CHAPTER 12 Numerical Methods for Constrained Optimum Design

Section 12.1 Basic Concepts Related to Numerical Methods

12.1

Answer True or False.

1. The basic numerical iterative philosophy for solving constrained and unconstrained problems is the same. *True*
2. Step size determination is a one-dimensional problem for unconstrained problems. *True*
3. Step size determination is a multidimensional problem for constrained problems. *False*
4. An inequality constraint $gi(\mathbf{x})\le 0$ is violated at $\mathbf{x}^{(k)}$ if $gi(\mathbf{x}^{(k)}) > 0$. *True*
5. An inequality constraint $gi(\mathbf{x})\le 0$ is active at $\mathbf{x}^{(k)}$ if $gi(\mathbf{x}^{(k)}) > 0$. *False*
6. An equality constraint $hi(\mathbf{x})=0$ is violated at $\mathbf{x}^{(k)}$ if $hi(\mathbf{x}^{(k)}) < 0$. *True*
7. An equality constraint is always active at the optimum. *True*
8. In constrained optimization problems, search direction is found using the cost gradient only. *False*
9. In constrained optimization problems, search direction is found using the constraint gradients only. *False*
10. In constrained problems, the descent function is used to calculate the search direction. *False*
11. In constrained problems, the descent function is used to calculate a feasible point. *False*
12. Cost function can be used as a descent function in unconstrained problems. *True*
13. One-dimensional search on a descent function is needed for convergence of algorithms. *True*
14. A robust algorithm guarantees convergence. *True*
15. A feasible set must be closed and bounded to guarantee convergence of algorithms. *True*
16. A constraint $x_1+x_2\le -2$ can be normalized as $(x_1+x_2)/(-2)\le 1.0$. *False*
17. A constraint $x^2_1+x^2_2\le 9$ is active at $x_1=3$ and $x_2=3$. *False*

Section 12.2 Linearization of the Constrained Problem

12.2

Answer True or False.

1. Linearization of cost and constraint functions is a basic step for solving nonlinear optimization problems. *True*
2. General constrained problems cannot be solved by solving a sequence of linear programming sub-problems. *False*
3. In general, the linearized sub-problem without move limits may be unbounded. *True*
4. The sequential linear programming method for general constrained problems is guaranteed to converge. *False*
5. Move limits are essential in the sequential linear programming procedure. *True*
6. Equality constraints can be treated in the sequential linear programming algorithm. *True*

12.3 ———

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

The beam design problem formulated in Section 3.8 at the point (b, d) = (250, 300) mm

Solution:

Writing the problem in the normalized form, we get:

$f(b,d) = bd;$

$g_1 = 2.4\times10^7/bd^2 - 1 \le 0$

$g_2 = 1.125\times10^5/bd - 1 \le 0$

$g_3 = d/2b - 1 \le 0, \text{ or } 1 - 2b/d \le 0$

$g_4 = 1 - b/10 \le 0$

$g_5 = b/1000 - 1 \le 0$

$g_6 = 1 - d/10 \le 0$

$g_7 = d/1000 - 1 \le 0$

Only f, g_1, g_2 and g_3 need to be linearized; other constraints are already linear.

b = 250 mm, d = 300 mm

$f(250,300) = 250\times300 = 75,000;\ \nabla f = [d, b] = [300, 250]$

$g_1(250,300) = 0.066667;\ \nabla g_1 = 2.4\times10^7[-1/b^2d^2, -2/bd^3] = [-0.004267, -0007111]$

$g_2(250,300) = 0.50;\ \nabla g_2 = 1.125\times10^5[-1/b^2d, -1/bd^2] = [-0.006, -0.005]$

$g_3(250,300) = -0.4 : \nabla g_3 = [-d/2b^2, 1/2b] = [-0.0024, 0.002]$

$\bar{f} = 75,000 + 300(b-250) + 250(d-300) = -75,000 + 300b + 250d$

$\bar{g}_1 = 0.066667 + (-0.004267)(b-250) + (-0.007111)(d-300)$

$\quad = -0.004267b - 0.007111d + 3.266747 \le 0$

$\bar{g}_2 = 0.50 + (-0.006)(b-250) + (-0.005)(d-300) = -0.006b - 0.005d + 3.5 \le 0$

$\bar{g}_3 = -0.4 - 0.0024(b-250) + 0.002(d-300) = -0.0024b + 0.002d - 0.4 \le 0$

12.4

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

The tubular column design problem formulated in Section 2.7 at the point (R, t) = (12, 4) cm. Let $P = 50kN$, $E = 210GPa$, $l = 500cm$, $\sigma_a = 250MPa$, and $\rho = 7850\, kg/m^3$.

Solution:

Substituting the given values the problem is formulated as:

$$f(R,t) = 2(7850\times10^{-6})(500)\pi Rt = 24.6615Rt, (kg)$$

$$g_1 = P/2\pi\sigma_a Rt - 1 = 50\times10^3/2\pi(25000)Rt - 1 = 0.31831/Rt - 1 \le 0$$

$$g_2 = 1 - \pi^3 ER^3 t/4l^2 P = 1 - \pi^3 \times 210\times10^5 R^3 t/(4\times500\times500\times50\times10^3) = 1 - 0.013023R^3 t \le 0$$

$$g_3 = 1 - R/5 \le 0$$

$$g_4 = R/100 - 1 \le 0$$

$$g_5 = 1 - 2t \le 0$$

$$g_6 = t/5 - 1 \le 0$$

Linearizing f, g_1 and g_2 about R = 12, t = 4, we get:

$$f(12,4) = 24.6615\times12\times4 = 1183.752;\ \nabla f(12,4) = [24.6615t, 24.6615R] = [98.646, 295.938]$$

$$g_1(12,4) = 0.31831/(12\times4) - 1 = -0.99337$$

$$\nabla g_1 = 0.31831[-1/R^2 t, -1/Rt^2] = [-0.0005526, -0.0016579]$$

$$g_2(12,4) = 1 - 0.013023(12^3\times4) = -89.01498$$

$$\nabla g_2 = [1 - 0.039069R^2 t, 1 - 0.013023R^3] = [-22.50374, -22.50374]$$

$$\bar{f} = 1183.752 + 98.646(R-12) + 295.938(t-4) = 98.646R + 295.938t - 1183.752$$

$$\bar{g}_1 = -0.99337 + (-0.0005526)(R-12) + (-0.0016579)(t-4)$$
$$= -0.0005526R - 0.0016579t - 0.98011 \le 0$$

$$\bar{g}_2 = -9.01498 + (-22.50374)(R-12) + (-22.50374)(t-4)$$
$$= -22.50374R - 22.50374t + 271.04486 \le 0$$

12.5

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

The wall bracket problem formulated in Section 4.9.1 at the point (A_1, A_2) = (150, 150) cm^2.

Solution:

Substituting the given values the problem is formulated as:

$f(\mathrm{A}_1, \mathrm{A}_2) = 50A_1 + 40A_2$

$g_1 = 125/A_1 - 1 \le 0$

$g_2 = 100/A_2 - 1 \le 0$

$g_3 = -A_1 \le 0$

$g_4 = -A_2 \le 0$

Linearizing f, g_1 and g_2 at A_1 = 150 and A_2 = 150, we get:

$f(150,150) = (50 \times 150) + (40 \times 150) = 13500;\ \nabla f(150,150) = [50, 40]$

$g_1(150,150) = 125/150 - 1 = -0.16667;\ \nabla g_1 = [-125/A_1^2, 0] = [-0.005555, 0]$

$g_2(150,150) = 100/150 - 1 = -0.33333;\ \ \nabla g_2 = [0, -100/A_2^2] = [0, -0.004444]$

$\bar{f} = 13500 + 50(A_1 - 150) + 40(A_2 - 150)$

$\bar{g}_1 = -0.16667 + (-0.005555)(A_1 - 150) = -0.005555A_1 + 0.6666 \le 0$

$\bar{g}_2 = -0.3333 + (-0.004444)(A_2 - 150) = -0.004444A_2 + 0.3333 \le 0$

12.6

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.1 at the point h = 12 m, A = 4000 m^2.

Solution:

Substituting the given values the problem is formulated as:

$$f(h, A) = (0.6h + 0.001A)$$

$$g_1 = 1 - 1.42857 \times 10^{-5} hA \le 0$$

$$g_2 = 0.0001A + 7.14286 \times 10^{-6} hA - 1 \le 0$$

$$g_3 = 1 - h/3.5 \le 0$$

$$g_4 = h/21 - 1 \le 0$$

$$g_5 = -A \le 0$$

Linearizing f, g_1 and g_2 about h = 12, A = 4000, we get:

$$f(12, 4000) = (0.6 \times 12) + (0.001 \times 4000) = 11.2;\ \nabla f(12, 4000) = [0.6, 0.001]$$

$$g_1(12, 4000) = 1 - (1.42857 \times 10^{-5}) \times (12 \times 4000) = 0.3143$$

$$\nabla g_1 = [1 - 1.42857 \times 10^{-5} A, 1 - 1.42857 \times 10^{-5} h] = [-0.057143, -1.7143 \times 10^{-4}]$$

$$g_2(12, 4000) = 0.0001 \times 4000 + 7.14286 \times 10^{-6} \times (12 \times 4000) - 1 = -0.25714$$

$$\nabla g_2 = [7.14286 \times 10^{-6} A, 0.0001 + 7.14286 \times 10^{-6} h] = [0.02857, 1.857 \times 10^{-4}]$$

$$\overline{f} = 11.2 + 0.6(h - 12) + 0.001(A - 4000)$$

$$\overline{g}_1 = 0.3143 + (-0.057143)(h - 12) + (-1.7143 \times 10^{-4})(A - 4000)$$

$$= -0.057143h - 1.7143 \times 10^{-4} A + 1.68574 \le 0$$

$$\overline{g}_2 = -0.25714 + 0.02857(h - 12) + 1.857 \times 10^{-4}(A - 4000) = 0.02857h + 1.857 \times 10^{-4} A - 1.34284 \le 0$$

12.7

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.3 at the point (R, H) = (6, 15) cm.

Solution:

Substituting the given values the problem is formulated as:

$f(R,H) = -\pi R^2 H$

$g_1 = \pi RH/450 - 1 \le 0$

$g_2 = 1 - R/5 \le 0$

$g_3 = R/20 - 1 \le 0$

$g_4 = -H \le 0$

$g_5 = H/20 - 1 \le 0$

Linearizing f and g_1 about R = 6, H = 15, we get:

$f(6,15) = -\pi \times 6^2 \times 15 = -1696.46;\ \nabla f(12,4) = [-2\pi RH, -\pi R^2] = [-565.487, -113.097]$

$g_1(6,15) = \pi \times 6 \times 15/450 - 1 = -0.3717;\ \nabla g_1 = [\pi H/450, \pi R/450] = [0.10472, 0.04189]$

$\overline{f} = -1696.46 + (-565.487)(R-6) + (-113.097)(H-15) = -565.487R - 113.097H + 3392.92$

$\overline{g}_1 = -0.3717 + (0.10472)(R-6) + (0.04189)(H-15) = 0.10472R + 0.04189H - 1.62837 \le 0$

The remaining constraints are linear, so their linearized form is same as the original form in terms of the design variables.

12.8

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.4 at the point R = 2 cm, N = 100.

Solution:

Substituting the given values the problem is formulated as:

$$f(N, R) = -2\pi lNR = -6.2832NR$$

$$g_1 = \pi NR^2 / 2000 - 1 = 1.5708 \times 10^{-3} NR^2 - 1 \le 0$$

$$g_2 = 1 - 2R \le 0$$

$$g_3 = -N \le 0$$

Linearizing f and g_1 about R = 2, N = 100, we get:

$$f(100, 2) = -6.2832 \times 100 \times 2 = -1256.64;$$

$$\nabla f(100, 2) = [-6.2832R, -6.2832N] = [-12.5664, -628.32]$$

$$g_1(100, 2) = (1.5708 \times 10^{-3}) \times 100 \times 2^2 - 1 = -0.3717$$

$$\nabla g_1 = [1.5708 \times 10^{-3} R^2, 3.1416 \times 10^{-3} NR] = [6.2832 \times 10^{-3}, 0.62832]$$

$$\overline{f} = -1256.64 + (-12.5664)(N - 100) + (-628.32)(R - 2) = -12.5664N - 628.32R + 1256.64$$

$$\overline{g}_1 = -0.3717 + (6.2832 \times 10^{-3})(N - 100) + (0.62832)(R - 2)$$

$$= 6.2832 \times 10^{-3} N + 0.62832R - 2.25666 \le 0$$

12.9

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.5 at the point (W, D) = (100, 100) m.

