Problem Solving Approach
Mathematics for Data Science - 1

Contents

1 Objective

This document aims to provide an initial approach towards solving the problems for the mathematics students registered in this course. Every problem can be solved in many ways, and there is no fixed way to solve all types of problems. This document is meant to show the proceedings towards the solution of some common types of problems.
Note:

- All the problems can not be solved using the approaches discussed in this document, but most of the questions asked in this program can be answered.
- This document is not meant to cover all the syllabus taught in this course.

2 Straight Lines

2.1 Important formulas

- Equation of a line having slope m and $Y-$intercept c:

$$y = mx + c$$

.

- Equation of a line passing through two points (x_1, y_1) and (x_2, y_2):

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

- Equation of a line passing through point (x_1, y_1) with slope m:

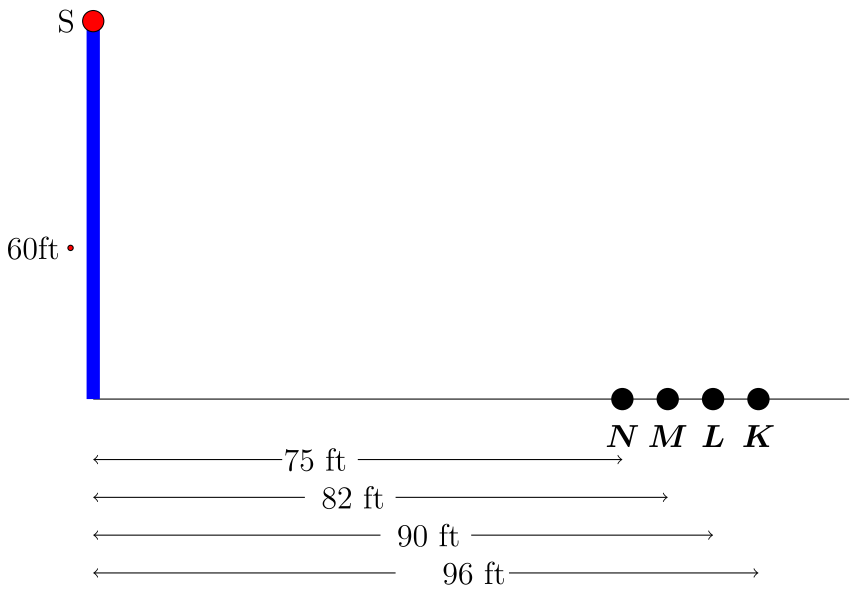
$$(y - y_1) = m(x - x_1)$$

- Equation of a line having $x-$intercept as a and $Y-$intercept as b:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Example:

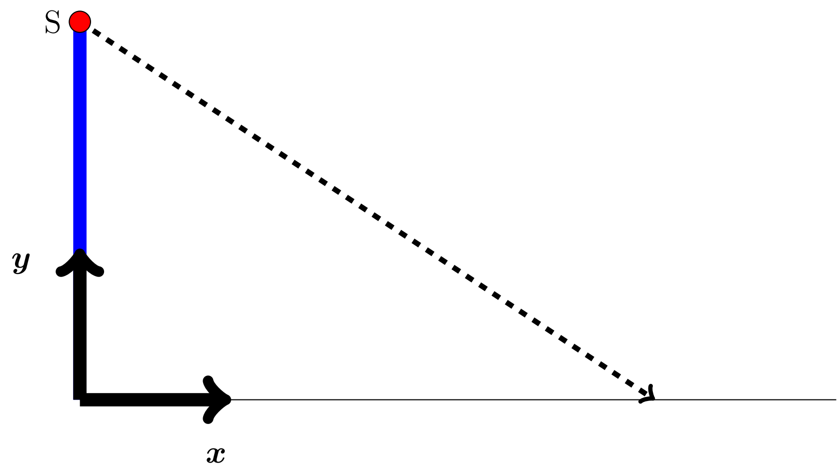
A sniper is sitting on top of a tower at a height of 60 ft. There are four workers K, L, M and N standing at a distance of 96 ft, 90 ft, 82 ft, and 75 ft respectively from the base of the tower. The heights of K, L, M, and N are 6 ft, 5.5 ft, 5.7 ft, and 5.2 ft respectively. The sniper misfires a bullet at an angle θ with the horizontal. Since the range covered by the bullet is short, the path of the bullet is assumed to be a straight-line path. If $\tan\theta = \frac{2}{3}$, choose the correct option.(Assume bullet can not penetrate human beings.)



