Differential equations and systems of differential equations

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Abstract

In this project problems associated with some second order ODEs, Sturm-Liouville problems, heat equation, wave equation, Laplace equation and autonomous linear system of differential equations are studied. The solution of Legendre equation provides us with the Legendre polynomials. These polynomials help us in solving Laplace equation in spherical coordinates. The PDEs mentioned above are solved by the method of Separation of Variables in the problems below. This method leads to solving the Sturm-Liouville problem. The model of a non-fatal epidemic for a constant population size have been discussed and analysed using autonomous linear systems.

1 Introduction

1.1 Background

Second order ODEs are used for modeling motion of springs, motion of bodies and many other physical, biological, chemical and economic systems. Also solutions of some specific ODEs provide us with orthogonal sets of functions which can be used for approximating a function. The classical PDEs, the heat equation, the Laplace equation and the wave equation model the flow of heat through conduction and other diffusion processes, steady state temperature distribution in a volume, and the motion of a vibrating string or membrane respectively. The autonomous linear systems can be used for interacting species, spread of contagious diseases and also for solving a higher order ordinary differential equation. The stability of these systems gives us an idea of the long term behaviour of the state variables.

1.2 About the problems

- The problem of aging spring models the motion of a spring, which loses it's elasticity over time.

- Some interesting Sturm-Liouville problems, which are an integral part of the separation of variables method are solved for singular and regular cases.
- The problems on wave equation are solved for the one dimensional case in which it represents the motion of a string.
- The problems on heat equation are solved for the temperatures in a rod with constant and variable end temperatures, in the absense and presence of internal sources.
- The solutions to problems on Laplace equation give the steady state temperature distributions for sphere and annular region.
- The measles epidemic model is for a non-fatal disease, and the recovered individuals are immune to the disease.

2 Main result : Problems and their solution

2.1 Aging spring model

Springs are crucial elements in mechanical systems. The motion of a spring depends on the spring constant, damping force and driving force. In the aging spring model discussed below damping and driving forces are not considered. The spring loses it's elasticity over time. This is modeled by the equations

$$my''(t) + ke^{-\epsilon t}y = 0 \quad \text{and} \quad y''(t) + (b^2 + a^2e^{-\epsilon t})y = 0$$

The first equation implies that the spring completely loses its elasticity as time increases. The term $e^{-\epsilon t}$ tends to 0 as t increases. The second model implies that the elasticity is not completely lost and that it decreases till a fixed value.

For solving the second equation we need to make some variable changes. Let $s = \alpha e^{\beta t}$. Applying the chain rule gives,

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{dy}{ds}\beta s$$

$$\frac{d^2y}{dt^2} = \frac{d}{ds}\left[\frac{dy}{ds}\beta s\right]\frac{ds}{dt} = \frac{d^2y}{ds^2}(\beta s)^2 + \frac{dy}{ds}\beta^2 s$$

Substituting these expressions in the equation gives,

$$y''\beta^2 s^2 + y'\beta^2 s + (b^2 + a^2(\frac{s}{\alpha})^{\frac{-\epsilon}{\beta}} y = 0$$

Dividing by β^2 ,

$$y''s^2 + y's + (\frac{b^2}{\beta^2} + \frac{a^2}{\beta^2}(\frac{s}{\alpha})^{\frac{-\epsilon}{\beta}})y = 0$$

Select the values of α and β such that the equation simplifies to a Bessel's equation. Let $\beta = \frac{-\epsilon}{2}$ and $\alpha = \frac{-a}{\beta}$

The equation is then, $s^2 y'' + \ y's + (\frac{b^2}{\beta^2} + s^2)y = 0$.

Here the order of this equation is $p = \frac{2ib}{\epsilon}$ and the independent variable is $s = \frac{2ae^{\frac{-\epsilon t}{2}}}{\epsilon}$

Thus the general solution is,

$$y = c_1 J_p(s) + c_2 Y_p(s)$$

Similarly solving the first model by the same method, we get the general solution as

$$y = c_1 J_0(2\alpha e^{\frac{-\epsilon t}{2}}) + c_2 Y_0(2\alpha e^{\frac{-\epsilon t}{2}}), \qquad \alpha = \sqrt{\frac{k}{m}}$$

