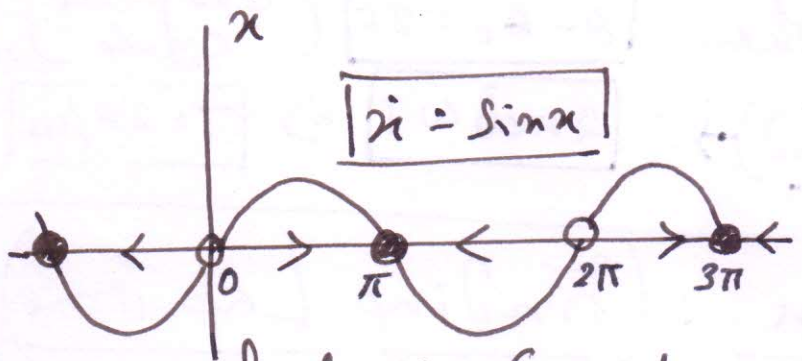
Nonlinear Dynamics: Flows on the Circle

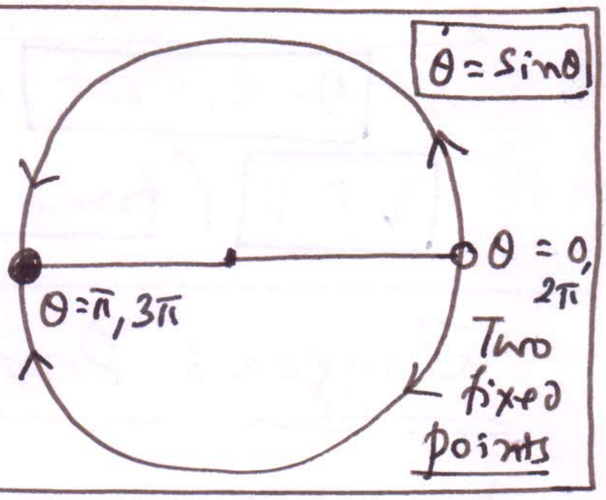
$\boxed{\dot{x} = f(x)}$ is a flow on the line.

$\boxed{\dot{\theta} = f(\theta)}$ is a flow on the circle.
(also a one-dimensional system).

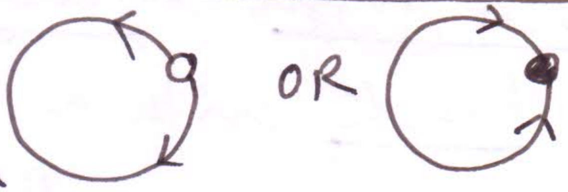
Example: $\boxed{\dot{x} = \sin x}$ and $\boxed{\dot{\theta} = \sin\theta}$



Infinite fixed points

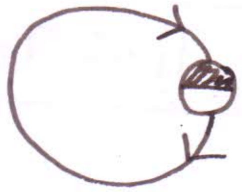


Impossible situations:



There cannot be an odd number of fixed points on the circle.

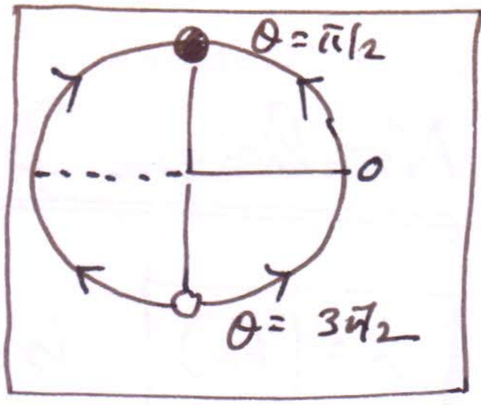
Exception: Odd number of half-stable points is allowed.



Nature of $f(\theta)$: 1/. $f(\theta)$ is real 2/. $f(\theta)$ is 2π periodic.

Example: $\boxed{\dot{\theta} = \theta}$ is NOT a 2π periodic function.

