Skip Lists

CMSC 420

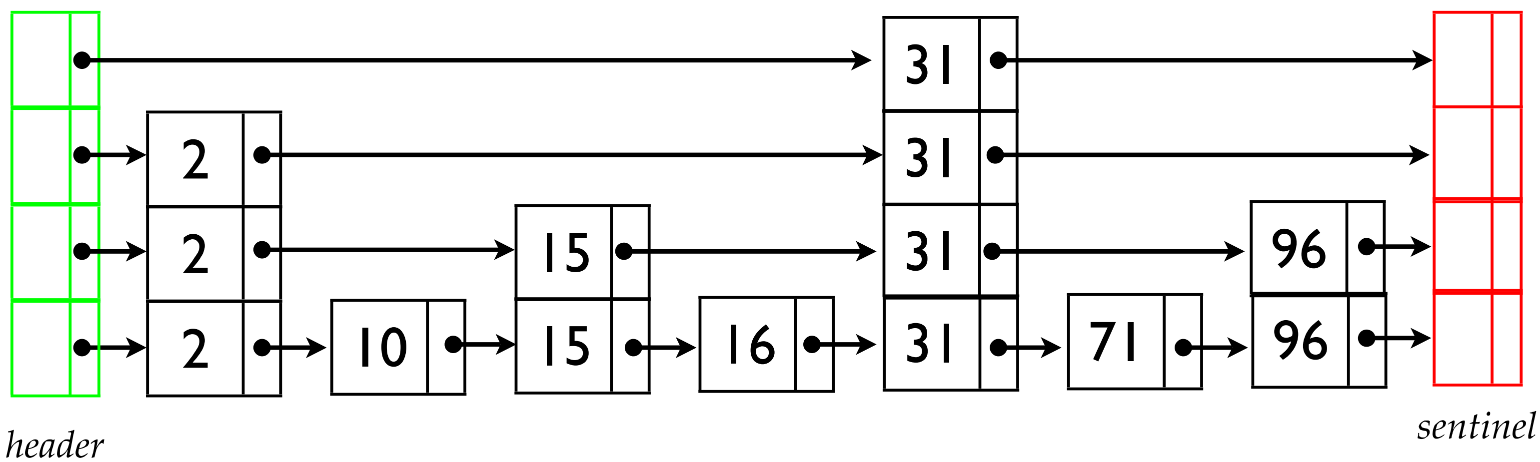
Linked Lists Benefits & Drawbacks

- Benefits:
  - Easy to insert & delete in O(1) time
  - Don’t need to estimate total memory needed
- Drawbacks:
  - Hard to search in less than O(n) time
    (binary search doesn’t work, eg.)
  - Hard to jump to the middle
- Skip Lists:
  - fix these drawbacks
  - good data structure for a dictionary ADT

Skip Lists

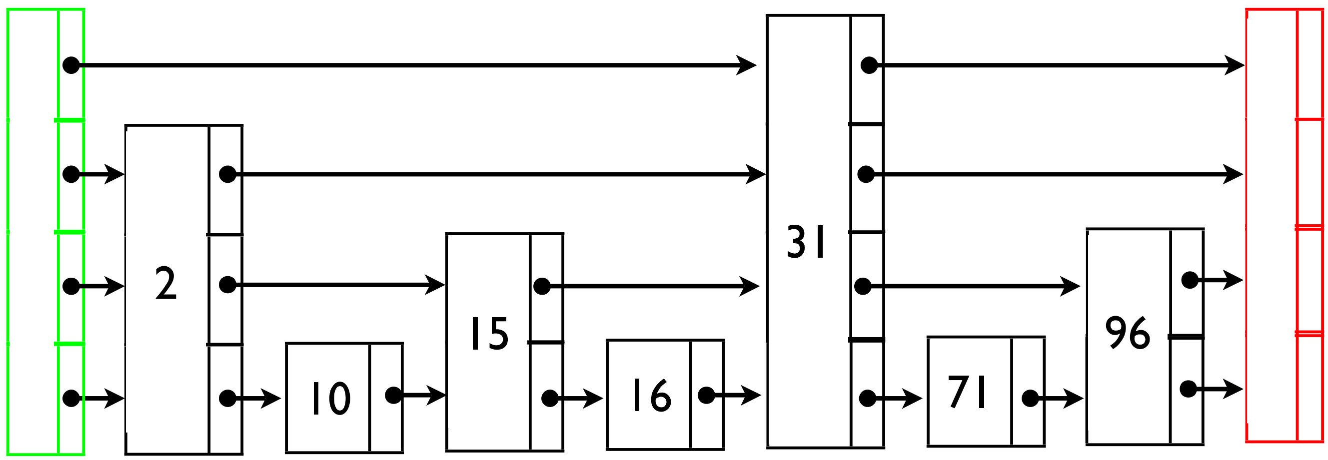
- Invented around 1990 by Bill Pugh
- Generalization of sorted linked lists – so simple to implement
- Expected search time is O(log n)
- *Randomized* data structure:
  - use random coin flips to build the data structure