Plotting for $J_0(100e^{\frac{-t}{100}}) - Y_0(100e^{\frac{-t}{100}})$

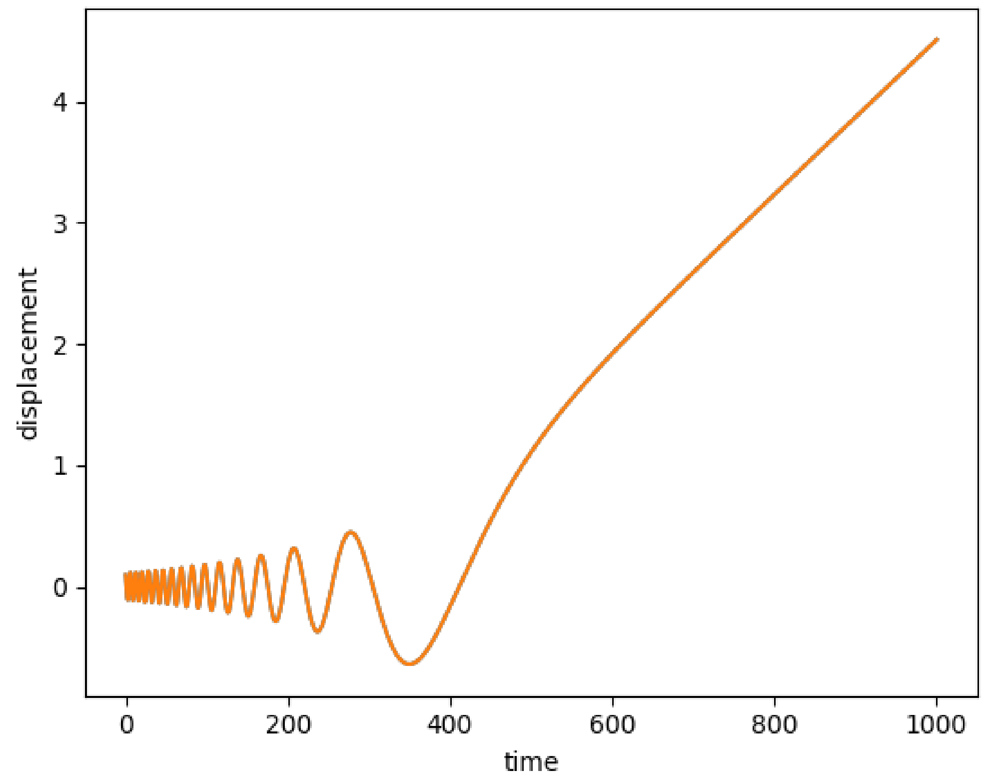

Fig 1 aging spring model

2.2 Sturm-Liouville problems

Let L be a symmetric operator. If the symmetric operator L acts on y in $C^2[a,b]$ and gives values in $C^0[a,b]$, then the problem of finding the eigenvalue represented as λ in the problems below is the Sturm-Liouville problem.

2.2.1 Singular Sturm-Liouville problems

Problem 1

$$x^4y'' + 2x^3y' = \lambda y \qquad y(1) = 0,\ y(2) = 0 \tag{1}$$

Substituting $x = \frac{1}{s}$ and applying chain rule we have,

$$\frac{dy}{dx} = \frac{dy}{ds}.\frac{ds}{dx} = \frac{dy}{ds}.(\frac{-1}{x^2})$$

$$\frac{d^2y}{dx^2} = \frac{d}{ds}(\frac{-1}{x^2}.\frac{dy}{ds}).\frac{ds}{dx} = \frac{d^2y}{ds^2}(\frac{1}{x^4}) + \frac{d}{ds}(-s^2)\frac{dy}{ds}.\frac{ds}{dx} = \frac{d^2y}{ds^2}(\frac{1}{x^4}) + \frac{2}{x^3}\frac{dy}{ds}$$

substituting expressions of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the equation we have

$$x^4[\frac{d^2y}{ds^2}(\frac{1}{x^4}) + \frac{2}{x^3}.\frac{dy}{ds}] + 2x^3[\frac{dy}{ds}(\frac{-1}{x^2})] = \lambda y$$

$$\frac{d^2y}{ds^2} = \lambda y$$

The above equation has a symmetric operator, considering the negative eigenvalue as $-k^2$ we have the general solution of y as,

$$y = C_1 sin\ \frac{k}{x} + C_2 cos\frac{k}{x}$$

We have the condition y(1) = 0 and y(2) = 0 . These conditions are satisfied for $C_2 = 0$ and $k = 2n\pi$

