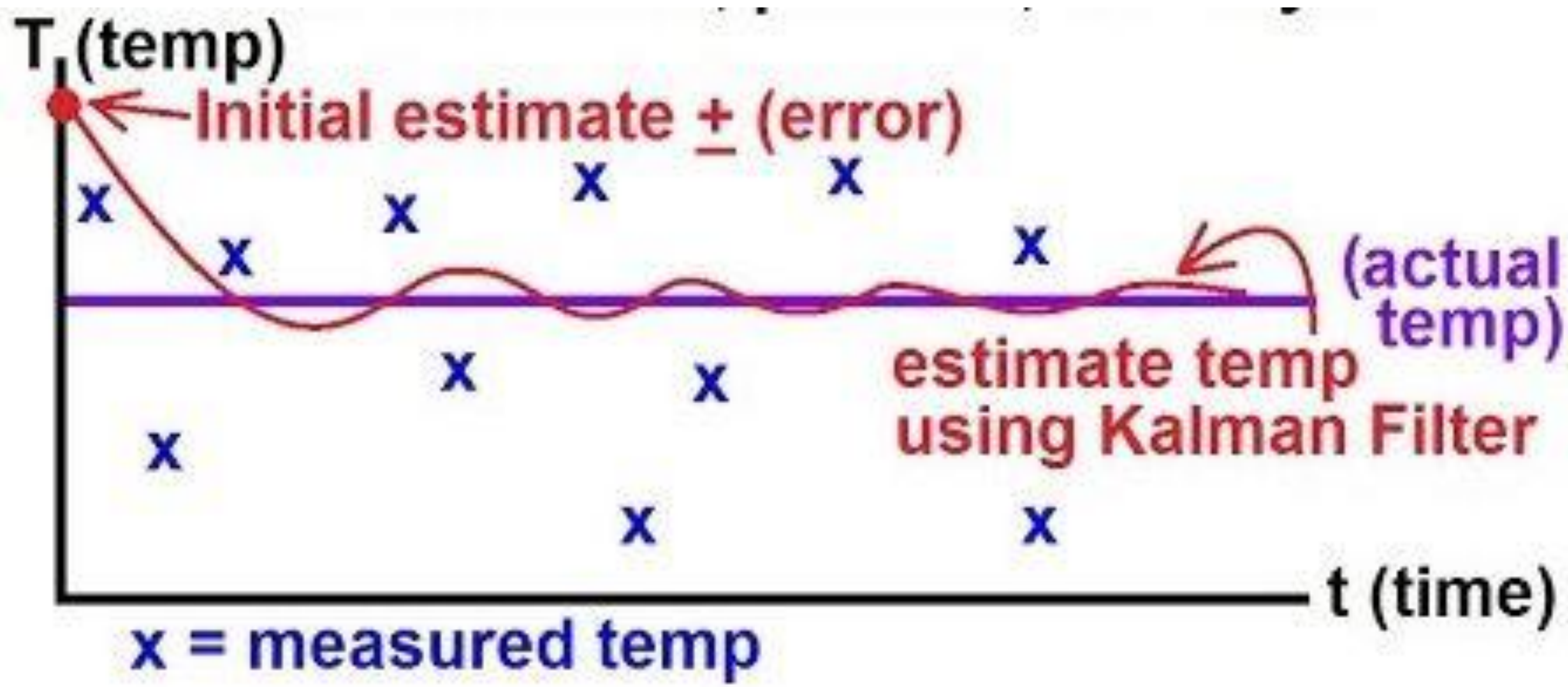
Kalman Filter

What is Kalman Filter

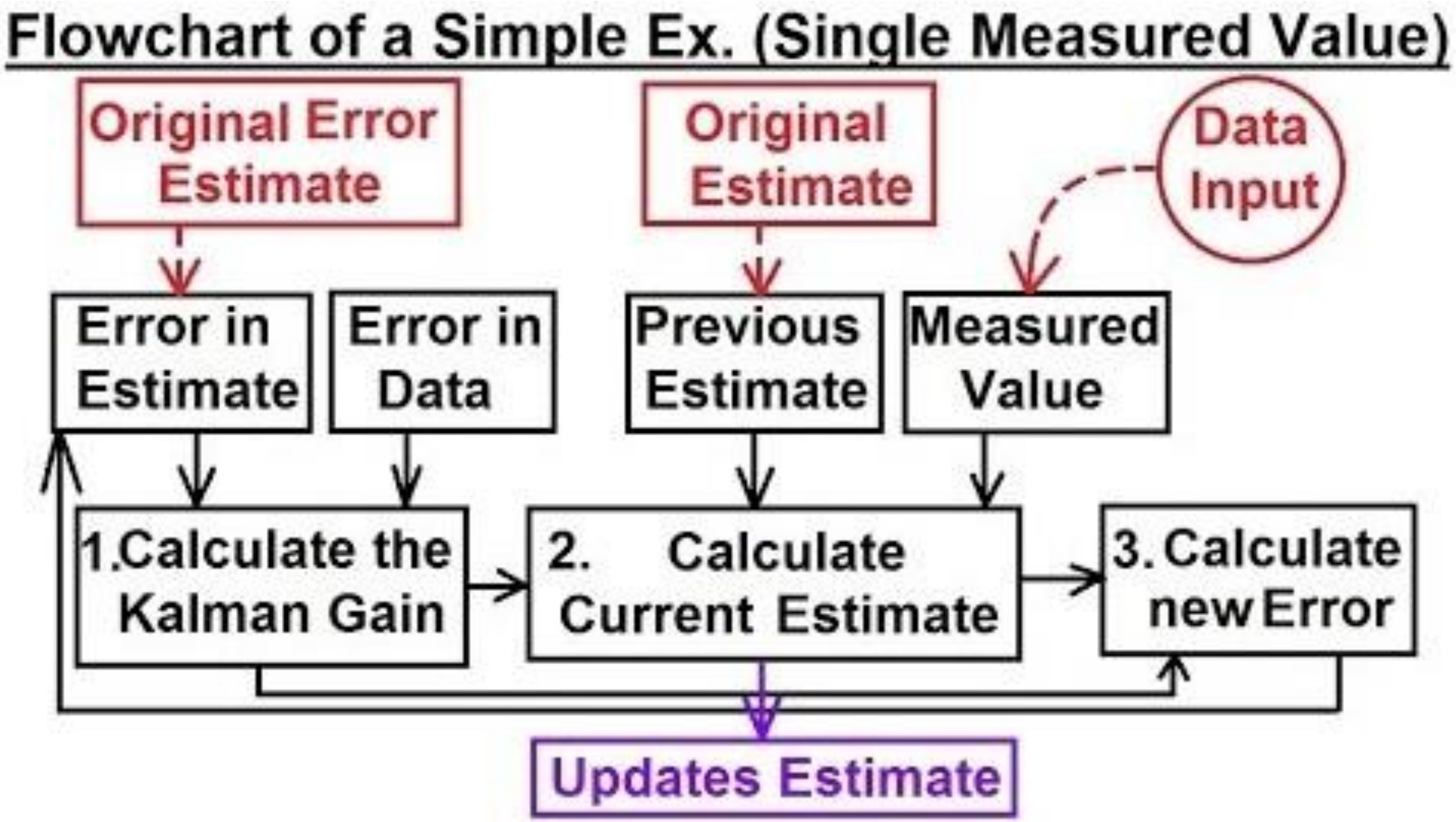
- Iterative mathematical process with set of equations and data input
- Quickly estimates true value like position, velocity etc. Of object being measured
- Here, measured value contains unpredicted or random error, uncertainty or variation
- Example: temperature may have error in measurement using thermometer
- Quickly converges towards actual temp. using KF


T (temp)
Initial estimate ± (error)
(actual temp)
estimate temp using Kalman Filter
t (time)
x = measured temp

History

- Kalman filter was first described and partially developed in technical papers by Swerling (1958), Kalman (1960) and Kalman and Bucy (1961).
- during a visit by Kálmán to the NASA Ames Research Center that Schmidt saw the applicability of Kálmán's ideas to the **nonlinear problem of trajectory estimation for the Apollo program** leading to its incorporation in the Apollo navigation computer
- Kalman filters have been vital in the implementation of the navigation systems of U.S. Navy nuclear ballistic missile submarines, and in the guidance and navigation systems of cruise missiles such as the U.S. Navy's Tomahawk missile and the U.S. Air Force's Air Launched Cruise Missile
