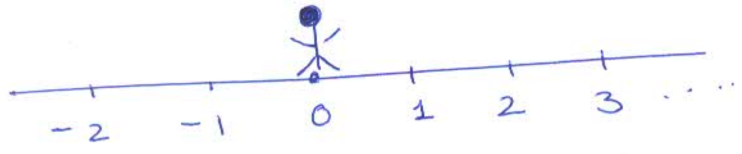
Random Walks

1) Introduction

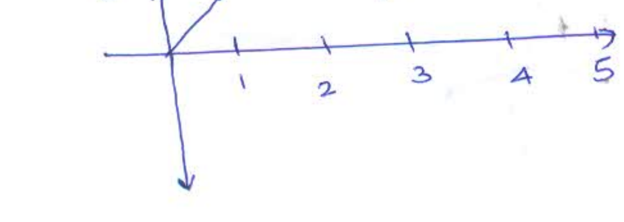


Starts with a person at origin on a lattice ($\mathbb{Z}$). A coin is flipped. If head occurs person moves 1-step to right otherwise one step to right.

$S_n \sim$ Position of random walker at time n

$$S_n = \sum_{i=1}^{n} X_i \quad \text{where} \quad X_i = \begin{cases} 1 & \text{if } H \\ -1 & \text{if } T \end{cases}$$

$P(H) = p$, $P(T) = 1 - p$. If $p = q = \frac{1}{2}$ then it will be called a symmetric random walk.



$$P(S_n = k) = \binom{n}{\frac{n+k}{2}} \frac{1}{2^n}$$

$\#H + \#T = n$

$\#H - \#T = k$

$\#H = \left(\frac{n+k}{2}\right)$

2) Random walks are quite important in engineering, physics and finance. Mathematicians also study properties of random walks.

Some properties of r.w. are

a) First passage time τ_m

$$\tau_m = \{ \min n : S_n = m \}$$

b) Maximum

$$M_n = \{ \max S_i \mid i = 0, \dots n \}$$

c) Return time

$$P(S_n = 0 \text{ for some } n \mid S_0 = 0)$$

We will be interested in a), b), c)

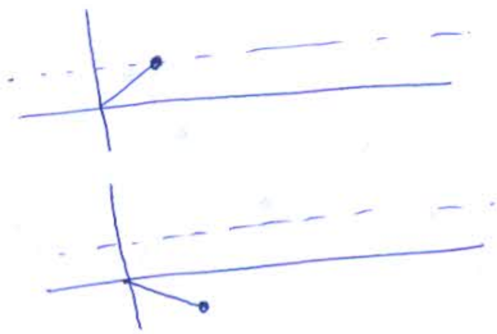
3) <u>First passage</u>

We are interested in the probability distribution $P(\tau_m = k)$

Let $m = 1$. (First time that a random walker hits level 1)

