Review: Robust Encoding of a Qubit in a Molecule[1]

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ABSTRACT

The review presented here attempts to summarize the original ideas of the authors of the mentioned paper. They have given quantum codes that can be encoded in rotational states of molecules and are also robust against small errors in their angular positions and momenta. The strategy is to generalise the Gottesman-Kitaev-Preskill (GKP) codes to more exotic Hilbert spaces. The analysis is mathematically rigorous and provides new grounds for encoding quantum information and robust quantum error correction.

Keywords: Quantum error correction, Group theory

1. Introduction

Constructing quantum error correcting codes inspired from classical error correction theory leads to codes which takes advantage of the redundancy introduced in the form of extra qubits. Typically, in such codes, n physical qubits encode k logical qubits ($k < n$) where the code protects against arbitrary errors on a maximum of $m < k$ qubits.
However, continuous Hilbert spaces in quantum mechanics, as opposed to discrete codewords available classically, introduces for us the notion of 'magnitude' of error which can be different than the classical notion. We can now think of codes embedded in a continuous state space which are protected against small shifts inside that space. For example, the position eigenspace of a particle on a ring is a continuum of states $\{|\phi\rangle : \phi \in [0, 2\pi)\}$ where we can define errors $\mathscr{E}$ to be shifts in the position $|\phi\rangle \xrightarrow{\mathscr{E}} |\phi + \delta\rangle$ with $|\delta|$ quantifying the magnitude of the shift.
Such continuous states arise in many practical settings and have potential for applications to quantum information processing. This work focuses on describing quantum error correcting codes embedded in the rotational states of molecules. Since only the rotational states will be considered, the molecules would be equivalent to rigid rotors for us. These codes are generalisations of the Gottesman-Kitaev-Preskill (GKP)[2] codes which are introduced in Section 2. I first consider the simpler settings of the planar and the linear rotor codes in Sections 3. and 4.1 to build up to the case of the completely asymmetric rigid rotor in Section 4.2 and conclude in Section 5. with some comments about more complex group spaces that can be used and further work that these codes demand before they can be realised in the lab.

2. Gentle introduction to GKP Codes

Consider a particle bound to move on a line. It's position eigenstates can be labeled by kets in the set $\{|q\rangle : q \in (-\infty, \infty)\}$. Now, consider the d states given by:

$$|j\rangle_q = \sum_{s=-\infty}^{\infty} |\alpha(j + ds)\rangle \tag{1}$$

where $j \in \{0, d-1\}$ and α is any positive real number.

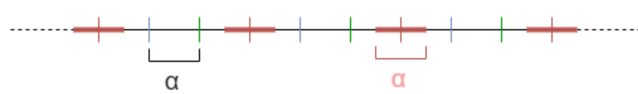

Fig. 1. Codestates(position basis) for d = 3, the red, blue and green ticks depict the three codestates. The red region depicts the region of shifts for which the errors are correctable if the initial state was the red state.

The GKP code uses these states as codewords to correct errors of the form

$$|q\rangle \rightarrow |q + \delta q\rangle \tag{2}$$

with $|\delta q| < \frac{\alpha}{2}$. This is done by projecting the error-ridden state onto its nearest codeword. These ideas can be made rigorous by putting them into the stabiliser formalism. The stabiliser generators for this code would be

$$\hat{S}_Z = e^{2\pi i\hat{q}/\alpha} \tag{3}$$
$$\hat{S}_X = e^{-id\hat{p}\alpha} \tag{4}$$

and the logical operators

$$\hat{\bar{S}}_Z = e^{2\pi i\hat{q}/d\alpha} \tag{5}$$
$$\hat{\bar{S}}_X = e^{-i\hat{p}\alpha} \tag{6}$$

Since $\bar{S_Z}$ translates p by $2\pi/d\alpha$ and $\bar{S_x}$ translates q by α, the errors that can be corrected unambiguously have to satisfy

$$|\delta q| < \alpha/2 \tag{7}$$
$$|\delta p| < \pi/d\alpha \tag{8}$$

or the error-ridden state would be projected onto the incorrect code word. This region (red line segments in fig.(1)) which denotes all states that are closer to one particular state than any other, is called the Voronoi cell of that particular state. If the error ridden state lies inside the Voronoi cell, then the original state which it emerged from can be identified uniquely and hence, the error can be corrected. This idea of the Voronoi cell is going to be useful throughout this review.
Equivalently, the codewords can be written in the momentum basis as

$$|j\rangle_p = \sum_{s=-\infty}^{\infty} |\frac{2\pi}{d\alpha}(j+ds)\rangle \tag{9}$$

These states are separated in momentum by $2\pi/d\alpha$ and hence, the correctable momentum shifts are those with magnitude less than $\pi/d\alpha$ (eqn.(8))

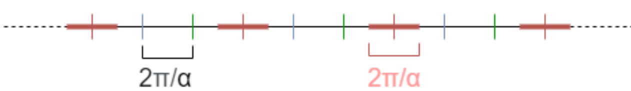

Fig. 2. Codestates(momentum basis) for d = 3, the red, blue and green ticks depict the three codestates. The red region depicts the region of shifts for which the errors are correctable if the initial state was the red state.

