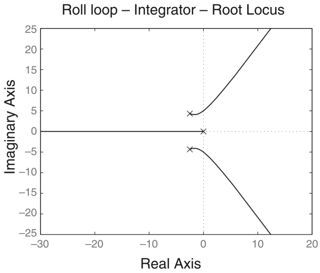
$$\phi = \left(\frac{s}{s^3 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi}) s^2 + k_{p_\phi} a_{\phi_2} s + a_{\phi_2} k_{i_\phi}} \right) d_{\phi_2} + \left(\frac{a_{\phi_2} k_{p_\phi} \left(s + \frac{k_{i_\phi}}{k_{p_\phi}} \right)}{s^3 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi}) s^2 + k_{p_\phi} a_{\phi_2} s + a_{\phi_2} k_{i_\phi}} \right) \phi^c$$

- **Steady-state error**: Zero for constant $d_{\phi 2}$, $\frac{A}{a_{\phi_2} k_{i_\phi}}$ for ramp inputs
- k_{i_ϕ} can be chosen using root-locus technique
- Characteristic equation of closed-loop system

$$s^3 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi}) s^2 + k_{p_\phi} a_{\phi_2} s + a_{\phi_2} k_{i_\phi} = 0$$

- How to draw root locus?

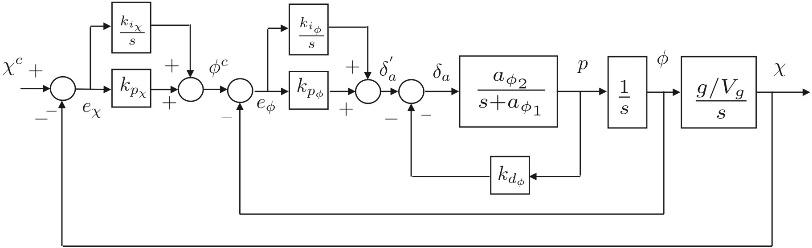
$$1 + k_{i_\phi} \left(\frac{a_{\phi_2}}{s\left(s^2 + (a_{\phi_1} + a_{\phi_2} k_{d_\phi}) s + k_{p_\phi} a_{\phi_2}\right)} \right) = 0$$

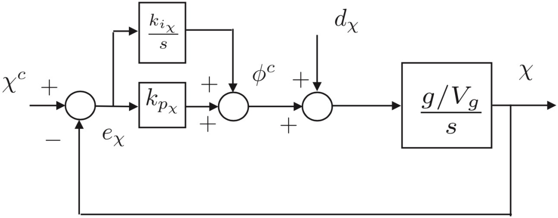


$$\boxed{\delta_a = k_{p_\phi}(\phi^c - \phi) + \frac{k_{i_\phi}}{s}(\phi^c - \phi) - k_{d_\phi} p}$$

Proper gain for better performance

What next? Course hold outer loop





- With proper tuning of gains, $H_{\phi/\phi^c} \approx 1$ over frequencies between 0 to ω_{n_ϕ}
- Objective: Design k_{p_χ} and k_{i_χ} s.t. $\chi \to \chi^c$ asymptotically
- Transfer functions relating χ, χ^c, and d_χ

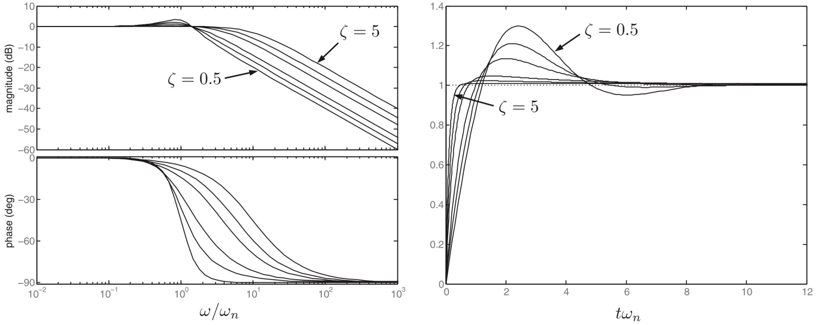
$$\chi = \frac{\frac{g}{V_g} s}{s^2 + k_{p_\chi} \frac{g}{V_g} s + k_{i_\chi} \frac{g}{V_g}} d_\chi + \frac{k_{p_\chi} \frac{g}{V_g} s + k_{i_x} \frac{g}{V_g}}{s^2 + k_{p_\chi} \frac{g}{V_g} s + k_{i_\chi} \frac{g}{V_g}} \chi^c$$