- ◯ All workers are safe.
- ◯ All the workers are safe except K.
- ◯ Only K and N are safe.
- ◯ No one is safe.
- ◯ Only K is safe.
- ◯ **All the workers are safe except M.**

Solution:

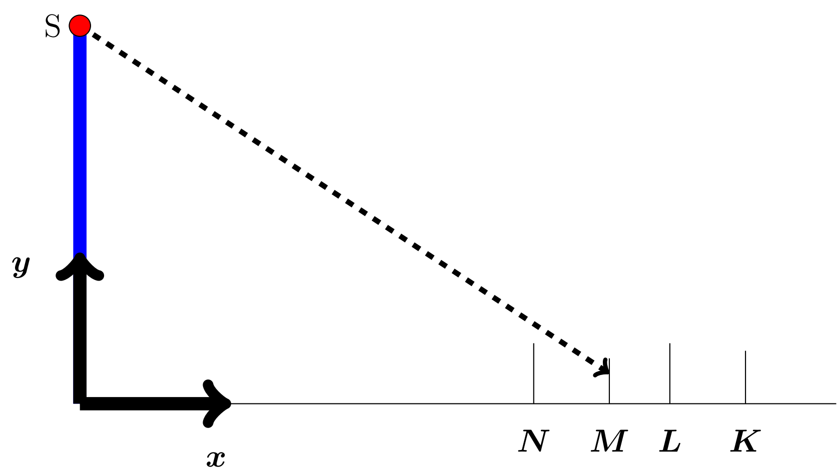
Let us treat height from ground as $Y-$ axis and horizontal distance on ground from tower base as $X-$ axis as shown in Figure.



From figure it is clear that if $x = 0 \longrightarrow y = 60$.
The path of bullet can be written using slope intercept form as $y = mx + c$.
Here $c = 60$ then the path of bullet will be $y = mx + 60$.
Now m will be the slope of line and it is known as $m = tan\ \theta$. According to figure, θ is from -ve $x-$ axis therefore $m = -tan\theta = -\frac{2}{3}$.
There the path of bullet will be $y = -\frac{2}{3}x + 60$.

Worker N is standing at 75 ft away from the tower base which means $x = 75$. Putting this value in the path of bullet $y = -\frac{2}{3} \times 75 + 60 = 10$, where the height of worker is 6 ft. Therefore the bullet will even not touch the worker N.

Similarly for M, $y = 5.333$ which is lower than the height of worker M. It means the bullet will hit worker M and then it will not cross $x = 82$, therefore we do not need to check for others.
Visualization of the above scenario can be scene below:



3 Quadratic

3.1 Important formulas

Let the quadratic function be $f(x) = ax^2 + bx + c$, $a \neq 0$ then,

- Coordinate of vertex: $(-b/2a, (-b^2/4a) + c)$.
- Equation of axis of symmetry: $x = -b/2a$
- $X-$intercepts or zeros: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Discriminant (D): $b^2 - 4ac$
  - $D = 0$: $f(x)$ has only one $X-$intercept (parabola will only touch the $X-$ axis).
  - $D > 0$: $f(x)$ has two $X-$intercepts (parabola will cross the $X-$ axis).
  - $D < 0$: $f(x)$ has no $X-$intercepts (parabola will always be above or lower to the $X-$ axis).
- Equation of a parabola having vertex (h, k): $(y - k) = a(x - h)^2$, where a is a non-zero constant.

3.2 Drawing a parabola

Drawing a parabola by finding the X-intercepts of a quadratic function (i.e., by finding the roots of the quadratic equation corresponding to the quadratic function) is quite useful and interesting. For example, to draw parabola represented by the quadratic function $f(x) = x^2 - 6x - 16$ we can proceed as:

(a) The coefficient of x^2 is positive which means the parabola opens upward (i.e. opens towards the positive direction of Y-axis) .

(b) As the parabola opens upward, we get the minimum value of function and the point where the minimum value is attained is nothing but the vertex of the parabola. The coordinate of the vertex is $(-\frac{b}{2a}, -\frac{b^2}{4a} + c) \equiv (-\frac{-6}{2\times 1}, -\frac{(-6)^2}{4\times 1} + (-16)) \equiv (3, -25)$.

(c) Now the next question is: will the parabola intersect the $X-$ axis, touches the $X-$ axis, or none of the above cases will occur? For that purpose, we can use the discriminant $\sqrt{b^2 - 4ac} = \sqrt{6^2 + 4 \times 16} > 0$ (no need of calculation). The discriminant is greater than zero which means the parabola intersects the X-axis.

(d) If a parabola intersects the $X-$ axis, it intersects the X-axis exactly at two points, which means the function has two distinct zeroes.

(e) For the accurate value of zeroes (or X- intercepts) we need to solve $f(x) = 0 \implies x^2 - 6x - 16 = 0$.

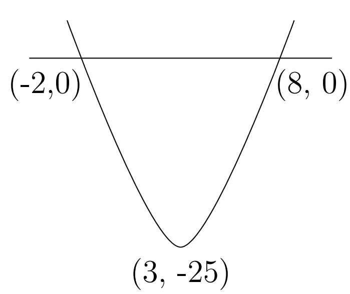
(f) Here $c = -16$ has two factors -8 and 2 which gives $-8 + 2 = -6$. Therefore, the above equation can be solved using factorization method i.e.,

$$x^2 - 6x - 16 = 0 \implies (x - 8)(x + 2) = 0$$

Hence the roots of the quadratic equation are 8 and -2.

(g) Therefore the parabola will intersect the $X-$ axis at $(8, 0)$ and $(-2, 0)$.

(h) Combining all the above information we can draw a rough diagram now as shown below.



We can conclude the following results after watching the above graph.