3. GKP code on a ring

With a sense of how error correcting codes can be embedded in infinite dimensional Hilbert spaces, we begin to define our molecular codes on rotational states. The codes presented here are associated with sequences of nested groups

$$H \subset K \subset G \tag{10}$$

wherein the elements of G describe the space of rotational states. The elements of the coset G/K describe the set of correctable errors and the elements of the coset K/H describe the set of codewords. Let's make this notion concrete by considering first the GKP code on a ring. This also describes the case of a planar rigid rotor.

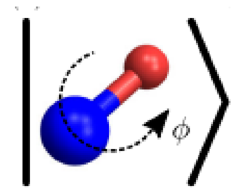

Fig. 3. Planar rotor with rotational states given by elements of U_1.[1]

A planar rotor is one which has a symmetry axis and is confined to rotate in a plane. The rotational states of such an object has a one-to-one correspondence with the elements of the group U_1 (group of all 1×1 unitary matrices). Equivalently, these states can be described by the position eigenstates of a particle confined on a ring $\{|\phi\rangle : \phi \in [0,2\pi)\}$ with the states $|\phi = 0\rangle$ and $|\phi = 2\pi\rangle$ identified.
The angular position and momentum shifts are equivalent to the application of operators

$$\hat{X}_\phi = e^{-i\phi\hat{L}} = \int_0^{2\pi} d\alpha\, |\phi+\alpha\rangle\langle\alpha| \tag{11}$$
$$\hat{Z}_l = e^{i\hat{\phi}l} = \sum_{k=-\infty}^{\infty} |k+l\rangle\langle k| \tag{12}$$

where the states are normalised as $\langle\phi|\phi'\rangle = \delta(\phi-\phi')$ and $\langle l|l'\rangle = \delta_{ll'}$ A general error on the state can be given by the error channel

$$\mathscr{E}(\rho) = \iint_0^{2\pi} d\phi d\phi' \sum_{l,l'\in\mathbb{Z}} \mathscr{E}_{\phi,l}^{\phi',l'} \hat{X}_\phi \hat{Z}_l \rho \hat{Z}_{l'}^\dagger \hat{X}_{\phi'}^\dagger \tag{13}$$

We want to embed logical states in this Hilbert space where we can protect the logical system against action of such error channels which have support only over sufficiently low values of ϕ, l. In other words, we want to recover ρ from $\mathscr{E}(\rho)$ if $\mathscr{E}$ has support over some restricted set of shifts only.

3.1 Logical qubit

A two dimensional code space can be constructed as follows:

$$|0\rangle = |\phi = 0\rangle + |\phi = 2\pi/3\rangle + |\phi = 4\pi/3\rangle \tag{14}$$
$$|1\rangle = |\phi = \pi/3\rangle + |\phi = \pi\rangle + |\phi = 5\pi/3\rangle$$

In the momentum basis, the codewords would be

$$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) = \sum_{s=-\infty}^{\infty}|l=6s\rangle \quad (15)$$

$$\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) = \sum_{s=-\infty}^{\infty}|l=3+6s\rangle$$

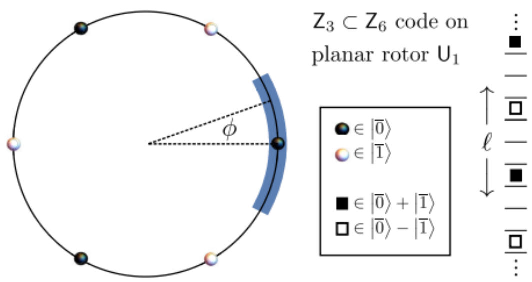

Fig. 4. A qubit encoded in the rotational states of a planar rotor. Equal superpositions of the black and white circles are the logical $|0\rangle$ and $|1\rangle$ states respectively.[1]

Since the code words are shifted by $\pi/3$ from each other in the angular position space, only the shifts that deviate the original state by an angle

$$|\delta\phi| < \pi/6 \quad (16)$$

can be unambiguously identified with the correct original state. The Voronoi cell for this code can be visualised as the blue band in fig.(4).
Similarly, the code words in the conjugate basis are shifted from each other by angular momentum = $3\hbar$, so the set of correctable angular momentum shifts satisfy

$$|\delta l| < 3/2 \implies |\delta l| \leq 1 \quad (17)$$

Here, the groups H, K, G (10) can be identified to be $G = U_1, K = \mathbb{Z}_6$ and $H = \mathbb{Z}_3$. The errors can be corrected by just projecting onto the nearest codeword.
To formalize this error correction procedure, we can define the stabilisers for this code. The codewords are invariant under rotations of integer multiples of $2\pi/3$. This tells us that they are +1 eigenstates of the operator

$$\hat{S}_X = \hat{X}_{2\pi/3} = e^{-i(2\pi/3)\hat{L}} \quad (18)$$

Similarly, the codewords, as written in the momentum basis, are superpositions of states with momentum shifted by multiples of $6\hbar$. Thus, a shift of 6 in l also acts trivially on the code words giving us the second generator of the stabiliser group.