Solution:

Substituting the given values the problem is formulated as:

$f(W, D) = 200W + 100D$

$g_1 = W/100 - 1 \le 0$

$g_2 = D/200 - 1 \le 0$

$g_3 = 1 - WD/10000 \le 0$

$g_4 = D/2W - 1 \le 0$ (or $1 - 2W/D \le 0$)

$g_5 = W/2D - 1 \le 0$ (or $1 - 2D/W \le 0$)

$g_6 = -W \le 0$

$g_7 = -D \le 0$

Only g_3, g_4, and g_5 need to be linearized.

W = 100, D = 100

$g_3(100,100) = 1 - 100 \times 100/10000 = 0;\ \nabla g_3 = [-D/10000, -W/10000] = [-0.01, -0.01]$

$g_4(100,100) = (100/2 \times 100) - 1 = -0.5;\ \ \nabla g_4 = [-D/2W^2, 1/2W] = [-0.005, 0.005]$

$g_5(100,100) = (100/2 \times 100) - 1 = -0.5;\ \ \nabla g_5 = [1/2D, -W/2D^2] = [0.005, -0.005]$

$\bar{g}_3 = 0 + (-0.01)(W - 100) + (-0.01)(D - 100) = -0.01W - 0.01D + 2 \le 0$

$\bar{g}_4 = -0.5 - 0.005(W - 100) + 0.005(D - 100) = -0.005W + 0.005D - 0.5 \le 0$

$\bar{g}_5 = -0.5 + 0.005(W - 100) - 0.005(D - 100) = 0.005W - 0.005D - 0.5 \le 0$

12.10

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.9 at the point (r, h) = (6, 16) cm

Solution:

Substituting the given values the problem is formulated as:

$$f(r,h) = \pi r^2 + 2\pi rh$$

$$h_1 = \pi r^2 h/600 - 1 = 0$$

$$g_1 = 1 - h/2r \le 0$$

$$g_2 = h/3r - 1 \le 0$$

$$g_3 = h/20 - 1 \le 0$$

$$g_4 = -h \le 0$$

$$g_5 = -r \le 0$$

Linearizing f, h_1, g_1 and g_2 about r = 6, h = 16, we get:

$$f(6,16) = \pi(6^2) + 2\pi(6\times16) = 716.283;\ \nabla f(6,16) = [2\pi r + 2\pi h, 2\pi r] = [138.2301, 37.699]$$

$$h_1(6,16) = \pi r^2 h/600 - 1 = 2.01593;\ \nabla h_1 = [\pi rh/300, \pi r^2/600] = [1.0053, 0.1885]$$

$$g_1(6,16) = 1 - 6/2\times16 = -0.33333\ \ \nabla g_1 = [h/2r^2, -1/2r] = [0.22222, -0.083333]$$

$$g_2(6,16) = 6/3\times16 - 1 = -0.11111;\ \nabla g_2 = [-h/3r^2, 1/3r] = [-0.14815, 0.05555]$$

$$\bar{f} = 716.283 + 138.2301(r-6) + 37.699(h-16) = 138.2301r + 37.699h - 716.283$$

$$\bar{h}_1 = 2.01593 + 1.0053(r-6) + 0.1885(h-16) = 1.0053r + 0.1885h - 7.0318 = 0$$

$$\bar{g}_1 = -0.33333 + (0.22222)(r-6) - 0.083333(h-16) = 0.22222r - 0.083333h - 0.33332 \le 0$$

$$\bar{g}_2 = -0.11111 - 0.14815(r-6) - 0.05555(h-16) = -0.14815r + 0.05555h - 0.11111 \le 0$$

12.11

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.10 at the point (b, h) = (5, 10) m

Solution:
Substituting the given values the problem is formulated as:

$$f(b,h) = (32/15)(2/b + 1/h)$$
$$g_1 = b/10 - 1 \le 0$$
$$g_2 = h/18 - 1 \le 0$$
$$g_3 = -b \le 0$$
$$g_4 = -h \le 0$$

Linearizing f about b = 5, h = 10, we get:

$$f(5,10) = (32/15) + (2/5 + 1/10) = 1.06667;$$
$$\nabla f(6,16) = [(32/15)(-2/b^2), (32/15)(-1/h^2)] = [-0.17067, -0.021333]$$
$$\bar{f} = 1.06667 - 0.17067(b-5) - 0.021333(h-10) = -0.17067b - 0.021333h + 2.13333$$

12.12

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.11 at the point, width = 5 m, depth = 5 m, and height = 5 m.

Solution:
Substituting the given values the problem is formulated as:

$$f(b,d,h) = 276.6749dh + 296.15656bh + 177.30077bd$$
$$g_1 = 1 - bdh/150 \le 0$$
$$g_2 = -b \le 0$$
$$g_3 = -d \le 0$$
$$g_4 = -h \le 0$$

Linearizing f and g_1 about b = 5, d = 5, and h = 5, we get:

$$f(5,5,5) = 276.6749 \times 5 \times 5 + 296.15656 \times 5 \times 5 + 177.30077 \times 5 \times 5 = 18753.306$$
$$\nabla f(5,5,5) = [296.15656h + 177.30077d, 276.6749h + 177.30077b, 276.6749d + 296.15656b]$$
$$= [2367.2867, 2269.8784, 2864.1573]$$
$$g_1(5,5,5) = 1 - 5 \times 5 \times 5/150 = 0.16667$$
$$\nabla g_1 = [-dh/150, -bh/150, -bd/150] = [-0.16667, -0.16667, -0.16667]$$
$$\bar{f} = 18753.306 + 2367.2867(b-5) + 2269.8784(d-5) + 2864.1573(h-5)$$
$$= 2367.2867b + 2269.8784d + 2864.1573h - 18753.306$$
$$\bar{g}_1 = 0.16667 - 0.16667(b-5) - 0.16667(d-5) - 0.16667(h-5)$$
$$= -0.16667b - 0.16667d - 0.16667h + 2.66667 \le 0$$

12.13

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.12 at the point D = 4 m and H = 8 m.

Exercise 2.12

Design a circular tank closed at both ends to have a volume of 250 m^3. The fabrication cost is proportional to the surface area of the sheet metal and is \$400/m^2. The tank is to be housed in a shed with a sloping roof. Therefore, height H of the tank is limited by the relation $H \le (10-D/2)$, where D is the tank's diameter. Formulate the minimum-cost design problem.

Solution:

Step 3: Definition of Design Variables

D = diameter of the tank in m

H = height of the tank in m

Step 4: Optimization Criterion

Optimization criterion is to minimize the cost, and the cost function is defined as

$Cost = 400(\pi D^2/2 + \pi DH)$

Step 5: Formulation of Constraints

Constraint: $\pi D^2 H/4 = 250$, m³

Constraint: $H \le 10 - D/2$, m

Explicit Design Variable Constraints:

$H \ge 0$, m; $D \ge 0$, m

Substituting the given values, the problem is transcribed into the standard and normalized form as as:

$$f(D,H) = 400(\pi D^2/2 + \pi DH)$$

$$h_1 = \pi D^2 H/1000 - 1 = 0$$

$$g_1 = D/20 + H/10 - 1 \le 0$$

$$g_2 = -D \le 0$$

$$g_3 = -H \le 0$$

Linearizing f and h_1 about D = 4 and H = 8, we get:

$$f(4,8) = 400\left(\pi 4^2/2 + \left(\pi \times 4 \times 8\right)\right) = 50265.482$$

$$\nabla f(4,8) = [400\pi D + 400\pi H, 400\pi D]$$

$$= [15079.645,\ \ 5026.548]$$

$$h_1(4,8) = \pi \times 4^2 \times 8/600 - 1 = -0.32979$$

$$\nabla h_1 = [\pi DH/300, \pi D^2/600]$$

$$= [0.3351,\ \ 0.08378]$$

Linearization in terms of the changes d_1 and d_2 to the design variables:

$\bar{f} = 50265.482 + 15079.645 d_1 + 5026.548 d_2$

$\bar{h}_1 = -0.32979 + 0.3351 d_1 + 0.08378 d_2 = 0$

Similarly linearization of the inequality constraints at the point (4, 8) is given as

$\bar{g}_1 = \frac{1}{20} d_1 + \frac{1}{10} d_2 \le 0$

$\bar{g}_2 = -4 - d_1 \le 0$

$\bar{g}_3 = -8 - d_2 \le 0$

Linearization in terms of the original design variables D and *H* at the point (4, 8):

$\bar{f} = 50265.482 + 15079.645(D-4) + 5026.548(H-8)$

$= 15079.645D + 5026.548H - 50265.482$

$\bar{h}_1 = -0.32979 + 0.3351(D-4) + 0.08378(H-8)$

$= 0.3351D + 0.08378H - 2.34043 = 0$

Since all the inequality constraints are already linear, their linearized form in terms of the original design variables will not change

12.14

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.13 at the point w = 10 m, d = 10 m, h = 4 m.

Solution:
Substituting the given values the problem is formulated as:

$$f(w,d,h) = 80w + 120d + 200h$$
$$g_1 = 1 - wdh/600 \le 0$$
$$g_2 = -w \le 0$$
$$g_3 = -d \le 0$$
$$g_4 = -h \le 0$$

Linearizing g_1 about w = 10, d = 10, and h = 4, we get:

$$g_1(10,10,4) = 1 - 10 \times 10 \times 4/600 = 0.33333$$
$$\nabla g_1 = [-dh/600, -wh/600, -wd/600] = [-0.06667, -0.06667, -0.16667]$$
$$\bar{g}_1 = 0.33333 - 0.06667(w-10) - 0.06667(d-10) - 0.16667(h-4)$$
$$= -0.06667w - 0.06667d - 0.16667h + 2.33333 \le 0$$

12.15

Formulate the following design problem, transcribe it into the standard form, create a linear approximation at the given point, and plot the linearized sub-problem on a graph.

Exercise 2.14 at the point P_1 = 2 and P_2 = 1.

Solution:
Substituting the given values the problem is formulated as:

$$f(P_1,P_2) = 2 - P_1 + 0.6P_2 + P_1^2 + P_2^2$$
$$g_1 = 1 - P_1/60 - P_2/60 \le 0$$
$$g_2 = -P_1 \le 0$$
$$g_3 = -P_2 \le 0$$

Linearizing f about P_1 = 2 and P_2 = 1.

$$f(2,1) = 2 - 2 + (0.6 \times 1) + 2^2 + 1^2 = 5.6$$
$$\nabla f = [-1 + 2P_1, 0.6 + 2P_2] = [3, 2.6]$$
$$\bar{g}_1 = 5.6 + 3(P_1 - 2) + 2.6(P_2 - 1) = 3P_1 + 2.6P_2 - 3$$

Section 12.3 Sequential Linear Programming Algorithm

Note that answers to the exercises in Sections 12.3 are not given in the text because the final results depend on how the constraints are normalized

12.16

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Beam design problem formulated in Section 3.8 at the point (b, d) = (250, 300) mm.

Solution:

Referring to Exercise 12.3, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [b^{(0)}, d^{(0)}] = [250, 300] mm;\ k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = 75000$

$b_1 = -g_1 = -0.066667;\quad b_2 = -g_2 = -0.50$

$b_3 = -g_3 = 0.40;\quad b_4 = -g_4 = 24$

$b_5 = -g_5 = 0.75;\quad b_6 = -g_6 = 29$

$b_7 = -g_7 = 0.7$

Step 3. $c_1 = \partial f / \partial b = 300; c_2 = \partial f / \partial d = 250$

$a_{11} = \partial g_1 / \partial b = -0.004267; a_{21} = \partial g_1 / \partial d = -0.007111$

$a_{12} = \partial g_2 / \partial b = -0.006; a_{22} = \partial g_2 / \partial d = -0.005$

$a_{13} = \partial g_3 / \partial b = -0.0024; a_{23} = \partial g_3 / \partial d = 0.002$

$a_{14} = \partial g_4 / \partial b = -0.10; a_{24} = \partial g_4 / \partial d = 0$

$a_{15} = \partial g_5 / \partial b = 0.001; a_{25} = \partial g_5 / \partial d = 0$

$a_{16} = \partial g_6 / \partial b = 0; a_{26} = \partial g_6 / \partial d = -0.10$

$a_{17} = \partial g_7 / \partial b = 0; a_{27} = \partial g_7 / \partial d = 0.001$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)}$ = 125, $\Delta_{1u}^{(0)}$ = 125; $\Delta_{2l}^{(0)}$ = 150, $\Delta_{2u}^{(0)}$ = 150.