Perfect Skip Lists



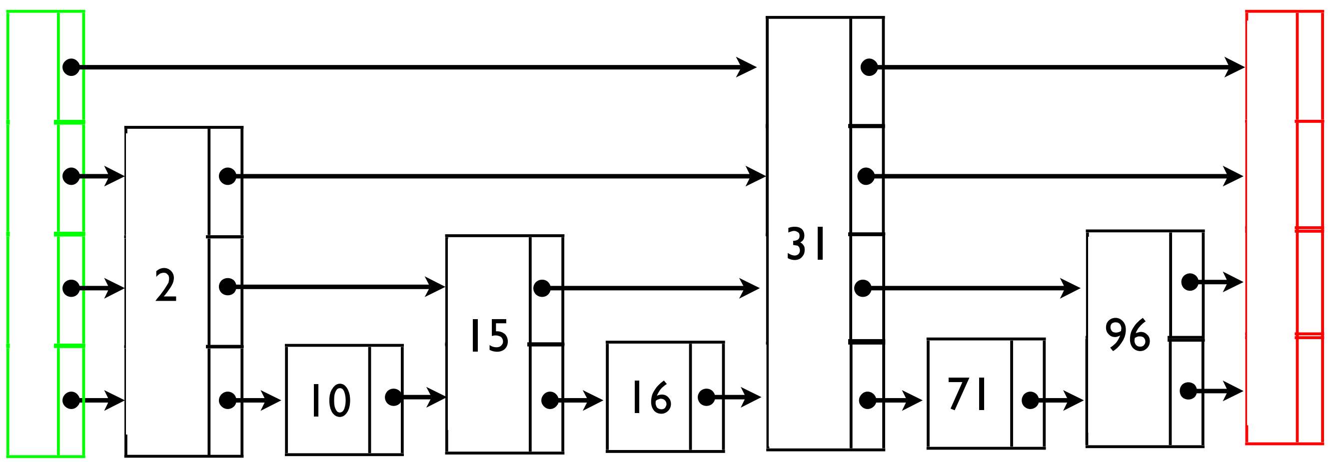
Perfect Skip Lists

- Keys in sorted order.
- O(log n) *levels*
- Each higher level contains 1/2 the elements of the level below it.
- Header & sentinel nodes are in every level