(a) For the domain $(-\infty, -2) \cup (8, \infty)$ the function has positive values and in the domain $(2, 8)$ the function has negative values.

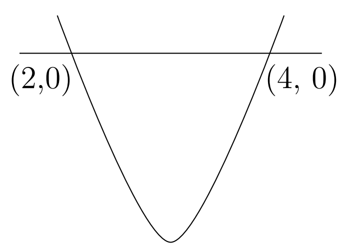
(b) The function $f(x)$ has the range $[-25, \infty)$ or $\mathbb{R} \setminus (-\infty, -25)$.

(c) The same concept can be applied to find the domain of a function f in which the functional value $f(x)$ is greater than some constant c. To solve this kind of questions we can write $f(x) = c$ and solve for x. For example to find the domain when the function $f(x)$ will have value strictly greater than -24, we will solve the equation $f(x) = -24$.

$$f(x) = -24$$

$$x^2 - 6x - 16 = -24$$
$$x^2 - 6x + 8 = 0$$
$$(x-4)(x-2) = 0$$

Using the above procedure we can draw a graph for the function $g(x) = x^2 - 6x + 8 = (x-4)(x-2)$ as shown below.



Now we can conclude that the $g(x)$ has the value strictly greater than zero in $(-\infty, 2) \cup (4, \infty)$. Which means $f(x)$ has the value strictly greater than -24 in $(-\infty, 2) \cup (4, \infty)$.

3.3 Finding the equation of a parabola

The equation of parabola can be found using many approaches but the simplest way is assuming it as a function $f(x) = ax^2 + bx + c$. Here we need to find the three parameters a, b, and c and can be found using the given conditions one by one.

(a) Choosing the axes and the origin solves half of the problems for example if parabola passes through origin, then $c = 0$ and we need to find only a and b.

(b) If parabola has the axis of symmetry at $Y-$ axis, then $b = 0$ and we need to find only a and c. (Finding a and c is much easier than finding a and b).

(c) Example: A gateway is constructed in the shape of a parabola. The road through the gateway has a width of 10m and the maximum height of the gateway is 5m as shown in Figure 1. A container truck of width 4m and uniform height tries to pass through the gateway. What is the maximum possible height of the truck so that it just touches the wall of the gateway?

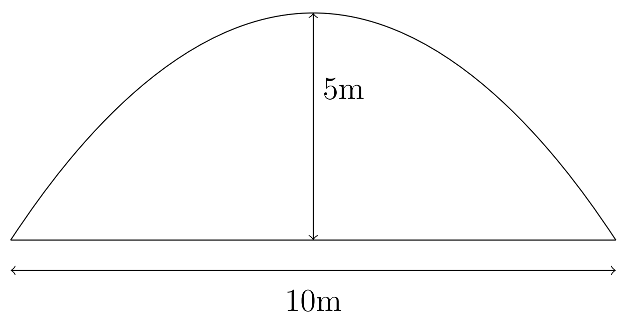
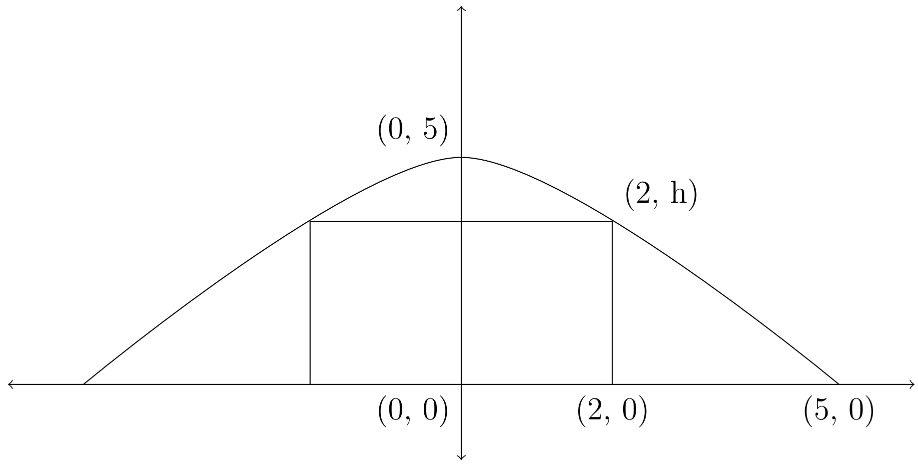


Figure 1

Solution: Let the equation of parabolic gate is represented by $f(x) = ax^2 + bx + c$, and if we assume that the axes is at the middle of gateway, the parabola will be symmetric about Y-axis which means $b = 0$ and $f(x) = ax^2 + bx$. We need to find only a and c now.
As the truck has to pass from the middle of gateway for maximum possible height of truck. The truck will take 2m on both sides of Y-axis. So we just need to find the height of truck we need to find the height of gateway at $x = 2$.



As the parabola passes through the point $(0, 5)$, then solving:

$$f(x) = ax^2 + c = 0$$

$$f(0) = a \times (0)^2 + c = 5$$

$$c = 5$$

$$f(x) = ax^2 + 5$$

The parabola also passes through the point $(5, 0)$. So,

$$f(5) = a \times (5)^2 + 5 = 0 \implies a = -\frac{1}{5}$$

Therefore

$$f(x) = -\frac{1}{5}x^2 + 5$$

$$f(2) = 4.2$$

So the maximum possible height of the truck would be 4.2 m.