Step 5. The LP sub-problem is defined as: $\overline{f} = 300d_1 + 250d_2$ subject to

$-0.004267d_1 - 0.007111d_2 \le -0.066667;$

$-0.006d_1 - 0.005d_2 \le -0.50\quad -0.0024d_1 - 0.002d_2 \le 0.40;$

$-0.10d_1 \le 24\quad 0.001d_1 \le 0.75;$

$-0.10d_2 \le 29\quad 0.001d_2 \le 0.70;$

$-125 \le d_1 \le 125\quad -150 \le d_2 \le 150$

Step 6. Solving the LP sub-problem, we get: $d_1 = 83.33334; d_2 = 0$

Step 7. Convergence criteria are not satisfied: $g_1 > \varepsilon_1, g_2 > \varepsilon_2; \|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = b^{(1)} = 250 + 83.33334 = 333.33334;\ x_2^{(1)} = d^{(1)} = 300 + 0 = 300$

Set k = 1, go to Step 2.

12.17

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Tubular column design problem formulated in Section 2.7 at the point (R, t) = (12, 4) cm. Let P = 50 kN, E = 210 GPa, l = 500 cm, σ_a = 250 MPa, and σ = 7850 kg/m^3.

Solution:

Referring to Exercise 12.4, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [R^{(0)}, t^{(0)}] = [12, 4]\text{cm};\ k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = 1183.752$

$b_1 = -g_1 = 0.99337;\quad b_2 = -g_2 = 89.01498$

$b_3 = -g_3 = 1.4;\quad b_4 = -g_4 = 0.88$

$b_5 = -g_5 = 7;\quad b_6 = -g_6 = 0.2$

Step 3. $c_1 = \partial f / \partial R = 98.646; c_2 = \partial f / \partial t = 295.938$

$a_{11} = \partial g_1/\partial R = -0.0005526; a_{21} = \partial g_1/\partial t = -0.001658$

$a_{12} = \partial g_2/\partial R = -22.50374; a_{22} = \partial g_2/\partial t = -22.50374$

$a_{13} = \partial g_3/\partial R = -0.2; a_{23} = \partial g_3/\partial t = 0$

$a_{14} = \partial g_4/\partial R = 0.01; a_{24} = \partial g_4/\partial t = 0$

$a_{15} = \partial g_5/\partial R = 0; a_{25} = \partial g_5/\partial t = -2$

$a_{16} = \partial g_6/\partial R = 0; a_{26} = \partial g_6/\partial t = 0.2$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 6; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 2$

Step 5. The LP sub-problem is defined as: $\overline{f} = 98.646d_1 + 295.938d_2$ subject to:

$-0.0005526d_1 - 0.001658d_2 \le 0.99337;\quad -22.50374d_1 - 22.50374d_2 \le 89.01498$

$-0.2d_1 \le 1.4;\quad 0.01d_1 \le 0.88$

$-2d_2 \le 7;\quad 0.2d_2 \le 0.2$

$-6 \le d_1 \le 6;\quad -2 \le d_2 \le 2$

Step 6. Solving the LP sub-problem, we get: $d_1 = -1.95556; d_2 = -2$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = R^{(1)} = 12 - 1.95556 = 10.04444;\ x_2^{(1)} = t^{(1)} = 4 - 2 = 2$

Set k = 1, go to Step 2.

12.18

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Wall bracket problem formulated in Section 4.9.1 at the point $(A_1, A_2) = (150, 150)$ cm^2.

Solution:

Referring to Exercise 12.5, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [A_1, A_2] = [150,150]\,\text{cm}^2$; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = 13500$

$b_1 = -g_1 = 0.16667;\qquad b_2 = -g_2 = 0.33333$

$b_3 = -g_3 = 150;\qquad b_4 = -g_4 = 150$

Step 3. $c_1 = \partial f/\partial A_1 = 50; c_2 = \partial f/\partial A_2 = 40$

$a_{11} = \partial g_1/\partial A_1 = -0.005555; a_{21} = \partial g_1/\partial A_2 = 0$

$a_{12} = \partial g_2/\partial A_1 = 0; a_{22} = \partial g_2/\partial A_2 = -0.004444$

$a_{13} = \partial g_3/\partial A_1 = -1; a_{23} = \partial g_3/\partial A_2 = 0$

$a_{14} = \partial g_4/\partial A_1 = 0; a_{24} = \partial g_4/\partial A_2 = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 75; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 75$

Step 5. The LP sub-problem is defined as: $\overline{f} = 50d_1 + 40d_2$ subject to:

$-0.005555d_1 \le 0.16667;\qquad -0.004444d_2 \le 0.33333$

$-d_1 \le 150;\qquad -d_2 \le 150$

$-75 \le d_1 \le 75;\qquad -75 \le d_2 \le 75$

Step 6. Solving the LP sub-problem, we get: $d_1 = -30; d_2 = -75$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = A_1^{(1)} = 150 - 30 = 120;\ x_2^{(1)} = A_2^{(1)} = 150 - 75 = 75$

Set $k = 1$, go to Step 2.

12.19

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.1 at the point h = 12 m, A = 4000 m^2.

Solution:

Referring to Exercise 12.6, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [h^{(0)}, A^{(0)}] = [12, 4000]\,\text{m}^2$; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001$.

Step 2. $f_0 = 11.2$

$b_1 = -g_1 = -0.3143$; $b_2 = -g_2 = 0.25714$

$b_3 = -g_3 = 2.42857$; $b_4 = -g_4 = 0.42857$

$b_5 = -g_5 = 4000$

Step 3. $c_1 = \partial f / \partial h = 0.6; c_2 = \partial f / \partial A = 0.001$

$a_{11} = \partial g_1 / \partial h = -0.057143; a_{21} = \partial g_1 / \partial A = -1.7143 \times 10^{-4}$

$a_{12} = \partial g_2 / \partial h = 0.02857; a_{22} = \partial g_2 / \partial A = 1.857 \times 10^{-4}$

$a_{13} = \partial g_3 / \partial h = -0.285714; a_{23} = \partial g_3 / \partial A = 0$

$a_{14} = \partial g_4 / \partial h = 0.047619; a_{24} = \partial g_4 / \partial A = 0$

$a_{15} = \partial g_5 / \partial h = 0; a_{25} = \partial g_5 / \partial A = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 6; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 2000$

Step 5. The LP sub-problem is defined as: $\bar{f} = 0.6d_1 + 0.001d_2$ subject to:

$-0.057143d_1 - 0.00017143d_2 \le -0.3143$;

$-0.02857d_1 + 0.0001857d_2 \le 0.25714$; $-0.285714d_1 \le 2.42857$

$0.047619d_1 \le 0.42857$; $-d_2 \le 4000$

$-6 \le d_1 \le 6$; $-2000 \le d_2 \le 2000$

Step 6. Solving the LP sub-problem, we get: $d_1 = 2.5; d_2 = 1000$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = h^{(1)} = 12 + 2.5 = 14.5$; $x_2^{(1)} = A^{(1)} = 4000 + 1000 = 5000$

Set k = 1, go to Step 2.

12.20

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.3 at the point (R, H) = (6, 15) cm.

Solution:

Referring to Exercise 12.7, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [R^{(0)}, H^{(0)}] = [6,15]\text{cm};\ k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = -1696.46$

$b_1 = -g_1 = 0.3717;\qquad b_2 = -g_2 = 0.2$

$b_3 = -g_3 = 0.7;\qquad b_4 = -g_4 = 15$

$b_5 = -g_5 = 0.25$

Step 3. $c_1 = \partial f/\partial R = -565.487; c_2 = \partial f/\partial H = -113.097$

$a_{11} = \partial g_1/\partial R = 0.10472; a_{21} = \partial g_1/\partial H = 0.04189$

$a_{12} = \partial g_2/\partial R = -0.2; a_{22} = \partial g_2/\partial H = 0$

$a_{13} = \partial g_3/\partial R = 0.05; a_{23} = \partial g_3/\partial H = 0$

$a_{14} = \partial g_4/\partial R = 0; a_{24} = \partial g_4/\partial H = -1$

$a_{15} = \partial g_5/\partial R = 0; a_{25} = \partial g_5/\partial H = 0.05$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 3; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 7.5$

Step 5. The LP sub-problem is defined as: $\bar{f} = -565.487d_1 - 113.097d_2$ subject to:

$0.10472d_1 + 0.04189d_2 \le 0.3717;$

$-0.2d_1 \le 0.2;\qquad 0.05d_1 \le 0.7$

$-d_2 \le 15;\qquad 0.05d_2 \le 0.25$

$-3 \le d_1 \le 3;\qquad -7.5 \le d_2 \le 7.5$

Step 6. Solving the LP sub-problem, we get: $d_1 = 3; d_2 = 1.3736$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = R^{(1)} = 6 + 3 = 9;\ x_2^{(1)} = H^{(1)} = 15 + 1.3736 = 16.3736.$

Set k = 1, go to Step 2.

12.21

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.4 at the point R = 2 cm, N = 100.

Solution:

Referring to Exercise 12.8, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [N^{(0)}, R^{(0)}] = [100, 2]$cm; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = -1256.64$

$b_1 = -g_1 = 0.3717;$ $\quad b_2 = -g_2 = 3$

$b_3 = -g_3 = 100;$

Step 3. $c_1 = \partial f / \partial N = -12.5664; c_2 = \partial f / \partial R = -628.32$

$a_{11} = \partial g_1 / \partial N = 6.2832 \times 10^{-3}; a_{21} = \partial g_1 / \partial R = 0.62832$

$a_{12} = \partial g_2 / \partial N = 0; a_{22} = \partial g_2 / \partial R = -2$

$a_{13} = \partial g_3 / \partial N = -1; a_{23} = \partial g_3 / \partial R = 0$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 50; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 1$

Step 5. The LP sub-problem is defined as: $\overline{f} = -12.566 d_1 - 628.32 d_2$ subject to:

$0.0062832 d_1 + 0.62832 d_2 \le 0.3717;$

$-2 d_2 \le 3;$ $\quad -d_1 \le 100$

$-50 \le d_1 \le 50;$ $\quad -1 \le d_2 \le 1$

Step 6. Solving the LP sub-problem, we get: $d_1 = 50; d_2 = 0.09156$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = N^{(1)} = 100 + 50 = 150;$ $x_2^{(1)} = R^{(1)} = 2 + 0.09156 = 2.09156.$

Set k = 1, go to Step 2.

12.22

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.5 at the point (W, D) = (100, 100) m.

Solution:

Referring to Exercise 12.9, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [W^{(0)}, D^{(0)}] = [100, 100]\,\text{m}$; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001$.

Step 2. $f_0 = 30000$

$b_1 = -g_1 = 0$; $\quad b_2 = -g_2 = 0.5$

$b_3 = -g_3 = 0$; $\quad b_4 = -g_4 = 0.5$

$b_5 = -g_5 = 0.5$; $\quad b_6 = -g_6 = 100$

$b_7 = -g_7 = 100$;

Step 3. $c_1 = \partial f / \partial W = 200; c_2 = \partial f / \partial D = 100$

$a_{11} = \partial g_1 / \partial W = 0.01; a_{21} = \partial g_1 / \partial D = 0$

$a_{12} = \partial g_2 / \partial W = 0; a_{22} = \partial g_2 / \partial D = 0.005$

$a_{13} = \partial g_3 / \partial W = -0.01; a_{23} = \partial g_3 / \partial D = -0.01$

$a_{14} = \partial g_4 / \partial W = -0.005; a_{24} = \partial g_4 / \partial D = 0.005$

$a_{15} = \partial g_5 / \partial W = 0.005; a_{25} = \partial g_5 / \partial D = -0.005$

$a_{16} = \partial g_6 / \partial W = -1; a_{26} = \partial g_6 / \partial D = 0$

$a_{17} = \partial g_7 / \partial W = 0; a_{27} = \partial g_7 / \partial D = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 50$.