$$\hat{S}_Z = \hat{Z}_6 = e^{i6\hat{\phi}} \quad (19)$$

We want to consider errors of the general form $\hat{X}_\phi \hat{Z}_l$. So we can compute the commutation relations

$$[\hat{X}_\phi \hat{Z}_l, \hat{S}_X] = (e^{i\frac{2\pi}{3}l}-1)\hat{S}_X \hat{X}_\phi \hat{Z}_l \quad (20)$$

$$[\hat{X}_\phi \hat{Z}_l, \hat{S}_X] = -(e^{i6\phi}-1)\hat{S}_Z \hat{X}_\phi \hat{Z}_l \quad (21)$$

telling us that the operators which commute with the stabilizer group

$$\hat{\bar{X}} = \hat{X}_{\pi/3} = e^{-i\frac{\pi}{3}\hat{L}} \quad (22)$$

$$\hat{\bar{Z}} = \hat{Z}_3 = e^{i3\hat{\phi}} \quad (23)$$

leave the code space invariant and form the set of logical operations for this code.
$\hat{S}_Z, \hat{S}_X$ are also the operators which lets us diagnose the errors. A measurement of $\hat{S}_Z$ is equivalent to measuring the position modulo $\pi/3$ and a measurement of $\hat{S}_X$ is equivalent to the measurement of angular momentum modulo 3. And since these two operators commute in the codespace, the diagnosis of position and momentum shifts can be done simultaneously. Subsequently, the error can be corrected by applying the appropriate shift of the form $\hat{Z}_l^\dagger \hat{X}_\phi^\dagger$ for the diagnosed values of ϕ, l.
In the general group theoretic structure introduced earlier, $G = U_1$ is the set of all rotational states. $G/K = U_1/\mathbb{Z}_6$ is the set of correctable errors. This is the same as dividing the ring into 6 equal parts and taking one part of it i.e. a $\pi/3$ segment of the complete circle - the Voronoi cell. Elements of $K/H = \mathbb{Z}_6/\mathbb{Z}_3 = \mathbb{Z}_2$ label the logical states $|0\rangle$ and $|1\rangle$.

4. Molecular codes

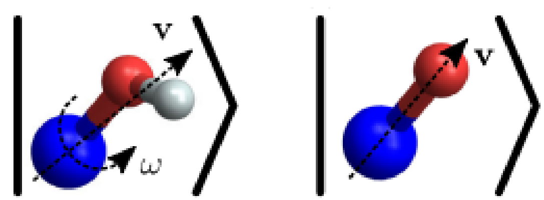

Fig. 5. Left: A general rigid rotor with position states in the space labeled by elements of $SO(3)$. Right: Linear rotor with position states labelled by elements of $S^2 = SO(3)/U_1$ [1]

The planar rotor had a symmetry axis and was confined to a plane. In general, a molecule's rotational states can be labelled by elements of the group $SO(3)$ which is the set of all 3×3 orthogonal matrices with determinant +1. It is also known as the 3D rotation group. The group elements can be specified by an axis **v** and angle of rotation **w**.(see fig.(5)). Here, we choose to label the position basis as

$$\{|R\rangle : R \in SO(3)\}$$

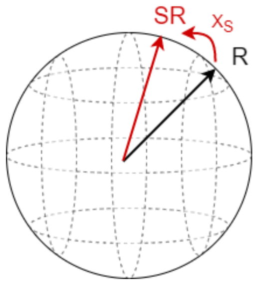

Fig. 6. State space is labeled by points in the solid sphere of radius π. Position shifts can be visualised as going from one point to another in this sphere.

The shifts in position are caused by the operator

$$\hat{X}_S : |R\rangle \rightarrow |SR\rangle \tag{24}$$

The number of parameters needed to specify an element of the rotation group are 3 (2 for the rotation axis and 1 for the angle of rotation). Consequently, the fourier conjugate basis or the angular momentum basis is also labelled by three indices, typically called l, m and n. The relation between the two bases is

$$|\overset{l}{mn}\rangle = \int_{SO(3)} dR \sqrt{\frac{2l+1}{8\pi^2}} D^l_{mn}(R) |R\rangle \tag{25}$$

$$|R\rangle = \sum_{l=0}^{\infty} \sum_{|m|,|n|\leq l} \sqrt{\frac{2l+1}{8\pi^2}} D^{l\star}_{mn}(R) |\overset{l}{mn}\rangle \tag{26}$$

where $D^l_{mn}(R)$ are the l-angular momentum irreducible representations of $SO(3)$. Some important properties of these matrix representations are:

- **Composition of rotations**:

$$D^l_{mn}(SR) = \sum_{|p|<l} D^l_{mp}(S) D^{\ell}_{pn}(R) \tag{27}$$

- **Inverse element**:

$$D^l_{mn}\left(R^{-1}\right) = D^{l\star}_{nm}(R) \tag{28}$$

- **Normalisation**:

$$\int_{SO_3} dR D^{l\star}_{mn}(R) D^{l'}_{m'n'}(R) = \frac{8\pi^2}{2l+1} \delta_{ll'} \delta_{mm'} \delta_{mn'} \tag{29}$$

and the fourier conjugate basis elements experience position shifts as

$$\hat{X}_R |\overset{l}{mn}\rangle = \sum_{|p|<l} D^{l\star}_{pm}(R) |\overset{l}{pn}\rangle \tag{30}$$

These properties are important to note for completeness but we shouldn't lose sight of the objective getting lost in their details. They just define the transformation from the position basis to the angular momentum basis for us. The D^l_{mn} matrix elements should be thought of in analogy to the fourier basis functions $e^{il\phi}$ as they appeared in the planar rotor case. And just like the functions $e^{il\phi}$, these matrix elements form a complete basis i.e. any function $f(R)$ can be expanded in terms of these matrix elements. Such an expansion is given by

$$f(R) = \sum_{l\geq 0} \sum_{|m|,|n|\leq l} \sqrt{\frac{2l+1}{8\pi^2}} f^l_{mn} D^l_{mn}(R) \tag{31}$$

where f^l_{mn} can be computed using the normalisation property to be

$$f^l_{mn} = \int_{SO(3)} dR \sqrt{\frac{2l+1}{8\pi^2}} D^{l\star}_{mn}(R) f(R) \tag{32}$$