$P(\tau_1 = 2j+1)$ for $j = 0, 1, 2 \dots$

<u>Note:</u> $P(\tau_1 = 2j) = 0$

$P(\tau_1 = 1) = \frac{1}{2}$ $\Longrightarrow$ 

We will solve the general case of

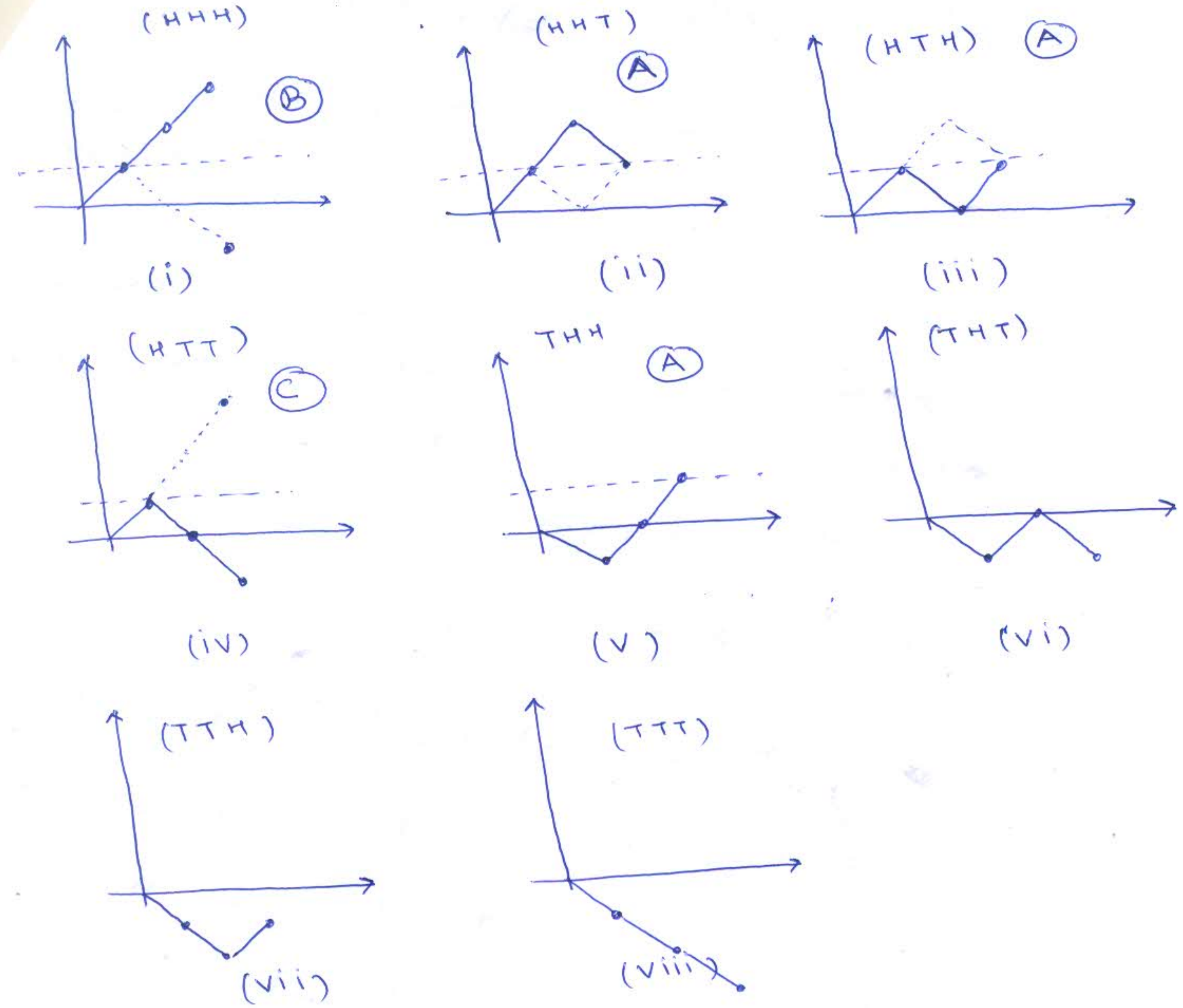
$P(\tau_1 = 2j+1)$. It is convenient to look at

$P(\tau_1 \leq 2j+1)$ (First time r. w. hits level 1 before time $2j+1$)

and observe that $P(\tau_1 = 2j+1) = P(\tau_1 \leq 2j+1) - P(\tau_1 \leq 2j-1)$

(why?)

Now consider the figures below



Consider all paths such that $\tau_1 \leq 2j+1$.

~~We~~ They are a union (disjoint) of paths of type A, B and C.

A ~ Paths that are at level 1 at $2j+1$

B ~ Paths that are strictly above level 1 at time $2j+1$

C ~ Paths that have touched or crossed level 1 at time before $2j+1$ but are at strictly below level 1 at $2j+1$

$P(\tau_1 \le 2j+1) = P(A) + P(B) + P(C)$

<u>Claim</u> $\# B = \# C$ (No. of paths of type B and C are equiv.)