- They are also used in the guidance and navigation systems of reusable launch vehicles and the attitude control and navigation systems of spacecraft which dock at the International Space Station

Kalman filter flow diagram



Kalman Gain

- Kalman gain =KG, to decide how much of the new measurement to use for new estimate
- Current estimate= EST_t, Previous estimate= EST_{t-1} , Error in estimate=E_{EST}
- Measurement=MEA, Error in measurement=E_{MEA}
- $KG = \frac{E_{EST}}{E_{EST}+E_{MEA}}$, 0<=KG<=1
- KG=1 => measurements are accurate, estimates are unstable
- KG~0 =>Measurements are inaccurate, estimates are stable
- If the error in measurements are small, we converge quickly however if error in measurements are large we converge slowly

Three calculations

- Calculate Kalman gain

$$KG = \frac{E_{EST}}{E_{EST}+E_{MEA}}, \ 0<=KG<=1$$

- Calculate current estimate

$$EST_t = EST_{t-1} + KG[MEA - EST_{t-1}]$$

- Calculate new estimate error

$$E_{EST_t} = \frac{(E_{MEA})\,(E_{EST_{t-1}})}{E_{MEA} + E_{EST_{t-1}}}$$

or $$E_{EST_t} = (1 - KG)\,E_{EST_{t-1}}$$

A simple example

- True temp.=72, initial estimate =68, initial estimate error=2 (assumed here), initial measurement=75, error in measurement=4
- $KG = \frac{E_{EST}}{E_{EST}+E_{MEA}}$=2/2+4=0.33
- EST_t= $EST_{t-1} + KG[MEA - EST_{t-1}]$