Step 5. The LP sub-problem is defined as: $\overline{f} = 200d_1 + 100d_2$ subject to:

$0.01d_1 \le 0$; $\quad 0.005d_2 \le 0.5$;

$-0.01d_1 - 0.01d_2 \le 0$; $\quad -0.005d_1 + 0.005d_2 \le 0.5$

$0.005d_1 - 0.005d_2 \le 0.5$ $\quad -d_1 \le 100$

$-50 \le d_1 \le 50$; $\quad -50 \le d_2 \le 50$

Step 6. Solving the LP sub-problem, we get: $d_1 = -50; d_2 = 50$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = W^{(1)} = 100 - 50 = 50$; $x_2^{(1)} = D^{(1)} = 100 + 50 = 150$.

Set k = 1, go to Step 2.

12.23

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.9 at the point (r, h) = (6, 16) cm.

Solution:

Referring to Exercise 12.10, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [r^{(0)}, h^{(0)}] = [6, 16]$cm; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = 716.283$

$e_1 = -h_1 = -2.01593;$ $b_1 = -g_1 = 0.33333$

$b_2 = -g_2 = 0.11111;$ $b_3 = -g_3 = 0.2$

$b_4 = -g_4 = 16;$ $b_5 = -g_5 = 6$

Step 3. $c_1 = \partial f / \partial r = 138.2301; c_2 = \partial f / \partial h = 37.699$

$n_{11} = \partial h_1 / \partial r = 1.0053; n_{21} = \partial h_1 / \partial h = 0.1885$

$a_{11} = \partial g_1 / \partial r = 0.22222; a_{21} = \partial g_1 / \partial h = -0.083333$

$a_{12} = \partial g_2 / \partial r = -0.14815; a_{22} = \partial g_2 / \partial h = 0.05555$

$a_{13} = \partial g_3 / \partial r = 0; a_{23} = \partial g_3 / \partial h = 0.05$

$a_{14} = \partial g_4 / \partial r = 0; a_{24} = \partial g_4 / \partial h = -1$

$a_{15} = \partial g_5 / \partial r = -1; a_{25} = \partial g_5 / \partial h = 0$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 3; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 8.$

Step 5. The LP sub-problem is defined as: $\overline{f} = 138.2301 d_1 + 37.699 d_2$ subject to:

$1.0053 d_1 + 0.1885 d_2 = 2.01593;$ $0.22222 d_1 - 0.083333 d_2 \le 0.33333;$

$-0.14815 d_1 + 0.05555 d_2 \le 0.11111;$ $0.05 d_2 \le 0.2$

$-d_2 \le 16$ $-d_1 \le 6$

$-3 \le d_1 \le 3;$ $-8 \le d_2 \le 8$

Step 6. Solving the LP sub-problem, we get: $d_1 = -0.86611; d_2 = -6.30981$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = r^{(1)} = 6 - 0.86611 = 5.13389;$ $x_2^{(1)} = h^{(1)} = 16 - 6.30981 = 9.69019.$

Set k = 1, go to Step 2.

12.24

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.10 at the point (b, h) = (5, 10) m.

Solution:

Referring to Exercise 12.11, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [b, h] = [5, 10]\,\text{m}$; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001$.

Step 2. $f_0 = 1.06667$

$b_1 = -g_1 = 0.5$; $b_2 = -g_2 = 0.44444$

$b_3 = -g_3 = 5$; $b_4 = -g_4 = 10$

Step 3. $c_1 = \partial f/\partial b = -0.17067; c_2 = \partial f/\partial h = -0.021333$

$a_{11} = \partial g_1/\partial b = 0.1; a_{21} = \partial g_1/\partial h = 0$

$a_{12} = \partial g_2/\partial b = 0; a_{22} = \partial g_2/\partial h = 0.055555$

$a_{13} = \partial g_3/\partial b = -1; a_{23} = \partial g_3/\partial h = 0$

$a_{14} = \partial g_4/\partial b = 0; a_{24} = \partial g_4/\partial h = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 2.5; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 5$.

Step 5. The LP sub-problem is defined as: $\overline{f} = -0.17067d_1 - 0.021333d_2$ subject to:

$0.1d_1 \le 0.5$; $0.055555d_2 \le 0.44444$;

$-d_1 \le 5$; $-d_2 \le 10$

$-2.5 \le d_1 \le 2.5$; $-5 \le d_2 \le 5$

Step 6. Solving the LP sub-problem, we get: $d_1 = 2.5; d_2 = 5$.

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = b^{(1)} = 5 + 2.5 = 7.5$; $x_2^{(1)} = h^{(1)} = 10 + 5 = 15$.

Set k = 1, go to Step 2.

12.25

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.11 at the point, width = 5 m, depth = 5 m, and height = 5 m.

Solution:

Referring to Exercise 12.12, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [b^{(0)}, d^{(0)}, h^{(0)}] = [5, 5, 5]\,\text{m}$; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001$.

Step 2. $f_0 = 18753.306$

$b_1 = -g_1 = -0.16667$; $\quad b_2 = -g_2 = 5$

$b_3 = -g_3 = 5$; $\quad b_4 = -g_4 = 5$

Step 3. $c_1 = \partial f / \partial b = 2367.2867; c_2 = \partial f / \partial d = 2269.8784; c_3 = \partial f / \partial h = 2864.1573$

$a_{11} = \partial g_1 / \partial b = -0.16667; a_{21} = \partial g_1 / \partial d = -0.16667; a_{31} = \partial g_1 / \partial h = -0.16667$

$a_{12} = \partial g_2 / \partial b = -1; a_{22} = \partial g_2 / \partial d = 0; a_{32} = \partial g_2 / \partial h = 0$

$a_{13} = \partial g_3 / \partial b = 0; a_{23} = \partial g_3 / \partial d = -1; a_{33} = \partial g_3 / \partial h = 0$

$a_{14} = \partial g_4 / \partial b = 0; a_{24} = \partial g_4 / \partial d = 0; a_{34} = \partial g_4 / \partial h = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = \Delta_{3l}^{(0)} = \Delta_{3u}^{(0)} = 2.5$

Step 5. The LP sub-problem is defined as: $\overline{f} = 2367.2867 d_1 + 2269.8784 d_2 + 2864.1573 d_3$

subject to: $-0.16667 d_1 - 0.16667 d_2 - 0.16667 d_3 \le -0.16667$;

$-d_1 \le 5$; $\quad -d_2 \le 5$; $\quad -d_3 \le 5$

$-2.5 \le d_1 \le 2.5$; $\quad -2.5 \le d_2 \le 2.5$ $\quad -2.5 \le d_3 \le 2.5$

Step 6. Solving the LP sub-problem, we get: $d_1 = 1; d_2 = 2.5; d_3 = -2.5$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = b^{(1)} = 5 + 1 = 6$; $x_2^{(1)} = d^{(1)} = 5 + 2.5 = 7.5$; $x_3^{(1)} = h^{(1)} = 5 - 2.5 = 2.5$.

Set k = 1, go to Step 2.

12.26

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.12 at the point, D = 4 m and H = 8 m.

Solution:

Referring to Exercise 12.13, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [D^{(0)}, H^{(0)}] = [4,8]\text{m};\ k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001.$

Step 2. $f_0 = 50265.482$

$e_1 = -h_1 = 0.32979;\quad b_1 = -g_1 = 0$

$b_2 = -g_2 = 4;\quad b_3 = -g_3 = 8$

Step 3. $c_1 = \partial f / \partial D = 15079.645; c_2 = \partial f / \partial H = 5026.548$

$n_{11} = \partial h_1 / \partial D = 0.3351; n_{21} = \partial h_1 / \partial H = 0.08378$

$a_{11} = \partial g_1 / \partial D = 0.05; a_{21} = \partial g_1 / \partial H = 0.1$

$a_{12} = \partial g_2 / \partial D = -1; a_{22} = \partial g_2 / \partial H = 0$

$a_{13} = \partial g_3 / \partial D = 0; a_{23} = \partial g_3 / \partial H = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 2; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 4$

Step 5. The LP sub-problem is defined as: $\overline{f} = 15079.645d_1 + 5026.548d_2$ subject to:

$0.3351d_1 + 0.08378d_2 = 0.32979;\quad 0.05d_1 + 0.1d_2 \le 0$

$-d_1 \le 4;\quad -d_2 \le 8$

$-2 \le d_1 \le 2;\quad -4 \le d_2 \le 4$

Step 6. Solving the LP sub-problem, we get: $d_1 = 1.98421; d_2 = -4$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = D^{(1)} = 4 + 1.98421 = 5.98421;\ x_2^{(1)} = H^{(1)} = 8 - 4 = 4$

Set k = 1, go to Step 2.

12.27

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.13 at the point, w = 10 m, d = 10 m, and h = 4 m.

Solution:

Referring to Exercise 12.14, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [w^{(0)}, d^{(0)}, h^{(0)}] = [10, 10, 4]$ m; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001$.

Step 2. $f_0 = 2800$

$b_1 = -g_1 = -0.33333$; $b_2 = -g_2 = 10$

$b_3 = -g_3 = 10$; $b_4 = -g_4 = 4$

Step 3. $c_1 = \partial f / \partial w = 80; c_2 = \partial f / \partial d = 120$

$a_{11} = \partial g_1 / \partial w = -0.06667; a_{21} = \partial g_1 / \partial d = -0.06667; a_{31} = \partial g_1 / \partial h = -0.16667$

$a_{12} = \partial g_2 / \partial w = -1; a_{22} = \partial g_2 / \partial d = 0; a_{32} = \partial g_2 / \partial h = 0$

$a_{13} = \partial g_3 / \partial w = 0; a_{23} = \partial g_3 / \partial d = -1; a_{33} = \partial g_3 / \partial h = 0$

$a_{14} = \partial g_4 / \partial w = 0; a_{24} = \partial g_4 / \partial d = 0; a_{34} = \partial g_4 / \partial h = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 5; \Delta_{3l}^{(0)} = \Delta_{3u}^{(0)} = 2$

Step 5. The LP sub-problem is defined as: $\overline{f} = 80d_1 + 120d_2 + 200d_3$ subject to:

$-0.06667d_1 - 0.06667d_2 - 0.16667d_3 \le -0.33333$

$-d_1 \le 10$; $-d_2 \le 10$; $-d_3 \le 4$

$-5 \le d_1 \le 5$; $-5 \le d_2 \le 5$; $-2 \le d_3 \le 2$

Step 6. Solving the LP sub-problem, we get: $d_1 = 5; d_2 = -5; d_3 = 2$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = w^{(1)} = 10 + 5 = 15$; $x_2^{(1)} = d^{(1)} = 10 - 5 = 5$; $x_3^{(1)} = h^{(1)} = 4 + 2 = 6$

Set k = 1, go to Step 2.

12.28

Complete one iteration of the sequential linear programming algorithm for the following problem (try 50 percent move limits and adjust them if necessary).

Exercise 2.14 at the point, $P_1 = 2$ and $P_2 = 1$.

Solution:

Referring to Exercise 12.15, first iteration is as follows:

Step 1. Starting point: $\mathbf{x}^{(0)} = [P_1, P_2] = [2,1]$; $k = 0, \varepsilon_1 = 0.001, \varepsilon_2 = 0.001$.

Step 2. $f_0 = 5.6$

$b_1 = -g_1 = -0.95$; $b_2 = -g_2 = 2$

$b_3 = -g_3 = 1$;

Step 3. $c_1 = \partial f / \partial P_1 = 3; c_2 = \partial f / \partial P_2 = 2.6$

$a_{11} = \partial g_1 / \partial P_1 = -0.016667; a_{21} = \partial g_1 / \partial P_2 = -0.016667$

$a_{12} = \partial g_2 / \partial P_1 = -1; a_{22} = \partial g_2 / \partial P_2 = 0$

$a_{13} = \partial g_3 / \partial P_1 = 0; a_{23} = \partial g_3 / \partial P_2 = -1$

Step 4. Select 50% move limits; $\Delta_{1l}^{(0)} = \Delta_{1u}^{(0)} = 1; \Delta_{2l}^{(0)} = \Delta_{2u}^{(0)} = 0.5$

Step 5. The LP sub-problem is defined as: $\overline{f} = 3d_1 + 2.6d_2$ subject to:

$-0.016667d_1 - 0.016667d_2 \le -0.95$

$-d_1 \le 2$; $-d_2 \le 1$;

$-1 \le d_1 \le 1$; $-0.5 \le d_2 \le 0.5$;

Step 6. Solving the LP sub-problem, we get: $d_1 = 1; d_2 = 0.5$

Step 7. Convergence criteria are not satisfied: $\|\mathbf{d}\| > \varepsilon_2$

Step 8. $x_1^{(1)} = P_1^{(1)} = 2 + 1 = 3$; $x_2^{(1)} = P_2^{(1)} = 1 + 0.5 = 1.5$

Set $k = 1$, go to Step 2.