3.4 Things to remember

(a) The quadratic equation can be solved using Squaring method, Factorization method, and Quadratic formula. But most of the time the factorization is recommended.

(b) Sometimes, using factorization method we can directly find the solution without writing much. For example to find the solution of $x^2 - 5x + 6 = 0$, we can directly say that $x = 3$ and $x = 2$ are the solutions as $-3 \times -2 = 6$ and $-3 + (-2) = -5$ (not even need to write the factor).

(c) Quadratic equation is not recommended to be solved by square method when the roots are not real.

(d) It is mandatory to think which one is the better root among the two roots of a quadratic equation (assuming there are two distinct roots) for a particular context of a given question. For example if the question has the reference of time, distance or age we can not accept negative solutions. If the question has reference of month number or number of people we can not accept the fractional values.

(e) Discriminant has a lot of applications whenever the question is related with the number of roots of a quadratic equation. Even without solving the equation, we can conclude many important properties of the roots using discriminant.

4 Polynomials

4.1 Finding roots

Note: Here, we assume that all the polynomial (discussed in this section) zeros are real with at least one rational zero (which is not the case for all the polynomials).

Points to remember:

- If $(x-a)$ is a factor of polynomial $p(x)$, then $p(a)$ will be zero.
- If $(x-a)$ is a factor of polynomial $p(x)$, then $p(x)/(x-a)$ will provide the other factor of $p(x)$.

4.2 Problems

1. Find the solutions for the equation $f(x) = x^3 + 9x^2 + 26x + 24 = 0$.

   Solution:
   $f(x)$ is a third-degree polynomial. For polynomial more than the second degree, we do not have any particular formula to find the factors. The best approach here will be that we can at least find one factor, then the other factor will be a quadratic function and we can solve it in our traditional way.

   Step 1: The constant term in $f(x)$ is 24. Factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. We will check one by one for each value.

   Step 2: One thing to notice here is that $f(x)$ does not have any negative coefficient therefore, any positive value of x will never make the function zero. Therefore, we will try for -1, -2, -3... and so on.

   Step 3:
   $$f(-1) \neq 0$$
   $$f(-2) = 0$$
   It means $x = -2$ is a solution for $f(x) = 0$ and therefore $(x + 2)$ will be a factor of $f(x)$.

   Step 4: We will divide $f(x)$ with this factor to get other factors as we have found one factor. On dividing, we will get the other factor as $x^2 + 7x + 12$, which also can be factorized as $(x + 3)(x + 4)$.

   Step 5: Finally we have found our solutions $x = -2, -3$, and -4.

2. Find the solutions for $f(x) = -x^3 + 13x^2 - 20x - 100 = 0$.

Solution:

Step 1: The factors of 100 are 1, 2, 4, 5, 10...and so on. We will check one by one for each value.

Step 2: For $x = 1, f(x)$ is not zero. So (x-1) is not a factor of f(x). Similarly we can find that for $x = 5, f(x)$ is zero which means $(x - 5)$ is a factor of f(x).

Step 3: We can use the same method as we did in question 1. Therefore after dividing with $(x - 5)$ we will get the other factor as $-x^2 + 8x + 20$. And after factorizing it we will get $(10 - x)(x + 2)$. Therefore, our solutions are $x = 5, 10$, and -2.

4.3 Graphing of Polynomial

There is no fixed way to solve all kinds of graphing problems. Each problem has its own way of solving it. But to get an idea of how to start, we can take some problems based on graphing and discuss the approach.

Problem 1: Discuss the graphical nature of polynomial $f(x) = -x^3 + 13x^2 - 20x - 100$

Solution:

Step 1: For drawing the graph of a polynomial we can start with the $X-$intercepts or zeros of the polynomial. We have discussed about finding the $X-$ intercepts earlier (see section on finding roots for your reference).

$$f(x) = -x^3 + 13x^2 - 20x - 100$$

$$f(x) = (x-5)(10-x)(x+2)$$

Step 2: We can come up with the following information about $f(x)$:

- As $f(x)$ is a polynomial of degree three, we can get a maximum of three zeros and two turning points. Between any two X-intercepts, there always be a turning point. (Remember this is not the only way to check the turning points. A polynomial can have turning points without having any zeros such as $x^2 + 1$.)
- For this case as we have three roots and all factors have multiplicities of 1 (odd), so there will be no turning point at intercepts.
- As $f(x)$ can have maximum two turning points and we have three zeros with odd multiplicities we will get one turning point between -2 and 5 and other between 5 and 10.
- The part between two intercepts will be positive or negative might be decided by the multiplicities of $X-$intercept, number of possible turning point, and the end behavior of the polynomials.

Step 3: Since $f(x)$ is an odd degree polynomial, then both of the end of the graph will be in different side of $Y-$ axis based on the nature of coefficient of the leading term.