So $\theta = 0 \Rightarrow \dot{\theta} = 0$, and $\theta = 2\pi \Rightarrow \dot{\theta} \neq 0$ (not unique at $\theta = 0, 2\pi, \ldots$)



Example : $\boxed{\dot{\theta} = \cos\theta}$

The fixed points are shifted by a phase of $\pi/2$.

Uniform Oscillator

$\theta \rightarrow$ phase, angle or phase angle.

$\boxed{\dot{\theta} = \omega \text{ (constant)}} \Rightarrow \boxed{\theta = \omega t + \theta_0}$.

$\Rightarrow \boxed{\theta - \theta_0 = \omega t}$. when $\boxed{\theta - \theta_0 = 2\pi}$ (one full cycle), $\boxed{t = T}$ (time period) $\rightarrow \therefore \boxed{2\pi = \omega T} \Rightarrow \boxed{T = 2\pi/\omega}$

Jogger's Problem : (Finding Lap Time)

1) Faster jogger Speedy runs on a circular track with a time period of T_1.

2) Slower jogger Pokey has a time period T_2.

$\therefore \boxed{T_1 < T_2}$. Also $\boxed{\dot{\theta}_1 = \omega_1}$ and $\boxed{\dot{\theta}_2 = \omega_2}$
(both have uniform jogging rate).

Phase difference, $\boxed{\phi = \theta_1 - \theta_2}$

$$\Rightarrow \dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2$$

$$\Rightarrow \boxed{\frac{d\phi}{dt} = \omega_1 - \omega_2} \Rightarrow \boxed{dt = \frac{d\phi}{\omega_1 - \omega_2}}$$

(P.T.O.)

(continued)

$$\Rightarrow \int_0^{T_{lap}} dt = \int_0^{2\pi} \frac{d\phi}{\omega_1-\omega_2}$$

Now $\boxed{\omega_1 = 2\pi/T_1}$ and $\boxed{\omega_2 = 2\pi/T_2}$

$$\Rightarrow \boxed{T_{lap} = \frac{2\pi}{\omega_1-\omega_2}}$$

One full lap implies 2π phase ~~and~~ difference

$$\Rightarrow \boxed{T_{lap} = \frac{1}{T_1^{-1}-T_2^{-1}} = \left(\frac{1}{T_1}-\frac{1}{T_2}\right)^{-1}}$$

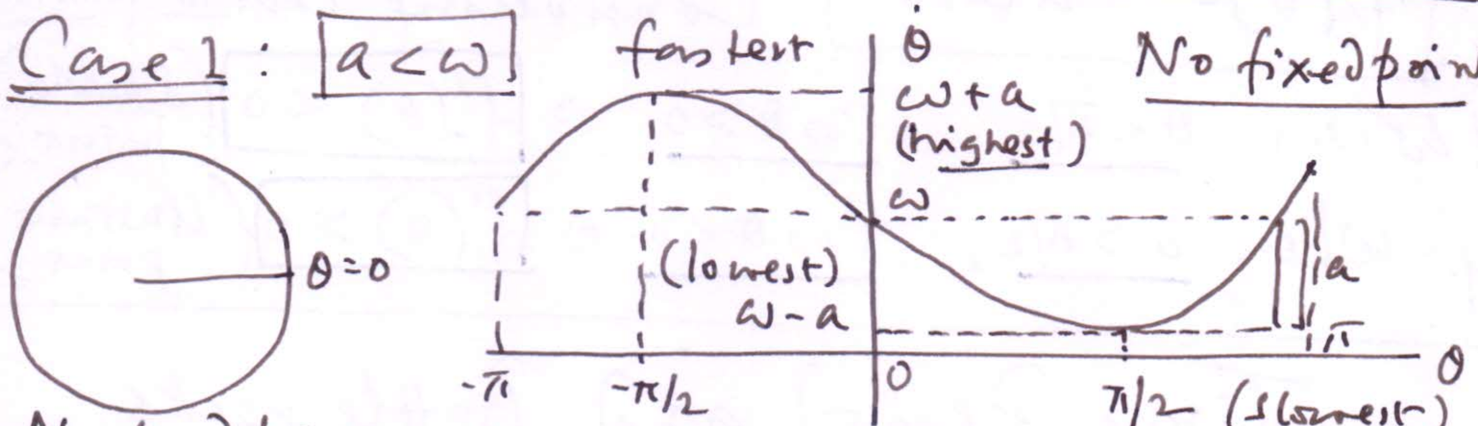
(Beats)

