Solution to Permutation Equations

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Problem: Fix an integer $n \geq 1$. For any permutation $x = (x_1, x_2, \ldots, x_n)$ of $1, 2, \ldots, n$, define $f(x) = y$ to be another permutation $(y_1, y_2, \ldots, y_n)$ as $y_1 = 1$ and for any $i \geq 2$, if $x_{y_{i-1}} \neq y_j$ for any $j < i$, then $y_i = x_{y_{i-1}}$ otherwise y_i is the minimum positive integer not in $y_1, y_2, \ldots, y_{i-1}$. For given L and R, find number of permutations y such that $L \leq g(y) \leq R$ where $g(y)$ is the number of permutations x such that $f(x) = y$.

Anton Lunyov

Solution: Let $A_n = (1, 2, \ldots, n)$. We use the following lemmas for the problem.

Lemma 1: Consider any permutation $(y_1, y_2, \ldots, y_n)$ of A_n. If $y_1 \neq 1$, then we have $g(y) = 0$. Otherwise, we have $g(y) = 2^k$ where k is the number of indices $i > 1$ such that $\{1, 2, \ldots, y_i - 1\} \subseteq \{y_1, y_2, \ldots, y_{i-1}\}$. Call all such indices *good* indices.

Proof: Fix a permutation y of A_n. We want to count the number of permutations x of A_n such that $f(x) = y$. Note that, if such an x exists, then we need to have $y_1 = 1$ by definition. So, if $y_1 \neq 1$, we clearly have $g(y) = 0$. Now, assume $y_1 = 1$.

For a permutation x, when we write $y = f(x)$, assume that we put a marker between y_{i-1} and y_i whenever $y_i \neq x_{y_{i-1}}$ i.e. y_i was chosen to be the least integer not in $y_1, y_2, \ldots, y_{i-1}$. Then, whenever there was a marker before y_i, we definitely have $\{1, 2, \ldots, y_i - 1\} \subseteq \{y_1, y_2, \ldots, y_{i-1}\}$ by the choice of y_i. Now, consider a permutation y with exactly k good indices. If x was such that $f(x) = y$, then the markers can only be placed before y_i for all i such that $\{1, 2, \ldots, y_i - 1\} \subseteq \{y_1, y_2, \ldots, y_{i-1}\}$. But this is precisely the definition of good indices. So, the markers can only occur before the good indices.

Now, we show that any set of markers(before the good indices) define a unique permutation x for which $f(x) = y$. Suppose the markers are placed before the good indices $i_1, i_2, \ldots, i_l$ where $l \leq k$. We define the permutation x as follows. If $i + 1 \neq i_t$ for some t, then define $x_{y_i} = y_{i+1}$. If $i + 1 = i_t$ for some t, then $x_{y_i} = y_{i_{t-1}+1}$ where we take $i_0 = 0$. Observe that as i ranges from 1 to n, x_i ranges from 1 to n. So, x is a permutation.

Consider $z = f(x)$. We use strong induction on i to prove that $z_i = y_i$ for all i. First of all, we have $z_1 = 1 = y_1$ and this forms the base case. Now, assume for some $i \leq n - 1$, we have that $z_j = y_j$ for all $1 \leq j \leq i$. Consider z_{i+1}.
If $i + 1 \neq i_t$ for some t, then $x_{z_i} = x_{y_i} = y_{i+1}$ is not present in $y_1, y_2, \ldots, y_i$ otherwise $i + 1$ will be a good index of y which is not the case. But $y_1, y_2, \ldots, y_i$ is the same as $z_1, z_2, \ldots, z_i$ by the induction hypothesis. So, $z_{i+1} = y_{i+1}$ in this case.

On the other hand, if $i+1 = i_t$ for some t, then $x_{z_i} = x_{y_i} = y_{i_{t-1}+1} = z_{i_{t-1}+1}$, by the induction hypothesis and this has already occured in the sequence $z_1, z_2, \ldots, z_i$ and so, z_{i+1} is the least number which has not yet occured. Since $i+1 = i_t$, we have that $i+1$ is a good index for y which means that y_{i+1} is the least number which has not yet occured in $y_1, y_2, \ldots, y_i$. That implies that $z_{i+1} = y_{i+1}$ because the sequence $y_1, y_2, \ldots, y_i$ is identical to $z_1, z_2, \ldots, z_i$ by the induction hypothesis. This completes the strong induction.

When we particularly choose $i = n$, we have $z_j = y_j$ for all $j \leq n$. Or in other words, $f(x) = z = y$. So, for different ways of placing the markers, we get different unique permutations x such that $f(x) = y$. Since there are 2^k ways to place the markers, we have exactly 2^k permutations x such that $f(x) = y$. So, $g(y) = 2^k$ as required. $\square$

Lemma 2: The number of permutations of $(y_1, y_2, \ldots, y_n)$ of A_n with $y_1 = 1$ and k good indices is equal to the number of permutations $(z_1, z_2, \ldots, z_n)$ of A_n with $z_1 = 1$ and exactly $k+1$ indices i(including $i = 1$) such that $z_j < z_i$ for all $j < i$. Call all such indices *nice* indices.

Proof: For a given permutation y with k good indices, consider the inverse permutation z of y defined by $z_{y_i} = i$ i.e. z_i is the index where i goes under the permutation y. Now, if i is any good index for y, then since $\{1, 2, \ldots, y_i - 1\} \subseteq \{y_1, y_2, \ldots, y_{i-1}\}$, it follows that all elements less than y_i go to indices lower than that of the index to which y_i goes(which is i). In other words, $z_u < z_{y_i}$ for all $u < y_i$. Also, note that since $i > 1$(by definition of a good index), we have that $y_i > 1$. So, for all good indices i of y, $y_i > 1$ is a nice index for z. That means there are at least $k+1$ nice indices(including 1) in z.