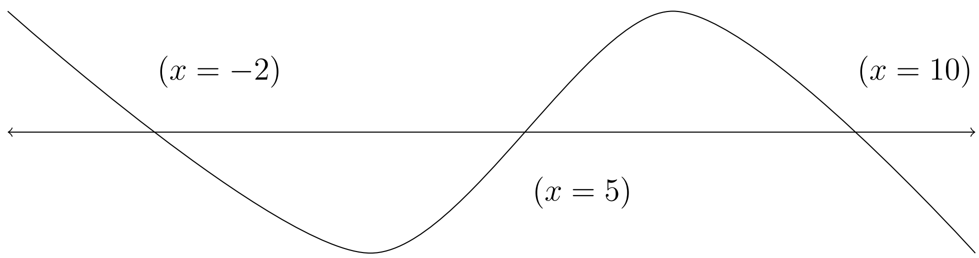
- The coefficient of the leading term is negative then for the graph after $x = 10$ (biggest zero) will be in negative side of $Y-$ axis.
- The coefficient of the leading term is negative then for the graph before $x = -2$ (smallest zero) will be in positive side of $Y-$ axis.
- If the polynomial had coefficient of the leading term positive then the graph after $x = 10$ (biggest zero) would have been in positive side of $Y-$ axis.

- If the polynomial had coefficient of the leading term positive then the graph before $x = -2$ (smallest zero) would have been in negative side of $Y-$ axis.

Step 4: As $f(x)$ will be negative after $x = 10$ then to get a turning point between $x = 5$ and $x = 10$, $f(x)$ will have to be positive between $x = 5$ and $x = 10$.

Step 5: Similarly, $f(x)$ will have to be negative between $x = -2$ and $x = 10$.

Step 5: After collecting all the information as discussed above, we can draw a rough diagram as shown below.



Problem 2: Discuss the graphical nature of polynomial $f(x) = x^4 - 11x^3 + 42x^2 - 64x + 32$

Solution:

Step 1: As factor of 32 are 1, 2, 4,.. and so on and we get $f(1) = 0$, which means $(x-1)$ is a factor of $f(x)$. After dividing $f(x)$ by $(x-1)$, we will get a polynomial of degree three as $g(x) = x^3 - 10x^2 + 32x - 32$. We will again repeat the process for $g(x)$ as we did for $f(x)$. Finally we can conclude that $f(x) = (x-1)(x-2)(x-4)^2$.

Step 2: We can come up with the following information about $f(x)$:

- As $f(x)$ is a four degree polynomial, we can get maximum four roots and three turning points. Between any two $X-$intercepts there always be a turning points. (Remember this is not the only way to check the turning points. A polynomial can have turning points without having any zeros such as $x^2 + 1$.)
- For this case as we have four roots but two roots are equal which means only three distinct roots. $(x-1)$ has multiplicities 1 (odd) therefore, no turning point on $x = 1$ (at $X-$intercept with odd degree graph crosses the $X-$axis) and similarly for $x = 2$.
- $(x-4)$ has multiplicities of 2 (even) therefore, one turning point will be on $x = 4$.
- As $f(x)$ can have maximum three turning points and we have three zeros where two are with odd multiplicities we will get one turning point between 1 and 2 and one between 2 and 4. One turning point will be on $x = 4$ because of even multiplicity.
- The part between two intercepts will be positive or negative might be decided by the multiplicities of $X-$intercept, number of possible turning point, and the end behavior of the polynomials.

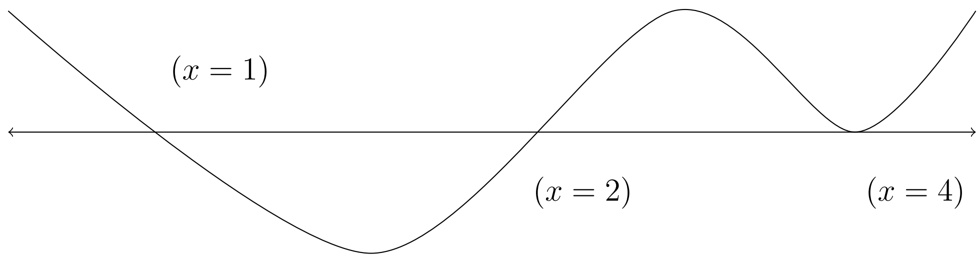
Step 3: Since $f(x)$ is an even degree polynomial, then both of the end of the graph will be in one side of $Y-$ axis based on the nature of coefficient of the leading term.

- The coefficient of the leading term is positive, then for the graph after $x = 4$ (biggest zero) will be in positive side of $Y-$ axis.
- The coefficient of the leading term is positive, then for the graph before $x = 1$ (smallest zero) will be in positive side of $Y-$ axis.
- If the polynomial had coefficient of the leading term negative, then the graph after $x = 4$ (biggest zero) would have been in negative side of $Y-$ axis.
- If the polynomial had coefficient of the leading term negative, then the graph before $x = 1$ (smallest zero) would have been in negative side of $Y-$ axis.