This gives the solution of Sturm-Liouville problem as

$$y = C_1 sin(\frac{2n\pi}{x})$$

The eigenvalue is $-(2n\pi)^2$

Problem 2

$$\frac{1}{x}[(xy')' - (\frac{p^2}{x})y] = \lambda y \qquad y(r) = 0 \tag{2}$$

Rearranging the terms we have,

$x^2y'' + xy' + (-\lambda x^2 - p^2)y$ let $\lambda = -k^2$ and choose $kx = s$

This substitution gives us the familiar Bessel's equation of order p as

$$s^2\frac{d^2y}{ds^2} + s\frac{dy}{ds} + (s^2 - p^2)y = 0$$

The solution of this equation is given as $J_p(kx)$ and since the given conditon is y(r) = 0 we can choose k to be $\frac{x_n}{r}$ where x_n are the roots of the Bessel's equation of given order p.

Thus the solution of the problem is $y = J_p(\frac{x_n x}{r})$

The eigenvalue of given problem is $-(\frac{x_n}{r})^2$

2.2.2 Sturm-Liouville problem associated with the wave equation

Porblem 3

Find the solution of the Sturm-Liouville problem associated with the wave equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad 0 \le x \le L, \quad t > 0 \tag{3a}$$

$$u(0,t) = 0, \qquad t > 0 \tag{3b}$$

$$u_x(L,t) = -hu(L,t), \qquad t > 0, \quad h > 0 \tag{3c}$$

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 < x < L \tag{3d}$$

Let $u = X(x)T(t)$ and substitute it in the PDE above

Thus, $X(x)T''(t) - c^2X''(x)T(t) = 0$

rearranging and equating to a constant λ we have

$X''(x) = \lambda X(x)$

From given boundary conditions we have X(0) = 0 and X'(L) + hX(L) = 0

This forms our Sturm-Liouville problem

The general solution for $\lambda = -k^2$ is $X(x) = C_1 sin(kx) + C_2 cos(kx)$

The condition X(0) = 0 gives $X(x) = C_1 sin(kx)$

Second condition gives $kC_1 cos(kL) + hAsin(kL) = 0$

Rearranging above equation gives $tan(kL) = \frac{-k}{h}$. Let kL = s

Thus, $\frac{s}{L} + htan(s) = 0$, which can be rearranged as $tan(s) = \frac{-s}{hL}$

Depending on the value of hL the solution can be found out for above equation. The solution lies in 2 quadrant and 4 quadrant. Since tan is periodic the above equation has infintely many roots. These roots of provide us with the required eigenvalues.

2.3 Problems on Heat equation

Problems solved below are for a rod of constant diffusivity and uniform cross section. Also the lateral surface is perfectly insulated. The initial and boundary temperatures of the rod are given. The problem is to find temperature distribution within the rod at later times. Assume the central axis of the rod to be the x-axis. Since the lateral walls are insulated and cross-sectional symmetry is present we can neglect temperature variation in y and z-axis. Thus we are left with the PDE and the boundary (BC) and initial conditions (IC) given as

$$u_t - Ku_{xx} = 0 \qquad 0 \leq x \leq L, \quad t > 0 \tag{4 a}$$

$$u(0,t) = g_1(x) \qquad u(L,t) = g_2(x) \qquad t \geq 0 \tag{4 b}$$

$$u(x,0) = f(x) \qquad 0 \leq x \leq L \tag{4 c}$$

2.3.1 Constant end temperatures

For constant end temperatures we have the equation (4b) transformed to ,

$u(0,t) = a, \quad u(L,t) = b$ where a and b are constants

Let us consider a = 10 and b= 20 and f(x) = 0 for the problem

Consider u(x,t) = A(x) + v(x,t) to be the solution of given equations where, A''(x) = 0, A(0) = 10 , A(1) = 20