- Using final value theorem, with constant d_χ and χ^c, $\chi \to \chi^c$
- Desired transfer function from χ^c to χ

$$H_\chi = \frac{2\zeta_\chi \omega_{n_\chi} s + \omega_{n_\chi}^2}{s^2 + 2\zeta_\chi \omega_{n_\chi} s + \omega_{n_\chi}^2}$$

- Gains selection: Based on desired ζ_χ and ω_{n_χ}
- Selection of ω_{n_χ} and ζ_χ

- Due to zero in numerator, standard intuition for selection of ζ does not hold.



- Larger ζ results in larger bandwidth and smaller overshoot.

- On comparing coefficients, we get

$$\omega_{n_\chi}^2 = k_{i_\chi} g/V_g \implies \boxed{k_{i_\chi} = \frac{\omega_{n_\chi}^2 V_g}{g}}$$

$$2\zeta_\chi \omega_{n_\chi} = k_{p_\chi} g/V_g \implies \boxed{k_{p_\chi} = \frac{2\zeta_\chi \omega_{n_\chi} V_g}{g}}$$

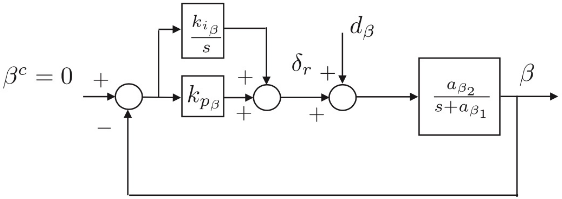
- To ensure sufficient bandwidth

$$\boxed{\omega_{n_\chi} = \frac{\omega_{n_\phi}}{W_\chi}, \quad W_\chi \geq 5}$$

- Large bandwidth separation is better How can this be ensured?
  - $\Rightarrow$ Slower response of χ-loop, lower ω_{n_χ}
  - $\Rightarrow$ Faster response of ϕ-loop, higher ω_{n_ϕ}
- More control demand for a faster response

$$\boxed{\phi^c = k_{p_\chi}(\chi^c - \chi) + \frac{k_{i_\chi}}{s}(\chi^c - \chi)}$$

- Can we maintain constant or zero sideslip angle?
- Yes, using rudder inputs



- Transfer function from β^c to β

$$H_{\beta/\beta^c}(s) = \frac{a_{\beta_2}(k_{p_\beta} s + k_{i_\beta})}{s^2 + (a_{\beta_1} + a_{\beta_2} k_{p_\beta})s + a_{\beta_2} k_{i_\beta}}, \quad H_{\beta/\beta^c}(0) = 1$$

- Characteristic equation for desired closed-loop poles

$$s^2 + 2\zeta_\beta \omega_{n_\beta} s + \omega_{n_\beta}^2 = 0$$

UAV Lateral Autopilot: Sideslip Hold

- On equating the coefficients of characteristic equations

$$2\zeta_\beta \omega_{n_\beta} = a_{\beta_1} + a_{\beta_2} k_{p_\beta}, \quad \omega_{n_\beta}^2 = a_{\beta_2} k_{i_\beta}$$

- Maximum error in sideslip ($e_\beta^{\max}$) and maximum allowable rudder deflection ($\delta_r^{\max}$)
- Gain with such constraints

$$k_{p_\beta} = \frac{\delta_r^{\max}}{e_\beta^{\max}} \,\mathsf{sign}(a_{\beta_2})$$

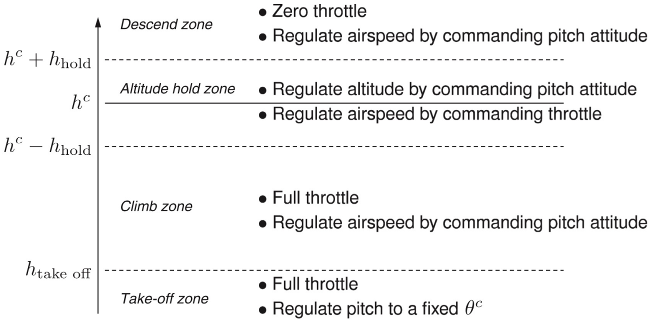
$$k_{i_\beta} = \frac{\omega_{n_\beta}^2}{a_{\beta_2}} = \frac{1}{a_{\beta_2}} \left(\frac{a_{\beta_1} + a_{\beta_2} k_{p_\beta}}{2\zeta_\beta} \right)^2$$

