1 Problem

Given two spheres centered at point vectors $\mathbf{p_1}$ and $\mathbf{p_2}$, at rest at time 0, acceleration vectors $\mathbf{a_1}$ and $\mathbf{a_2}$, and radii R_1 and R_2, we want to find out whether they will ever touch.

2 Solution

First, we can replace "acceleration" with "velocity" (i.e., assume that the given accelerations are actually velocities).

Proof. If the given vectors $\mathbf{a_1}$ and $\mathbf{a_2}$ were actually velocities, then at time t ($t \geq 0$), the two spheres are now centered at $\mathbf{p_1} + \mathbf{a_1}t$ and $\mathbf{p_2} + \mathbf{a_2}t$. But if they were accelerations, then at time u ($u \geq 0$), they are at $\mathbf{p_1} + \frac{1}{2}\mathbf{a_1}u^2$ and $\mathbf{p_2} + \frac{1}{2}\mathbf{a_2}u^2$.

Then for every time $t \geq 0$, there is a corresponding time $u \geq 0$ such that the state we are in at time t if we assume that $\mathbf{a_1}$ and $\mathbf{a_2}$ are velocities is the same as the state we are in at time u if we assume that $\mathbf{a_1}$ and $\mathbf{a_2}$ are accelerations. Moreover, this u is unique, and is given by $u = \sqrt{2t}$.

The converse also holds, i.e., for every $u \geq 0$, there exists a unique time $t \geq 0$, such that the state at time u assuming accelerations is the same as the state at time t assuming velocities. This time, the unique t is given by $t = \frac{1}{2}u^2$.

This means that the spheres will touch when $\mathbf{a_1}$ and $\mathbf{a_2}$ were velocities if and only if they will touch when $\mathbf{a_1}$ and $\mathbf{a_2}$ were accelerations, but possibly at different moments in time. $\square$

Now, given two spheres at points $\mathbf{p_1}$ and $\mathbf{p_2}$, velocities $\mathbf{v_1}$ and $\mathbf{v_2}$, and radii R_1 and R_2, will they ever touch? After a few moments' thought, we see that this is equivalent to the problem "Given two points at $\mathbf{p_1}$ and $\mathbf{p_2}$, with velocities $\mathbf{v_1}$ and $\mathbf{v_2}$, will there ever be a time when the distance between them is less than or equal to $R_1 + R_2$?". This is because the distance between two spheres is equal to the distance between their centers minus $R_1 + R_2$. So, our strategy is to find the minimum distance between the points, and check whether it is greater than $R_1 + R_2$, or not.

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At time $t \geq 0$, the points are at the positions $\mathbf{p_1} + \mathbf{v_1}t$ and $\mathbf{p_2} + \mathbf{v_2}t$. The distance between them is:

$$\begin{aligned} |(\mathbf{p_1} + \mathbf{v_1}t) - (\mathbf{p_2} + \mathbf{v_2}t)| &= |(\mathbf{p_1} - \mathbf{p_2}) + (\mathbf{v_1} - \mathbf{v_2})t| \\ &= |\mathbf{p} + \mathbf{v}t| \end{aligned}$$

where $\mathbf{p} = \mathbf{p_1} - \mathbf{p_2}$ and $\mathbf{v} = \mathbf{v_1} - \mathbf{v_2}$. Here, $|\mathbf{v}|$ denotes $\sqrt{v_x^2 + v_y^2 + v_z^2}$, or more simply, $\sqrt{\mathbf{v} \cdot \mathbf{v}}$ where "$\cdot$" is the dot product.

Therefore, we are trying to minimize the function $D(t) = |\mathbf{p} + \mathbf{v}t|$ with the constraint $t \geq 0$.

Now this can be done using algebra/calculus, but let's reason geometrically here (some may find it more elegant):

First, remove the condition $t \geq 0$. The set of points $L = \{\mathbf{p} + \mathbf{v}t : t \in \mathbb{R}\}$ forms a line in 3D space, and minimizing $|\mathbf{p} + \mathbf{v}t|$ means finding the point in the line L that is nearest to the origin.

Consider the question first in 2D. For a line in a plane, the answer is obviously the point Q such that OQ is perpendicular to L. To see this, suppose R is a point on L other than Q (see figure 1). Then OQR is a right triangle with OQ as a leg and OR as the hypotenuse. By the Pythagorean theorem, $|OQ| < |OR|$.