A''(x) = 0 implies solution of the type A(x) = Bx + C

A(0) = 10 gives C = 10 and A(1) = 20 gives B = 10 , thus A(x) = 10x + 10

we have v(x,t) = u(x,t) -10x -10

Solving for v(x,t) through the equations,

$$v_t - Kv_{xx} = 0 \quad 0 < x < 1, \quad t \geq 0$$

$$v(0,t) = 0 \quad v(1,t) = 0 \quad t \geq 0$$

$$v(x,0) = -A(x), \quad 0 \leq x \leq 1$$

will give the solution as u = A(x) + v

Solving for v : Let v = X(x)T(t) and substitute it in the PDE above,

$$X(x)T'(t) - KX''(x)T(t) = 0$$

Rearranging and equating to a constant λ we have the Sturm- Liouville problem $X''(x) = \lambda X(x), \quad X(0) = 0, \quad X(1) = 0$ and the equation $T'(t) = \lambda KT(t)$

Consider the eigenvalue $\lambda = -k^2$,

This gives the general solution as $X(x) = C_1 sin(kx) + C_2 cos(kx)$

The boundary conditions give the final solution as $X(x) = C_1 sin(n\pi x)$,

The value of λ is $-(n\pi)^2$

Therefore, $T(t) = Ae^{-K(n\pi)^2 t}$ and

$$v(x,t) = C_n sin(n\pi x).e^{-K(n\pi)^2 t}$$

$$v(x,0) = C_n sin(n\pi x) = -10x - 10$$

The Fourier Sine Series of -10-10x is

$$2\int_0^1 (-10 - 10x) sin(n\pi x) dx.sin(n\pi x) \; = (\frac{(-1)^n 40}{n\pi} - \frac{20}{n\pi}) sin(n\pi x)$$

This gives $v(x,t) = \sum_{n=1}^{\infty} (\frac{(-1)^n 40}{n\pi} - \frac{20}{n\pi}).sin(n\pi x).e^{-K(n\pi)^2 t}$

This gives the solution u(x,t) = A(x) + v(x,t)

$$u(x,t) = 10x + 10 \; - \sum_{n=1}^{\infty} (\frac{20}{n\pi})[1 - 2(-1)^n] sin(n\pi x) e^{-K(n\pi)^2 t}$$

After a long time the exponential term tends to 0 and the final steady state temperature is given by 10x + 10

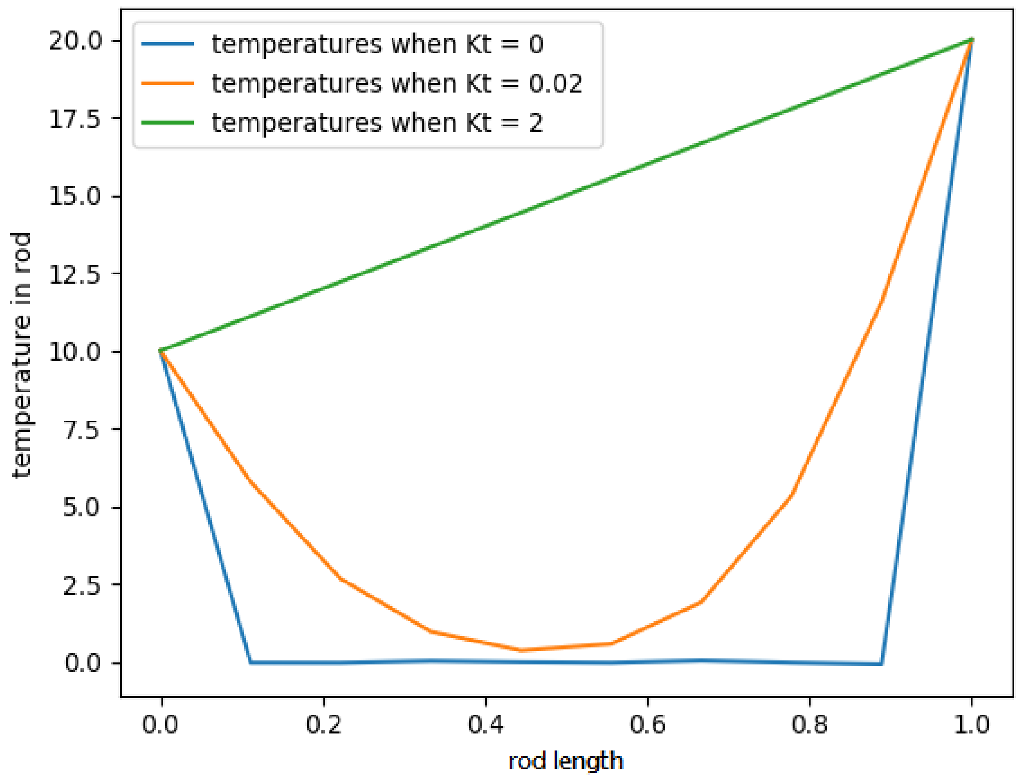

Fig 2 temperatures in rod for constant end temperatures

2.3.2 Variable end temperatures

Suppose the temperatures at the end of the rod are some funtions of time. Then the temperature distribution in the rod also changes accordingly. For solving this kind of PDE we use a method known as shifting of data.