<u>Proof:</u> As shown in figure draw a reflected path from the first time that the r.w. hits level 1

Paths of type C are in 1-1 correspondence with the reflected paths (Paths of type B).

This is called reflection principle.

$$\therefore P(\tau_1 \le 2j+1) = P(A) + 2P(B)$$

$$= P(S_{2j+1} = 1) + 2P(S_{2j+1} > 1)$$

$$= P(S_{2j+1} = 1) + P(S_{2j+1} > 1) + P(S_{2j+1} < -1)$$

$$\therefore P(\tau_1 \le 2j+1) = 1 - P(S_{2j+1} = -1)$$

$$\therefore P(\tau_1 = 2j+1) = P(\tau_1 \le 2j+1) - P(\tau_1 \le 2j-1)$$

$$= P(S_{2j-1} = -1) - P(S_{2j+1} = -1)$$

$$P(S_{2j-1} = -1) = \binom{2j-1}{j-1} \frac{1}{2^{2j-1}} \quad \left(\begin{matrix} j \text{ tails} \\ j-1 \text{ heads} \end{matrix}\right)$$

$$P(S_{2j+1} = -1) = \binom{2j+1}{j} \frac{1}{2^{2j+1}} \quad \left(\begin{matrix} j+1 \text{ tails} \\ j \text{ heads} \end{matrix}\right)$$

$$P(\tau_1 = 2j+1) = P(S_{2j-1} = -1) - P(S_{2j+1} = -1) \xrightarrow{\text{simplify}} \left(\frac{1}{2}\right)^{2j+1} \frac{(2j)!}{j!(j+1)!}$$

✓

Maximum of random walker in n-tosses

$$M_n = \max_{0 \le i \le n} S_i$$

What is the connection between τ_m and M_n?

$$P(\tau_m = n) = P(M_{n-1} = m-1,\ S_{n-1} = m-1,\ S_n = m)$$

We will exploit this to find an alternate derivation of τ_m for all m.

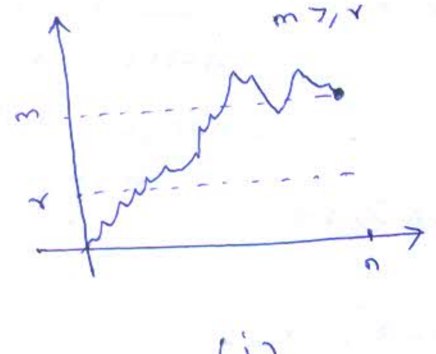
Again note that

$$P(M_n = r) = P(M_n \geq r) - P(M_n \geq r+1) \quad (*)$$

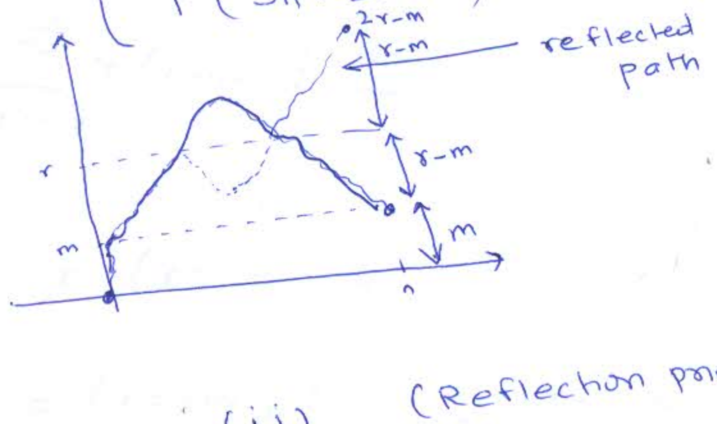
Lets focus on $P(M_n \geq r)$

a). Lemma 1:

$$P(M_n \geq r,\ S_n = m) = \begin{cases} P(S_n = m) & \text{if } m \geq r \\ P(S_n = 2r - m) & \text{if } m < r \end{cases}$$