Section 12.5 Search Direction Calculation: The QP Subproblem

Solve the following QP problems using KKT optimality conditions.

Note: QP problems in this section are solved using the linear programming method for quadratic programming problems presented in Section 9.5.

12.29

$f(\mathbf{x}) = (x_1 - 3)^2 + (x_2 - 3)^2 = x_1^2 - 6x_1 + x_2^2 - 6x_2 + 18$ (ignore the constant 18) subject to $x_1 + x_2 \le 5$, $x_1 \ge 0$, $x_2 \ge 0$, identifying: $\mathbf{c} = \begin{bmatrix} -6 \\ -6 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = [5]$;

Introducing slack variable s, Lagrange multipliers u, ζ_1, ζ_2 we have the following system of equations:

$$\begin{bmatrix} 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}; \text{ with } us = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Introduce the artificial variables Y_1, Y_2, Y_3, and the artificial cost as

$w = 17 - 3x_1 - 3x_2 - 2u + \zeta_1 + \zeta_2 - s$. The solution is given in Table E12.29.

The optimum solution is: $x_1^* = 5/2$, $x_2^* = 5/2$, $u^* = 1$, $f^* = 1/2$.

12.30

$f(\mathbf{x}) = (x_1 - 1)^2 + (x_2 - 1)^2 = x_1^2 - 2x_1 + x_2^2 - 2x_2 + 2$ (ignore the constant 2) subject to $x_1 + 2x_2 \le 6$, $x_1 \ge 0$, $x_2 \ge 0$, the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = [6]$;

Introducing slack variable s and Lagrange multipliers u, ζ_1, ζ_2 we obtain the following system of equations:

$$\begin{bmatrix} 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}; \text{ with } us = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Introduce the artificial variables Y_1, Y_2, Y_3, and the artificial cost as:

$w = 10 - 3x_1 - 4x_2 - 3u + \zeta_1 + \zeta_2 - s$. The solution is given in Table E12.30.

The optimum solution is: $x_1^* = 1$, $x_2^* = 1$, $u^* = 0$, $f^* = 0$.

12.31

$f(\mathbf{x}) = x_1^2 - 2x_1 + x_2^2 - 2x_2 + 2$, subject to $x_1 + 2x_2 \le 2$, $x_1, x_2 \ge 0$; Ignoring the constant 2, the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = [2]$;

Introducing slack variable s and Lagrange multipliers u, ζ_1, ζ_2 we obtain the following system of equations:

$$\begin{bmatrix} 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix};\ \text{with } us = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Introduce the artificial variables Y_1, Y_2, Y_3, and the artificial cost as:

$w = 6 - 3x_1 - 4x_2 - 3u + \zeta_1 + \zeta_2 - s$. The solution is given in Table E12.31.

The optimum solution is: $x_1^* = 4/5$, $x_2^* = 3/5$, $u^* = 2/5$, $f^* = 1/5$.

12.32

$f(\mathbf{x}) = x_1^2 + x_2^2 - x_1x_2 - 3x_1$, subject to $x_1 + x_2 \le 3$, $x_1 \ge 0$, $x_2 \ge 0$, the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = [3]$;

Introducing slack variable s and Lagrange multipliers u, ζ_1, ζ_2 we obtain the following system of equations:

$$\begin{bmatrix} 2 & -1 & 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix};\ \text{with } us = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Introduce the artificial variables Y_1, Y_2, Y_3, and the artificial cost as:

$w = 6 - 2x_1 - 2x_2 - 2u + \zeta_1 + \zeta_2 - s$. The solution is given in Table E12.32.

The optimum solution is: $x_1^* = 2$, $x_2^* = 1$, $u^* = 0$, $f^* = -3$.

12.33

$f(\mathbf{x}) = x_1^2 - 2x_1 + x_2^2 - 4x_2 + 4$, subject to $x_1 + x_2 \le 4$, $x_1 \ge 0$, $x_2 \ge 0$; the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = [4]$;

Introducing the slack variable s and Lagrange multipliers u, ζ_1, ζ_2 we obtain the system of equations as:

$$\begin{bmatrix} 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}; \text{ with } us = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Introduce the artificial variables Y_1, Y_2, Y_3, and the artificial cost as:

$w = 10 - 3x_1 - 3x_2 - 2u + \zeta_1 + \zeta_2 - s$. The solution is given in Table E12.33.

The optimum solution is: $x_1^* = 1$, $x_2^* = 2$, $u^* = 0$, $f^* = -1$.

12.34

$f(\mathbf{x}) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$, subject to $x_1 + x_2 = 4$, $x_1, x_2 \ge 0$, the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 8 & -5 \\ -5 & 6 \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{e} = [4]$;

Introducing the Lagrange multipliers ζ_1, ζ_2, y_1, z_1 (all are positive quantities), we obtain the system of equations as:

$$\begin{bmatrix} 8 & -5 & -1 & 0 & 1 & -1 \\ -5 & 6 & 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \zeta_1 \\ \zeta_2 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix};$$

$(y_1 - z_1)$ is the Lagrange multiplier of the equality constraint; $\zeta_1 x_1 = 0$, $\zeta_2 x_2 = 0$. Introduce artificial variables Y_1, Y_2, Y_3, and artificial cost as:

$w = 12 - 4x_1 - 2x_2 + \zeta_1 + \zeta_2 - 2y_1 + 2z_1$. The solution is given in Table E12.34.

The optimum solution is $x_1^* = 13/6$, $x_2^* = 11/6$, $v^* = y_1 - z_1 = -1/6$, $f^* = -25/3$.

12.35

$f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2$, subject to $x_1 + x_2 - 4 = 0$, $x_1 - x_2 - 2 = 0$, $x_1, x_2 \geq 0$; the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathbf{e} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$;

Introducing Lagrange multipliers $\zeta_1, \zeta_2, y_1, y_2, z_1, z_2$ (all are positive quantities), we obtain the system of equations as:

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \zeta_1 \\ \zeta_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

; and, $(y_1 - z_1)$ and $(y_2 - z_2)$ are Lagrange multipliers for the equality constraints respectively; $\zeta_1 x_1 = 0$, $\zeta_2 x_2 = 0$. Introduce artificial variables Y_1, Y_2, Y_3, Y_4 and artificial cost as $w = 10 - 4x_1 - 2x_2 + \zeta_1 + \zeta_2 - 2y_1 + 2z_1$. The solution is given in Table E12.35.

The optimum solution is: $x_1^* = 3$, $x_2^* = 1$, $v_1^* = (y_1 - z_1) = -2$, $v_2^* = (y_2 - z_2) = -2$, $f^* = 2$.

12.36

$f(\mathbf{x}) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$; subject to $x_1 + x_2 \leq 4$, $x_1 \geq 0$, $x_2 \geq 0$; the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -8 \\ 0 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 8 & -5 \\ -5 & 6 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = [4]$;

Introducing the slack variable s and Lagrange multipliers u, ζ_1, ζ_2 we obtain the following system of equations:

$$\begin{bmatrix} 8 & -5 & 1 & -1 & 0 & 0 \\ -5 & 6 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$$

; with $us = 0$, $\zeta_1 x_1 = 0$, $\zeta_2 x_2 = 0$.

Introduce the artificial variables Y_1, Y_2, Y_3, and the artificial cost as

$w = 12 - 4x_1 - 2x_2 - 2u + \zeta_1 + \zeta_2 - s$. The solution is given in Table E6.36.

The optimum solution is: $x_1^* = 48/23$, $x_2^* = 40/23$, $u^* = 0$, $f^* = -192/23$.

12.37

$f(\mathbf{x}) = x_1^2 + x_2^2 - 4x_1 - 2x_2$ subject to $-x_1 - x_2 \le -4$, $x_1 \ge 0$, $x_2 \ge 0$; the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\mathbf{b} = [-4]$;

Introducing the slack variable s and Lagrange multipliers u, ζ_1, ζ_2 we obtain the following system of equations:

$$\begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \\ \zeta_1 \\ \zeta_2 \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}; \text{ with } us = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Multiply the third row by – 1 and introduce artificial variables Y_1, Y_2, Y_3, and artificial cost as $w = 10 - 3x_1 - 3x_2 + 2u + \zeta_1 + \zeta_2 + s$. The solution is given in Table E12.37.

The optimum solution is: $x_1^* = 5/2$, $x_2^* = 3/2$, $u^* = 1$, $f^* = -9/2$.

12.38

$f(\mathbf{x}) = 2x_1^2 - 6x_1x_2 + 9x_2^2 - 18x_1 + 9x_2$ subject to $x_1 - 2x_2 \le 10$, $4x_1 - 3x_2 \le 20$, $x_1, x_2 \ge 0$; the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -18 \\ 9 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 4 & -6 \\ -6 & 18 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

Introduce slack variables s_1, s_2 and Lagrange multipliers u_1, u_2, ζ_1, ζ_2, we obtain the following system of equations:

$$\begin{bmatrix} 4 & -6 & 1 & 4 & -1 & 0 & 0 & 0 \\ -6 & 18 & -2 & -3 & 0 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ \zeta_1 \\ \zeta_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \\ 10 \\ 20 \end{bmatrix}; \text{ with } u_1 s_1 = 0,\ u_2 s_2 = 0,\ \zeta_1 x_1 = 0,\ \zeta_2 x_2 = 0.$$

Multiply the second row by –1, and introduce artificial variables Y_1, Y_2, Y_3, Y_4 and artificial cost as $w = 57 - 15x_1 + 29x_2 - 3u_1 - 7u_2 + \zeta_1 - \zeta_2 - s_1 - s_2$. The solution is given in Table E10 .38.

The optimum solution is: $x_1^* = 63/10$, $x_2^* = 26/15$, $u_1^* = 0$, $u_2^* = 4/5$, $f^* = -2845/50$.

12.39

$f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2$ subject to $x_1 + x_2 - 4 \le 0$, $2 - x_1 \le 0$, $x_1 \le 0$, $x_2 \ge 0$; the following quantities are identified: $\mathbf{c} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$, $\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$; Introducing slack variables s_1, s_2 and Lagrange multipliers u_1, u_2, ζ_1, ζ_2, we obtain the following system of equations:

$$\begin{bmatrix} 2 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ \zeta_1 \\ \zeta_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ -2 \end{bmatrix}$$

; with $u_1 s_1 = 0$, $u_2 s_2 = 0$, $\zeta_1 x_1 = 0$, $\zeta_2 x_2 = 0$. Multiply the fourth row by –1, and introduce artificial variables Y_1, Y_2, Y_3, Y_4 and artificial cost as $w = 10 - 4x_1 - 3x_2 - 2u_1 + u_2 + \zeta_1 + \zeta_2 - s_1 + s_2$. The solution is given in Table E12.39. The optimum solution is: $x_1^* = 2$, $x_2^* = 1$, $u_1^* = 0$, $u_2^* = 2$, $f^* = -1$.

12.40

$f(\mathbf{x}) = 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 - x_1x_3 - 0.8x_2x_3$; subject to $-1.3x_1 - 1.2x_2 - 1.1x_3 \le -1.15$; $x_1 + x_2 + x_3 = 1$, $x_1 \le 0.7$, $x_2 \le 0.7$, $x_3 \le 0.7$, $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$; the following quantities are identified:

$$\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 4 & -0.8 \\ -1 & -0.8 & 2 \end{bmatrix}, \ \mathbf{N} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} -1.3 & 1 & 0 & 0 \\ -1.2 & 0 & 1 & 0 \\ -1.1 & 0 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -1.15 \\ 0.7 \\ 0.7 \\ 0.7 \end{bmatrix}, \ \mathbf{e} = [1]$$

Introducing slack variables s_1, s_2, s_3, s_4, and Lagrange multipliers u_1, u_2, u_3, u_4, and $v_1 = y_1 - z_1$, ζ_1, ζ_2, ζ_3, we obtain the following system of equations.

$$
\begin{bmatrix}
4 & 2 & -1 & -1.3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
2 & 4 & -0.8 & -1.2 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
-1 & -0.8 & 2 & -1.1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\
-1.3 & -1.2 & -1.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ y_1 \\ z_1
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ -1.2 \\ 0.7 \\ 0.7 \\ 0.7 \\ 1
\end{bmatrix}
$$

with $u_i s_i = 0$, $i = 1$ to 4; $\zeta_i x_i = 0$, $i = 1$ to 3.