Let the given functions for this problem be $g_1(t) = sint, \;\; g_2(t) = 0$

$$w(x,t) = u(x,t) - v(t) \tag{5}$$

Inserting w in the given PDE and boundary and initial conditions gives,

$$w_t - Kw_{xx} \ = (x-1)sint\,,$$

$$w(0,t) \ = \ w(1,t) = 0\,,$$

$$w(x,0) = f(x) - v(x,0) = f(x) \qquad v(x,0) = 0$$

By the method of eigenfunction expansion we can expect the solution w to be of the form $\sum_{n=1}^{\infty} W_n(t).sin(n\pi x)$ The Fourier Sine series of $(x-1)sint$ is

$$\sum_{n=1}^{\infty} 2\int_0^1 (x-1).sin(n\pi x)sintdx.sin(n\pi x) \ = \frac{-2}{n\pi}sint$$

Inserting the Fourier sine series and the expected solution in the PDE, we have the equation,

$W_n'(t) + K(n\pi)^2 W_n(t) = \ \frac{-2}{n\pi}sint$ This is a linear first order ODE, using the integrating factor $e^{K(n\pi)^2 t}$ we have solution as

$$W_n(t) = e^{-K(n\pi)^2 t}[\frac{-2}{n\pi}\int_0^t e^{K(n\pi)^2\rho} sin\rho d\rho + C]$$

from the initial condition we have

$$W_n(0) = 2\int_0^1 f(x)sin(n\pi x)dx = -C$$

where $2\int_0^1 f(x)sin(n\pi x)dx$ represents the coefficient of Fourier sine series.

This gives solution

$$u = \sum_{n=1}^{\infty} e^{-K(n\pi)^2 t}[2\int_0^1 f(x)sin(n\pi x)dx - \frac{-2}{n\pi}\int_0^t e^{K(n\pi)^2\rho} sin\rho d\rho]sin(n\pi x) + (1-x)sint$$

2.3.3 Internal Source

Consider that there is an internal source in the rod. This condition will give a driven PDE. In this problem, the rod is of unit length and the PDE and boundary and initial conditions are choosen as follows

$$u_t - Ku_{xx} = 3e^{-2t} + x, \quad 0 \leq \ x \leq 1, \quad t > 0$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0$$

$$u(x,0) = 0, \qquad 0 \leq x \leq 1$$

Since for the homogeneous condition we have the solution of the from $\sum_{n=1}^{\infty}\{C_n e^{-K(n\pi)^2 t}\}sin(n\pi x)$

We can expect the solution of the above equation to be of the form $\sum_{n=1}^{\infty} U_n(t) sin(n\pi x)$

Substituting the expected solution in the PDE we get the equation as,

$$\sum_{n=1}^{\infty}\{U_n'(t) + K(n\pi)^2 U_n(t)\}sin(n\pi x) = 3e^{-2t} + x$$

Finding the Fourier sine series (FSS) of the function will enable us to equate both sides of the equation

$$FSS[3e^{-2t} + x] = \sum_0^{\infty} 2\int_0^1 (3e^{-2t} + x) sin(n\pi x) dx sin(n\pi x)$$

$$\int_0^1 (3e^{-2t} + x) sin(n\pi x) dx = e^{-2t}(\frac{3-3(-1)^n}{n\pi}) + (\frac{-(-1)^n}{n\pi})$$

Inserting this in the above equation,

$$U_n'(t) \ + K(n\pi)^2 U_n(t) = 2\{e^{-2t}(\frac{3-3(-1)^n}{n\pi}) + (\frac{-(-1)^n}{n\pi})\}$$