  = 68+0.33(75-68)=70.33
- $E_{EST_t} = (1 - KG)\ E_{EST_{t-1}}$

  = (1-0.33)2=1.33

2nd iteration

- $KG = \frac{E_{EST}}{E_{EST}+E_{MEA}}$=1.33/1.33+4=0.25

- EST_t= $EST_{t-1} + KG[MEA - EST_{t-1}]$ *(2nd measurement=71)*

  = 70.33+0.25(71-70.33)=70.50

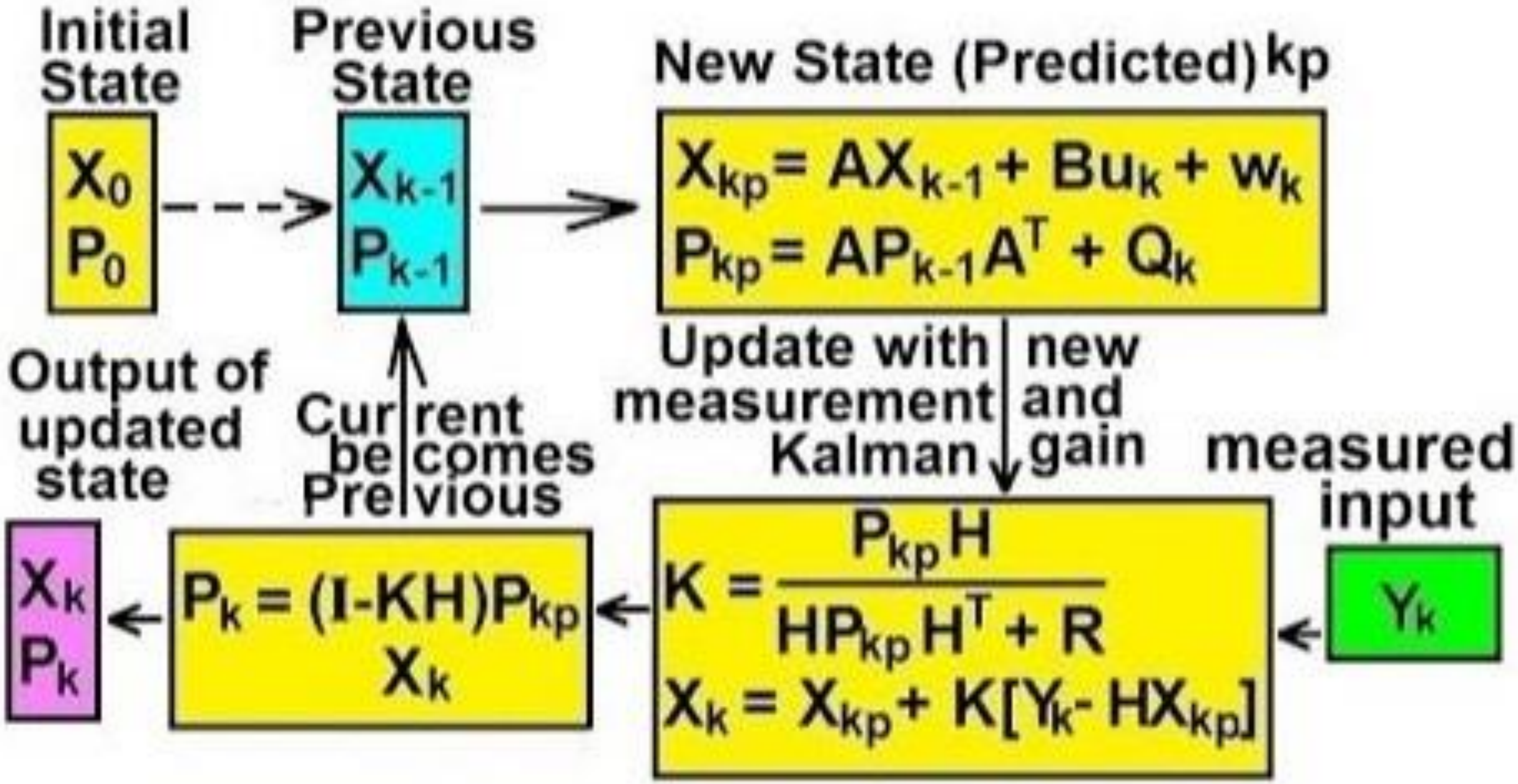
- $E_{EST_t} = (1 - KG)\ E_{EST_{t-1}}$

  = (1-0.25)1.33=1.00

Four Iterations

	MEA	E_{MEA}	EST_t	$E_{EST_{t-1}}$	**KG**	E_{EST_t}
t-1		4	68	2		
t	75	4	70.33	2	0.33	1.33
t+1	71	4	70.50	1.33	0.25	1.00
t+2	70	4	70.40	1	0.20	0.80
t+3	74	4	71	0.8	0.17	0.66