This now equips us to write any operator in the momentum basis as

$$\hat{f} = \int_{SO(3)} dR |R\rangle f(R) \langle R| = \sum_{l\geq 0} \sum_{|m|,|n|\leq l} \sqrt{\frac{2l+1}{8\pi^2}} f^l_{mn} \hat{D}^l_{mn} \tag{33}$$

4.1 Linear rotor - S^2 space

Before describing the codes for a general rigid rotor, there is yet another slightly simpler space we can consider that will be useful. Let us now allow our planar rotor to not be confined to a plane anymore. It still has an axis of symmetry but can now rotate about in the space described by the surface of a sphere. The position states of this space can be labeled by the elements of the coset space $S^2 = SO(3)/U_1$. U_1 is "quotiented out" because all rotations about the symmetry axis result in equivalent states for the molecule.
The state basis can be labeled by $\{|v\rangle : v = (\theta, \phi), \theta \in [0, \pi), \phi \in [0, 2\pi)\}$ The v here is the same as in fig. (5) where it labels the axis of rotation or a point on the surface of a sphere. The position shifts are described in the same way as the $SO(3)$ framework(eqn.(24)).

$$\hat{X}_R : |v\rangle \rightarrow |Rv\rangle \tag{34}$$

But notice that there are now many operators $\hat{X}_R$ which act trivially on some corresponding basis states, namely, the rotations about axis v act trivially on the basis state $|v\rangle$

$$\hat{X}_{R=(v,w)} |v\rangle = |v\rangle \ \forall \ w \tag{35}$$

We define another operator $\hat{X}_P$ here which takes every point v to its antipodal point -v

$$\hat{X}_P |v\rangle = |-v\rangle \tag{36}$$

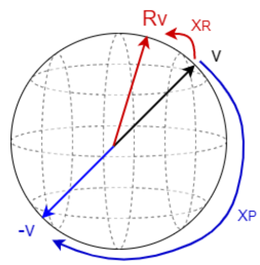

Fig. 7. State space is labeled by points on the surface of a sphere. Position shifts can be visualised as going from one point to another in this sphere. Inversion takes any point to its antipodal point.

Special cases of the D^l_{mn} matrix elements when restricted to the surface are the matrix elements of the spherical harmonics Y^l_m. These give us the fourier conjugate basis elements and operators for the linear rotor. The following relations hold for the spherical harmonics, and should be looked at in analogy to the $SO(3)$ case.

- **Change of basis** (analogous to eqn. (25, 26)):

$$|v\rangle = \sum_{l=0}^{\infty}\sum_{|m|\le l} Y^{l\star}_m(v)\,|{}^l_m\rangle \quad (37)$$

$$|{}^l_m\rangle = \int_{S^2} dv Y^l_m(v)\,|v\rangle \quad (38)$$

- **Normalisation** (analogous to eqn. (29)):

$$\int_{S^2} dv Y^{l\star}_m(v) Y^{l'}_{m'}(v) = \delta_{ll'}\delta_{mm'} \quad (39)$$

4.1.1 Logical qubit

Finally, we are ready to describe the error correcting code. The associated groups for this code are $G = S^2$[§], $K = \mathbb{Z}_{2N}$, $H = \mathbb{Z}_N$. The two code states are

$$|0\rangle = \frac{1}{\sqrt{N}}\sum_{h\in\mathbb{Z}_N}|\frac{\pi}{2},\frac{2\pi}{N}h\rangle \quad (40)$$

$$|1\rangle = \frac{1}{\sqrt{N}}\sum_{h\in\mathbb{Z}_N}|\frac{\pi}{2},\frac{2\pi}{N}h+\frac{\pi}{N}\rangle$$

where $|\frac{\pi}{2},\alpha\rangle$ is the state with polar angle $\theta = \pi/2$ and azimuthal angle $\phi = \alpha$. Polar angle $\pi/2$ just corresponds to points on the equator.

§ S^2 is technically not a group but for analogy with the other group spaces discussed here, we can discuss the error correction procedure on this coset space as if it were a group.

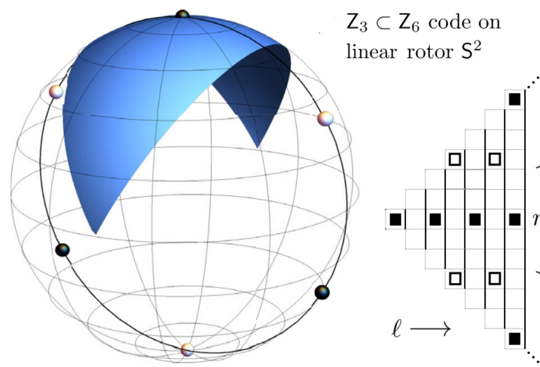

Fig. 8. A qubit encoded in the rotational states of a linear rotor. $N = 3$ here. Equal superpositions of the black and white circles are the logical $|0\rangle$ and $|1\rangle$ states respectively.[1]