This is a linear equation of first order. Multiplying on both sides by the integrating factor $e^{K(n\pi)^2 t}$ will give us the solution

$$U_n(t) = e^{-K(n\pi)^2 t}\{ \int \frac{e^{K(n\pi)^2 t-2t}}{n\pi}(3-3(-1)^n) - \frac{(-1)^n}{(n\pi)}e^{K(n\pi)^2 t}dt\}$$

$$U_n(t) = e^{-K(n\pi)^2 t}\{ \frac{e^{K(n\pi)^2 t-2t}}{[K(n\pi)^2-2]n\pi}(3-3(-1)^n) - \frac{(-1)^n}{K(n\pi)^3}e^{K(n\pi)^2 t} + C\}$$

The initial condition gives u(x, 0) = 0 which implies

$$U_n(0) = \frac{3-3(-1)^n}{K(n\pi)^2-2} - \frac{(-1)^n}{(Kn\pi)^3} + C \ = \ 0$$

Substituting this value of C in the equation of $U_n(t)$ we have the solution

$$u(x,t) = \sum_{n=1}^{\infty} 2\{\frac{(-1)^n}{K(n\pi)^3}(e^{-K(n\pi)^2 t} - 1) + \frac{(3(-1)^n-3)}{n\pi[K(n\pi)^2-2]}(e^{-K(n\pi)^2 t} - e^{-2t})\}sin(n\pi x)$$

2.4 Steady State temperatures

By steady state temperatures we mean that the temperature does not vary with time, although it may vary from point to point within the body. The Laplace equation models steady-state temperatures in a body of constant material diffusivity.

2.4.1 Steady state temperatures in a sphere

The Laplace equation in spherical coordinates is as follows. This equation is used for the steady state temperatures in a spherical volume. Here u is the variable representing temperature.

$$\frac{1}{\rho^2}(\rho^2 u_\rho)_\rho + \frac{1}{\rho^2 sin\phi}(sin\phi u_\phi)_\phi + \frac{1}{\rho^2 sin^2\phi}u_{\theta\theta} = 0, \quad 0 \le \rho \le 1, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi$$

$$u(1,\theta,\phi) = f(\theta,\phi)$$

For the problems considered here it is assumed that f and u are independent of θ .

Using the method of separartion of variables and considering $u(\rho,\phi) = R(\rho)\Phi(\phi)$ we get the separated ODEs as

$$\rho^2 R'' + 2\rho R' + \lambda R = 0, \qquad\qquad 0 < \rho < 1$$

$$sin\phi\Phi'' + (cos\phi)\Phi' - (\lambda sin\phi)\Phi = 0, \qquad 0 < \phi < \ \pi$$

Changing the variable as $s = cos\phi$ we have the singular Sturm- Liouville problem as

$$(1-s^2)\frac{d^2\Phi}{ds^2} - 2s\frac{d\Phi}{ds} - \lambda\Phi = 0, \quad -1 < s < 1, \Phi \ \ in \ \ C^2[-1,1]$$

If we replace λ by -n(n+1) then the ODE is precisely the Legendre equation and has one of the solutions as the Legendre polynomial $P_n(s)$.

Our other separated equation changes to

$$\rho^2 R'' + 2\rho R' - n(n+1)R = 0, \ \ R^2 \ \ in \ \ C^2[0,1], n = 0,1,..$$

This is an Euler equation with general solution as $R(\rho) = A\rho^n + B\rho^{-n-1}, \ \ n = 0,1,2,..$ Since ρ takes the value 0, we must take B = 0 in the above equation. This gives the solution as $R(\rho) = A\rho^n, \quad n = 0,1,2,\ldots$.

The final solution is given by $u(\rho,\phi) = \sum_0^\infty A_n \rho^n P_n(cos\phi)$

inserting $\rho \ = 1$ for the boundary condition we have $f(\phi) = \sum_0^\infty A_n P_n(cos\phi)$

Changing variable x to cosΦ and let g be a function on |x| ≤1 such that g(cosΦ) =f(Φ) and for all Φ such that 0 ≤ Φ ≤ π , then we have

$$g(x) = \sum_0^\infty A_n P_n(x), \quad |x| \leq 1$$

Using Fourier Euler formula on appropriate legendre basis gives us the value of A_n as

$A_n = \frac{<g,P_n>}{||P_n||^2} = \frac{2n+1}{2}\int_{-1}^{1} g(x)P_n(x)dx, \quad n = 0, 1, 2....$