(i)



(ii) (Reflection principle)

Now we give alternative derivation of $P(\tau_m = n)$

$$P(\tau_m = n) = P(M_{n-1} = S_{n-1} = m-1,\ S_n = m)$$

$$= \frac{1}{2}\Big[P(M_{n-1} \geq m-1,\ S_{n-1} = m-1) - P(M_{n-1} \geq m,\ S_{n-1} = m-1)\Big]$$

we have used $(*)$

then applying lemma 1 we get.

$$P(\tau_m=n) = \frac{1}{2}\left[P(S_{n-1}=m-1) - P(S_n = 2m-(m-1))\right]$$

$$= \frac{1}{2}\left[P(S_{n-1}=m-1) - P(S_n=m+1)\right]$$

Further simplification

$$\boxed{P(\tau_m=n) = \frac{m}{n} P(S_n=m)}$$

Now onward towards $P(M_n=r)$

b) <u>Lemma 2</u> $P(M_n \geq r) = P(S_n=r) + 2P(S_n \geq r+1)$

$$P(M_n \geq r) = P(M_n \geq r, S_n \geq r) + P(M_n \geq r, S_n < r)$$

$$= \sum_{k=r}^{\infty} P(M_n \geq r, S_n = k) + \sum_{j=-\infty}^{r-1} P(M_n \geq r, S_n = j)$$

$$= \sum_{k=r}^{\infty} P(S_n=k) + \sum_{j=-\infty}^{r-1} P(S_n = 2r-j)$$

(By lemma 1)

$$= \cancel{\Sigma}\ P(S_n=r) + \sum_{k=r+1}^{\infty} P(S_n=k) + \sum_{u=r+1}^{\infty} P(S_n=u)$$

(Change of variable $u=2r-j$)

$$P(M_n \geq r) = P(S_n=r) + 2P(S_n \geq r+1)$$

c) <u>~~Lemma 3~~ Theorem</u> $P(M_n=r) = P(S_n=r) + P(S_n=r+1)$

$$P(M_n=r) = P(M_n \geq r) - P(M_n \geq r+1)$$

From lemma 2 we get

$$P(M_n=r) = \left[P(S_n=r)+2P(S_n \geq r+1)\right] - \left[P(S_n=r+1) + 2P(S_n \geq r+2)\right]$$

$$= P(S_n=r) + 2\left[P(S_n \geq r+1) - P(S_n \geq r+2)\right] - P(S_n=r+1)$$

$$P(M_n=r) = P(S_n=r) + P(S_n=r+1)]$$

Return to origin of symmetric RW (1-D, 2-D, 3-D)

$u_{2m} \sim$ Probability of equalization

$$u_{2m} = P(S_{2m}=0 \mid S_0=0)$$

$f_{2k} \sim$ Probability of first return to origin occuring at time $2k$

$$f_{2k} = P(S_2 \neq 0, S_4 \neq 0, \ldots, S_{2k-2} \neq 0, S_{2k}=0 \mid S_0=0)$$

$W_{2n} := \sum_{i=1}^{n} f_{2i}$ is the probability

that the random walker returns to origin occurs no later than $2n$

$W^* := \lim_{n\to\infty} W_{2n} = \sum_{i=1}^{\infty} f_{2i} \equiv$ Probability of eventual return.

Thm 1: $u_{2m} = \binom{2m}{m} \frac{1}{2^{2m}}$

Proof: obvious.

