Ito Calculus and Ito's lemma

To solve the option pricing problem in the continuous domain one needs a calculus called Ito calculus. The reason that the Ito calculus is different from regular calculus is because Brownian motion paths are not differentiable. We will not go into depth of Ito calculus, but focus on the main result called Ito's lemma.

We define the Ito integral of a process $\Delta(t)$ with respect to a Brownian motion as

$$\int_0^t \Delta(t)\, dW(t) := \lim_{n \to \infty} \sum_{j=0}^{n-1} \Delta(t_j)\left(W(t_{j+1}) - W(t_j)\right)$$

Usually the formulas of Ito calculus are more useful in the ~~differentiable~~ differential form although they always have to be interpreted as integrals. Ito's lemma is a result about integrating functions of a Brownian motion. All the results are in differential form.

Ito's lemma :

- Let $F(W(t))$ then

$$dF = \frac{\partial F}{\partial W} dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} dt \qquad (1)$$

$F(t, W(t))$

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial W} dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} dt \qquad -(2)$$

- $F(t, S(t))$ where $dS = \mu S dt + \sigma S dW$

then $\qquad -(3)$

$$dF = \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 F}{\partial S^2}\right) dt + \sigma S \frac{\partial F}{\partial S} dW(t)$$

Lets take some examples

1) Let $F(W(t)) = W^2(t)$ then

$$dF = \frac{\partial F}{\partial W} dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} dt$$

Thus,

$$dF = 2W\,dW + \frac{1}{2} 2 . dt$$

$$dF = 2W\,dW + dt$$

This means

$$W^2(t) = 2\int W(t)\,dW(t) + t \qquad \text{(Integrating)}$$

Ito's integral. defined earlier.

$F(W(t)) = W^4(t)$

Again

$$dF = \frac{\partial F}{\partial W} dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} dt$$

Therefore,

$$dF = 4W^3\,dW + 6W^2\,dt$$

or in integral form

$$W^4(t) = 4\int_0^t W^3\,dW + 6\int_0^t W^2\,dt \quad \text{(Integrals)}$$

3) $F(t, W(t)) = \cancel{W^4(t)}\, e^{\alpha W(t) - \frac{1}{2}\alpha^2 t}$

We will use Ito's lemma (2)

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial W}dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}dt$$

$$= -\frac{1}{2}\alpha^2 F\,dt + \alpha F\,dW + \frac{\alpha}{2}\frac{\partial}{\partial W}e^{\alpha W(t) - \frac{1}{2}\alpha^2 t}\,dt$$

$$= \cancel{-\frac{1}{2}\alpha^2 F\,dt} + \alpha F\,dW + \cancel{\frac{\alpha^2}{2}F\,dt}$$

$$dF = \alpha F\,dW$$

$$\therefore\quad F = \alpha\int F\,dW$$

4) $F(S(t)) = \ln S(t)$ (Log stock price)

Using Ito's formula (3)

$$dF = \left(\frac{\partial F}{\partial t} + \mu S\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right)dt + \sigma S\frac{\partial F}{\partial S}\,dW(t)$$

$$= \left(0 + \mu S \cdot \frac{1}{S} + \frac{1}{2}\sigma^2 S^2 \cdot -\frac{1}{S^2}\right)dt + \sigma S \frac{1}{S} dWt$$

$$dF = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW(t)$$

$$d \ln S = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW(t)$$

Integrating (assuming μ and σ to be constant)

$$\ln(S(T)) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(W(T) - W(t)\right)$$

OR

$$S(T) = S(t)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(W(T) - W(t)\right)}$$

Thus stock price follows GBM.