Kalman Filter: The Multi-Dimension Model (Matrix Format)



- X – state matrix, typically contains position, velocity etc..
- P- State covariance matrix- represents error in the estimate or process
- U – control variable matrix (e.g. drone : gravity controls motion)
- W – predicted state noise matrix (error)
- Q – Process noise covariance matrix
- R – Sensor noise covariance matrix (measurement error)

Multi-Dimension Model : State matrix

- $X_k = A\,X_{k-1} + BU_k + W_k$
- $x = x_0 + \dot{x}t + \frac{1}{2}\ddot{x}t^2$, Δt=time for one cycle
- $A\,X_{k-1}$: to take care of update in state matrix (position and velocity) because of previous position and velocity, BU_k: control variable , to take care of change in position and velocity because of control variable , here acceleration
- X –state matrix, A: state transition matrix, U – control variable matrix, W – noise in process
- $X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $X = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix}$, $X = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \\ z \\ \dot{z} \end{bmatrix}$

Example

- $X_k = A\,X_{k-1} + BU_k + W_k$

Rising fluid in a tank

$$X = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} y + \Delta t \\ \dot{y} \end{bmatrix}$$

B= $\begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$

U=[0]

BU= $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

falling object

$$X = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} y + \Delta t \\ \dot{y} \end{bmatrix}$$

B= $\begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$

U= [g]

BU= $\begin{bmatrix} \frac{1}{2} g\Delta t^2 \\ g\Delta t \end{bmatrix}$

vehicle motion in 1-D

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

$$AX = \begin{bmatrix} x + \Delta t \\ \dot{x} \end{bmatrix}$$

B= $\begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$

U=[a]

BU= $\begin{bmatrix} \frac{1}{2} a\Delta t^2 \\ a\Delta t \end{bmatrix}$

Falling object example

- $X_k = A\,X_{k-1} + BU_k + W_k$
- $Y_k = C\,X_k + Z_k$
- Y: observation, Z: measurement noise, C: to convert X to the form of Y
- Let y_{k-1}=20, $\dot{y_{k-1}}$=0, Δt=1
- Then $X_k = A\,X_{k-1} + BU_k + W_k = \begin{bmatrix} 15.1 \\ -9.8 \end{bmatrix}$, assuming W_k=[0]
- If we want only position C=$[1 \;\; 0]$ so that CX= $[1 \;\; 0]$ $\begin{bmatrix} y \\ \dot{y} \end{bmatrix}$=[y]
- Both position and velocity: C= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, CX $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} y \\ \dot{y} \end{bmatrix}$= $\begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

Two dimension

- $X_k = A\, X_{k-1} + BU_k + W_k$
- $X = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $AX = \begin{bmatrix} x + \Delta t \dot{x} \\ y + \Delta t \dot{y} \\ \dot{x} \\ \dot{y} \end{bmatrix}$

Two dimension (control variable matrix)

- $X_k = A\,X_{k-1} + BU_k + W_k$
- U: control variable matrix
- $BU = \begin{bmatrix} \frac{1}{2}\Delta t^2 & 0 \\ 0 & \frac{1}{2}\Delta t^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Delta t^2 a_x \\ \frac{1}{2}\Delta t^2 a_y \\ \Delta t a_x \\ \Delta t a_y \end{bmatrix}$

Two dimension (converting from prev. state to current state)

- $X_k = A\,X_{k-1} + BU_k + W_k$
- $X_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \dot{x}_{k-1} \\ \dot{y}_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\Delta t^2 & 0 \\ 0 & \frac{1}{2}\Delta t^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix}$

$$= \begin{bmatrix} x_{k-1} + \Delta t \dot{x}_{k-1} + \frac{1}{2}\Delta t^2\, a_x \\ y_{k-1} + \Delta t \dot{y}_{k-1} + \frac{1}{2}\Delta t^2\, a_y \\ \dot{x}_{k-1} + \Delta t a_x \\ \dot{y}_{k-1} + \Delta t a_y \end{bmatrix}$$

3-D case

- $X_k = A\,X_{k-1} + BU_k + W_k$=

$$\begin{bmatrix} 1 & 0 & 0 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta t & 0 \\ 0 & 0 & 1 & 0 & 0 & \Delta t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0.5\Delta t^2 & 0 & 0 \\ 0 & 0.5\Delta t^2 & 0 \\ 0 & 0 & 0.5\Delta t^2 \\ \Delta t & 0 & 0 \\ 0 & \Delta t & 0 \\ 0 & 0 & \Delta t \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

