# Turán's theorem

From Wikipedia, the free encyclopedia

In graph theory, **Turán's theorem** is a result on the number of edges in a $K_{r+1}$-free graph.

An $n$-vertex graph that does not contain any $(r + 1)$-vertex clique may be formed by partitioning the set of vertices into $r$ parts of equal or nearly equal size, and connecting two vertices by an edge whenever they belong to two different parts. We call the resulting graph the Turán graph $T(n, r)$. Turán's theorem states that the Turán graph has the largest number of edges among all $K_{r+1}$-free $n$-vertex graphs.

Turán graphs were first described and studied by Hungarian mathematician Paul Turán in 1941, though a special case of the theorem was stated earlier by Mantel in 1907.

## Contents



## Formal statement

> **Turán's Theorem.** Let $G$ be any graph with $n$ vertices, such that $G$ is $K_{r+1}$ -free. Then the number of edges in $G$ is at most
>
> $$\frac{r-1}{r} \cdot \frac{n^2}{2} = \left(1 - \frac{1}{r}\right) \cdot \frac{n^2}{2}.$$

An equivalent formulation is the following:

> **Turán's Theorem (Second Formulation).** Among the $n$-vertex simple graphs with no $(r + 1)$-cliques, $T(n, r)$ has the maximum number of edges.

## Proof

Let $G$ be an $n$-vertex simple graph with no $(r + 1)$-clique and with the maximum number of edges.

> **Overview:** The proof consists of two claims about $G$, which we outline, before proving. The first claim is that $G$ must be a complete r-partite graph (although it's stated more technically below). In other words, we can partition the vertex set into $r$ subsets $S_1, S_2, \ldots, S_r$ such that if two

vertices are in different sets, $S_i$ and $S_j$, then they have an edge between them, but if they are in the same set, then they have no edge between them. The second claim is that the sizes of these sets $S_i$ differ from each other by at most 1. For example, if we want the graph on 23 vertices with the most edges that does not contain a triangle, then we partition the vertices into sets $S_1$ and $S_2$, with $|S_1| = 12$ and $|S_2| = 11$. We add all the edges between the two sets, so the graph will have 11*12 = 132 edges. This matches with the theorem, which says that $G$ will have at most $\frac{1}{2}\frac{23^2}{2} = \frac{23^2}{4} = 132.25$ edges.

**Claim 1:** Graph $G$ does not contain any three vertices $u, v, w$ such that $G$ contains edge $uv$, but contains neither edge $uw$ nor $vw$. (This claim is equivalent to the relation x~y iff x not connected to y being an equivalence relation. ~ is always reflexive and symmetric, but only in special cases is it transitive. ~ is not transitive precisely when we have $u, v, w$ with $u \sim w$ and $w \sim v$ without $u \sim v$.)

Assume the claim is false. Construct a new $G$-vertex simple graph $G'$ that contains no $(r + 1)$-clique but has more edges than $G$, as follows:

Case 1: $d(w) < d(u)$ or $d(w) < d(v)$.

Assume that $d(w) < d(u)$. Delete vertex $w$ and create a copy of vertex $u$ (with all of the same neighbors as $u$); call it $u'$. Any clique in the new graph contains at most one vertex among $\{u, u'\}$. So this new graph does not contain any $(r + 1)$-clique. However, it contains more edges:

$$|E(G')| = |E(G)| - d(w) + d(u) > |E(G)|.$$

Case 2: $d(w) \geq d(u)$ and $d(w) \geq d(v)$

Delete vertices $u$ and $v$ and create two new copies of vertex $w$. Again, the new graph does not contain any $(r + 1)$-clique. However it contains more edges:

$$|E(G')| = |E(G)| - (d(u) + d(v) - 1) + 2d(w) \geq |E(G)| + 1.$$

This proves Claim 1.

The claim proves that one can partition the vertices of $G$ into equivalence classes based on their non-neighbors; i.e. two vertices are in the same equivalence class if they are nonadjacent. This implies that $G$ is a complete multipartite graph (where the parts are the equivalence classes).

**Claim 2:** The number of edges in a complete $k$-partite graph is maximized when the size of the parts differs by at most one.

If $G$ is a complete $k$-partite graph with parts *A* and *B* and $|A| > |B| + 1$, then we can increase the number of edges in $G$ by moving a vertex from part *A* to part $B$. By moving a vertex from part *A* to part *B*, the graph loses $|B|$ edges, but gains $|A| - 1$ edges. Thus, it gains at least edge. This proves Claim 2.

This proof shows that the Turan graph has the maximum number of edges. Additionally, the proof shows that the Turan graph is the *only* graph that has the maximum number of edges.

# Mantel's theorem

As a special case of Turán's theorem, for $r = 2$, one obtains:

> **Mantel's Theorem.** The maximum number of edges in an $n$-vertex triangle-free graph is $\lfloor n^2/4 \rfloor$.

In other words, one must delete nearly half of the edges in $K_n$ to obtain a triangle-free graph.

A strengthened form of Mantel's theorem states that any hamiltonian graph with at least $n^2/4$ edges must either be the complete bipartite graph $K_{n/2,n/2}$ or it must be pancyclic: not only does it contain a triangle, it must also contain cycles of all other possible lengths up to the number of vertices in the graph (Bondy 1971).

Another strengthening of Mantel's theorem states that the edges of every $n$-vertex graph may be covered by at most $\lfloor n^2/4 \rfloor$ cliques which are either edges or triangles. As a corollary, the intersection number is at most $\lfloor n^2/4 \rfloor$ (Erdős, Goodman & Pósa 1966).

# See also

- Extremal graph theory
- Erdős–Stone theorem
- A probabilistic proof of Turán's theorem
- Turán's method in analytic number theory

# References


- Aigner, Martin; Ziegler, Günter M. (1998), *Proofs from THE BOOK*, Berlin, New York: Springer-Verlag.
- Bondy, J. A. (1971), "Pancyclic graphs I", *Journal of Combinatorial Theory, Series B* **11** (1): 80–84, doi:10.1016/0095-8956(71)90016-5 (http://dx.doi.org/10.1016%2F0095-8956%2871%2990016-5).
- Erdős, Paul; Goodman, A. W.; Pósa, Louis (1966), "The representation of a graph by set intersections" (http://www.renyi.hu/~p_erdos/1966-21.pdf), *Canadian Journal of Mathematics* **18** (1): 106–112, doi:10.4153/CJM-1966-014-3 (http://dx.doi.org/10.4153%2FCJM-1966-014-3),

MR 0186575 (https://www.ams.org/mathscinet-getitem?mr=0186575).

- Turán, Paul (1941), "On an extremal problem in graph theory", *Matematikai és Fizikai Lapok* (in Hungarian) **48**: 436–452.
- West, Douglas Brent (1999) [1996], *Introduction to Graph Theory* (2nd ed.), Prentice Hall, ISBN 978-0-13-014400-3.

Retrieved from "http://en.wikipedia.org/w/index.php?title=Turán%27s_theorem&oldid=627506536"

Categories: Extremal graph theory | Theorems in graph theory

---