Also, for each nice index $i > 1$ of z, since $z_j < z_i$ for all $j < i$, it follows that $\{1, 2, \ldots, y_{z_i} - 1\} = \{1, 2, \ldots, i-1\} \subseteq \{y_1, y_2, \ldots, y_{z_i - 1}\}$ because $y_{z_j} = j$ and $z_j \leq z_i - 1$ for $j < i$ and hence z_i is a good index of y. So, there is a bijection between the permutations with exactly k good indices and the permutations with exactly $k+1$ nice indices and it follows that their cardinalities are the same, which is what we wanted to show. $\square$

Lemma 3: The number of permutations of $(y_1, y_2, \ldots, y_n)$ of A_n with $y_1 = 1$ and $k+1$ nice indices is equal to the number of permutations $(z_1, z_2, \ldots, z_{n-1})$ of A_{n-1} with exactly k nice indices.

Proof: Given that the permutation $(y_1, y_2, \ldots, y_n)$ with $y_1 = 1$ has $k+1$ nice indices(including 1), consider the sequence $z = (y_2 - 1, y_3 - 1, \ldots, y_n - 1)$ i.e. $z_i = y_{i+1} - 1$. Since $(y_1, y_2, \ldots, y_n)$ is a permutation of $(1, 2, \ldots, n)$ and $y_1 = 1$, we have that $(y_2, y_3, \ldots, y_n)$ is a permutation of $(2, 3, \ldots, n)$ and so, z is a permutation of $(1, 2, \ldots, n-1) = A_{n-1}$. Moreover, if $i > 1$ is any nice index of y, then since $y_j < y_i \Longrightarrow y_j - 1 < y_i - 1$ for all $1 < j < i$, we have that z_{i-1} is a nice index of z. Note that we only consider nice indices $i > 1$. There are k such indices and so, z has at least k nice indices. Also, if z has a nice index i, then since $y_{j+1} - 1 = z_j < z_i = y_{i+1} - 1$, we have $y_j < y_{i+1}$ for all $1 < j < i+1$. We also have $1 = y_1 < y_{i+1}$. So, z has exactly k nice indices and we have a bijection $(y_1 = 1, y_2, \ldots, y_n) \longrightarrow (y_2 - 1, y_3 - 1, \ldots, y_n - 1)$ between the set of permutations of A_n with exactly $k+1$ good indices and leading element 1 and the set of permutations of A_{n-1} with exactly k good indices and this proves the lemma. $\square$

Lemma 4: The number of permutations of $(y_1, y_2, \ldots, y_n)$ of A_n with k nice indices is equal to the number of permutations $(z_1, z_2, \ldots, z_n)$ of A_n such that its cycle decomposition has exactly k cycles.

Proof: Take a permutation y of A_n whose cycle decomposition has exactly k cycles. Write its cycle decomposition such that each cycle is written with the largest element first and cycles are written in increasing order of their largest elements. Now, after writing such a cycle decomposition, if we remove the brackets, we get a unique permutation z of A_n. Note that z has k indices i such that $z_j < z_i$ for all $j < i$ and these indices marks the beginning of a new cycle. By definition, these are precisely the nice indices of z. If z had another nice index which wasn't the beginning of a new cycle, then consider the cycle to which that element belongs. The cycle begins with an element which is smaller than this number which is a contradiction to how we constructed the permutation z. So, z has exactly k nice indices. On the other hand, if we take a permutation z with k nice indices, we can uniquely recover the permutation y by inserting brackets before every nice index and after we do that, we just read off the cycle decompositon of y. So, this gives a bijection between the set of permutations of A_n with exactly k cycles and the set of permutations of A_n with exactly k nice indices. □

Let us see how to solve the problem using the above lemmas.

First of all, note that for any permutation y with $y_1 \neq 1$, we have $g(y) = 0$. There are $(n-1)(n-1)!$ such permutations. So, when $L = 0$, we add this to the answer. Now, we consider only permutations with $y_1 = 1$. In order to have $L \leq g(y) \leq R$, we need to have $L \leq 2^k \leq R$. where k is the number of good indices in y. There are at most 60 values of k. We fix k and find the number of permutations y with exactly k good indices and then we just sum the values obtained for each k.

By lemma 2, this is equal to the number of permutations y with $y_1 = 1$ and exactly $k+1$ nice indices and by lemma 3, this is equal to the number of permutations of A_{n-1} with exactly k nice indices and by lemma 4, this is equal to the number of permutations of A_{n-1} with exactly k cycles. By definition, this is the unsigned stirling number of the first kind $c(n-1,k)$. There is a very simple recursive formula for these numbers. $c(0,0) = 1, c(n,0) = c(0,n) = 0$ for all $n \geq 1$ and $c(n+1,k) = nc(n,k) + c(n,k-1)$. For a proof, refer the wikipedia page. Then, the answer is

$$((n-1)(n-1)! \text{ if } L = 0 \text{ else } 0) + \sum_{L \leq 2^k \leq R} c(n-1,k)$$

Note that, using the recurive formula, we can compute the answer for each test case in $O(\log R)$ time complexity with a precomputation of $O(\max(n) \log \max(R))$ time complexity. ■