- Output of sideslip loop (rudder deflection)

$$\boxed{\delta_r = -k_{p_\beta} \beta - \frac{k_{i_\beta}}{s} \beta}$$

UAV Longitudinal Autopilot

- Complicated longitudinal autopilot as compared to lateral one
- Role of airspeed in the longitudinal dynamics
- **Objective**: Regulate airspeed and altitude using throttle and elevator inputs
- Regulation of altitude and airspeed: Role of altitude error



- **Take-off zone**: full throttle is commanded and $\theta \to \theta^c$ using the elevator.
- **Climb zone**: to maximize the climb rate

  $\Rightarrow$ To maximize the climb rate, full throttle is commanded and airspeed is regulated using pitch angle.

  $\Rightarrow$ If airspeed increases above nominal value, then aircraft is caused to pitch up, leading to increase in climb rate and decrease in airspeed.

  $\Rightarrow$ If the airspeed drops below nominal value, aircraft is pitched down, resulting in increase of airspeed but also decreases climb rate.

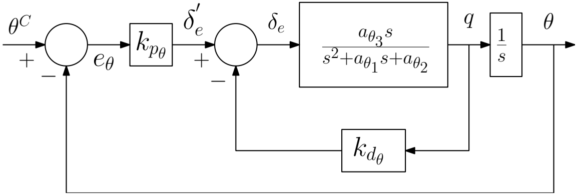
  $\Rightarrow$ Regulating airspeed using pitch attitude effectively avoids stall conditions.

  $\Rightarrow$ No regulation of airspeed with pitch altitude immediately after take-off

UAV Longitudinal Autopilot

- **Descend zone**: Similar to climb zone except zero throttle
  - ⇒ Avoidance of stall conditions by regulating airspeed using pitch angle
  - ⇒ Maximization of descent rate at a given airspeed
- **Altitude hold zone**
  - ⇒ Airspeed: by adjusting throttle
  - ⇒ Altitude: by commanding pitch attitude.
- Longitudinal autopilot: Necessary feedback loops
  - ⇒ Pitch attitude hold using elevator
  - ⇒ Airspeed hold using throttle
  - ⇒ Airspeed hold using pitch attitude
  - ⇒ Altitude hold using pitch attitude

- Pitch attitude hold loop: Similar to roll attitude hold loop



- Closed-loop transfer function

$$H_{\theta/\theta^c}(s) = \frac{k_{p_\theta} a_{\theta_3}}{s^2 + (a_{\theta_1} + k_{d_\theta} a_{\theta_3})s + (a_{\theta_2} + k_{p_\theta} a_{\theta_3})}$$

- What about DC gain of this transfer function? nonunity DC gain
- Desired canonical second order transfer function

$$\frac{k_{\theta_{DC}} \omega_{n_\theta}^2}{s^2 + 2\zeta_\theta \omega_{n_\theta} s + \omega_{n_\theta}^2}$$

- Equating coefficients of characteristic equation

$$\omega_{n_\theta}^2 = a_{\theta_2} + k_{p_\theta} a_{\theta_3}, \quad 2\zeta_\theta \omega_{n_\theta} = a_{\theta_1} + k_{d_\theta} a_{\theta_3}$$

- Gain k_{p_θ} chosen to satisfy actuator constraint, then

$$k_{p_\theta} = \frac{\delta_e^{\max}}{e_\theta^{\max}} \,\mathsf{sign}(a_{\theta_3})$$

- Consideration of sign of a_{θ_3} as it depends on $C_{m_{\delta e}}$ (typically <0)
- Bandwidth limit of pitch loop

$$\omega_{n_\theta} = \sqrt{a_{\theta_2} + \frac{\delta_e^{\max}}{e_\theta^{\max}} |a_{\theta_3}|}$$