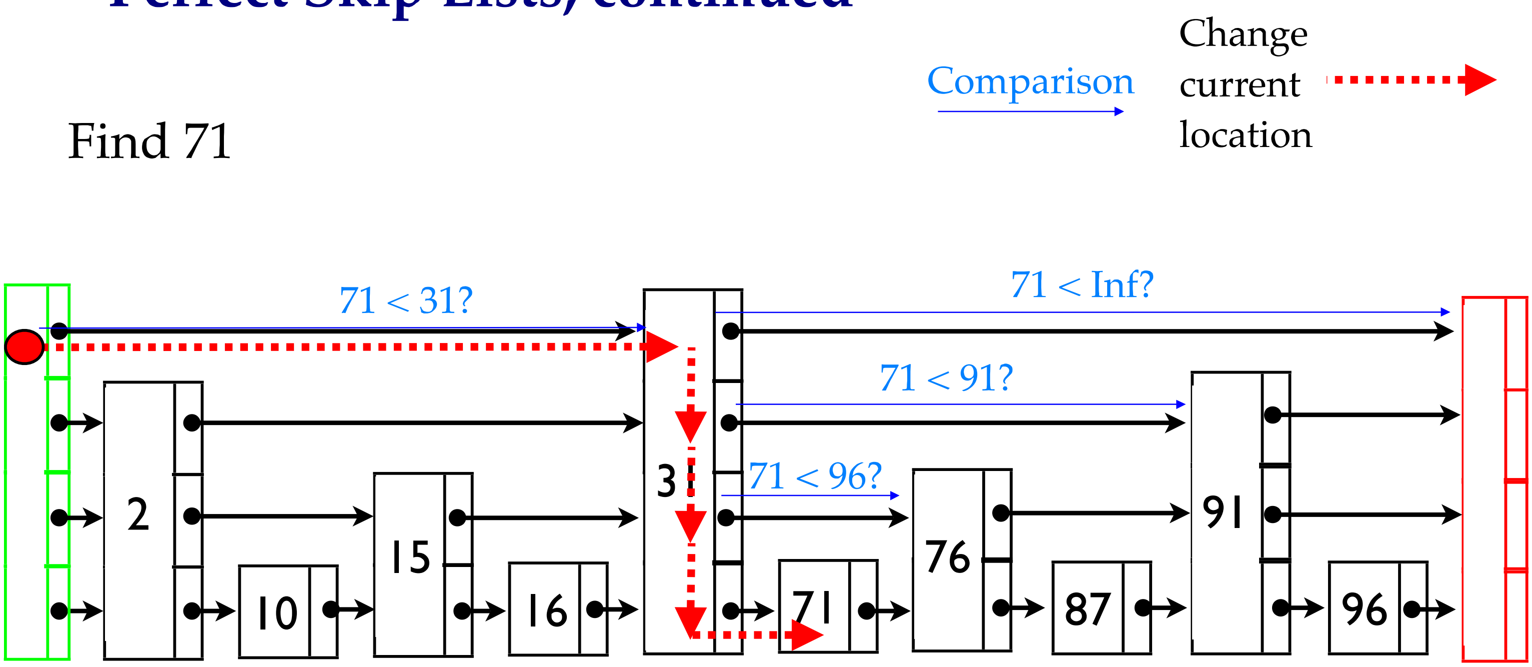
Perfect Skip Lists, continued

- Nodes are of variable size:
  - contain between 1 and O(log n) pointers
- Pointers point to the start of each node
  (picture draws pointers horizontally for visual clarity)
- Called <u>*skip lists*</u> because higher level lists let you skip over many items



Perfect Skip Lists, continued

Find 71



When search for k:

- If k = key, done!
- If k < next key, go down a level
- If k ≥ next key, go right

In other words,

- To find an item, we scan along the shortest list until we would "pass" the desired item.