The optimum solution is: $x_1^* = 0.241507$, $x_2^* = 0.184076$, $x_3^* = 0.574317$, $u_1^* = 0$, $u_2^* = 0$, $u_3^* =$ 0,

$u_4^* = 0$, $v_1^* = y_1 - z_1 = -0.7599$, $f^* = 0.3799$.

Table E12.29

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	**2**	0	1	–1	0	0	1	0	0	6	
Y_2	0	2	1	0	–1	0	0	1	0	6	Initial
Y_3	1	1	0	0	0	1	0	0	1	5	
	–**3**	–3	–2	1	1	–1	0	0	0	w –17	
x_1	1	0	1/2	–1/2	0	0	1/2	0	0	3	
Y_2	0	2	1	0	–1	0	0	1	0	6	lst
Y_3	0	**1**	–1/2	1/2	0	1	–1/2	0	1	2	iteration
	0	–**3**	–1/2	–1/2	1	–1	3/2	0	0	w – 8	
x_1	1	0	1/2	–1/2	0	0	1/2	0	0	3	
Y_2	0	0	**2**	–1	–1	–2	1	1	–2	2	2nd
x_2	0	1	–1/2	1/2	0	1	–1/2	0	1	2	iteration
	0	0	–**2**	1	1	2	0	0	3	w – 2	
x_1	1	0	0	–1/4	1/4	1/2	1/4	–1/4	1/2	5/2	
u	0	0	1	–1/2	–1/2	–1	1/2	1/2	–1	1	3rd
x_2	0	1	0	1/4	–1/4	1/2	–1/4	1/4	1/2	5/2	iteration
	0	0	0	0	0	0	1	1	1	w – 0	

Table E12.30

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	2	0	1	–1	0	0	1	0	0	2	
Y_2	0	**2**	2	0	–1	0	0	1	0	2	Initial
Y_3	1	2	0	0	0	1	0	0	1	6	
	–3	–**4**	–3	1	1	–1	0	0	0	w –10	
Y_1	**2**	0	1	–1	0	0	1	0	0	2	
x_2	0	1	1	0	–1/2	0	0	1/2	0	1	lst
Y_3	1	0	–2	0	1	1	0	–1	1	4	iteration
	–**3**	0	1	1	–1	–1	0	2	0	w – 6	
x_1	1	0	1/2	–1/2	0	0	1/2	0	0	1	
x_2	0	1	1	0	–1/2	0	0	1/2	0	1	2nd
Y_3	0	0	–5/2	1/2	1	**1**	–1/2	–1	1	3	iteration
	0	0	5/2	–1/2	–1	–**1**	3/2	2	0	w – 3	
x_1	1	0	1/2	–1/2	0	0	1/2	0	0	1	
x_2	0	1	1	0	–1/2	0	0	1/2	0	1	3rd
s	0	0	–5/2	1/2	1	1	–1/2	–1	1	3	iteration
	0	0	0	0	0	0	1	1	1	w – 0	

Table E12.31

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	2	0	1	–1	0	0	1	0	0	2	
Y_2	0	**2**	2	0	–1	0	0	1	0	2	Initial
Y_3	1	2	0	0	0	1	0	0	1	2	
	–3	– **4**	–3	1	1	–1	0	0	0	w – 6	
Y_1	2	0	1	–1	0	0	1	0	0	2	
x_2	0	1	1	0	–1/2	0	0	1/2	0	1	lst
Y_3	**1**	0	–2	0	1	1	0	–1	1	0	iteration
	–**3**	0	1	1	–1	–1	0	2	0	w –2	
Y_1	0	0	**5**	–1	–2	–2	1	2	–2	2	
x_2	0	1	1	0	–1/2	0	0	1/2	0	1	2nd
x_1	1	0	–2	0	1	1	0	–1	1	0	iteration
	0	0	–**5**	1	2	2	0	–1	3	w – 2	
u	0	0	1	–1/5	–2/5	–2/5	1/5	2/5	–2/5	2/5	
x_2	0	1	0	1/5	–1/10	2/5	–1/5	1/10	2/5	3/5	3rd
x_1	1	0	0	–2/5	1/5	1/5	2/5	–1/5	1/5	4/5	iteration
	0	0	0	0	0	0	1	1	1	w – 0	

Table E12.32

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	2	–1	1	–1	0	0	1	0	0	3	
Y_2	–1	**2**	1	0	–1	0	0	1	0	0	Initial
Y_3	1	1	0	0	0	1	0	0	1	3	
	–2	–**2**	–2	1	1	–1	0	0	0	$w - 6$	
Y_1	**3/2**	0	3/2	–1	–1/2	0	1	1/2	0	3	
x_2	–1/2	1	1/2	0	–1/2	0	0	1/2	0	0	lst
Y_3	3/2	0	–1/2	0	1/2	1	0	–1/2	1	3	iteration
	–**3**	0	–1	1	0	–1	0	1	0	$w - 6$	
x_1	1	0	1	–2/3	–1/3	0	2/3	1/3	0	2	
x_2	0	1	1	–1/3	–2/3	0	1/3	2/3	0	1	2nd
Y_3	0	0	–2	1	1	**1**	–1	–1	1	0	iteration
	0	0	2	–1	–1	–**1**	2	2	0	$w - 0$	
x_1	1	0	1	–2/3	–1/3	0	2/3	1/3	0	2	
x_2	0	1	1	–1/3	–2/3	0	1/3	2/3	0	1	3rd
s	0	0	–2	1	1	1	–1	–1	1	0	iteration
	0	0	0	0	0	0	1	1	1	$w - 0$	

Table E12.33

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	**2**	0	1	–1	0	0	1	0	0	2	
Y_2	0	2	1	0	–1	0	0	1	0	4	Initial
Y_3	1	1	0	0	0	1	0	0	1	4	
	–**3**	–3	–2	1	1	–1	0	0	0	w –10	
X_1	1	0	1/2	–1/2	0	0	1/2	0	0	1	
Y_2	0	**2**	1	0	–1	0	0	1	0	4	lst
Y_3	0	1	–1/2	1/2	0	1	–1/2	0	1	3	iteration
	0	–**3**	–1/2	–1/2	1	–1	3/2	0	0	w –7	
x_1	1	0	1/2	–1/2	0	0	1/2	0	0	1	
x_2	0	1	1/2	0	–1/2	0	0	1/2	0	2	2nd
Y_3	0	0	–1	1/2	1/2	**1**	–1/2	–1/2	1	1	iteration
	0	0	1	–1/2	–1/2	–**1**	3/2	3/2	0	w –1	
x_1	1	0	1/2	–1/2	0	0	1/2	0	0	1	
x_2	0	1	1/2	0	–1/2	0	0	1/2	0	2	3rd
s	0	0	–1	1/2	1/2	1	–1/2	–1/2	1	1	iteration
	0	0	0	0	0	0	1	1	1	w – 0	

Table E12.34

	x_1	x_2	ζ_1	ζ_2	y_1	z_1	Y_1	Y_2	Y_3	**D**	
Y_1	**8**	–5	–1	0	1	–1	1	0	0	8	
Y_2	–5	6	0	–1	1	–1	0	1	0	0	Initial
Y_3	1	1	0	0	0	0	0	0	1	4	
	-4	–2	1	1	–2	2	0	0	0	w –12	
x_1	1	–5/8	–1/8	0	1/8	–1/8	1/8	0	0	1	
Y_2	0	**23/8**	–5/8	–1	13/8	–13/8	5/8	1	0	5	lst
Y_3	0	13/8	1/8	0	–1/8	1/8	–1/8	0	1	3	iteration
	0	–**9/2**	1/2	1	–3/2	3/2	1/2	0	0	w – 8	
x_1	1	0	– 6/23	– 5/23	11/23	– 11/23	6/23	5/23	0	48/23	
x_2	0	1	–5/23	–8/23	13/23	–13/23	5/23	8/23	0	40/23	2nd
Y_3	0	0	11/23	13/23	–24/23	**24/23**	–11/23	–13/23	1	4/23	iteration
	0	0	–11/23	–13/23	24/23	–**24/23**	34/23	36/23	0	w – 4/23	
x_1	1	0	–1/24	1/24	0	0	1/24	–1/24	11/24	13/6	
x_2	0	1	1/24	–1/24	0	0	–1/24	1/24	13/24	11/6	3rd
z_1	0	0	11/24	13/24	–1	1	–11/24	–13/24	23/24	1/6	iteration
	0	0	0	0	0	0	1	1	1	w – 0	

Table E12.35

	x_1	x_2	ζ_1	ζ_2	y_1	y_2	z_1	z_2	Y_1	Y_2	Y_3	Y_4	**D**
Y_1	**2**	0	–1	0	1	1	–1	–1	1	0	0	0	2
Y_2	0	2	0	–1	1	–1	–1	1	0	1	0	0	2 Init.
Y_3	1	1	0	0	0	0	0	0	0	0	1	0	4
Y_4	1	–1	0	0	0	0	0	0	0	0	0	1	2
	– **4**	– 2	1	1	– 2	0	2	0	0	0	0	0	w – 10
x_1	1	0	–1/2	0	1/2	1/2	–1/2	–1/2	1/2	0	0	0	1
Y_2	0	**2**	0	–1	1	–1	–1	1	0	1	0	0	2 1st
Y_3	0	1	1/2	0	–1/2	–1/2	1/2	1/2	–1/2	0	1	0	3 Itr.
Y_4	0	–1	1/2	0	–1/2	–1/2	1/2	1/2	–1/2	0	0	1	1
	0	–**2**	–1	1	0	2	0	–2	2	0	0	0	w – 6
x_1	1	0	–1/2	0	1/2	1/2	–1/2	–1/2	1/2	0	0	0	1
x_2	0	1	0	–1/2	1/2	–1/2	–1/2	1/2	0	1/2	0	0	1 2nd
Y_3	0	0	1/2	1/2	–1	0	**1**	0	–1/2	–1/2	1	0	2 Itr.
Y_4	0	0	1/2	–1/2	0	–1	0	1	–1/2	1/2	0	1	2
	0	0	–1	0	1	1	–**1**	–1	2	1	0	0	w – 4
x_1	1	0	–1/4	1/4	0	1/2	0	–1/2	1/4	–1/4	1/2	0	2
x_2	0	1	1/4	–1/4	0	–1/2	0	1/2	–1/4	1/4	1/2	0	2 3rd
z_1	0	0	1/2	1/2	–1	0	1	0	–1/2	–1/2	1	0	2 Itr.
Y_4	0	0	1/2	–1/2	0	–1	0	**1**	–1/2	1/2	0	1	2
	0	0	–1/2	1/2	0	1	0	–**1**	3/2	1/2	1	0	w – 2
x_1	1	0	0	0	0	0	0	0	0	0	1/2	1/2	3
x_2	0	1	0	0	0	0	0	0	0	0	1/2	–1/2	1 4th
z_1	0	0	1/2	1/2	–1	0	1	0	–1/2	–1/2	1	0	2 Itr.
z_2	0	0	1/2	–1/2	0	–1	0	1	–1/2	1/2	0	1	2
	0	0	0	0	0	0	0	0	1	1	1	1	w – 0

Table E12.36

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	**8**	–5	1	–1	0	0	1	0	0	8	
Y_2	–5	6	1	0	–1	0	0	1	0	0	Initial
Y_3	1	1	0	0	0	1	0	0	1	4	
	– **4**	–2	–2	1	1	–1	0	0	0	w – 12	
x_1	1	–5/8	1/8	–1/8	0	0	1/8	0	0	1	
Y_2	0	**23/8**	13/8	–5/8	–1	0	5/8	1	0	5	lst
Y_3	0	13/8	–1/8	1/8	0	1	–1/8	0	1	3	iteration
	0	– **9/2**	–3/2	1/2	1	–1	1/2	0	0	w – 8	
x_1	1	0	11/23	–6/23	–5/23	0	6/23	5/23	0	48/23	
x_2	0	1	13/23	–5/23	–8/23	0	5/23	8/23	0	40/23	2nd
Y_3	0	0	–24/23	11/23	13/23	**1**	–11/23	–13/23	1	4/23	iteration
	0	0	24/23	–11/23	–13/23	–**1**	34/23	36/23	0	w – 4/23	
x_1	1	0	11/23	–6/23	–5/23	0	6/23	5/23	0	48/23	
x_2	0	1	13/23	–5/23	–8/23	0	5/23	8/23	0	40/23	3rd
s	0	0	–24/23	11/23	13/23	1	–11/23	–13/23	1	4/23	iteration
	0	0	0	0	0	0	1	1	1	0	