Non-Uniform Oscillator

$\boxed{\dot\theta = \omega - a\sin\theta}$ with $a>0$, $\omega>0$.

Case I: $\boxed{a<\omega}$

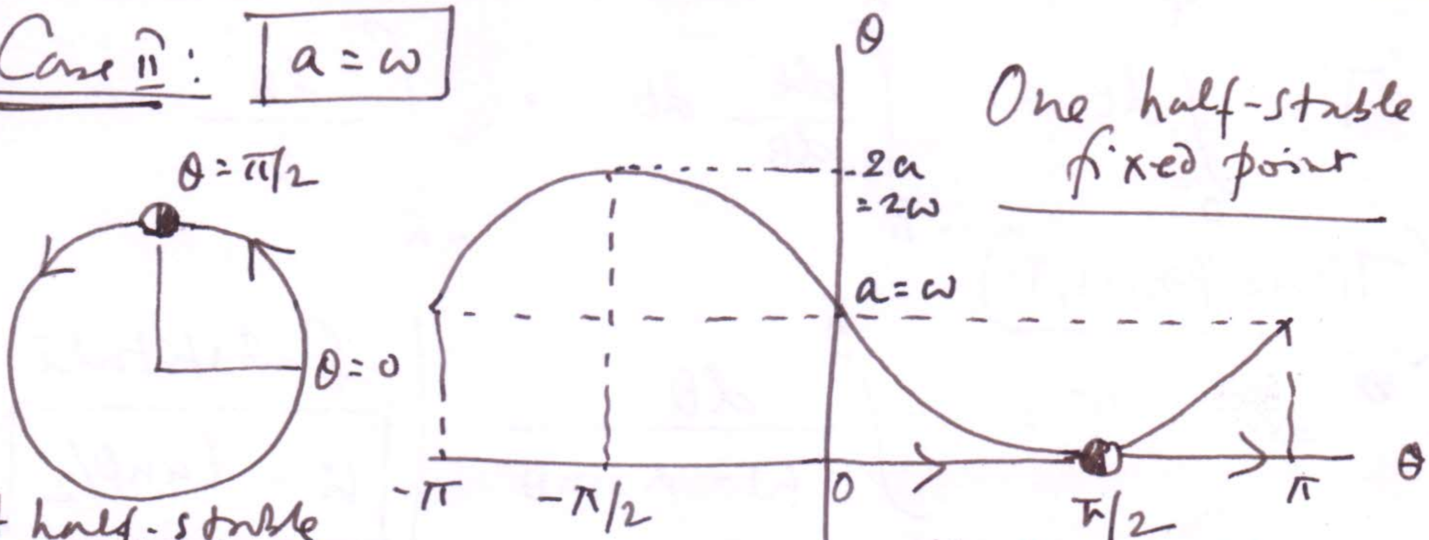
No fixed point



No fixed point on the circle.