Boundary Condition 1. Consider the surface temperatures as $f = cos2\phi - sin^2\phi$ considering $x = cos\phi$ we have an equivalent function as $g(x) = 3x^2 - 2$

from previous formula we have $g(x) = \sum_0^\infty A_n P_n(x)$

But since we have $P_0(x) = 1$ and $P_2(x) = \frac{3x^2+1}{2}$ we can take the coefficients $A_0 = -1$, $A_2 = 2$ and other coefficients to be 0 and form a particular solution for g(x) as $-1P_0(x) + 2P2(x)$

We have the final solution as $u = -1 + 2P_2(x)\rho^2$

Boundary Condititon 2. $f(\phi) = \ 1, \quad 0 < \phi < \pi/2$

$= 0, \quad \pi/2 < \phi < \pi$

General solution for the spherical case is $u = \sum_0^\infty A_n P_n(cos\phi)\rho^n$ and $f(\phi) = \sum_0^\infty A_n P_n(cos\phi)$

Let $x = cos\phi$ which gives $g(x) = \ 0, \quad -1 < x < 0$

$= 1, \quad 0 < x < 1$

Thus $g(x) = \sum_0^\infty A_n P_n(x)$

and $A_n \ = \frac{2n+1}{2}\int_{-1}^{1} g(x)\ P_n(x)dx$

$A_n = 0, \ \ for \ -1 < x < 0$

$A_n = \frac{2n+1}{2}\int_0^1 P_n(x)dx \;\; for \;\;\; 0 < x < 1$

$= \frac{2n+1}{2}\{ \; 1.P_n'(x)|_0^1 - \int \; xP_n'(x)dx\}$

$= \frac{2n+1}{2}\{\frac{P_n'(x)|_0^1 - P_{n-1}}{1+n} \;\}$

Our final solution is $u(\phi, \rho) = \;\; \sum_0^\infty\{\frac{2n+1}{2} \; \frac{P_n'(1)-P_n'(0)-P_{n-1}}{1+n}\}P_n(cos\phi).\rho^n$

2.4.2 Steady state temperatures in a ring

For circular regions the Laplace equation is written in polar form. For the ring there are two boundary conditions one from inside and one from outside

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad\;\; 0 \le \theta \le 2\pi, \; 0 \le r \le R$$

$$u(R, \theta) = f(\theta)$$

$$u(\rho, \theta) = g(\theta)$$

Let $u = R(r)\Theta(\theta)$, substituting this value in the PDE we get the equation

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) \; + \; \frac{1}{r^2}R(r)\Theta''(\theta) \; = 0$$

Separating the variables and equating to a constant λ we have the two equations $\Theta''(\theta) = \lambda\Theta$ and $r^2R''(r) + rR'(r) - \lambda R = 0$

The equation Θ''(θ) = λΘ(θ) must satisfy periodic boundary conditions to represent steady state temperature distribution. This condition gives the solution

$$\Theta_n(\theta) \; = A_n cos(n\theta) + B_n sin(n\theta)$$

Inserting the value of λ as $-n^2$ in the other equation we have the Euler's equation $r^2R''(r) + rR'(r) \; + n^2R(r) = 0$

The general solution of the Euler's equation for all n is

$R(r) = A + Blnr \qquad n = 0$ and

$$R(r) = Ar^n + Br^{-n}, \qquad n = 1, 2, 3....$$

Since r is never 0 in case of a ring both the equations are valid in this case

This gives the general solution of temperature in a ring as

$u(r,\theta) = (A + Blnr)(A_n \cos n\theta + B_n \sin n\theta) + \sum_0^\infty (Ar^n + Br^{-n})(A_n cosn\theta + B_n sinn\theta)$

2.5 Problem on Vibrating string

The motion of a vibrating string is modeled by the wave equation in one dimension. It is assumed that the motion is only in transverse direction and only small deflections from the equilibrium position are considered. Also the density of the string and the tension along it are constants. Various kinds boundary conditions can be assumed, but for the considered problem, fixed boundary conditions are applied. The string is considered to be of finite lenght L . These physical conditions are considered by the following equations. Also the damping effect due to the air resistance is included as b^2u_t

$$u_{tt} + b^2 u_t - a^2 u_{xx} = 0, \quad 0 < x < L, t > 0$$

$$u(0,t) = u(L,t) = 0 \quad t \geq 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0 \quad 0 \leq x \leq L$$

where $b^2 < \frac{2\pi a}{L}$ and a are positive constants and f belongs to PC[0,L]

