[redacted]

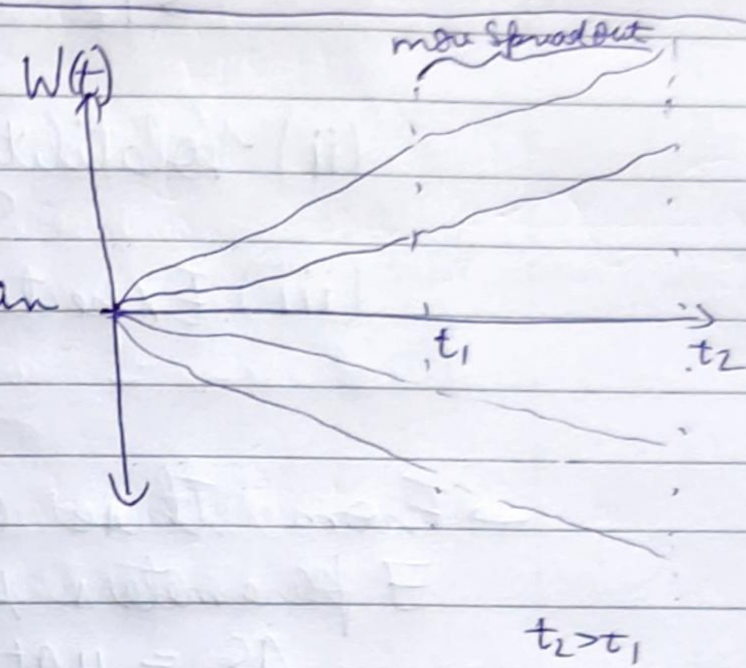
→ Continuous R.V.

$$\lim_{n \to \infty} W^{(n)}(t) = W(t)$$

↓ discrete R.V.

↓ What is probability distribution of $W(t)$ for fixed t?

CF



→ For fixed t, $W(t)$ follows the probability distribution of Gaussian distribution

$\therefore W(t)$ is $\underset{\text{mean}\ \ \ \text{variance}}{\overset{\text{normal random var.}}{N}(0, t)}$

Properties of $W(t)$

1) $W(0) = 0$
2) $E(W(t)) = 0$, $Var(W(t)) = t$
3) $W(t) = N(0, t)$
4) $0 = t_0 < t_1 < t_2 < \dots < t_m$

   $W(t_1) - W(t_0)$, $W(t_2) - W(t_1) \dots W(t_m) - W(t_{m-1})$

   are independent of each other
5) $W(t) - W(s) = N(0, t-s)$; $t > s$
6) $W(t)$ is continuous everywhere but differentiable nowhere

log Normal distribution of Geometric Brownian motion (GBM) (avg. Value is exp.)

$\boxed{\text{Model of stock price} \sim e^{W(t)}}$

There might be some parameters like:
$e^{\mu t + \sigma W(t)}$

Geometric Brownian Motion

Some Observations from stock market

(i) Returns are normally distributed

$$\left.\frac{\Delta S}{S}\right|_t = \frac{S(t+\Delta t) - S(t)}{S} \sim N(\mu \Delta t, \sigma^2 \Delta t)$$

(ii) Volatility σ is independent of stock price

(iii) Expected return $\mathbb{E}\left(\frac{\Delta S}{S}\right) = \mu \Delta t$ is also independent of stock price

→ From these observations one can say that $\exists$ parameters μ & σ such that

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma Z \sqrt{\Delta t} \text{ ; where } Z \sim N(0,1)$$

Taking $\mathbb{E}$ on both sides → $\mathbb{E}\left(\frac{\Delta S}{S}\right) = \mu \Delta t$

Taking Var " " " → $\mathrm{Var}\left(\frac{\Delta S}{S}\right) = \sigma^2 \Delta t$

⇒ A stock with expected return $\mu = 15\%$ p.a. and volatility $\sigma = 30\%$ per annum is given by

$$\frac{\Delta S}{S} = 0.15 \Delta t + 0.3 \sqrt{\Delta t}\, Z$$

[Take $\Delta t = 0.0192$ (per week) (252 Work days)]

$$\therefore \frac{\Delta S}{S} = (0.15)(0.0192) + (0.3)\sqrt{0.0192}\, Z$$

$$\boxed{\Delta S = \underbrace{0.002885}_{\text{Non-random path}} + \underbrace{0.0416\, S Z}_{\text{random path cause of Gaussian } Z}}$$

→ Estimation of $\mu \Delta t$: can be found by taking average of $\frac{\Delta S}{S}$ values.

QFT-II → 1 year
string theory →
M theory →

CF
Stochastic Calcus

Stock return $\frac{\Delta S}{S} = \mu \Delta t + \sigma \sqrt{\Delta t}\, \underbrace{Z}_{N(0,1)} \longrightarrow N(\mu \Delta t, \overbrace{\sigma \sqrt{\Delta t}}^{\text{Std. dev}})$

$$\Delta S = \mu S \Delta t + \sigma \sqrt{\Delta t}\, Z$$

Going to infinitesimal form

$$dS = \mu S dt + \sigma S \underbrace{dW}_{\text{Brownian motion}}$$

Ito Calculus:

$W(t)$ is Brownian motion

Ito's lemma different forms

$F(W(t))$

$$dF = \frac{\partial F}{\partial W} dW + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} dt$$

$F(t, W(t))$

$$dF = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial W} dW + \frac{1}{2} \frac{\partial^2 F}{\partial W^2} dt$$

→ $F(t, S(t))$; $dS = \mu S dt + \sigma S dW$

$S(t)$ is stock price

then:

$\frac{1}{S}$

$\frac{-1}{S^2}$

$$dF = \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW$$

$12W^2$ $\quad e^{\tau W}\ e^{-\frac{1}{2}\alpha^2 t}$

Example: $F(W(t)) = W^2(t)$

$$dF = 2W\,dW + \frac{1}{2}(2dt)$$

$$\boxed{dF = 2W\,dW + dt}$$

$F(W(t)) = W^4(t)$

$$dF = 4W^3 dW + \frac{1}{2}(12W^2)\,dt$$

$$\boxed{dF = 4W^3 dW + 6W^2 dt}$$

$$F(t, W(t)) = e^{\alpha W(t) - \frac{1}{2}\alpha^2 t}$$

$e^{\alpha x(t)}$

$$\therefore dF = -\frac{1}{2}\alpha^2 e^{\alpha W(t) - \frac{1}{2}\alpha^2 t} \cancel{\left(\alpha W(t) - \frac{1}{2}\alpha^2 t\right)}\,dt$$

$$+ \alpha\, e^{\alpha W(t) - \frac{1}{2}\alpha^2 t} \cancel{\left(\alpha W(t) - \frac{1}{2}\alpha^2 t\right)}\,dW$$

$e^{\sin x}$

$$+\frac{1}{2}\left[\alpha e^{\alpha W(t) - \frac{1}{2}\alpha^2 t} + \alpha^2 e^{\alpha W(t) - \frac{1}{2}\alpha^2 t}\right.$$

$\sin x \cos x$

?

$$= -\frac{1}{2}\alpha^2 F$$

$$dF = \alpha F\, dW$$

$\longrightarrow$ $F(S(t)) = \ln S(t)$

$$d(\ln(S(t))) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dW$$

Integrating:

$$\ln\left(\frac{S(T)}{S(t)}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W(T) - W(t))$$

GBM Nature of Stock price

$$S(T) = S(t)\, e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W(T) - W(t))}$$

$$S(t) = S(0)\, e^{\left(\mu - \sigma^2/2\right)t \pm \sigma W(t)}$$