- Selection of gain k_{d_θ}

$$k_{d_\theta} = \frac{2\zeta_\theta \omega_{n_\theta} - a_{\theta_1}}{a_{\theta_3}}$$

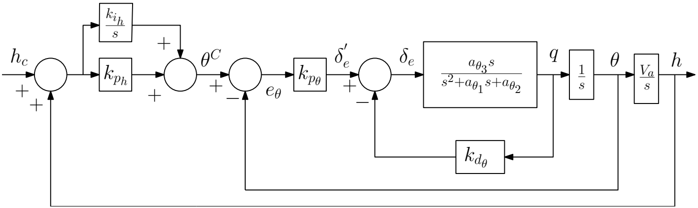
- What about DC gain of inner loop transfer function as $k_{p_\theta} \to \infty$?

$$k_{\theta_{DC}} = \frac{k_{p_\theta} a_{\theta_3}}{a_{\theta_2} + k_{p_\theta} a_{\theta_3}} \to 1$$

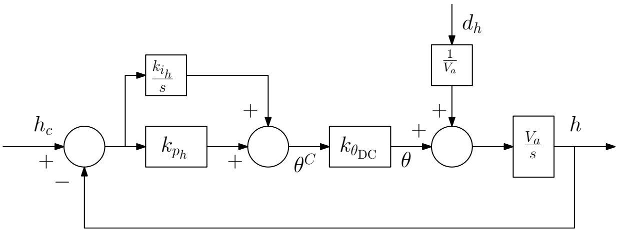
- What about the DC gain value with nominal value of k_{p_θ}?
- Significantly less than one, for typical gain values of k_{p_θ}
- Integral feedback to ensure unit DC gain
- Drawback: Limitation on bandwidth by use of integral term
- Elevator deflection input (Output of pitch attitude hold loop)

$$\boxed{\delta_e = k_{p_\theta}(\theta^c - \theta) - k_{d_\theta} q}$$

- **Altitude-hold autopilot**: Successive-loop-closure strategy with pitch-attitude-hold autopilot as an inner loop



- How to deal with attitude hold design?
- Can we replace the inner loop with unity, as before? No Why?
- DC gain of $K_{\theta_{DC}}$, $\theta = K_{\theta_{DC}}\theta^c$



$$h(s) = \left(\frac{K_{\theta_{DC}} V_a k_{p_h} \left(s + k_{i_h}/k_{p_h}\right)}{s^2 + K_{\theta_{DC}} V_a k_{p_h} s + K_{\theta_{DC}} V_a k_{i_h}} \right) h^c(s) + $$
$$+ \left(\frac{s}{s^2 + K_{\theta_{DC}} V_a k_{p_h} s + K_{\theta_{DC}} V_a k_{i_h}} \right) d_h(s)$$

- What about DC gain and disturbance rejection?
- DC gain of one and rejection of constant disturbance
- Independent closed-loop transfer function on vehicle parameters, except V_a

UAV Longitudinal Autopilot: Altitude Hold using Pitch

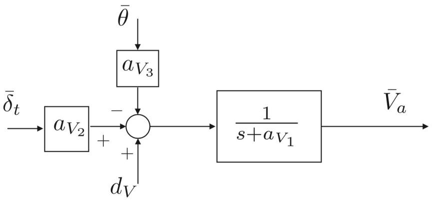
- Selection of gains k_{p_h} and k_{i_h} to ensure $\omega_{n_h} = \dfrac{\omega_{n_\theta}}{W_h}$, $W_h \in [5 \quad 15]$
- Desired closed-loop transfer function $\dfrac{\omega_{n_h}^2}{s^2 + 2\zeta_{n_h}\omega_{n_h}s + \omega_{n_h}^2}$
- Equating coefficients

$$2\zeta_{n_h}\omega_{n_h} = K_{\theta_{DC}} V_a k_{p_h} \implies k_{p_h} = \frac{2\zeta_{n_h}\omega_{n_h}}{K_{\theta_{DC}} V_a}$$