Step 4: As $f(x)$ will be positive after $x = 4$, then to get a turning point between $x = 2$ and $x = 4$, $f(x)$ will have to be negative between $x = 2$ and $x = 4$ but curve will have one turning point at $x = 4$ there $f(x)$ will be positive between $x = 2$ and $x = 4$.

Step 5: Similarly, $f(x)$ will have to be negative between $x = 1$ and $x = 2$.

Step 6: After collecting all the information as discussed above we can draw a rough diagram as shown below.



4.4 Important Formulas

Note: Take all the parameter ($a, b, m, n, ...$ and so on) in the interval where the functions are well defined.

1. $log_a m + log_a n = log_a mn$
2. $log_a m \times log_a n \neq log_a m + log_a n$
3. $log_a m - log_a n = log_a (m/n)$
4. $\frac{log_a m}{log_a n} \neq log_a m - log_a n$
5. $m^{log_a n} = n^{log_a m}$
6. $a^m \times a^n = a^{(m+n)}$
7. $log_a n = \frac{1}{log_n a}$

4.5 Problems

1. Find the values of x satisfying the equation $(m+1)^{log_9(x^2-x+1)} \times (x+1)^{log_9(m+1)} = 1$.

Solution: To solve these types of questions we can follow the following steps:

Step 1: Before finding the value of x we need to find the domain of x and accepted values of m.

- $x^2 - x + 1$ is a quadratic function therefore, domain of $x^2 - x + 1$ is $\mathbb{R}$.
- $x^2 - x + 1$ is an input for the log function and we know that domain of $log\ x$ is $x > 0$. Therefore, $x^2 - x + 1$ should be greater than zero.
- $x^2 - x + 1$ has discriminant value, $1 - 4 = -3$, which is negative. It means $x^2 - x + 1$ does not have any real roots therefore, $x^2 - x + 1$ will always be positive. It means the domain for function $(m+1)^{log_9(x^2-x+1)}$ will be $\mathbb{R}$ and simultaneously for function $(m+1)^{log_9(x^2-x+1)}$ to be defined $m+1$ should be greater than zero which means m should be greater than -1.
- Similarly we can check for $(x+1)^{log_9(m+1)}$ and conclude that m and x both should always be greater than -1.

Step 2: Given equation

$$(m+1)^{log_9(x^2-x+1)} \times (x+1)^{log_9(m+1)} = 1$$

- LHS is the multiplication of two exponential functions. To solve these types of problems, the first possible way is making the base same. If we apply the formula $c^{log_a d} = d^{log_a c}$, then we can write the equation as

$$(m+1)^{log_9(x^2-x+1)} \times (m+1)^{log_9(x+1)} = 1$$

- After making the base same, we can apply formula $a^m \times a^n = a^{(m+n)}$. Now the expression will be

$$(m+1)^{log_9(x^2-x+1)+log_9(x+1)} = 1$$

- To get 1 in LHS too, we need to make the exponent term zero therefore, we will get as

$$log_9(x^2-x+1) + log_9(x+1) = 0$$

- We have formula $log_a m + log_a n = log_a mn$, and if we apply then,

$$log_9(x^2-x+1)(x+1) = 0$$

- Value of $log\ x$ is zero when $x = 1$ therefore our expression reduces to

$$(x^2-x+1)(x+1) = 1$$

$$x^3 - x^2 + x + x^2 - x + 1 = 1$$

$$x^3 = 0 \implies x = 0$$

- $x = 0$ lies in the interval $x > -1$, therefore the solution is $x = 0$.

2. Find the value of x satisfying the equation $18^x - 12^x - (2 \times 8^x) = 0$.

Solution:

Step 1: $18^x - 12^x - (2 \times 8^x)$ is an exponential function therefore, the domain will be $\mathbb{R}$.

Step 2: Given equation

$$18^x - 12^x - (2 \times 8^x) = 0$$

- LHS has only one common thing i.e., exponent x, but we can not apply any formula until bases are same. If we observe properly we can understand that 2 is a common factor for every term.

$$18^x - 12^x - (2 \times 8^x) = 0$$
$$2^x 9^x - 2^x 6^x - 2^x (2 \times 4^x) = 0$$

- As 2^x is a positive function, we can divide by it in both sides and we will get the expression as

$$9^x - 6^x - (2 \times 4^x) = 0$$

- Now we do not have any other common factor, then we can factorize each terms as

$$3^{2x} - 2^x 3^x - (2 \times 2^{2x}) = 0$$

- At this point assuming $3^x = a$ and $2^x = b$ will be a good idea. And our expression reduces to

$$a^2 - ab - 2b^2 = 0, \qquad a > 0,\ b > 0$$

- The middle term $-ab$ can be written as $-2ab + ab$ and the benefit of this step is shown below.

$$a^2 - 2ab + ab - 2b^2 = 0$$
$$a(a - 2b) + b(a - 2b) = 0$$
$$(a + b)(a - 2b) = 0$$

- $a + b$ can not be zero because both are positive and addition of positive numbers are always positive.
- If $a - 2b = 0$, then $a = 2b$ which means,

$$3^x = 2 \times 2^x$$

- Now we can take log on both sides and we will get

$$xlog3 = log2 + xlog2$$

.