Table E12.37

	x_1	x_2	u	ζ_1	ζ_2	s	Y_1	Y_2	Y_3	**D**	
Y_1	**2**	0	–1	–1	0	0	1	0	0	4	
Y_2	0	2	–1	0	–1	0	0	1	0	2	Initial
Y_3	1	1	0	0	0	–1	0	0	1	4	
	–**3**	–3	2	1	1	+1	0	0	0	w – 10	
x_1	1	0	–1/2	–1/2	0	0	1/2	0	0	2	
Y_2	0	**2**	–1	0	–1	0	0	1	0	2	lst
Y_3	0	1	1/2	1/2	0	–1	–1/2	0	1	2	iteration
	0	–**3**	1/2	–1/2	1	1	3/2	0	0	w – 4	
x_1	1	0	–1/2	–1/2	0	0	1/2	0	0	2	
x_2	0	1	–1/2	0	–1/2	0	0	1/2	0	1	2nd
Y_3	0	0	**1**	1/2	1/2	–1	–1/2	–1/2	1	1	iteration
	0	0	–**1**	–1/2	–1/2	1	3/2	3/2	0	w – 1	
x_1	1	0	0	–1/4	1/4	–1/2	1/4	–1/4	1/2	5/2	
x_2	0	1	0	1/4	–1/4	–1/2	–1/4	1/4	1/2	3/2	3rd
u	0	0	1	1/2	1/2	–1	–1/2	–1/2	1	1	iteration
	0	0	0	0	0	0	1	1	1	w – 0	

Table E12.38

	x_1	x_2	u_1	u_2	ζ_1	ζ_2	s_1	s_2	Y_1	Y_2	Y_3	Y_4	**D**
Y_1	4	–6	1	4	–1	0	0	0	1	0	0	0	18
Y_2	**6**	–18	2	3	0	1	0	0	0	1	0	0	9 Init.
Y_3	1	–2	0	0	0	0	1	0	0	0	1	0	10
Y_4	4	–3	0	0	0	0	0	1	0	0	0	1	20
	-15	29	–3	–7	1	–1	–1	–1	0	0	0	0	w – 57
Y_1	0	6	–1/3	2	–1	–2/3	0	0	1	–2/3	0	0	12 I
x_1	1	–3	1/3	1/2	0	1/6	0	0	0	1/6	0	0	3/2 Itr
Y_3	0	1	–1/3	–1/2	0	–1/6	1	0	0	–1/6	1	0	17/2
Y_4	0	**9**	–4/3	–2	0	–2/3	0	1	0	–2/3	0	1	14
	0	–**16**	2	1/2	1	3/2	–1	–1	0	5/2	0	0	w – 69/2
Y_1	0	0	5/9	**10/3**	–1	–2/9	0	–2/3	1	–2/9	0	–2/3	8/3 II
x_1	1	0	–1/9	–1/6	0	–1/18	0	1/3	0	–1/18	0	1/3	37/6Itr
Y_3	0	0	–5/27	–5/18	0	–5/54	1	–1/9	0	–5/54	1	–1/9	125/18
x_2	0	1	– 4/27	–2/9	0	–2/27	0	1/9	0	–2/27	0	1/9	14/9
	0	0	–10/27	**-55/18**	1	17/54	–1	7/9	0	71/54	0	16/9	w –173/18
u_2	0	0	1/6	1	–3/10	–1/15	0	–1/5	3/10	–1/15	0	–1/5	4/5 III
x_1	1	0	–1/12	0	–1/20	–1/15	0	3/10	1/20	–1/15	0	3/10	63/10Itr
Y_3	0	0	–5/36	0	–1/12	–1/9	**1**	–1/6	1/12	–1/9	1	–1/6	43/6
x_2	0	1	–1/9	0	–1/15	– 4/45	0	1/15	1/15	– 4/45	0	1/15	26/15
	0	0	5/36	0	1/12	1/9	–**1**	1/6	11/12	10/9	0	7/6	w – 43/6
u_2	0	0	1/6	1	–3/10	–1/15	0	–1/5	3/10	–1/15	0	–1/5	4/5 IV
x_1	1	0	–1/12	0	–1/20	–1/15	0	3/10	1/20	–1/15	0	3/10	63/10Itr
s_1	0	0	–5/36	0	–1/12	–1/9	1	–1/6	1/12	–1/9	1	–1/6	43/6
x_2	0	1	–1/9	0	–1/15	– 4/45	0	1/15	1/15	– 4/45	0	1/15	26/15

	0	0	0	0	0	0	0	0	1	1	1	1	w-0

Table E12.39

	x_1	x_2	u_1	u_2	ζ_1	ζ_2	s_1	s_2	Y_1	Y_2	Y_3	Y_4	**D**
Y_1	**2**	0	1	–1	–1	0	0	0	1	0	0	0	2
Y_2	0	2	1	0	0	–1	0	0	0	1	0	0	2 Init
Y_3	1	1	0	0	0	0	1	0	0	0	1	0	4
Y_4	1	0	0	0	0	0	0	–1	0	0	0	1	2
	-4	–3	–2	1	1	1	–1	1	0	0	0	0	w –10
x_1	1	0	1/2	–1/2	–1/2	0	0	0	1/2	0	0	0	1
Y_2	0	**2**	1	0	0	–1	0	0	0	1	0	0	2 I
Y_3	0	1	–1/2	1/2	1/2	0	1	0	–1/2	0	1	0	3 Itr
Y_4	0	0	–1/2	1/2	1/2	0	0	–1	–1/2	0	0	1	1
	0	–**3**	0	–1	–1	1	–1	1	2	0	0	0	w – 6
x_1	1	0	1/2	–1/2	–1/2	0	0	0	1/2	0	0	0	1
x_2	0	1	1/2	0	0	–1/2	0	0	0	1/2	0	0	1 II
Y_3	0	0	–1	1/2	1/2	1/2	1	0	–1/2	–1/2	1	0	2 Itr
Y_4	0	0	–1/2	**1/2**	1/2	0	0	–1	–1/2	0	0	1	1
	0	0	3/2	–**1**	–1	–1/2	–1	1	2	3/2	0	0	w – 3
x_1	1	0	0	0	0	0	0	–1	0	0	0	1	2
x_2	0	1	1/2	0	0	–1/2	0	0	0	1/2	0	0	1 III
Y_3	0	0	–1/2	0	0	1/2	**1**	1	0	–1/2	1	–1	1 Itr
u_2	0	0	–1	1	1	0	0	–2	–1	0	0	2	2
	0	0	1/2	0	0	–1/2	–**1**	–1	1	3/2	0	2	w – 1
x_1	1	0	0	0	0	0	0	-1	0	0	0	1	2
x_2	0	1	1/2	0	0	–1/2	0	0	0	1/2	0	0	1 IV
s_1	0	0	–1/2	0	0	1/2	1	1	0	–1/2	1	–1	1 Itr
u_2	0	0	–1	1	1	0	0	–2	–1	0	0	2	2
	0	0	0	0	0	0	0	0	1	1	1	1	w – 0

12.41

Referring to Exercises 12.3 and 12.16, the QP subproblem at the point (250, 300) is defined as:

$\bar{f} = 300d_1 + 250d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $-0.004267d_1 - 0.007111d_2 \le -0.066667$

$-0.006d_1 - 0.005d_2 \le -0.5$;

$-0.0024d_1 + 0.002d_2 \le 0.4$

$-0.1d_1 \le 24$; $0.001d_1 \le 0.75$

$-0.1d_2 \le 29$; $0.001d_2 \le 0.70$.

Solution of the QP is given as **d** = (49.18033, 40.9836), with **u** = (0,58196.7, 0, 0, 0, 0, 0)

12.42

Referring to Exercises 12.4 and 12.17, the QP subproblem at the point (12, 4) is defined as:

$\bar{f} = 98.646d_1 + 295.938d_2 + 0.5(d_1^2 + d_2^2)$

subject to $0.0005526d_1 - 0.001658d_2 \le 0.99337$;

$-22.50374d_1 - 22.50374d_2 \le 89.01498$

$-0.2d_1 \le 1.4$; $0.01d_1 \le 0.88$

$-2d_2 \le 7$; $0.2d_2 \le 0.2$.

Solution of the QP is given as **d** = (– 0.45556, – 3.5), with **u** = (0, 4.4, 0, 0, 97.1, 0)

12.43

Referring to Exercises 12.5 and 12.18, the QP subproblem at the point (150, 150) is defined as:

$\bar{f} = 50d_1 + 40d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $-0.005555d_1 \le 0.16667$

$-0.004444d_2 \le 0.33333$

$-d_1 \le 150$; $-d_2 \le 150$.

Solution of the QP is given as **d** = (– 30.00306, – 40), with **u** = (3599.8, 0, 0, 0)

12.44

For the following problem, obtain the quadratic programming subproblem, plot it on a graph, and obtain the search direction for the problem.

Exercise 2.1 at the point $h = 12$m, $A = 4000$m^2.

Exercise 2.1

A 100 × 100m lot is available to construct a multistory office building. At least 20,000m^2 total floor space is needed. According to a zoning ordinance, the maximum height of the building can be only 21m, and the area for parking outside the building must be at least 25 percent of the total floor area. It has been decided to fix the height of each story at 3.5m. The cost of the building in millions of dollars is estimated at $0.6h + 0.001A$, where A is the cross-sectional area of the building per floor and h is the height of the building. Formulate the minimum cost design problem.

The problem is formulated in Exercise 2.1 as follows:

Step 2: Data and Information Collection

Area of the lot =100×100 = 10,000 m^2

Area available for parking = (10,000 – A), m^2

Total floor area = (number of floors)×A = $\frac{h}{3.5}A$, m^2

Step 3: Definition of Design Variables

A = cross-sectional area of the building for each floor, m^2

h = height of the building, m

Step 4: Optimization Criterion

Optimization criterion is to minimize $ cost, and the cost function is defined as

Cost = (0.6h + 0.001A), million dollars (1)

Step 5: Formulation of Constraints

Floor Space Constraint:

$hA/3.5 \geq 20{,}000,\ \text{m}^2$ (2)

Parking Constraint:

$(10{,}000 - A) \geq 0.25hA/3.5,\ \text{m}^2$ (3)

Explicit Design Variable Constraints:

$h \geq 3.5$, m (4)

$h \leq 21$, m (5)

$A \geq 0,\ \text{m}^2$ (6)

$A \leq 10000,\ \text{m}^2$ (7)

Substituting the given values the problem is formulated in the standard and normalized form as:

$$f(h, A) = (0.6h + 0.001A)$$

$$g_1 = 1 - 1.42857\times10^{-5}hA \leq 0$$

$$g_2 = 0.0001A + 7.14286\times10^{-6}hA - 1 \leq 0$$

$g_3 = 1 - h/3.5 \le 0$

$g_4 = h/21 - 1 \le 0$

$g_5 = -\dfrac{A}{10000} \le 0$

$g_6 = \frac{A}{10000} - 1 \le 0$

Linearizing f, g_1 and g_2 about h = 12, A = 4000, we get:

$f(12, 4000) = (0.6 \times 12) + (0.001 \times 4000) = 11.2;\ \nabla f(12, 4000) = [0.6, 0.001]$

$g_1(12, 4000) = 1 - (1.42857 \times 10^{-5}) \times (12 \times 4000) = 0.3143$

$\nabla g_1 = [1 - 1.42857 \times 10^{-5} A, 1 - 1.42857 \times 10^{-5} h] = [-0.057143, -1.7143 \times 10^{-4}]$

$g_2(12, 4000) = 0.0001 \times 4000 + 7.14286 \times 10^{-6} \times (12 \times 4000) - 1 = -0.25714$

$\nabla g_2 = [7.14286 \times 10^{-6} A, 0.0001 + 7.14286 \times 10^{-6} h] = [0.02857, 1.857 \times 10^{-4}]$

Referring to Exercises 12.6 and 12.19, the QP subproblem at the point (12, 4000) is defined as:

$\overline{f} = 0.6d_1 + 0.001d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $\bar{g}_1 = - 0.057143d_1 - 0.00017143d_2 + 0.3143 \le 0$

$\bar{g}_2 = - 0.02857d_1 + 0.0001857d_2 - 0.25714 \le 0$

$\bar{g}_3 = - -0.285714d_1 - 2.42857 \le 0$

$\bar{g}_4 = - 0.047619d_1 - 0.42857 \le 0$

$\bar{g}_5 = - \frac{d_2}{10000} - 0.4 \le 0.$

$\bar{g}_6 = \frac{d_2}{10000} - 0.6 \le 0.$

Solution of the QP is given as $\mathbf{d}$ = (5.49918, 0.0173), with $\mathbf{u}$ = (106.7, 0, 0, 0, 0)

12.45

Referring to Exercises 12.7 and 12.20, the QP subproblem at the point (6, 15) is defined as:

$\bar{f} = -565.487d_1 - 113.097d_2 + 0.5(d_1^2 + d_2^2)$

subject to $0.10472d_1 + 0.04189d_2 \le 0.3717$;

$-0.2d_1 \le 0.2$; $0.05d_1 \le 0.7$

$-d_2 \le 15$; $0.05d_2 \le 0.25$.