The codewords can be written in the fourier conjugate basis as

$$\frac{|0\rangle+|1\rangle}{\sqrt{2}} = \sqrt{2N}\sum_{l=0}^{\infty}\sum_{|2pN|\le l} Y^l_{2pN}(\pi/2,0)\,|{}^{\;l}_{2pN}\rangle \quad (41)$$

$$\frac{|0\rangle-|1\rangle}{\sqrt{2}} = \sqrt{2N}\sum_{l=0}^{\infty}\sum_{|(2p+1)N|\le l} Y^l_{(2p+1)N}(\pi/2,0)\,|{}^{\;\;\;\;\;\;\;l}_{(2p+1)N}\rangle$$

4.1.2 Error Correction

Without proof, the conditions([3] Thm 10.1) which the set of correctable position shifts satisfy are

$$\langle 0|\hat{X}_R^\dagger\hat{X}_R|0\rangle = \langle 1|\hat{X}_R^\dagger\hat{X}_R|1\rangle \quad (42)$$

$$\langle 0|\hat{X}_R^\dagger\hat{X}_R|1\rangle = 0 \quad (43)$$

Intuitively, the first condition says that the combined action of the error and its correction should be identical on every word in the codespace. The second condition prevents us from obtaining a codeword orthogonal to the original encoded state.

The condition in eqn. (42) is satisfied automatically if we consider N to be odd. This is because in such a case, the two code words would be related by just an inversion operation

$$\hat{X}_P\,|0\rangle = |1\rangle\,,\ \hat{X}_P\,|1\rangle = |0\rangle \quad (44)$$

and since the inversion operation commutes with every rotation,

$$\langle 0|\hat{X}_R^\dagger\hat{X}_R|0\rangle = \langle 0|\hat{X}_R^\dagger\hat{X}_R\hat{X}_P|1\rangle = \langle 0|\hat{X}_P\hat{X}_R^\dagger\hat{X}_R|1\rangle = \langle 1|\hat{X}_R^\dagger\hat{X}_R|1\rangle$$

The second condition gives us the restrictions we desire on how “small” R should be i.e. what the set of correctable shifts $\{\hat{X}_R\}$ is.

The answer is easy to see intuitively. Remember the Voronoi cell in the U_1 space which was an arc spanning $\pi/3$ radians (fig(4)). The error-ridden state after a shift lying anywhere in that arc could be reliably projected on the nearest codeword. Here, the Voronoi cell is a spherical tile with the edges separated by an angle $\pi/3$ (blue region in fig.(8)). The idea is the same, the shift will be correctable as long as it can be identified with its nearest codeword unambiguously. This means the error on a codeword should cause it to shift inside a region that is closest to that word than any other. That is exaactly what the Voronoi cell is. Thus, the position shifts lying inside the Voronoi cell are correctable.

Let's consider momentum shifts now. Shifts in momenta in the S^2 space are given by the operators

$$\hat{Y}^l_m = \int_{S^2} dv \,|v\rangle\, Y^l_m(v)\,\langle v| \tag{45}$$

Our codewords in momentum space (eqn.(41)) have support only over those values of m which are integer multiples of N and the two codewords are shifted in their m values by the value N. Also, the theory of addition of angular momenta dictates that a momentum shift $\hat{Y}^{l_1}_{m_1}$ acting on a state with angular momentum (l_2, m_2) produces a superposition of states with

$$l \in \{|l_1 - l_2|, |l_1 - l_2| + 1, \cdots, |l_1 + l_2| - 1, |l_1 + l_2|\}$$

and

$$m = m_1 + m_2$$

Putting all this information, we can conclude that if the error operator satisfy

$$l_1 < N/2$$

then the shifts in m would be unambiguously diagnosed as well and the error correction would be successful.

Unlike the U_1 code, however, this code fails to correct position and momenta shifts acting simultaneously even if those errors are correctable on their own. This is because some elements of the codestates would be acted upon trivially by some rotation shifts as indicated in eqn.(35). To see why this leads to code failure, remember that the errors should not differentiate between the codewords i.e. it is necessary that

$$\langle 0|\hat{Y}^l_m \hat{X}_R|0\rangle = \langle 1|\hat{Y}^l_m \hat{X}_R|1\rangle \tag{46}$$

for correctable combinations of shifts. But suppose now that the rotation applied has a fixed point $|v\rangle$ as an element (i.e. part of the superposition states) in $|0\rangle$. Then $|-v\rangle$ would be a part of $|1\rangle$ since it contains all antipodal points of $|0\rangle$. $|0\rangle$ can be written as a superposition of the form $\frac{1}{\sqrt{N}}|v\rangle + \frac{1}{\sqrt{N}}\sum_i |a_i\rangle$. Then

$$\langle 0|\hat{Y}^l_m \hat{X}_R|0\rangle = \langle 0| \left(\int_{S^2} \frac{dv'}{N}\,|v'\rangle\, Y^l_m(v')\,\langle v'|\right)\hat{X}_{(v,w)}|0\rangle$$

$$= \int_{S^2} \frac{dv'}{N}\left(\langle v|v'\rangle\, Y^l_m(v')\,\langle v'|\hat{X}_{(v,w)}|v\rangle + \sum_{i,j}\langle a_j|v'\rangle\, Y^l_m(v')\,\langle v'|\hat{X}_{(v,w)}|a_i\rangle\right)$$

$$= \int_{S^2} dv'_N \,\langle v|v'\rangle\, Y^l_m(v')\,\langle v'|\hat{X}_{(v,w)}|v\rangle = \frac{Y^l_m(v)}{N}$$

Similarly,

$$\langle 1|\hat{Y}^l_m \hat{X}_R|1\rangle = \frac{(-1)^l Y^l_m(v)}{N} \neq \langle 0|\hat{Y}^l_m \hat{X}_R|0\rangle$$

Thus, the error correction condition is not satisfied when both types of error occur simultaneously. Notice that this problem arises only when there are some fixed points of the rotation group in the codewords. Else

$$\langle 1|\hat{Y}^l_m \hat{X}_R|1\rangle = \langle 0|\hat{Y}^l_m \hat{X}_R|0\rangle = 0$$

and we would be able to correct errors of this kind. We will see that this is the case in the rigid rotor codes as discussed in section 4.2.