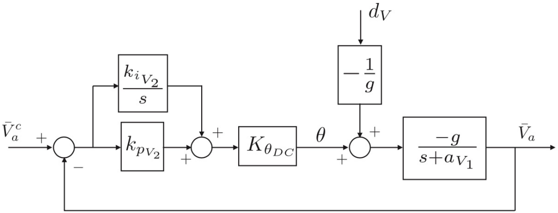
$$\omega_{n_h}^2 = K_{\theta_{DC}} V_a k_{i_h} \implies k_{i_h} = \frac{\omega_{n_h}^2}{K_{\theta_{DC}} V_a}$$

- Pitch command inputs (Output of altitude-hold-with pitch loop)

$$\boxed{\theta^c = k_{p_h}(h^c - h) + \frac{k_{i_h}}{s}(h^c - h)}$$



- Disturbance rejection requires integrator



UAV Longitudinal Autopilot: Airspeed Hold using Commanded Pitch

- Transfer function in Laplace domain

$$\bar{V}_a(s) = -\left(\frac{K_{\theta_{DC}} g k_{p_{V_2}} \left(s + k_{i_{V_2}}/k_{p_{V_2}}\right)}{s^2 + \left(a_{V_1} - K_{\theta_{DC}} g k_{p_{V_2}}\right) s - K_{\theta_{DC}} g k_{i_{V_2}}}\right) \bar{V}_a^c(s) + \left(\frac{s}{s^2 + \left(a_{V_1} - K_{\theta_{DC}} g k_{p_{V_2}}\right) s - K_{\theta_{DC}} g k_{i_{V_2}}}\right) d_V(s)$$

- What about DC gain and disturbance rejection?
- DC gain of 1 and disturbance rejection
- To hold constant airspeed, $\theta \to \alpha \neq 0$.
- How will this be generated?
- Integrator will windup to a command value of proper α
- Bandwidth requirement

$$\omega_{n_{V_2}} = \frac{\omega_{n_\theta}}{W_{V_2}}, \quad W_{V_2} > 5$$

- How to obtain controller gains?
- Using the placement of closed-loop poles at desired locations

$$s^2 + 2\zeta_{V_2}\omega_{n_{V_2}} s + \omega_{n_{V_2}}^2 = 0$$

- Equating coefficients of characteristic equation

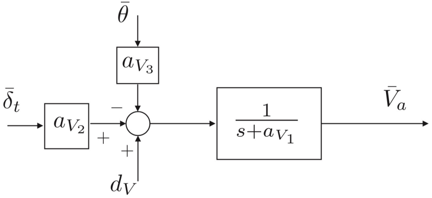
$$\omega_{n_{V_2}}^2 = -K_{\theta_{DC}} g k_{i_{V_2}}, \quad 2\zeta_{V_2}\omega_{n_{V_2}} = a_{V_1} - K_{\theta_{DC}} g k_{p_{V_2}}$$

- Gains selection

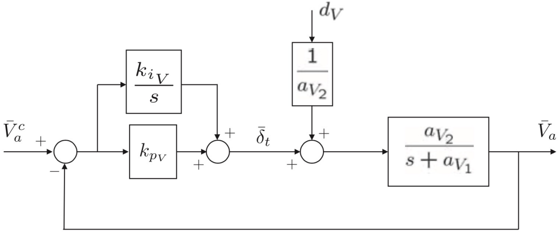
$$\boxed{k_{i_{V_2}} = -\frac{\omega_{n_{V_2}}^2}{K_{\theta_{DC}} g}, \quad k_{p_{V_2}} = \frac{a_{V_1} - 2\zeta_{V_2}\omega_{n_{V_2}}}{K_{\theta_{DC}} g}}$$

- Pitch angle command (output of airspeed hold using pitch loop)

$$\boxed{\theta^c = k_{p_{V_2}} \left(V_a^c - V_a\right) + \frac{k_{i_{V_2}}}{s} \left(V_a^c - V_a\right)}$$



- Block diagram for this design



- Transfer function with proportional (P) control

$$\bar{V}_a(s) = \left(\frac{a_{V_2} k_{p_V}}{s + (a_{V_1} + a_{V_2} k_{p_V})} \right) \bar{V}_a^c(s) + \left(\frac{1}{s + (a_{V_1} + a_{V_2} k_{p_V})} \right) d_V(s)$$