3-D case

$$= \begin{bmatrix} x_{k-1} + \dot{x}_{k-1}\Delta t \\ y_{k-1} + \dot{y}_{k-1}\Delta t \\ z_{k-1} + \dot{z}_{k-1}\Delta t \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} a_x 0.5\Delta t^2 \\ a_y 0.5\Delta t^2 \\ a_z 0.5\Delta t^2 \\ a_x \Delta t \\ a_y \Delta t \\ a_z \Delta t \end{bmatrix} = \begin{bmatrix} x_{k-1} + \dot{x}_{k-1}\Delta t + a_x 0.5\Delta t^2 \\ y_{k-1} + \dot{y}_{k-1}\Delta t + a_y 0.5\Delta t^2 \\ z_{k-1} + \dot{z}_{k-1}\Delta t + a_z 0.5\Delta t^2 \\ \dot{x} + a_x \Delta t \\ \dot{y} + a_y \Delta t \\ \dot{z} + a_z \Delta t \end{bmatrix}$$

Covariance matrix

$$P_k = AP_{k-1}A^T$$

$$K_k = \frac{P_k H^T}{HP_k H^T + R}$$

- P: State covariance matrix (error in the estimate)
- Q: Process covariance matrix (keeps the state covariance matrix from becoming too small or zero)
- R: measurement covariance matrix (error in the measurement)
- K: Kalman Gain (weight factor based on comparing the error in the estimate to the error in the measurement)
- If R$\rightarrow 0$, then K $\rightarrow 1$; adjusts primarily with measurement
- If R$\rightarrow large$, then K $\rightarrow 0$; adjusts primarily with the predicted state
- If P $\rightarrow 0$; measurement updates are mostly ignored

Covariance matrix

- 1-D : $$\left[\frac{\sum (\bar{x} - x_i)^2}{N} \right]$$
- 2-D: $$\begin{bmatrix} \frac{\sum (\bar{x} - x_i)^2}{N} & \frac{\sum (\bar{x} - x_i)(\bar{y} - y_i)}{N} \\ \frac{\sum (\bar{y} - y_i)(\bar{x} - x_i)}{N} & \frac{\sum (\bar{y} - y_i)^2}{N} \end{bmatrix} = \begin{bmatrix} \sigma_x{}^2 & \sigma_x \sigma_y \\ \sigma_y \sigma_x & \sigma_y{}^2 \end{bmatrix}$$
- 3-D : $$\begin{bmatrix} \sigma_x{}^2 & \sigma_x \sigma_y & \sigma_x \sigma_z \\ \sigma_y \sigma_x & \sigma_y{}^2 & \sigma_y \sigma_z \\ \sigma_z \sigma_x & \sigma_z \sigma_y & \sigma_z{}^2 \end{bmatrix}$$

- x_i: individual measurement
- $\bar{x}$: avg. of measurement

Covariance matrix (example)

- 5 data sets of three measurements (col1: L, col2: W, col3: H)

$$A = \begin{bmatrix} 4 & 2 & 0.6 \\ 4.2 & 2.1 & 0.59 \\ 3.9 & 2 & 0.6 \\ 4.3 & 2.1 & 0.62 \\ 4.1 & 2.3 & 0.63 \end{bmatrix}$$

- Avg: $\begin{bmatrix} 4.1 & 2.1 & 0.6 \end{bmatrix}$

Covariance matrix (example)

- Covariance matrix:

$$\begin{bmatrix} {\sigma_L}^2 & \sigma_L\sigma_W & \sigma_L\sigma_H \\ \sigma_W\sigma_L & {\sigma_W}^2 & \sigma_W\sigma_H \\ \sigma_H\sigma_L & \sigma_H\sigma_W & {\sigma_H}^2 \end{bmatrix} = \begin{bmatrix} 0.02 & 0.006 & 0.0026 \\ 0.006 & 0.012 & 0.002 \\ 0.0026 & 0.002 & 0.0006 \end{bmatrix}$$

- For measurement matrix A, deviation matrix a = A-[1]A(1/n); here n=5; deviation matrix is deviation from average
- Then cov. matrix= $a^T a$

State covariance matrix

- $X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$; x_0=50m, $\dot{x_0} = v_0 = 5m/sec$, a_0=2m/s^2,$\sigma_x = 0.5\ m, \sigma_{\dot{x}}$=0.2 m/s,

$$P = \begin{bmatrix} {\sigma_x}^2 & \sigma_x \sigma_{\dot{x}} \\ \sigma_{\dot{x}} \sigma_x & {\sigma_{\dot{x}}}^2 \end{bmatrix}$$

- σ_x^2=0.5*0.5=0.25, $\sigma_{\dot{x}}^2$=0.2*0.2=0.04, $\sigma_x\ \sigma_{\dot{x}}$=0.5*0.2=0.1. Hence,