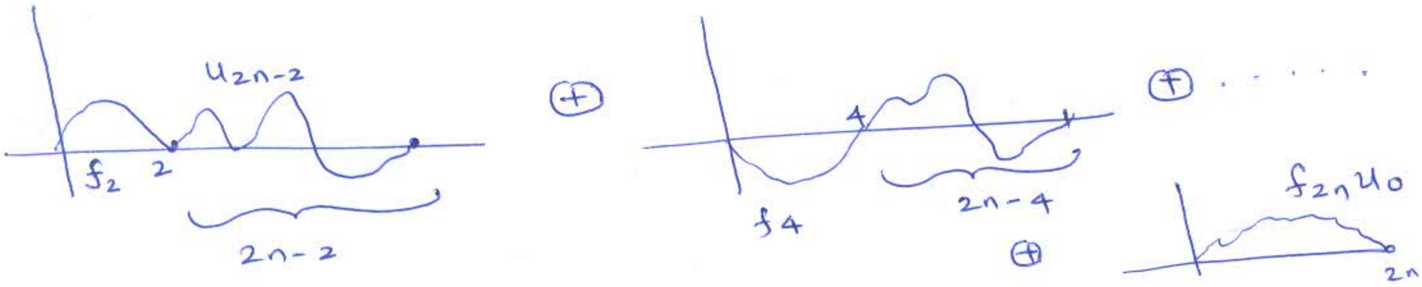
Thm 2: For $n \geq 1$

$$u_{2n} = f_0 u_{2n} + f_2 u_{2n-2} + \cdots + f_{2n} u_0 \qquad \left(\begin{array}{l} u_0 = 1 \\ f_0 = 0 \end{array} \right.$$

Proof: $u_{2n} \sim$ Probability that R.W. returns to origin in $2n$ steps.

Paths that come back to the origin in $2n$ steps are made of the following disjoint paths

Paths that return for first time at time 2 and then are at origin in remaining $2n-2$ steps $\oplus$ Paths that return to origin for first time in 4 steps and then are back at origin in remaining $2n-4$ steps $\oplus$ . . . .



Hence $u_{2n} = f_0 u_{2n} + f_2 u_{2n-2} + \cdots + f_{2n} u_0$

Next, we introduce the concept of generating functions.

$$U(x) = \sum_{m=0}^{\infty} u_{2m} x^m \quad , \quad F(x) = \sum_{m=0}^{\infty} f_{2m} x^m$$

Thm 3 : $U(x) = 1 + U(x) F(x)$

Proof : Directly follows from Thm 2. (check!)

From Thm 3 $\quad F(x) = \dfrac{U(x) - 1}{U(x)}$

We are interested in $W^* = F(1) = \dfrac{U(1)-1}{U(1)}$

Now if $U(1) < \infty$ then $F(1) < 1$

and $U(1) = \infty$ then $F(1) = 1$

(Needs to be technically justified)

Therefore return to origin probability depends on $U(1) = \sum_{m=0}^{\infty} u_{2m}$

If $\sum_{m=0}^{\infty} u_{2m} < \infty \Rightarrow W^* < 1$ otherwise $W^* = 1$.

$$u_{2m} = \binom{2m}{m}\frac{1}{2^{2m}}$$

We use Stirlings appx. to $m! \sim \left(\frac{m}{e}\right)^m \cdot \sqrt{2\pi m}$

we get $u_{2m} \sim \dfrac{1}{\sqrt{\pi m}}$

$$\sum_{m=0}^{\infty} u_{2m} \sim \sum_{m=0}^{\infty} \frac{1}{\sqrt{\pi m}} = \infty .$$

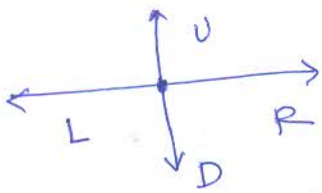
$\therefore$ $\cancel{P}$ $W^* = 1$ for 1-D.

For 2-D.

check.