Case II: $\boxed{a=\omega}$

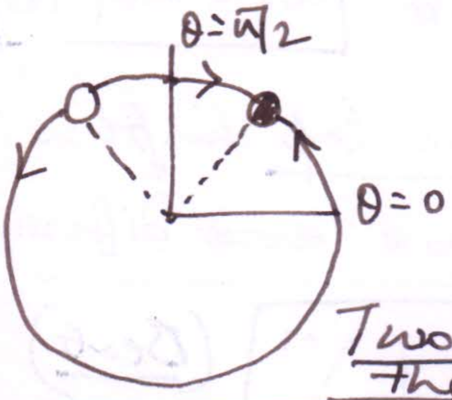
One half-stable fixed point



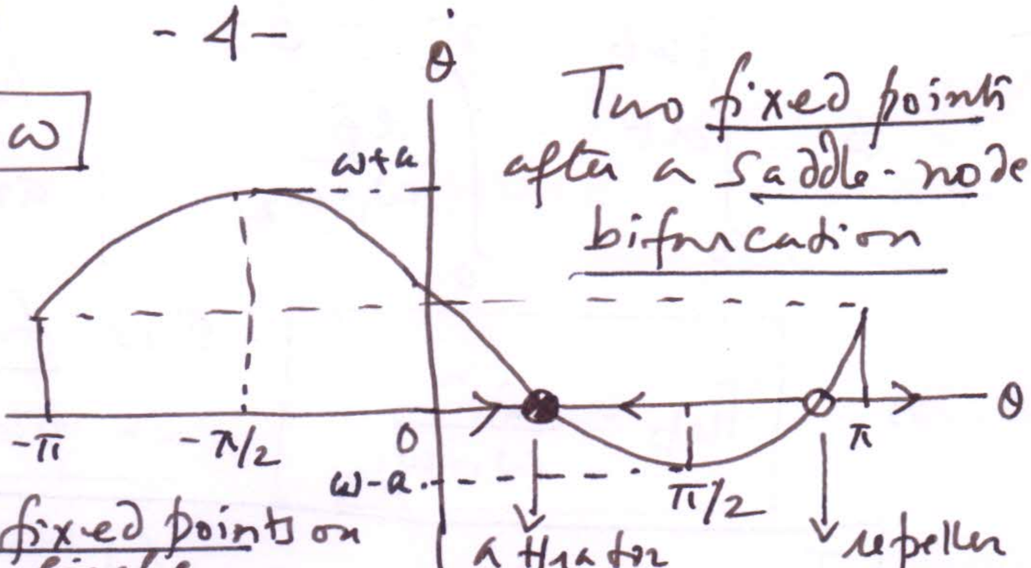
A half-stable fixed point on the circle

Half-stable

Case III : $\boxed{a > \omega}$



Two fixed points on the circle.



Two fixed points after a saddle-node bifurcation

Linear Stability Analysis

$\boxed{\dot{\theta} = \omega - a\sin\theta}$ is the form $\boxed{\dot{\theta} = f(\theta)}$ $(a > 0)$.

$\boxed{f'(\theta) = -a\cos\theta}$ Considering Case III $(a > \omega)$,

1. When $\theta < \pi/2$, $\cos\theta > 0 \Rightarrow \boxed{f'(\theta) < 0}$ (Stable point).
2. When $\theta > \pi/2$, $\cos\theta < 0 \Rightarrow \boxed{f'(\theta) > 0}$ (Unstable point).

Time Period and Bottlenecks

$$T = \int_0^T dt = \int_{-\pi}^{\pi} \frac{dt}{d\theta}\, d\theta = \int_{-\pi}^{\pi} \frac{d\theta}{\dot{\theta}}$$

(Time Period, T)

$$\Rightarrow \quad T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta}$$

Substitute $\boxed{u = \tan\theta/2}$

$\Rightarrow$ $\boxed{u = \frac{2\sin\theta/2}{2\cos\theta/2} \cdot \frac{\cos\theta/2}{\cos\theta/2}}$ $\Rightarrow$ $\boxed{\sin\theta = 2u\cos^2\theta/2}$

Further $\boxed{du = \frac{1}{2}\sec^2\theta/2\, d\theta}$ $\Rightarrow$ $\boxed{d\theta = \frac{2\,du}{\sec^2\theta/2}}$