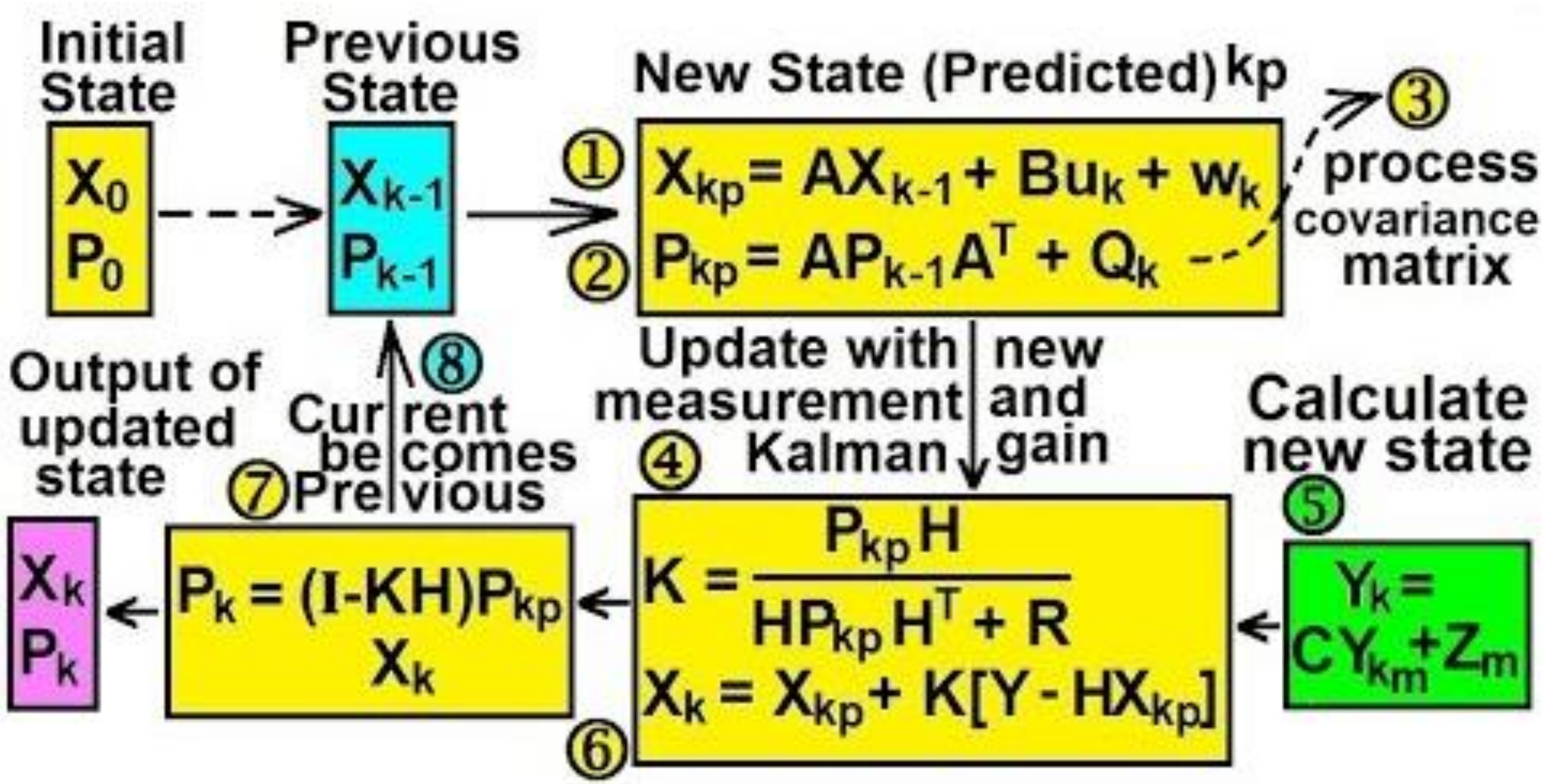
$$P = \begin{bmatrix} 0.25 & 0.1 \\ 0.1 & 0.04 \end{bmatrix}$$

State covariance matrix

- If the estimate error of one variable is completely independent of the other variable then covariance elements =0
- No adjustments are made to the estimates of one variable due to the process error of the other variable
- Approximating ,

$$P = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.04 \end{bmatrix}$$

Example for 2-D KF



Example

- Given, x_0=4000m, v_{0x}=280 m/s, y_0=3000m, v_{0y}=120 m/s
- Observations:

Position	Velocity
x_0=4000	v_{0x}=280
x_1=4260	v_{1x}=282
x_2=4550	v_{2x}=285
x_3=4860	v_{3x}=286
x_4=5110	v_{4x}=290

- Initial conditions: a_x=2m/s*s, v_x=280 m/s, Δt=1s, Δx=25m
- Process errors in process covariance matrix: Δx=20m, Δv=5 m/s,
- Observation error: Δx=25m, Δv_x=6 m/s