2-D

$$U^{(2)}_{2m} = \sum_{k=0}^{2m} \frac{2m!}{k!\,k!\,(m-k)!\,(m-k)!}$$



$$P(L) = P(U) = P(D) = P(D) = \frac{1}{4}$$

For return to origin $L = R$, $U = D$

Thm 4: $$U^{(2)}_{2m} = \binom{2m}{m}^2 \frac{1}{4^{2m}}$$

Proof:

$$u^{(2)}_{2m} = \sum_{k=0}^{2m} \frac{2m!\;m!\;m!}{m!\,m!\,(m-k)!\,(m-k)!\,k!\,k!} \cdot \frac{1}{4^{2m}}$$

$$= \sum_{k=0}^{2m} \binom{2m}{m} \cdot \binom{m}{k} \binom{m}{k} \cdot \frac{1}{4^{2m}}$$

$$= \binom{2m}{m}\frac{1}{4^{2m}} \sum_{k=0}^{2m} \binom{m}{k} \cdot \binom{m}{m-k}$$

Now consider the coefficient of x^m in the binomial expansion of $(1+x)^{2m} = (1+x)^m (1+x)^m$

$$LHS = \binom{2m}{m}, \quad RHS = \sum_{k=0}^{2m} \binom{m}{k}\binom{m}{m-k}$$

$$\therefore \binom{2m}{m} = \sum_{k=0}^{2m} \binom{m}{k}\binom{m}{m-k}$$

$$\therefore u_{2m}^{(2)} = \binom{2m}{m}^2 \frac{1}{4^{2m}}$$

Now again we use Stirlings appx. to get

$$u_{2m}^{(2)} \sim \frac{1}{\pi m}$$

$$\sum_{m=0}^{\infty} u_{2m}^{(2)} \sim \sum_{m=0}^{\infty} \frac{1}{\pi m} = \infty$$

In 3-D however a similar analysis leads to $u_{2m}^{(3)} \sim \frac{c}{n^{3/2}}$

and $\sum_{m=0}^{\infty} u_{2m}^{(3)} < \infty$ and hence

$w^* < 1.$

Scaled random walk

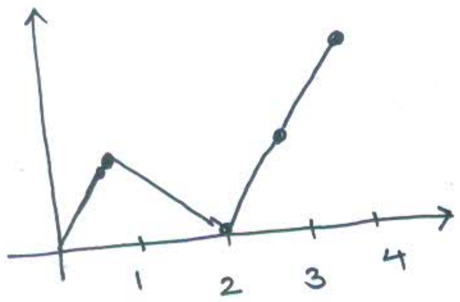
Position of a random walker at time K

$$S_K = \sum_{i=1}^{K} X_i \quad \text{where} \quad X_i = \begin{cases} 1 & \text{if H occurs} \\ -1 & \text{if T occurs.} \end{cases}$$

To go to the scaled random walk we toss the coin more and more frequently (instead of 1 coin toss per unit time, we toss n times per unit time). To avoid blow-up we also scale the r.w. appropriately.

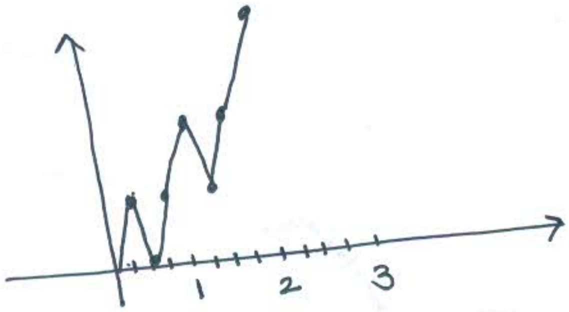
If we toss the coin n times per unit interval, the position of the random walker in t units of time (assuming nt is an integer) is

$$S_{nt} := \sum_{i=1}^{nt} X_i$$



Regular r.w.

(i)



R.W. with coin being tossed 4 times per unit ime.

(ii)

To avoid the blow-up of the random walkers position we scale by a factor of $\sqrt{n}$. So the scaled random walk denoted by $W^{(n)}(t)$ is given by $W^{(n)}(t) = \dfrac{S_{nt}}{\sqrt{n}} = \dfrac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}$

Here n denotes the number of coin tosses per unit time and t denotes the current time.