Substitute for $\sin\theta$ and $d\theta$ in the integral,

$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta} = \int_{-\pi}^{\pi} \frac{2\,du}{\sec^2\theta/2\left(\omega - 2au\cos^2\theta/2\right)}$$

$$\Rightarrow \boxed{T = \int_{-\pi}^{\pi} \frac{2\,du}{\omega\sec^2\theta/2 - 2au}}$$

When $\theta = \pm\pi$, $u = \tan\theta/2$ gives $u \rightarrow \pm\infty$. (Limits of the integral)

Further $\boxed{\sec^2\theta/2 = 1 + \tan^2\theta/2 = 1 + u^2}$

$$\Rightarrow \boxed{T = \int_{-\infty}^{\infty} \frac{2\,du}{\omega(1+u^2) - 2au} = \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{du}{(1+u^2) - 2au/\omega}}$$

Now considering $\boxed{1+u^2 - 2au/\omega}$, we write it as,

$$1 + u^2 - 2u\left(\frac{a}{\omega}\right) + \frac{a^2}{\omega^2} - \frac{a^2}{\omega^2} = \left(u - \frac{a}{\omega}\right)^2 + \left[1 - \left(\frac{a}{\omega}\right)^2\right]$$

Next we substitute $\boxed{z = u - \frac{a}{\omega}}$ and $\boxed{\alpha^2 = 1 - \left(\frac{a}{\omega}\right)^2}$

$\Rightarrow$ $\boxed{dz = du}$.

$$T = \frac{2}{\omega}\int_{-\infty}^{\infty} \frac{dZ}{Z^2+\alpha^2} = \frac{2}{\omega\alpha}\int_{-\infty}^{\infty} \frac{d(Z/\alpha)}{(Z/\alpha)^2+1}$$

$$\Rightarrow T = \frac{2}{\omega\alpha} \arctan\left(\frac{Z}{\alpha}\right)\Bigg|_{-\infty}^{\infty} = \frac{2}{\alpha\omega}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega\sqrt{1-a^2/\omega^2}} = \frac{2\pi}{\sqrt{\omega^2-a^2}}} \rightarrow \text{Time Period to go from } -\pi \text{ to } \pi$$

Limiting Cases: 1/. When $a = 0$, $\boxed{T = 2\pi/\omega}$.

2/. When $a \to \omega$, $T = \dfrac{2\pi}{\sqrt{(\omega+a)(\omega-a)}} \simeq \dfrac{\sqrt{2}\,\pi}{\sqrt{\omega(\omega-a)}}$

(approximate $\omega + a \simeq 2\omega$)

$\Rightarrow T \to \infty$ (becomes very large).

This is the BOTTLENECK TIME SCALE.

Saddle-Node Bifurcation near the Bottleneck

Start with $\boxed{\dot{\theta} = \omega - a\sin\theta}$.

The bifurcation occurs at $\pi/2$ for $\boxed{a = \omega}$.

We write $\boxed{\theta = \pi/2 + \phi}$ where ϕ is a small angle ($\phi \ll 1$).

$$\Rightarrow \dot{\theta} = \dot{\phi} = \omega - a\sin(\pi/2 + \phi)$$

$$\Rightarrow \dot{\phi} = \omega - a\sin[\pi/2 - (-\phi)] = \omega - a\cos(-\phi)$$

$\Rightarrow \dot{\phi} = \omega - a\cos\phi$ $\left(\because \underline{\cos\phi = \cos(-\phi)} \right)$

Now $\dot{\phi} \simeq \omega - a\left(1 - \frac{\phi^2}{2} + \cdots\right)$ | Expansion for small ϕ.

$$\Rightarrow \boxed{\dot{\phi} = (\omega - a) + \frac{a}{2}\phi^2}$$

Rescale $\boxed{\phi = kx}$

$$\Rightarrow k\dot{x} = (\omega - a) + \frac{a}{2}k^2x^2$$

$$\Rightarrow \boxed{\dot{x} = \frac{\omega - a}{k} + \frac{a}{2}kx^2}$$

Set $k = \frac{2}{a}$.

$$\therefore \boxed{\dot{x} = \frac{a}{2}(\omega - a) + x^2}$$

Writing $\boxed{\lambda = \frac{a}{2}(\omega - a)}$

we get,

$\boxed{\dot{x} = \lambda + x^2} \longrightarrow$ The normal form of a saddle-node bifurcation.