- Nonunity DC gain and no step disturbance rejection
- What do you expect with proportional and integral (PI) type feedback?
- Transfer function with PI feedback control

$$\begin{aligned} \bar{V}_a(s) = &\left(\frac{a_{V_2} (k_{p_V} s + k_{i_V})}{s^2 + (a_{V_1} + a_{V_2} k_{p_V}) s + a_{V_2} k_{i_V}} \right) \bar{V}_a^c(s) \\ &+ \left(\frac{s}{s^2 + (a_{V_1} + a_{V_2} k_{p_V}) s + a_{V_2} k_{i_V}} \right) d_V(s) \end{aligned}$$

- Unity DC gain and step disturbance rejection
- Gains selection using pole placement approach
- Equating coefficients

$$2\zeta_V \omega_{n_V} = a_{V_1} + a_{V_2} k_{p_V}, \quad \omega_{n_V}^2 = a_{V_2} k_{p_V}$$

UAV Longitudinal Autopilot: Airspeed Hold using Throttle

- Controller gains selection

$$\boxed{k_{i_V} = \frac{\omega_{n_V}^2}{a_{V_2}}, \quad k_{p_V} = \frac{2\zeta_V \omega_{n_V} - a_{V_1}}{a_{V_2}}}$$

$$\boxed{\bar{\delta}_t = k_{p_V}\left(\bar{V}_a^c - \bar{V}_a\right) + \frac{k_{i_V}}{s}\left(\bar{V}_a^c - \bar{V}_a\right)}$$

- Can we implement the controller without the knowledge of trim velocity?
- Yes, fortunately, we can do so. How?
- As $\bar{V}_a^c = V_a^c - V_a^\star$ and $\bar{V}_a = V_a - V_a^\star$, error signal

$$\boxed{e = \bar{V}_a^c - \bar{V}_a = V_a^c - V_a}$$

- Throttle command $\delta_t = \delta_t^\star + \bar{\delta}_t$
- Throttle input (output of airspeed hold)

$$\boxed{\delta_t = \delta_t^\star + k_{p_V}\left(V_a^c - V_a\right) + \frac{k_{i_V}}{s}\left(V_a^c - V_a\right)}$$

$$\delta_t = 0$$
$$\theta^c = \left(k_{p_{V_2}} + \frac{k_{i_{V_2}}}{s}\right)(V_a^c - V_a)$$
Descend zone

$h < h^c + h_{\text{hold}}$

$h \geq h^c + h_{\text{hold}}$

$$\delta_t = \delta_t^* + \left(k_{p_V} + \frac{k_{i_V}}{s}\right)(V_a^c - V_a)$$
$$\theta^c = \left(k_{p_h} + \frac{k_{i_h}}{s}\right)(h^c - h)$$
Altitude hold zone

$h < h^c - h_{\text{hold}}$

$h \geq h^c - h_{\text{hold}}$

$$\delta_t = 1$$
$$\theta^c = \left(k_{p_{V_2}} + \frac{k_{i_{V_2}}}{s}\right)(V_a^c - V_a)$$
Climb zone

$h < h_{\text{take off}}$

$h \geq h_{\text{take off}}$

$$\delta_t = 1$$
$$\theta^c = \theta_{\text{take off}}$$
Take-off zone

UAV Longitudinal Autopilot: Altitude Control State Machine

- Longitudinal autopilot: Control of motions in body $\boldsymbol{i}^b - \boldsymbol{k}^b$ plane
- Four basic designs together complete overall longitudinal autopilot
- **Climb zone**: $\delta_t = 1$ and airspeed hold from commanded pitch mode to control airspeed, thus avoiding stall conditions.
- Vehicle climbs at its maximum possible climb rate until it is close to altitude set point.
- **Descend zone**: $\delta_t = 0$ and airspeed hold from commanded pitch mode.
- Descend at a steady rate until vehicle reaches altitude hold zone.
- **Altitude hold zone**: Airspeed-from-throttle mode to regulate airspeed.
- Altitude-from-pitch mode is used to regulate altitude around h^c.
- Pitch attitude control loop is active in all four zones.

Reference

1. Randal Beard and Timothy W. McLain, "Small Unmanned Aircraft: Theory and Practice", Princeton University Press, 2012.
2. John Anderson Jr., "Introduction to Flight", McGraw-Hill Education, Sixth Edition, 2017.