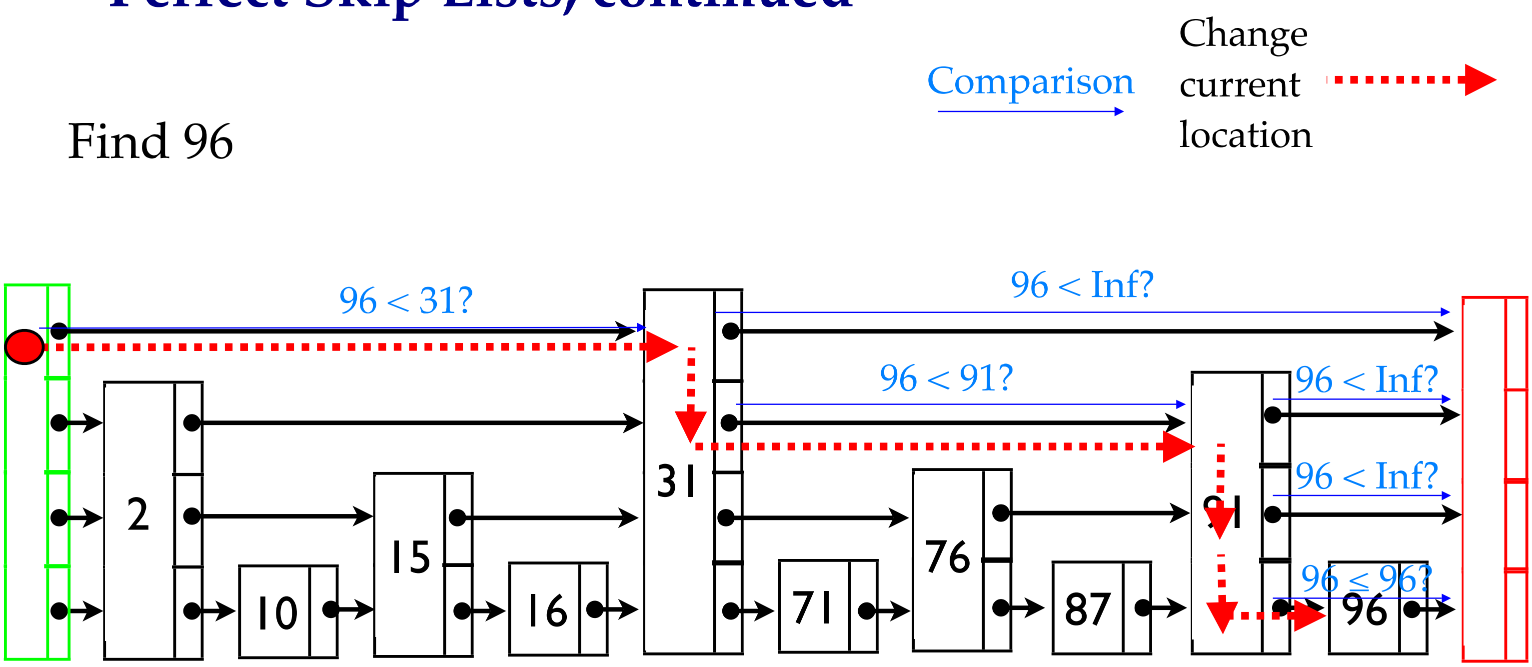
- At that point, we drop down to a slightly more complete list at one level lower.

- Remember: sorted sequential searching...

```
for(i = 0; i < n; i++)
    if(X[i] >= K) break;
if(X[i] != K) return FAIL;
```

Perfect Skip Lists, continued

Find 96



When search for k:

If k = key, done!

If k < next key, go down a level

If k ≥ next key, go right

Search Time:

- O(log n) levels --- because you cut the # of items in half at each level

- Will visit at most 2 nodes per level:
  If you visit more, then you could have done it on one level higher up.

- Therefore, search time is O(log n).

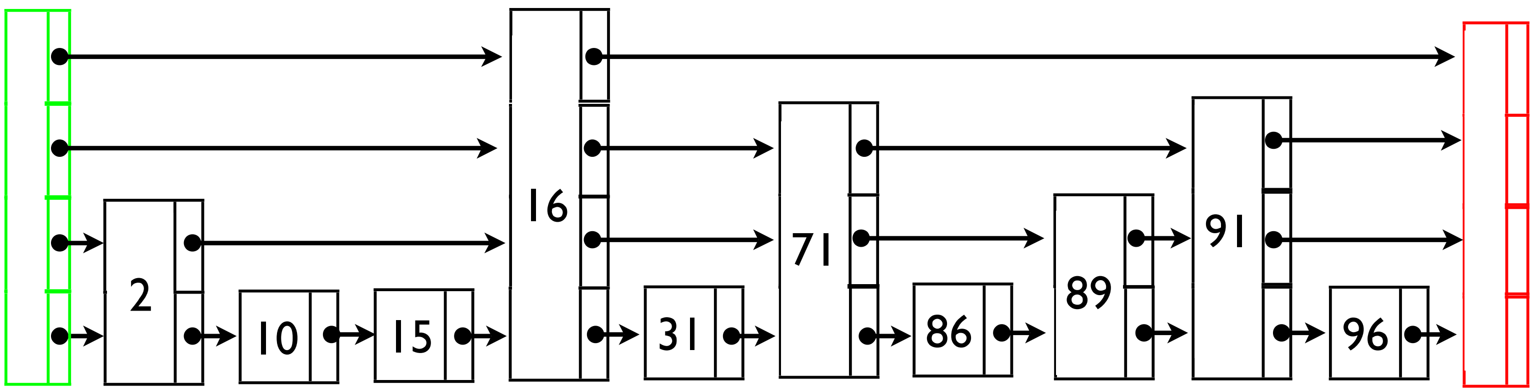
Insert & Delete

- Insert & delete might need to rearrange the entire list
- Like Perfect Binary Search Trees, Perfect Skip Lists are *too* structured to support efficient updates.
- Idea:
  - Relax the requirement that each level have exactly half the items of the previous level
  - Instead: design structure so that we *expect* 1/2 the items to be carried up to the next level
  - Skip Lists are a *randomized* data structure: the same sequence of inserts / deletes may produce different structures depending on the outcome of random coin flips.