- After solving,

$$x = \frac{log2}{log3 - log2}$$

5 Functions and composite functions

5.1 Domain

Domain of a function is the values of x, where the function is defined. For example $log\ x$ is defined for $x > 0$ then the domain of $log\ x$ is $x > 0$ or $(0, \infty)$.

Domain of composite function let us say for $f(g(x))$ is the value of x where $g(x)$ and $f(g(x))$ both are defined. To find the domain of composite functions like $f(g(x))$ we can follow the procedure given below.

- Find the domain of $g(x)$. Let us say it is a set and we call it as g_d.
- Find the values of $g(x)$ for which $f(g(x))$ is defined and let us call it as f_{gd}.
- Find the intersection of g_d and f_{gd}. The intersection set is the domain of given composite function.

5.2 Problems

1. Find the domain of $h(x)$ where $h(x) = e^{(x^2-3x-10)}$.

   Solution:

   Step 1: Let us write $h(x) = f(g(x))$, then $g(x) = x^2 - 3x - 10$ and $f(x) = e^x$.

   Step 2: Find the domain of $g(x) = x^2 - 3x - 10$ i.e., g_d.

   - $g(x)$ is a quadratic function which is defined for all real value of x. Therefore, $g_d = \mathbb{R}$ or $x \in \mathbb{R}$.

   Step 3: The domain of $f(x)$ is also $\mathbb{R}$ as $f(x)$ is an exponential function.

   - As $f(x)$ is defined for $x \in \mathbb{R}$, similarly $f(g(x))$ will be defined for $g(x) \in \mathbb{R}$.
   - Values of $g(x)$ where $f(g(x))$ is defined i.e., $f_{gd} = \mathbb{R}$.

   Step 4: The intersection of g_d and f_{gd} which is $g_d \cap f_{gd} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. Therefore, the domain for $h(x) = e^{(x^2-3x-10)}$ is set of all real numbers.

2. Find the domain of $h(x)$ where $h(x) = log(x^2 - 3x - 10)$.

   Solution:

   Step 1: Let us write $h(x) = f(g(x))$ then $g(x) = x^2 - 3x - 10$ and $f(x) = log\ x$.

   Step 2: The domain of $g(x) = x^2 - 3x - 10$ i.e., $g_d = \mathbb{R}$ or $x \in \mathbb{R}$.

   Step 3: The domain of $f(x)$ is values of x greater than zero i.e., $x > 0$. similarly $f(g(x))$ will be defined when $g(x) > 0$.

   - If
   $$g(x) > 0$$
   $$x^2 - 3x - 10 > 0$$
   $$(x - 5)(x + 2) > 0$$
   - If we plot a graph for $g(x)$ which we have already discussed, there will be one turning point between $x = -2$ and $x = 5$.
   - The coefficient of leading term is positive, parabola will be open upward which means $g(x)$ will be positive after $x = 5$ (the largest zero), will be positive before $x = -2$ (the smallest zero), and will be negative between $x = -2$ and $x = 5$.
   - $g(x) > 0$ when $x \in (-\infty, -2) \cup (5, \infty)$. Which means $f_{gd} = (-\infty, -2) \cup (5, \infty)$.

   Step 4:
   $$g_d \cap f_{gd} = \mathbb{R} \cap (-\infty, -2) \cup (5, \infty)$$
   $$g_d \cap f_{gd} = (-\infty, -2) \cup (5, \infty)$$

   Therefore, the domain for $h(x) = log(x^2 - 3x - 10)$ is $(-\infty, -2) \cup (5, \infty)$.

5.3 Graphing

As we discussed in polynomial graphing that there is no fixed way how to solve the graphing based questions. But again to get some initial approaching idea, we can take some questions and discuss it in detail.

Problem 1: Discuss the graphical nature of $h(x) = log(x^2 - 3x - 10)$

Solution:

Step 1: A composite function is also a function. Therefore, it must be defined for a particular set of values of x which is called the domain of the composite function. Here we can write $h(x) = f(g(x))$ where $g(x) = x^2 - 3x - 10$ and $f(x) = log\ x$.

- As we discussed above that the domain for $h(x)$ will be $(-\infty, -2) \cup (5, \infty)$. Therefore the graph will be drawn in the domain portion only.
- What about $x = -2$ and $x = 5$? As $log\ x$ is undefined for $x = 0$ and it works as an assymptote for $log\ x$. Similarly at $x = -2$, $g(x) = 0$ and $h(0)$ is undefined therefore, $x = -2$ will also work as an assymptote for $h(x)$.