4.1.3 Stabilisers and Logical operators

The two logical states are superpositions of states that differ by $2\pi/N$ angular positions in the equatorial circle (fig. 8). Thus, rotations of $2\pi/N$ about the z-axis leave the logical $|0\rangle$ and $|1\rangle$ invariant. So, $\hat{X}_{R=(z,2\pi/N)}$ acts trivially on the codespace.

The other stabiliser for this code is the operator

$$\hat{Y}^{2N}_{2N} = \int_{S^2} dv\, C(2N)\sin^{2N}(\hat{\theta})e^{i2N\hat{\phi}}\,|v\rangle\langle v| \tag{47}$$

where $C(x) = \frac{(-1)^x}{2^x x!}\sqrt{\frac{(2x+1)!}{4\pi}}$. We can check that this operator stabilises the code

$$\hat{Y}^{2N}_{2N}|0\rangle = \frac{1}{\sqrt{N}}\sum_{h\in\mathbb{Z}_N}\left(\int_{S^2} dv\, C(2N)\sin^{2N}(\hat{\theta})e^{i2N\hat{\phi}}\,|v\rangle\langle v|\right)|\frac{\pi}{2}, \frac{2\pi}{N}h\rangle$$

$$= \frac{C(2N)}{\sqrt{N}}\sum_{h\in\mathbb{Z}_N} e^{i2N(2\pi/N)h}\,|\frac{\pi}{2}, \frac{2\pi}{N}h\rangle = C(2N)\,|0\rangle$$

and

$$\hat{Y}^{2N}_{2N}|1\rangle = \frac{C(2N)}{\sqrt{N}}\sum_{h\in\mathbb{Z}_N} e^{i2N\{(2\pi/N)h+\pi/N\}}\,|\frac{\pi}{2}, \frac{2\pi}{N}h + \frac{\pi}{N}\rangle$$

$$= C(2N)\,|1\rangle$$

Thus, the stabiliser generators of this code are

$$\hat{S}_X = \hat{X}_{R=(z,2\pi/N)} \tag{48}$$

$$\hat{S}_Z = \frac{1}{C(2N)}\hat{Y}^{2N}_{2N} \tag{49}$$

For the logical operators, we're already seen that the inversion operator switches between the two codewords and hence, acts as the X operator on the logical qubit. The equivalent of the Z operator on our logical qubit is

$$\hat{Y}_N^N = \int_{S^2} dv C(N) \sin^N(\hat{\theta}) e^{iN\hat{\phi}} |v\rangle \langle v| \tag{50}$$

We can check its action on the logical qubit states

- $\hat{Y}_N^N |0\rangle = \frac{C(N)}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} e^{iN(2\pi/N)h} |\frac{\pi}{2}, \frac{2\pi}{N}h\rangle = C(N)|0\rangle$
- $\hat{Y}_N^N |1\rangle = \frac{C(N)}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} e^{iN(2\pi/Nh+\pi/N)} |\frac{\pi}{2}, \frac{2\pi}{N}h + \frac{\pi}{N}\rangle$
  $= e^{i\pi}C(N)|1\rangle = -C(N)|1\rangle$

Thus, $\hat{Y}_N^N$ §§ does act as the Z operator on the logical qubit. So, we have our set of logical operators for the encoded qubit:

$$\hat{\hat{X}} = \hat{X}_P \tag{51}$$

$$\hat{\hat{Z}} = \frac{1}{C(N)} \hat{Y}_N^N \tag{52}$$

With the understanding of planar and linear rotor codes, we can now go on to describe the error correction codes in the rotational state space of a completely asymmetric molecule.

4.2 General Rigid Rotor - $SO(3)$ space

The angular positions of the general rigid rotor can be labelled by elements of the group $SO(3)$. We can describe the position shifts in this space in analogy with the U_1 and the S^2 space for better understanding.

The **general error operator**, which in the case of U_1 was $\hat{X}_\phi \hat{Z}_l$ is now given by

$$\hat{E} = \int_{SO(3)} dS \sum_{l=0}^{\infty} \sum_{|m|,|n|\leq l} E^l_{mn}(S) \hat{X}_S \hat{D}^l_{mn} \tag{53}$$

The **general error channel** of the form of eqn.(13) can now be described by

$$\mathscr{E}(\rho) = \iint_{SO(3)} dS dS' \sum_{l,m,n} \sum_{l',m',n'} \mathscr{E}^{l\ \ l'}_{mnm'n'}(S,S') \hat{X}_S \hat{D}^l_{mn} \rho \hat{D}^{\dagger l'}_{m'n'} \hat{X}^\dagger_{S'} \tag{54}$$

and we aim to correct the errors on ρ for sufficiently constrained position shifts (S) and momentum shifts (l,m,n).