Step 1. predicted state

- $X_{k_P} = AX_{k-1} + BU_k + W_k$

$$= \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ v_{0x} \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \begin{bmatrix} a_{x_0} \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4000 \\ 280 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 4281 \\ 282 \end{bmatrix}$$

2. Initial process covariance matrix

- Δx=20m, Δv_x=5 m/s

$$P_{k-1} = \begin{bmatrix} \Delta x^2 & \Delta x \Delta v \\ \Delta v \Delta x & \Delta v^2 \end{bmatrix} = \begin{bmatrix} 400 & 100 \\ 100 & 25 \end{bmatrix}$$

- We can approximate as
- $$\begin{bmatrix} 400 & 0 \\ 0 & 25 \end{bmatrix}$$

3. Predicted process covariance matrix

- $P_{kp} = AP_{k-1}A^T + Q_k$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 400 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 425 & 25 \\ 25 & 25 \end{bmatrix} \approx \begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix}$$

4. Calculate the Kalman Gain

$$K = \frac{P_{kp} H^T}{H P_{kp} H^T + R}$$

$$= \frac{\begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 625 & 0 \\ 0 & 36 \end{bmatrix}} = \frac{\begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix}}{\begin{bmatrix} 1050 & 0 \\ 0 & 61 \end{bmatrix}} = \begin{bmatrix} 0.405 & 0 \\ 0 & 0.41 \end{bmatrix}$$

5. New observation

- $Y_k = CY_{km} + Z_k$ (assume $Z_k = 0$)

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4260 \\ 282 \end{bmatrix} + 0 = \begin{bmatrix} 4260 \\ 282 \end{bmatrix}$$

6. Calculate the current state

- $X_k = X_{kp} + K[Y_k - HX_{kp}]$
- We have already found,

$$X_{kp} = \begin{bmatrix} 4281 \\ 282 \end{bmatrix}, Y_k = \begin{bmatrix} 4260 \\ 282 \end{bmatrix}, K = \begin{bmatrix} 0.405 & 0 \\ 0 & 0.41 \end{bmatrix}$$

- So, $X_k = \begin{bmatrix} 4272.5 \\ 282 \end{bmatrix}$

7. Update the process covariance matrix

- $P_k = (I - KH)P_{kp}$

$$P_{kp} = \begin{bmatrix} 425 & 0 \\ 0 & 25 \end{bmatrix}, K = \begin{bmatrix} 0.405 & 0 \\ 0 & 0.41 \end{bmatrix}$$

- Hence,

$$P_k = \begin{bmatrix} 253 & 0 \\ 0 & 14.8 \end{bmatrix}$$

8. Current becomes previous for next iteration

- We have, $X_k = \begin{bmatrix} 4272.5 \\ 282 \end{bmatrix}$ $P_k = \begin{bmatrix} 253 & 0 \\ 0 & 14.8 \end{bmatrix}$

- Hence for next iteration,
- $X_{k-1} = \begin{bmatrix} 4272.5 \\ 282 \end{bmatrix}$ $P_{k-1} = \begin{bmatrix} 253 & 0 \\ 0 & 14.8 \end{bmatrix}$

2nd iteration

- Step 1,2. $X_{k_P} = AX_{k-1} + BU_k + W_k$, use step 8 of previous iteration for X_{k-1} and P_{k-1}
- So, $X_{kp} = \begin{bmatrix} 4555.5 \\ 284 \end{bmatrix}$
- Step 3. $P_{kp} = AP_{k-1}A^T + Q_k$

$$P_{kp} = \begin{bmatrix} 257.8 & 14.8 \\ 14.8 & 14.8 \end{bmatrix}$$

2^{nd} iteration

- 4. Kalman Gain:

$$K = \frac{P_{kp}H^T}{HP_{kp}H^T + R} = \begin{bmatrix} 0.300 & 0 \\ 0 & 0.291 \end{bmatrix}$$

- 5.New observation:

$$Y_k = CY_{km} + Z_k = \begin{bmatrix} 4550 \\ 285 \end{bmatrix}$$

- 6. Current state: $X_k = X_{kp} + K[Y_k - HX_{kp}]$

$$= \begin{bmatrix} 4553.8 \\ 284.3 \end{bmatrix}$$

2^{nd} iteration

- 7. Update the process covariance matrix: $P_k = (I - KH)P_{kp}$

$$= \begin{bmatrix} 187.5 & 0 \\ 0 & 10.5 \end{bmatrix}$$

8. Current becomes previous for next iteration:

So, for the next iteration,

$$P_{k-1} = \begin{bmatrix} 187.5 & 0 \\ 0 & 10.5 \end{bmatrix} \qquad X_{k-1} = \begin{bmatrix} 4553.8 \\ 284.3 \end{bmatrix}$$