Time Scales near the Saddle-Node Bifurcation

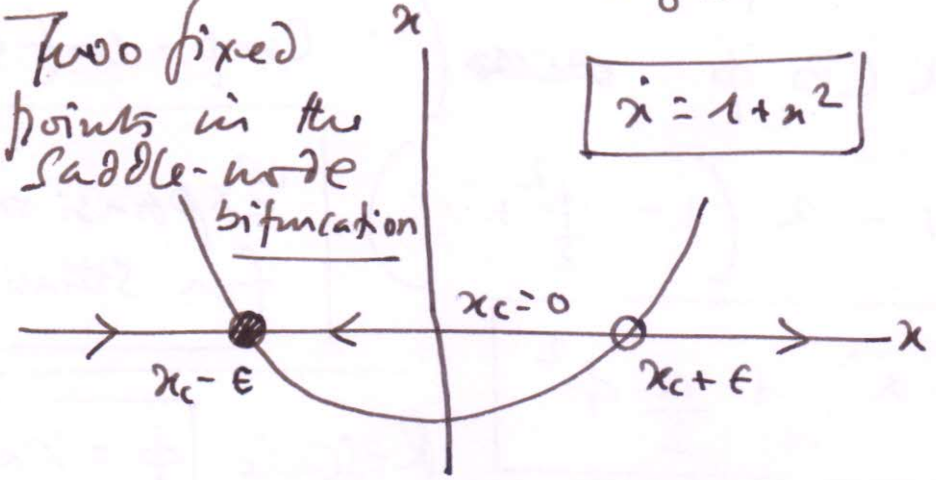
$\boxed{\dot{x} = f(x, \lambda) = \lambda + x^2}$ The bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{\lambda = 0}$.

$$\Rightarrow \boxed{\frac{dx}{dt} = 1 + x^2} \Rightarrow dt = \frac{dx}{1 + x^2}$$

$$\Rightarrow \tau = \int dt = \int_{x_c - \epsilon}^{x_c + \epsilon} \frac{dx}{1 + x^2}$$

| Timescale to travel through x_c

Two fixed points in the saddle-node bifurcation



The bifurcation takes place at $x_c = 0$ for $\lambda = 0$. $\pm\epsilon$ is a slight displacement from x_c.

$$\therefore \quad \boxed{\tau = \frac{1}{\sqrt{\lambda}} \arctan\left(\frac{x}{\sqrt{\lambda}}\right)\Bigg|_{x_c-\epsilon}^{x_c+\epsilon}}$$

Note $x_c = 0$.

$$\Rightarrow \quad \tau = \frac{1}{\sqrt{\lambda}}\left[\arctan\left(\frac{\epsilon}{\sqrt{\lambda}}\right) - \arctan\left(\frac{-\epsilon}{\sqrt{\lambda}}\right)\right]$$

Now $\boxed{\lambda = \frac{a}{2}(\omega - a)} \Rightarrow$ when $a \to \omega$, $\lambda \to 0$, and $\pm\epsilon/\sqrt{\lambda} \to \pm\infty$

$$\therefore \quad \tau \simeq \frac{1}{\sqrt{\lambda}} \cdot \left[\pi/2 - (-\pi/2)\right] \simeq \frac{\pi}{\sqrt{\lambda}} \to \infty$$

The time scale becomes very large.