§§ $\hat{Y}_N^N$ is not actually a unitary operation on the whole state space because the magnitude of Y_N^N is less than 1 for $\theta \neq \pi/2$. So, it is not achievable in the laboratory. But for purposes of correction, if we have diagnosed the error and know the axis our state is oriented against, we can choose the logical $\hat{\hat{Z}}$ appropriately and introduce the required phase in our logical states.

4.2.1 Logical qubit

As in the linear rotor case, we consider the groups $K = \mathbb{Z}_{2N}, H = \mathbb{Z}_N$ to encode our qubit. The codewords are

$$|0\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} |\frac{2\pi}{N}h, z\rangle \tag{55}$$

$$|1\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} |\frac{2\pi}{N}h + \frac{\pi}{N}, z\rangle$$

where $|\alpha, z\rangle$ represents the rotation element R which rotates about the z-axis by an angle α

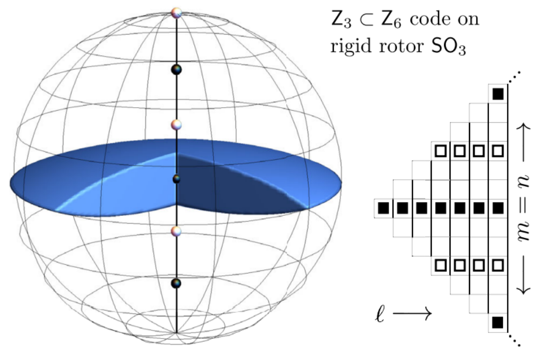

Fig. 9. A qubit encoded in the rotational states of a rigid rotor. $N = 3$ here. Equal superpositions of the black and white circles are the logical $|0\rangle$ and $|1\rangle$ states respectively.[1]

Equivalently, the code states in the momentum basis are

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \sqrt{\frac{N(2l+1)}{4\pi^2}} \sum_{|2pN|\leq l}^{l} |2pN, 2pN\rangle \tag{56}$$

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \sqrt{\frac{N(2l+1)}{4\pi^2}} \sum_{|(2p+1)N|\leq l}^{l} |(2p+1)N, (2p+1)N\rangle$$

These logical states are reminiscent of eqns.(40, 41) of the linear rotor codes because these are just the generalisations of those codes to the full rotation group.

4.2.2 Error Correction

We can now describe the error correction procedure on the defined logical states. Consider position shifts first. As mentioned earlier, the set of correctable errors are labeled by the elements of the coset G/K. Here, $SO(3)/\mathbb{Z}_{2N}$ would be represented by regions in the $SO(3)$ sphere such as that shown by the blue tile in fig.(9). Unsurprisingly, this is also the Voronoi cell for one of the constituent states of the logical $|0\rangle$ state. We again observe that the set of correctable position shifts are those which can be associated unambiguously with its nearest codeword. Thus, all the shifts that lie in this Voronoi cell can be corrected. The procedure to do so is the following. Let $\bar{S}$ be an element in this set of correctable shifts. Then,

$$\hat{X}_{\bar{S}}\,|R\rangle = |\bar{S}R\rangle$$

We can first diagnose the shift by measuring $\bar{S}$ and then correct it by applying $\hat{X}^{\dagger}_{\bar{S}}$ to the state and recover the original code state.

Next, we consider momentum shifts. Similar to the linear rotor, these are given by the operators

$$\hat{D}^{l}_{mn} = \int_{SO(3)} dR\,|R\rangle\,D^{l}_{mn}(R)\,\langle R| \tag{57}$$

In this case, the selection rules given by the theory of angular momentum dictates that the action of $\hat{D}^{l_1}_{m_1 n_1}$ on the state $|m_2^{l_2} n_2\rangle$ produces a superposition of states with

$$l \in \{|l_1 - l_2|, |l_1 - l_2| + 1, \cdots, |l_1 + l_2| - 1, |l_1 + l_2|\}$$

and

$$m = m_1 + m_2,\ \ n = n_1 + n_2$$

Also notice that the values of m and n in the angular momentum basis representation of the codewords differ by integer multiples of N. Thus for $l_1 < N/2$, the errors in m, n would be unambiguously diagnosed and could subsequently, be corrected. So the correctable momentum shifts are given by

$$l_1 < N/2$$

just as in the case of the planar and the linear rotor.

We also note that this codespace doesn't have any fixed points under action of any element of $SO(3)$(except the identity - zero rotations). So, this code allows simultaneous position and momentum shifts to be corrected as well.

4.2.3 Stabilisers and Logical operators

The two logical states as defined are superpositions of states that are labelled by elements of $SO(3)$ differing by $2\pi/N$ rotations about the z-axis(fig. 9). Thus, rotations of $2\pi/N$ about the z-axis leave the logical $|0\rangle$ and $|1\rangle$ invariant. So, $\hat{X}_{S=(z,2\pi/N)}$ acts trivially on our codespace.