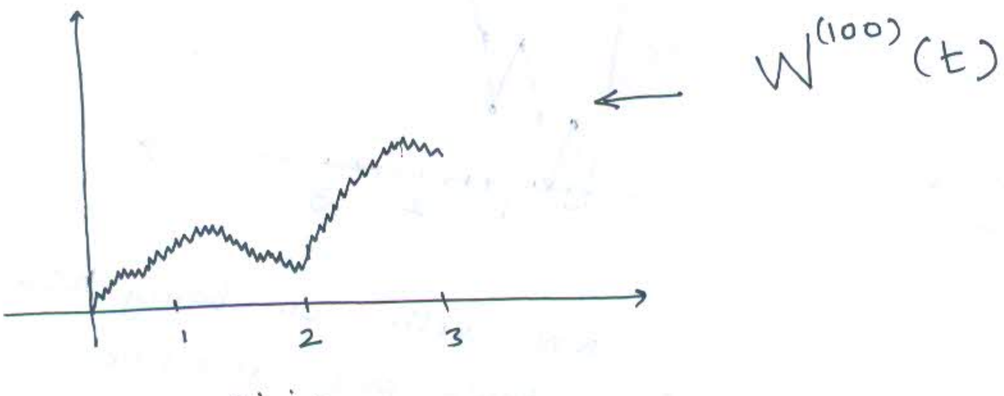
Lets take an example, we would like to calculate $P\left(W^{(100)}(0.25) = 0.1\right)$

The values of $W^{100}(0.25)$ range from -2.5 (all tails) to 2.5 (all heads)

$(-2.5, -2.3, \ldots\ldots -0.1, 0.1, 0.3, \ldots\ldots, 2.5)$

The value 0.1 occurs when there are 13 H and 12 tails so $P\left(W^{(100)}(0.25) = 0.1\right) = \binom{25}{13}\dfrac{1}{2^{25}}$

$\cong 0.155$.



(iii)

The mathematical way of arriving at a Brownian motion is taking the limit as $n \to \infty$

$$W(t) := \lim_{n\to\infty} W^{(n)}(t) = \lim_{n\to\infty} \frac{S_{nt}}{\sqrt{n}}$$

Expectation and variance of a scaled r.w.

$$W^{(n)}(t) = \frac{\sum_{i=1}^{nt} \cancel{S_t} X_i}{\sqrt{n}}$$

(linearity of expectation)

$$\mathbb{E}\left(W^{(n)}(t)\right) = \mathbb{E}\left(\frac{\sum_{i=1}^{nt} \cancel{S_i} X_i}{\sqrt{n}}\right) = \frac{\sum_{i=1}^{nt} \mathbb{E}(\cancel{S_i} X_i)}{\sqrt{n}} = 0$$

$$\mathrm{Var}\left(W^{(n)}(t)\right) = \mathrm{Var}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{n}}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{nt} \mathrm{Var}(X_i) \quad (\text{Since } X_i \text{ are ind.})$$

$$= \frac{1}{n}\sum_{i=1}^{nt} 1 = \frac{1}{n} nt = t$$

Thus $\mathbb{E}\left(W^{(n)}(t)\right) = 0$, $\mathrm{Var}\left(W^{(n)}(t)\right) = t$

We defined the Brownian motion as the limit of the scaled r.w. as $n \to \infty$. Lets look at some of the properties of the Brownian motion. These properties carry over from the scaled walks

Properties of $W(t)$ (Brownian motion)

(i) $W(0) = 0$

(ii) $\mathbb{E}(W(t)) = 0$, $\mathrm{Var}(W(t)) = t$

(iii) $W(t) \sim N(0, t)$ (This is a consequence of the central limit thm)

Normal random variable

(iv) Ind. increments $0 = t_0 < t_1 < t_2 < t_3 \ldots < t_m$

$W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$, ... $W(t_m) - W(t_{m-1})$

are independent of each other

(v) $W(t) - W(s) \sim N(0, t-s)$

(vi) $W(t)$ is continuous everywhere but differentiable nowhere.