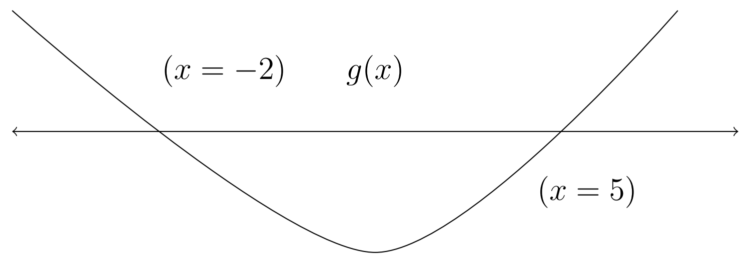
Step 2: As $log\ x$ has $X-$ intercept at $x = 1$ similarly, $log\ x$ will have $X-$intercept at $g(x) = 1$. If

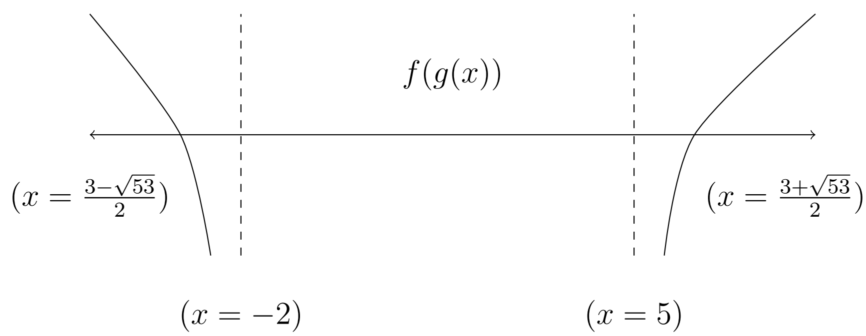
$$g(x) = 1$$
$$x^2 - 3x - 10 = 1$$
$$x^2 - 3x - 11 = 0$$
$$x = \frac{3 \pm \sqrt{53}}{2}$$

Step 3: As $log\ x$ is negative between $x = 0$ and $x = 1$ similarly, $log(g(x))$ will be negative between $g(x) = 0$ and $g(x) = 1$.

- Here we have two x values for $g(x) = 0$ and $g(x) = 1$. Therefore,
- $h(x)$ will be negative between $x = \frac{3-\sqrt{53}}{2}$ and $x = -2$ in left side.
- $h(x)$ will be negative between $x = 5$ and $x = \frac{3+\sqrt{53}}{2}$ in right side.

Step 4: Applying all the information we extracted above we will get a rough diagram as shown below.





5.4 Observations

- Function $g(x)$ is a quadratic function and it is not one to one in its domain. But if we restrict our domain from $(5, \infty)$, then $g(x)$ is one to one function because $g(x)$ is one to one function in the domain $[1.5, \infty)$. Similarly log(x) is also one to one in the domain of $(5, \infty)$ (because $logx$ is one to one function for domain $x > 0$). Composite function $h(x) = log(g(x))$ is also one to one in the same domain. Composite function of one to one functions always be one to one function.

- $g(x)$ is symmetric around $x = 1.5$, similarly $log(g(x))$ is also symmetric around $x = 1.5$.

6 One to One Functions

6.1 Definitions

A function is a one to one function if there are distinct output values for distinct input values. Whether a function is one to one or not can be checked in following ways:

- If for $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$ for all x_1 and x_2 in the defined domain then the function is one to one.
- If function is strictly increasing or in other words if $x_1 > x_2, \implies f(x_1) > f(x_2)$ for all x_1 and x_2 in the defined domain then the function is one to one.
- If function is strictly decreasing or in other words if $x_1 < x_2, \implies f(x_1) > f(x_2)$ for all x_1 and x_2 in the defined domain then the function is one to one.

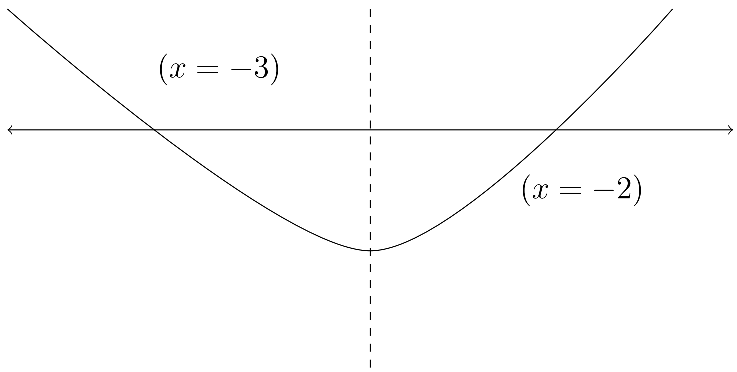
6.2 Problems

1. Find the values of x for which the function $f(x) = 5x + 9$ is one to one function.

   Solution:
   The given function is a linear function and the linear functions are always one to one functions. (Note that we are not talking about the discrete linear functions.)

2. Find the values of x for which the function $f(x) = x^2 + 5x + 6$ is one to one function.

   Solution:
   If we draw the graph of $f(x)$ we can see that it fails the horizontal line test. Which means the function is not one to one in its whole domain.



   But if we see on either side of curve with respect to the axis of symmetry, it is strictly increasing or decreasing. And we know that a function is one to one if it is strictly increasing or decreased in a particular domain.

   Now we can say that $f(x)$ is one to one if $x > -2.5$ (as $x = -2.5$ is the axis of symmetry). Or we can also say that $f(x)$ is one to one if $x < -2.5$.

   Note:

   - A quadratic function is one to one function when considered separately on either side of axis of symmetry.
   - A polynomial always be one to one after the largest $X-$ intercept and before the smallest $X-$ intercept.