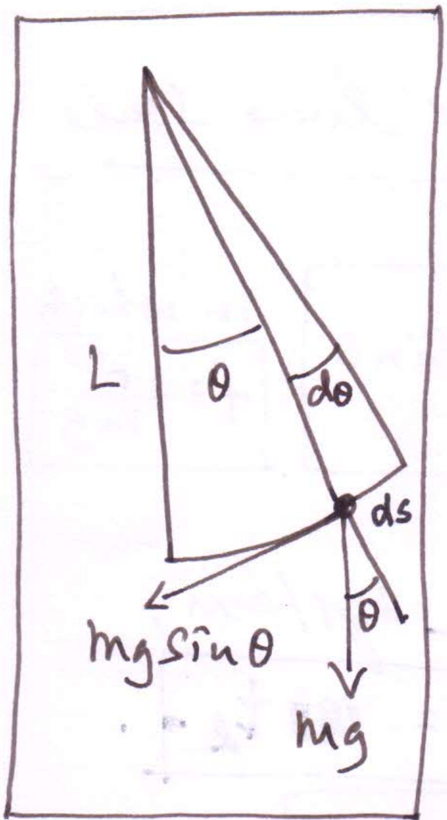
Hence, when $a \simeq \omega$, we can write,

$$\boxed{\tau \simeq \frac{\sqrt{2}\,\pi}{\sqrt{\omega}\sqrt{\omega - a}}}$$

This is the same as the bottle neck time.

$\therefore$ The bottle neck time is a consequence of the saddle-node bifurcation when $a \to \omega$.

Overdamped Pendulum



$$\boxed{ds = L\,d\theta} \Rightarrow \frac{ds}{dt} = L\frac{d\theta}{dt} = L\dot{\theta}$$

But $v = \dfrac{ds}{dt} = L\dot{\theta}$ $\qquad \boxed{\ddot{\theta} = \dfrac{d^2\theta}{dt^2}}$

Now $\boxed{\text{Force} = m\dfrac{dv}{dt} = mL\ddot{\theta}}$

Hence $\boxed{mL\ddot{\theta} = -mg\sin\theta}$

for an undamped pendulum.

For a pendulum that is damped, a ~~periodic~~ regular force is necessary to maintain the oscillations.

$$\therefore \quad \boxed{mL\ddot{\theta} = -mg\sin\theta + (F - b\dot{\theta})}$$

$F \rightarrow$ Force, $\quad b\dot{\theta} \longrightarrow$ Damping force.

$\therefore [b] = \dfrac{MLT^{-2}}{T^{-1}} = MLT^{-1} \rightarrow$ Dimension of b.

We define $\boxed{t = \tau T}$ where T is ~~the~~ a constant relevant time scale.

Now dividing throughout by mg we get,

$$\boxed{\frac{L}{gT^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{mgT}\frac{d\theta}{d\tau} = \frac{F}{mg} - \sin\theta}$$

We define $\boxed{t_{os}^2 = \frac{L}{g}}$ (Oscillatory time scale)

and $\boxed{t_d = \frac{b}{mg}}$ (Damping time scale)

$$\therefore \quad \boxed{\left(\frac{t_{os}^2}{T^2}\right)\frac{d^2\theta}{d\tau^2} + \left(\frac{t_d}{T}\right)\frac{d\theta}{d\tau} = f - \sin\theta}$$

in which $\boxed{f = \frac{F}{mg}}$

For a damping dominated system, the natural time scale $\boxed{T = t_d}$.

$$\therefore \quad \boxed{\left(\frac{t_{os}}{t_d}\right)^2 \frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau} = f - \sin\theta}$$

$\because \boxed{t_{os} \ll t_d} \Rightarrow \left(\frac{t_{os}}{t_d}\right)^2 \ll 1$. We approximate,

$$\boxed{\frac{d\theta}{d\tau} \simeq f - \sin\theta}$$

by neglecting the second-order derivative.

The above equation has the same form as $\boxed{\dot{\theta} = \omega - a\sin\theta}$ $\Rightarrow$ Bottleneck time scales are inherent to overdamped systems.

Large damping: Due to 1/. large b (damping coefficient).
2/. Small m (small inertial effects).
in the formula $\boxed{t_d = b/mg}$.