Randomization

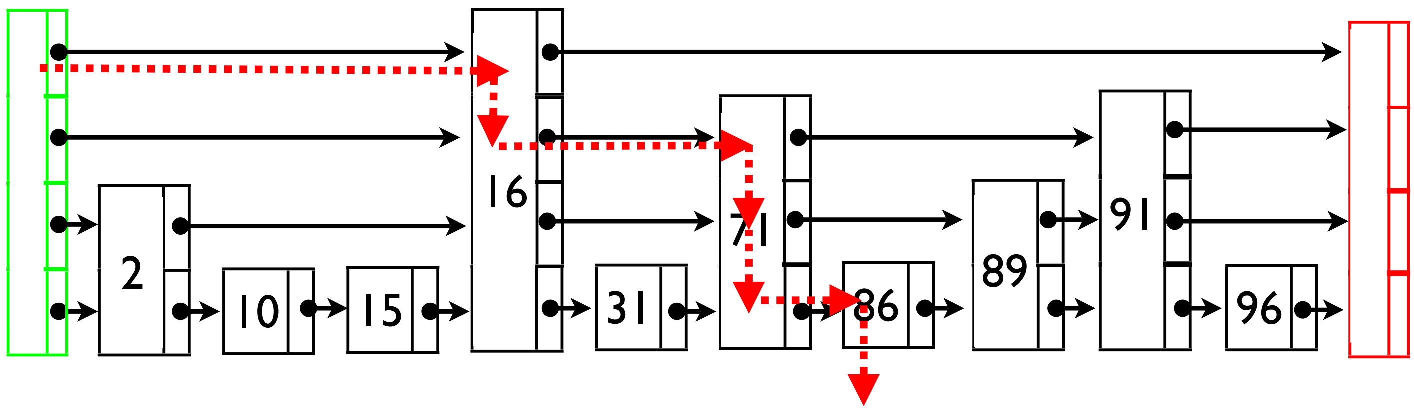
- Allows for some imbalance (like the +1 -1 in AVL trees)
- Expected behavior (over the random choices) remains the same as with perfect skip lists.
- Idea: Each node is promoted to the next higher level with probability 1/2
  - Expect 1/2 the nodes at level 1
  - Expect 1/4 the nodes at level 2
  - ...
- Therefore, expect # of nodes at each level is the same as with perfect skip lists.
- Also: expect the promoted nodes will be well distributed across the list

Randomized Skip List:



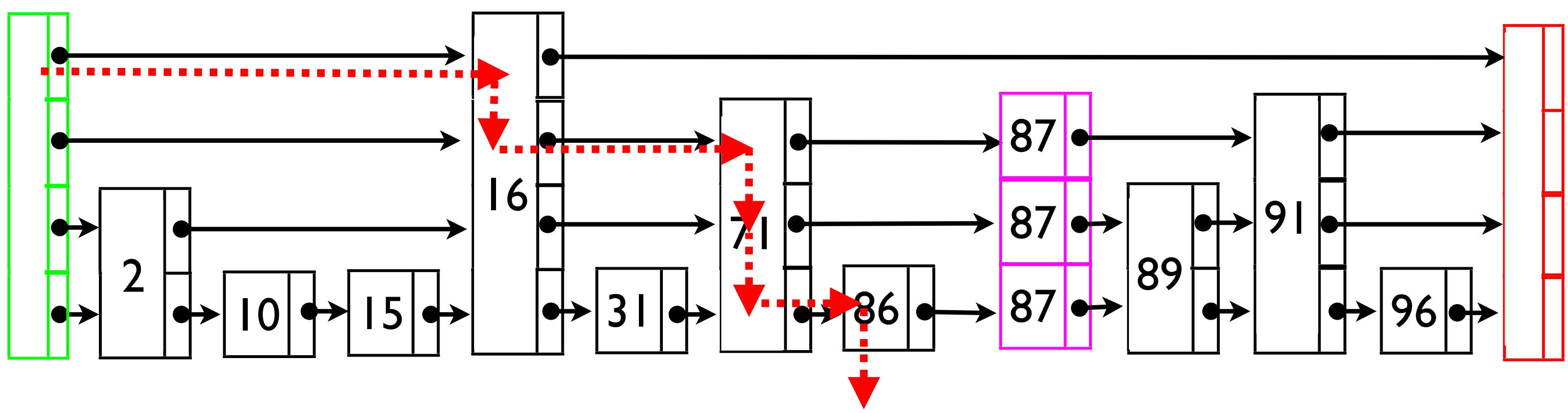
Insertion:

Insert 87



Insertion:

Insert 87

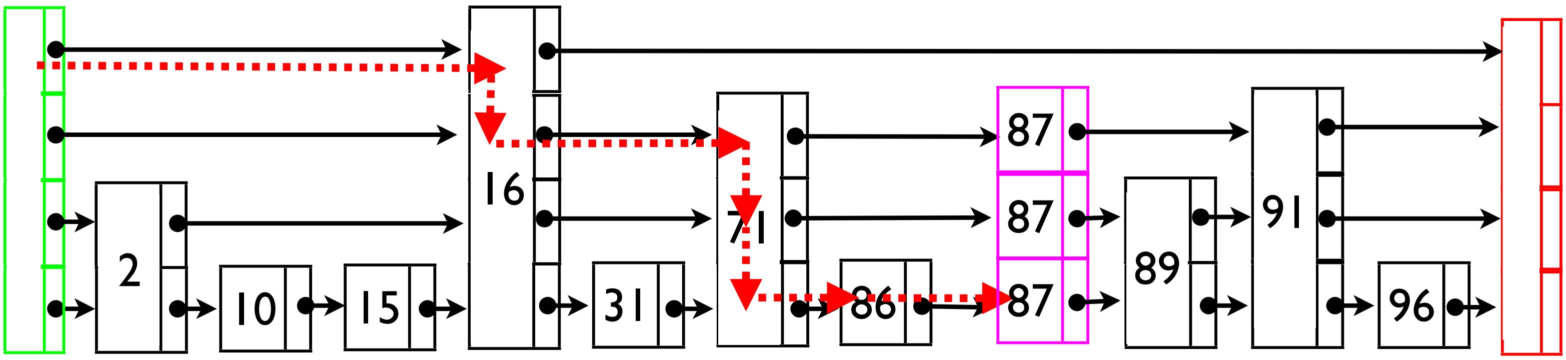



```
Find k
Insert node in level 0
let i = 1
while FLIP() == “heads”:
    insert node into level i
    i++
```

Just insertion into a linked list after last visited node in level *i*

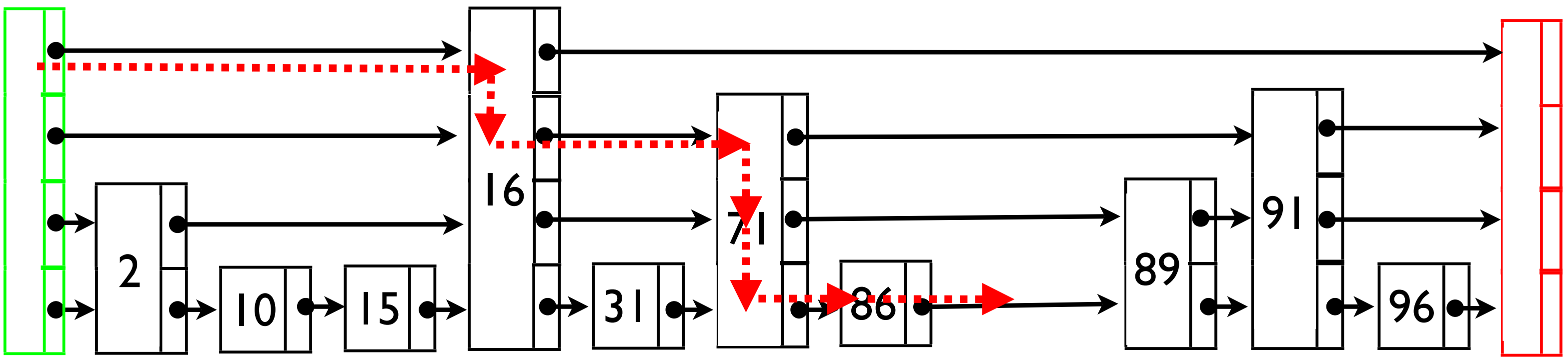
Deletion:

Delete 87



Deletion:

Delete 87



There are no "bad" sequences:

- We expect a randomized skip list to perform about as well as a perfect skip list.
- With some *very* small probability,
  - the skip list will just be a linked list, or
  - the skip list will have every node at every level
  - These *degenerate* skip lists are very unlikely!
- Level structure of a skip list is independent of the keys you insert.
- Therefore, there are no "bad" key sequences that will lead to degenerate skip lists

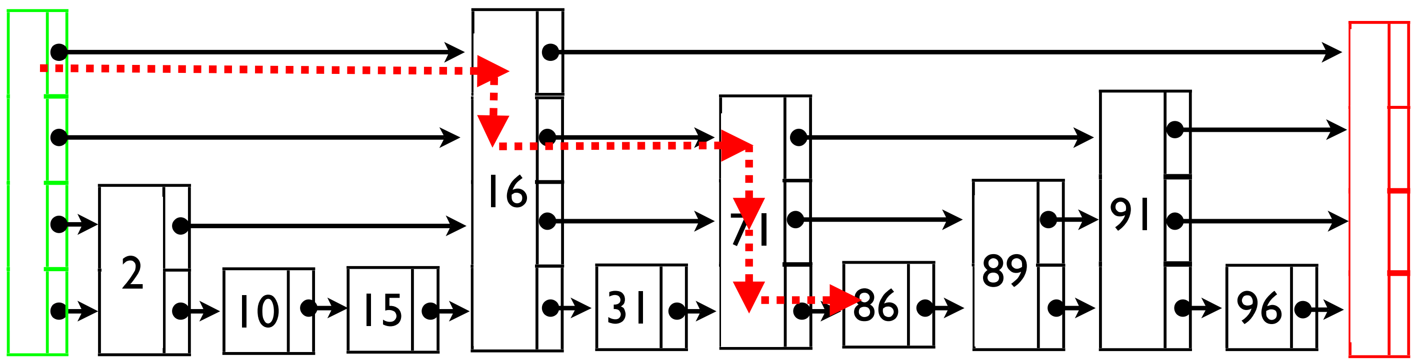
Skip List Analysis

- Expected number of levels = O(log n)
  - E[# nodes at level 1] = n/2
  - E[# nodes at level 2] = n/4
  - ...
  - E[# nodes at level log n] = 1

- Still need to prove that # of steps at each level is small.

Backwards Analysis

Consider the *reverse* of the path you took to find *k*:



Note that you *always* move up if you can.
(because you always enter a node from its topmost level when doing a find)

Analysis, continued...

- What's the probability that you can move up at a give step of the reverse walk?

0.5

- Steps to go up j levels =

  Make one step, then make either

  $C(j-1)$ steps if this step went up [Prob = 0.5]

  $C(j)$ steps if this step went left [Prob = 0.5]

- Expected # of steps to walk up j levels is:

$$C(j) = 1 + 0.5C(j-1) + 0.5C(j)$$

Analysis Continue, 2

- Expected # of steps to walk up j levels is:

$$C(j) = 1 + 0.5C(j\text{-}1) + 0.5C(j)$$

So:

$$2C(j) = 2 + C(j\text{-}1) + C(j)$$

$$C(j) = 2 + C(j\text{-}1)$$

Expected # of steps at each level = 2

- Expanding $C(j)$ above gives us: $C(j) = 2j$

- Since $O(\log n)$ levels, we have $O(\log n)$ steps, expected

Implementation Notes

- Node structures are of variable size
- But once a node is created, its size won't change
- It's often convenient to assume that you know the maximum number of levels in advance
  (but this is not a requirement).