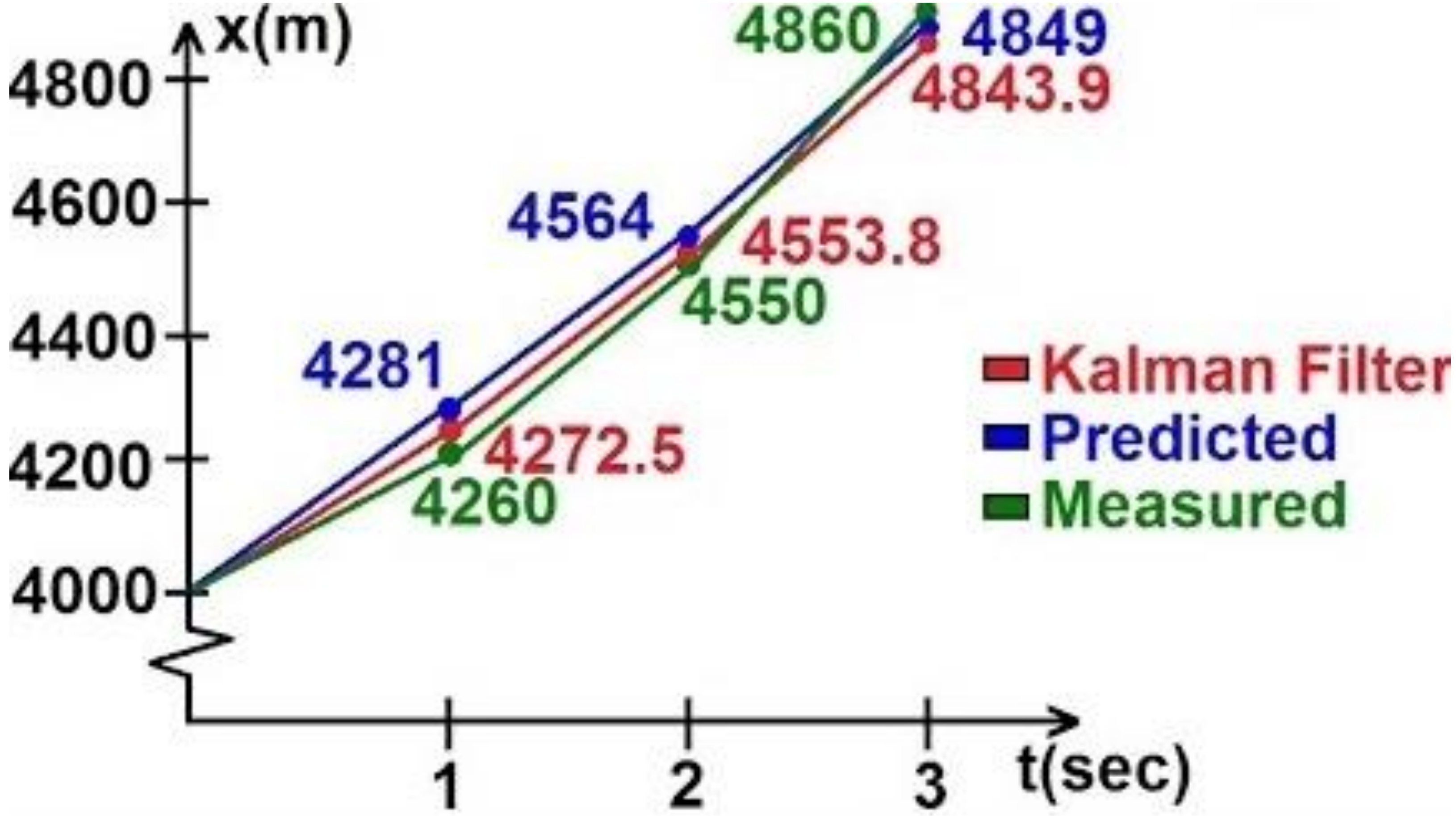
3rd iteration

- Repeating the same way:
- 1,2. $X_{kp} = \begin{bmatrix} 4839.1 \\ 286.3 \end{bmatrix}$
- 3. $P_{kp} = \begin{bmatrix} 187.5 & 10.5 \\ 10.5 & 10.5 \end{bmatrix} \approx \begin{bmatrix} 187.5 & 0 \\ 0 & 10.5 \end{bmatrix}$
- 4. $K = \dfrac{P_{kp} H^T}{H P_{kp} H^T + R} = \begin{bmatrix} 0.231 & 0 \\ 0 & 0.226 \end{bmatrix}$
- 5.Measurement: $Y_k = \begin{bmatrix} 4860 \\ 286 \end{bmatrix}$

- 6. $X_k = \begin{bmatrix} 4843.9 \\ 286.2 \end{bmatrix}$

- 7. $P_k = \begin{bmatrix} 144.2 & 0 \\ 0 & 81 \end{bmatrix}$

Plot of x vs t for 3 iteration



Plot of v vs t for 3 iteration