Solution of the QP is given as **d** = (9.54975, – 15), with **u** = (5308.8, 0, 0, 94.3, 0)

12.46

Referring to Exercises 12.8 and 12.21, the QP subproblem at the point (100, 2) is defined as:

$\bar{f} = -12.5664d_1 - 628.32d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $0.0062832d_1 + 0.62832d_2 \le 0.3717$

$-2d_2 \le 3$; $-d_1 \le 100$.

Solution of the QP is given as **d** = (6, 0.53158), with **u** = (999.2, 0, 0)

12.47

Referring to Exercises 12.9 and 12.22, the QP subproblem at the point (100, 100) is defined as:

$\bar{f} = 200d_1 + 100d_2 + 0.5(d_1^2 + d_2^2)$

subject to $0.01d_1 \le 0$; $0.005d_2 \le 0.5$

$-0.01d_1 - 0.01d_2 \le 0$;

$-0.005d_1 + 0.005d_2 \le 0.5$

$0.005d_1 - 0.005d_2 \le 0.5$

$-d_1 \le 100$; $-d_2 \le 100$.

Solution of the QP is given as **d** = (– 50, 50), with **u** = (0, 0, 15000, 0, 0, 0, 0)

12.48

Referring to Exercises 12.10 and 12.23, the QP subproblem at the point (6, 16) is defined as:

$\bar{f} = 138.2301d_1 + 37.699d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $1.0053d_1 + 0.1885d_2 = -2.01593$

$0.22222d_1 - 0.083333d_2 \le 0.33333$

$-0.14815d_1 + 0.05555d_2 \le 0.11111$

$0.05d_2 \le 0.2$

$-d_2 \le 16$; $-d_1 \le 6$

Solution of the QP is given as **d** = (– 0.86614, – 6.30966), with **u** = (– 148.4, 53.4, 0, 0, 0, 0)

12.49

Referring to Exercises 12.11 and 12.24, the QP subproblem at the point (5, 10) is defined as:

$\bar{f} = -0.17067d_1 - 0.021333d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $0.1d_1 \le 0.5$; $0.055555d_2 \le 0.44444$

$-d_1 \le 5$; $-d_2 \le 10$

Solution of the QP is given as **d** = (0.17067, 0.02133), with **u** = (0, 0, 0, 0)

12.50

Referring to Exercises 12.12 and 12.25, the QP subproblem at the point (5, 5, 5) is defined as:

$\bar{f} = 2367.2867d_1 + 2269.8784d_2 + 2864.1573d_3 + 0.5(d_1^2 + d_2^2 + d_3^2)$

subject to $-0.16667d_1 - 0.16667d_2 - 0.16667d_3 \le -0.16667$

$-d_1 \le 5$; $-d_2 \le 5$; $-d_3 \le 5$

Solution of the QP is given as **d** = (– 5, 11, – 5), with **u** = (13685.2, 81.4, 0, 578.3)

12.51

Referring to Exercises 12.13 and 12.26, the QP subproblem at the point (4, 8) is defined as:

$\bar{f} = 15079.645d_1 + 5026.548d_2 + 0.5(d_1^2 + d_2^2)$;

subject to $0.3351d_1 + 0.08378d_2 = 0.32979$

$0.05d_1 + 0.1d_2 \le 0$

$-d_1 \le 4$; $-d_2 \le 8$

Solution of the QP is given as **d** = (2.98427, – 8), with **u** = (– 45009.3, 0, 0, 1247.7)

12.52

Referring to Exercises 12.14 and 12.27, the QP subproblem at the point (10, 10, 4) is defined as:

$\bar{f} = 80d_1 + 120d_2 + 200d_3 + 0.5(d_1^2 + d_2^2 + d_3^2)$

subject to $-0.06667d_1 - 0.06667d_2 - 0.16667d_3 \le -0.33333$

$-d_1 \le 10$; $-d_2 \le 10$; $-d_3 \le 4$

Solution of the QP is given as **d** = (2.0711, – 10, 5.17159), with **u** = (1231, 0, 27.9, 0)

12.53

Referring to Exercises 12.15 and 12.28, the QP subproblem at the point (2, 1) is defined as:

$\bar{f} = 3d_1 + 2.6d_2 + 0.5(d_1^2 + d_2^2)$

subject to $-0.016667d_1 - 0.016667d_2 \le -0.95$

$-d_1 \le 2$; $-d_2 \le 1$

Solution of the QP is given as **d** = (28.29943, 28.69943), with **u** = (1877.9, 0, 0)

Section 12.7 The Constrained Steepest-descent Method

12.54

1. The constrained steepest-descent (CSD) method, when there are active constraints, is based on using the cost function gradient as the search direction. *False*
2. The constrained steepest-descent method solves two subproblems: the search direction and step size determination. *True*
3. The cost function is used as the descent function in the CSD method. *False*
4. The QP subproblem in the CSD method is strictly convex. *True*
5. The search direction, if one exists, is unique for the QP subproblem in the CSD method. *True*
6. Constraint violations play no role in step size determination in the CSD method. *False*
7. Lagrange multipliers of the subproblem play a role in step size determination in the CSD method. *True*
8. Constraints must be evaluated during line search in the CSD method. *True*

12.55

For the following problem, calculate the descent function values Φ_0, Φ_1, *and* $\Phi 2$ *at the trial step sizes* $\alpha = 0$, δ *and* 2.618δ *(let* $R_0 = 1$, *and* $\delta = 0.1$*).*

Refer to Exercise 12.3 for detailed formulation; Step size α 0.125

12.56

Refer to Exercise 12.4 for detailed formulation; Step size □ 1.0

12.57

Refer to Exercise 12.5 for detailed formulation; Step size □ 1.0

12.58

Refer to Exercise 12.6 for detailed formulation; Step size 1.0

QP Subproblem:

Minimize

$$\bar{f} = 0.6d_1 + 0.001d_2 + \frac{1}{2}\left({d_1}^2 + {d_2}^2\right)$$

Subject to

$$\bar{g}_1 = -0.057d_1 - 0.0001714d_2 + 0.3143 \le 0$$

$$\bar{g}_2 = 0.0286d_1 + 0.0001857d_2 - 0.257 \le 0$$

$$\bar{g}_3 = -0.2857d_1 - 2.43 \le 0$$

$$\bar{g}_4 = 0.0476d_1 - 0.429$$

$$\bar{g}_5 = -\frac{d_2}{10000} - 0.4 \le 0.$$

$$\bar{g}_6 = \frac{d_2}{10000} - 0.6 \le 0.$$

$\bar{g}_1$ is active at the optimum solution given as

d = (5.514, 0.0174), u^* = (107.269, 0, 0, 0, 0, 0), $\bar{f}^*$ = 18.51.

Descent Function Values:

Initial values at $x^{(0)}$ include

$$f_0 = 11.2 \text{ and } V_0 = 0.3143$$

The necessary condition must be checked in which r_0 is calculated, giving

$$r_0 = \sum_{i=1}^{m} {u_i}^{(0)} = 107.269$$

The necessary condition is then met if

$$R = \max(R_0, r_0) = \max(1, 107.269) = 107.269$$

Thus the descent function is given by

$$\phi_0 = f_0 + RV_0 = 11.2 + (107.269)(0.3143) = 44.92$$

Now we calculate the descent function with $\delta = 0.1$

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 12 \\ 4000 \end{bmatrix} + (0.1)\begin{bmatrix} 5.514 \\ 0.0174 \end{bmatrix} = \begin{bmatrix} 12.5514 \\ 4000.00174 \end{bmatrix}$$

Then we re-calculate the cost function and constraints at the new given point

$f = 0.6(12.5514) + 0.001(4000.00174) = 11.53$

$g_1 = 1 – 1.429*10^{-5} (12.5514*4000.00174) = 0.2826 > 0$ (Violated)

$g_2 = -4000.00174 \le 0$ (Inactive)

$g_3 = 7.143*10^{-6} (12.5514*4000.00174) + 0.0001(4000.00174) – 1 = -0.2414 \le 0$ (Inactive)

$g_4 = 1 – 12.5514/3.5 = -2.59 \le 0$ (Inactive)

$g_5 = 12.5514/21 – 1 = -0.4023 \le 0$ (Inactive)

$\bar{g}_6 = \frac{d_2}{10000} - 0.6 < 0$ (Inactive)

The maximum constraint violation is then calculated

$$V_1 = \max(0.2826, -4000.00174, -0.2414, -2.59, -0.4023) = 0.2826$$

The new descent function is then calculated as

$$\phi_1 = f_1 + RV_1 = 11.53 + (107.269)(0.2826) = 41.84$$

The final descent function now needs to be determined. The new step size is

$$\alpha_1 = 2.618(0.1) = 0.2618$$

The next trial point is then calculated

$$\boldsymbol{x}^{(2)} = \begin{bmatrix} 12 \\ 4000 \end{bmatrix} + (0.2618)\begin{bmatrix} 5.514 \\ 0.0174 \end{bmatrix} = \begin{bmatrix} 13.444 \\ 4000.0046 \end{bmatrix}$$

Then we re-calculate the function and constraints at the new given point

$f = 0.6(13.444) + 0.001(4000.0046) = 12.066$

$g_1 = 1 – 1.429*10^{-5} (13.444*4000.0046) = 0.2315 > 0$ (Violated)

$g_2 = -4000.0046 \le 0$ (Inactive)

$g_3 = 7.143*10^{-6} (13.444*4000.0046) + 0.0001(4000.0046) – 1 = -0.2159 \le 0$ (Inactive)

$g_4 = 1 – 13.444/3.5 = -2.841 \le 0$ (Inactive)

$g_5 = 13.444/21 – 1 = -0.3598 \le 0$ (Inactive)

The maximum constraint violation is then calculated

$$V_2 = \max(0.2315, -4000.0046 -0.2159, -2.841, -0.3598) = 0.2315$$

The new descent function is then calculated as

$$\phi_2 = f_2 + RV_2 = 12.066 + (107.269)(0.2315) = 36.90$$

Since $\phi_2 < \phi_1$, the minimum descent function has not yet been reached and a ϕ_3 needs to be calculated following the same process as above.

12.59

Refer to Exercise 12.7 for detailed formulation; Step size = 0.5

12.60

Refer to Exercise 12.8 for detailed formulation; Step size = 1.0

12.61

Refer to Exercise 12.9 for detailed formulation; Step size = 0.5

12.62

Refer to Exercise 12.10 for detailed formulation; Step size = 1.0

12.63

Refer to Exercise 12.11 for detailed formulation; Step size = 1.0

12.64

Refer to Exercise 12.12 for detailed formulation; Step size = 0.25

12.65

Refer to Exercise 12.13 for detailed formulation; Step size = 0.5

12.66

Refer to Exercise 12.14 for detailed formulation; Step size = 0.25

12.67

Refer to Exercise 12.15 for detailed formulation; Step size = 0.25