The other stabiliser for this code is the operator

$$\hat{D}^{2N}_{2N,2N} = \int_{SO(3)} dR\,D^{2N}_{2N,2N}\,|R\rangle\,\langle R| \tag{58}$$

We can use the following identity for rotations about the z-axis

$$D^{l}_{mn}(w,z) = \delta_{mn} e^{imw} \tag{59}$$

to check that it really stabilises the code

- $\hat{D}^{2N}_{2N,2N}\,|0\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} e^{i2N(\frac{2\pi}{N}h)}\,|\frac{2\pi}{N}h, z\rangle = |0\rangle$
- $\hat{D}^{2N}_{2N,2N}\,|1\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} e^{i2N(\frac{2\pi}{N}h+\frac{\pi}{N})}\,|\frac{2\pi}{N}h + \frac{\pi}{N}, z\rangle = |1\rangle$

Thus, $\hat{D}^{2N}_{2N,2N}$ acts trivially on the codespace. The set of stabiliser generators for this code is

$$\begin{aligned} \hat{S}_X &= \hat{X}_{S=(z,2\pi/N)} && (60) \\ \hat{S}_Z &= \hat{D}^{2N}_{2N,2N} && (61) \end{aligned}$$

For the logical operators, we notice that each component of the two logical states differ by a π/N rotation about the z-axis. Thus, a π/N rotation about the z-axis $\hat{X}_{\pi/N,z}$ would act as the logical X operator on the encoded states. The other logical operator is given by

$$\hat{D}^{N}_{N,N} = \int_{SO(3)} dR\,D^{N}_{N,N}\,|R\rangle\,\langle R| \tag{62}$$

Again, we can check its action on the logical qubit states

- $\hat{D}^{N}_{N,N}\,|0\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} e^{iN(2\pi/N)h}\,|\frac{2\pi}{N}h, z\rangle = |0\rangle$
- $\hat{D}^{N}_{N,N}\,|1\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} e^{iN(2\pi/Nh+\pi/N)}\,|\frac{2\pi}{N}h + \frac{\pi}{N}, z\rangle$
  $= e^{i\pi}\,|1\rangle = -\,|1\rangle$

Thus, $\hat{D}^{N}_{N,N}$ [§§§] does act as the Z operator on the logical qubit. So, we have our set of logical operators for the encoded qubit:

$$\begin{aligned} \hat{\hat{X}} &= \hat{X}_{\frac{\pi}{N},z} && (63) \\ \hat{\hat{Z}} &= \hat{D}^{N}_{N,N} && (64) \end{aligned}$$

5. Final comments and future work

1. **More general subgroups**: The codes considered in this review were all of the type $\mathbb{Z}_N \subset \mathbb{Z}_{2N}$ but the procedure can be extended to other subgroups as well with the same idea of identifying G/K as the set of correctable errors and the elements of K/H as labels for the logical states. Some of these non-abelian subgroup spaces (such as the dihedral groups $D_3 \subset D_6$) have been considered in the original paper but omitted here for simplicity. Those codes still correct errors that lie within their Voronoi cells but the geometry of those cells become more complicated for more complex subgroups. An example is shown in fig.(10)

[§§§] As in the linear rotor space, $\hat{D}^{N}_{N,N}$ is not actually a unitary operation on the whole state space because its magnitude is less than 1 for rotations about axes other than z. So, it is not achievable in the laboratory. But again, we can choose the logical $\hat{\hat{Z}}$ appropriately for error correction after diagnosing the orientation of the error ridden state

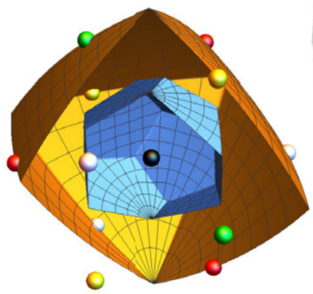

Fig. 10. $SO(3)$code with T(tetrahedral group)$\subset$ I(icosahedral group)$\subset$ $SO(3)$. Equal superpositions of the five different colours are the five encoded logical states. Blue region is the Voronoi cell surrounding the identity element[1]

2. **Qudit codes**: All the explicit constructions in this review have been for qubits i.e. 2 logical states. But the codes can easily be extended to qudits with d logical states by considering the subgroups $\mathbb{Z}_N \subset \mathbb{Z}_{dN}$. For example: the logical states of the linear rotor codes would then be
$$|j\rangle = \frac{1}{\sqrt{N}} \sum_{h\in\mathbb{Z}_N} |\frac{\pi}{2}, \frac{2\pi}{N}h + \frac{2\pi}{dN}j\rangle \tag{65}$$
where $j \in \{0, 1, \cdots d-1\}$ where the logical $|j\rangle$ state is, in general, shifted from the logical $|j-1\rangle$ state by an angle $\frac{2\pi}{dN}$
3. **Groups other than rotation**: Although the interest here was in encoding information in molecules' rotational states, the theory developed can be extended to other groups whose physical realizations may not even be possible. The group G can be any general group with appropriate subgroups and the codewords, stabilisers, logical operators can be defined in similar ways. This claim is backed by the theory of harmonic analysis in group theory which demonstrates a generalised version of fourier analysis, allowing us to define the logical states and operators for more exotic groups too in the generalised momentum bases and perform similar error correction procedures. This is also briefly discussed in the original paper but omitted in this review.
4. **Future work**: To realise this kind of error correction, it is important to establish that molecules that can be used in the lab actually experience noise of the kind that the codes protect against. The first step to that is analysing noise models where the molecule is not made to interact with anything but the environment. If the noise is found to be local (i.e. there are no abrupt changes in position and momenta) in such models, then we can successfully use the error correction methods described here. One such noise model is the generalisation of Brownian motion [4]. This noise model is already found to be local in the U_1 case and further work is ongoing to establish the same result for the $SO(3)$ rigid rotor.

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