But is this still true in 3D (and higher dimensions)?

To answer this, first find a plane passing through the origin and L (such a plane

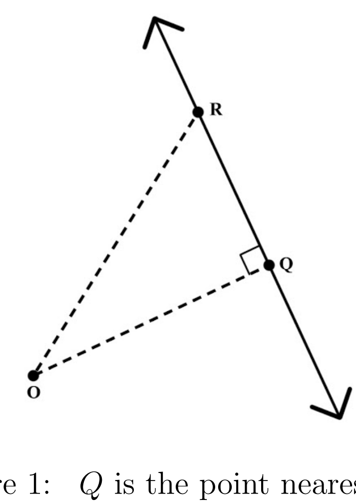

Figure 1: Q is the point nearest to O

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exists, because L is determined by two points, and three points determine a plane). Then in this plane, the argument above holds. Therefore, the answer to the question is yes!

So to find the answer, we simply have to find a point Q of the form $Q = \mathbf{p} + \mathbf{v}t$ such that the line OQ is perpendicular to line L. Now, line L is parallel to the vector $\mathbf{v}$, and OQ is parallel to the vector $\mathbf{p} + \mathbf{v}t$. Two vectors are perpendicular if and only if their dot product is zero, so using basic properties of dot product:

$$\begin{aligned}
\mathbf{v} \cdot (\mathbf{p} + \mathbf{v}t) &= 0 \\
(\mathbf{v} \cdot \mathbf{p}) + (\mathbf{v} \cdot (\mathbf{v}t)) &= 0 && \text{(distributivity)} \\
(\mathbf{v} \cdot \mathbf{p}) + (\mathbf{v} \cdot \mathbf{v})t &= 0 && \text{(scalar multiplication)} \\
t &= -\frac{\mathbf{v} \cdot \mathbf{p}}{\mathbf{v} \cdot \mathbf{v}}
\end{aligned}$$

Therefore, the point closest to the origin is $\mathbf{p} + \mathbf{v}t$ where $t = -\dfrac{\mathbf{v} \cdot \mathbf{p}}{\mathbf{v} \cdot \mathbf{v}}$.

Now, what happens if we add the restriction $t \geq 0$? Then we simply check if the t that we obtained above is ≥ 0, and if not, we use $t = 0$, i.e. choose the point P as the point closest to the origin. This is because if Q is the point described above and R is some point in our ray (see figure 2), then OQR is a right triangle with hypotenuse OR, so by Pythagorean's theorem $OR = \sqrt{OQ^2 + QR^2}$. Since we want to minimize this quantity and OQ is fixed, this means we want to minimize QR, i.e.

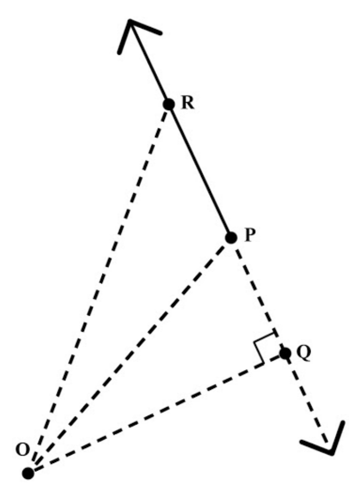

Figure 2: P is the point nearest to O

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we R want to be as close to Q as possible. Since larger t means farther from Q, this means that we should choose $t = 0$, i.e. the point P itself.

3 Summary

We summarize the algorithm here. Given $\mathbf{p_1}$, $\mathbf{p_2}$, $\mathbf{a_1}$, $\mathbf{a_2}$, R_1 and R_2:

1. Let $\mathbf{p} = \mathbf{p_1} - \mathbf{p_2}$ and $\mathbf{v} = \mathbf{a_1} - \mathbf{a_2}$.
2. Let $t = \max\left(-\frac{\mathbf{v} \cdot \mathbf{p}}{\mathbf{v} \cdot \mathbf{v}}, 0\right)$.
3. If $(\mathbf{p} + \mathbf{v}t) \cdot (\mathbf{p} + \mathbf{v}t) \leq (R_1 + R_2)^2$, output "YES", otherwise, output "NO".

4 Credits

I would like to thank **Lyra** for spending some of her time creating the nice images, despite her very busy schedule.

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