Let u (x,t) = X(x)T(t), and substitute it in the above PDE

$$T''(t)X(x) + b^2T'(t)X(x) - a^2X''(x)T(t) = 0$$

Separating the variables and equating to a constant gives us two equations

$$T''(t) + b^2T'(t) = \lambda T(t)\ and\ \ a^2X''(x) = \lambda X(x)$$

The boundary conditions give the Sturm -Liouville problem as

$$X''(x) = \frac{\lambda}{a^2}X(x)\ , \qquad X(0) = X(L) = 0$$

The solution of this problem is $X(x) = sin\frac{n\pi x}{L}$

Substituting the value of λ in the second ODE, we get

$$T''(t) + b^2T'(t) + (\frac{n\pi a}{L})^2T(t) = 0$$

This is a constant coefficient second order equation. The characteristic polynomial has complex roots. Thus the general solution is given by,

$$T_n(t) = e^{-b^2}(Ccos\tau t + Dsin\tau t)\ , \qquad \tau = \frac{\sqrt{\frac{2n\pi a}{L} - b^4}}{2}$$

The solution of the wave equation is

$$u(x,t) = \sum_0^\infty sin\frac{n\pi x}{L}\ \{e^{\frac{b^2}{2}}(C_n cos\tau t + D_n sin\tau t)\}$$

$$u(x,0) = f(x) = \sum_0^\infty sin\frac{n\pi x}{L} e^{\frac{-b^2}{2}}(C_n)$$

Finding the Fourier Sine series for f(x) gives the value $C_n = \frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}dx$

$u_t(x,0) = 0 = sin\frac{n\pi x}{L}e^{\frac{-b^2}{2}}D_n$ this gives $D_n = 0$

Thus the solution of the damped spring is given by the equations,

$$u = C_n sin\frac{n\pi x}{L}\{e^{\frac{-b^2}{2}}\}(cos\tau t), \qquad \tau = \frac{\sqrt{\frac{n\pi a}{L}2 - b^4}}{2}$$

$$C_n = \frac{2}{L}\int_0^L f(x) sin \frac{n\pi x}{L} dx$$

2.6 Measles Epidemic Model

Problem : Build a model for the disease with given discription

The disease is mild and everyone eventually recovers in 4 days .Once recovered the person develops immunity to the disease. The population can be assumed to be constant during the period. The diseased people are infective till they recover. A typical susceptible person meets 0.3% of the infected population every day and the disease is transmitted to 1 out of 6 people.

Since the population is assumed to be constant it can be divided into susceptibles (S), infectives(I) and recovered(R).

The rate of decline in susceptibles is proportional to the number of susceptibles as well as number of infectives because more number of infectives will spread the disease quickly and more number of susceptibles provide more targets for the infectives .

The constant of proportionality can be considered as the fraction of susceptibles that come into contact and the chances of getting infected. The rate of increase in the infectives is equal to the difference of decline in susceptibles and increase in rate of recovery.

The rate of recovery is directly proportional to the number of infectives. And the rate constant can be assumed to be the the reciprocal of the period of infection.

This gives us the system of differential equations as

$$S' = -aSI,\ I' = aSI - bI,\ \ R' = bI$$

Since the recovery time is 4 days we can take the constant b to be 0.25 with unit of time as 1 day. 0.3% come in contact and 16.667% out of them get the disease .Thus, constant a can be taken to be 0.003 * 0.166667 = 0.0005. A disease becomes an epidemic when more people get infected than are recovered, this implies $I' > 0$ which gives threshold value for susceptibles as $S > \frac{b}{a}$. In the above example the threshold value turns out to be 500. If the number of susceptibles is more than 500, the disease will become an epidemic.

Dividing the rate function I' by S' gives the equation, $\frac{dI}{dS} = -1 + \frac{b}{aS}$

Separarting the variables and solving we get the equation $I = \frac{I_0}{e^{S_0}} S_0{}^{\frac{b}{a}} \left(e^s S^{\frac{b}{a}}\right)$ where the initial population of infectives and susceptibles are given by $I_0 \ and \ S_0$ .This relation tells us the number of infectives for a certain number of susceptibles.

Bibliography

Robert L. Borrelli & Courtney S. Coleman , Differential Equations: A modeling perspective