Hysteresis in Subcritical Pitchfork Bifurcation

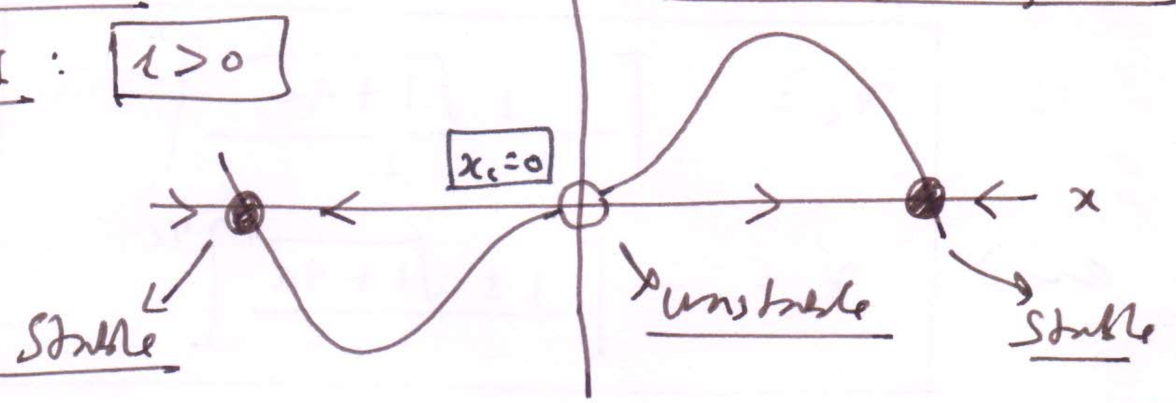
$$\boxed{\dot{x} = \mu x + x^3 - x^5}$$ → Fifth-order equation

1/. The fifth-degree term stabilizes.

2/. Symmetry is maintained for $\boxed{x \to -x}$.

Dynamics:

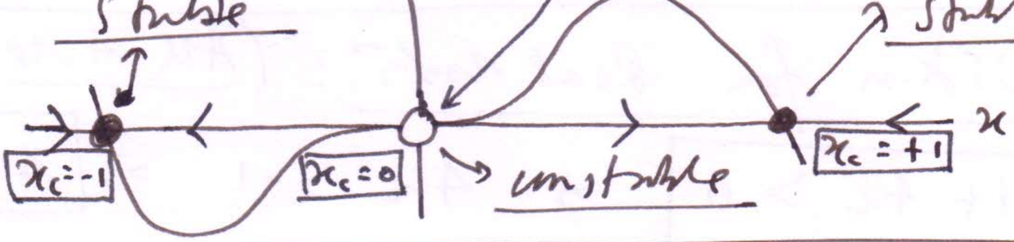
Case I : $\boxed{\mu > 0}$ Three fixed points



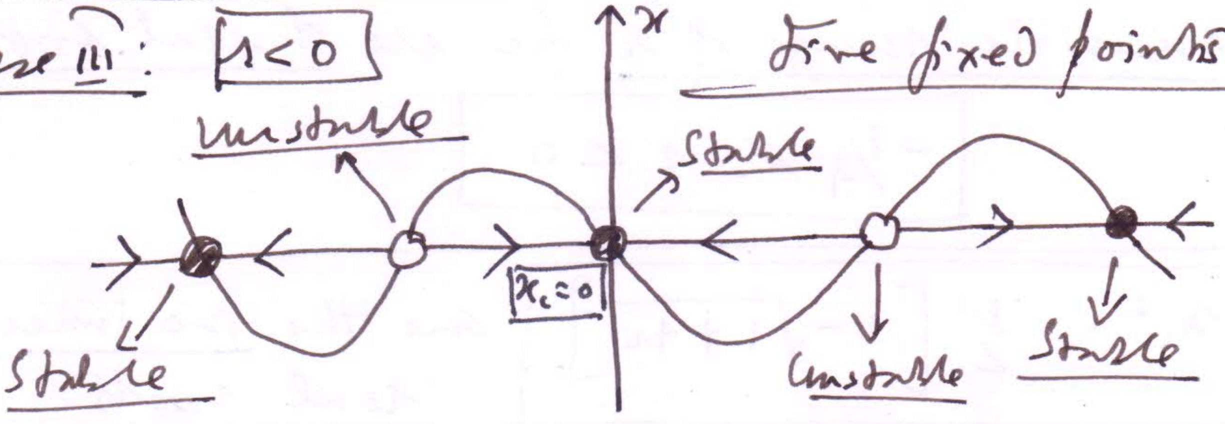
Case II : $\boxed{\mu = 0}$ $\boxed{\dot{x} = x^3(1 - x^2)}$ ⇒ $\dot{x} = 0$ $\boxed{x^3 = 0}$ and $\boxed{x = \pm 1}$

(Three fixed points coincide at $x_c = 0$)

~~Threeee~~ Three fixed points



Case III : $\boxed{\mu < 0}$ Five fixed points



Hysteresis for $r<0$:

$$\boxed{\dot{x} = rx + x^3 - x^5}$$

For fixed points $\boxed{\dot{x}=0}$

$$x(r + x^2 - x^4) = 0 \Rightarrow \boxed{x_c = 0}$$

and $\boxed{x_c^4 - x_c^2 - r = 0}$

$$\boxed{x_c^2 = \frac{1 \pm \sqrt{1+4r}}{2}}$$

Biquadratic

$$\Rightarrow \boxed{x_c = \left[\frac{1 \pm \sqrt{1+4r}}{2}\right]^{1/2} \quad \text{and} \quad x_c = -\left[\frac{1 \pm \sqrt{1+4r}}{2}\right]^{1/2}}$$

4 roots for $\pm$ signs.

$$\boxed{x_c^2 = \frac{1}{2}\left[1 + \sqrt{1+4r}\right]}$$

are the two outer most ~~roots~~ real roots.

Condition for Real roots: (All 4 real roots)

$$\boxed{1 + 4r > 0} \Rightarrow 4r > -1 \Rightarrow \boxed{r > -1/4}.$$

Admissible range of r for all 4 real roots:

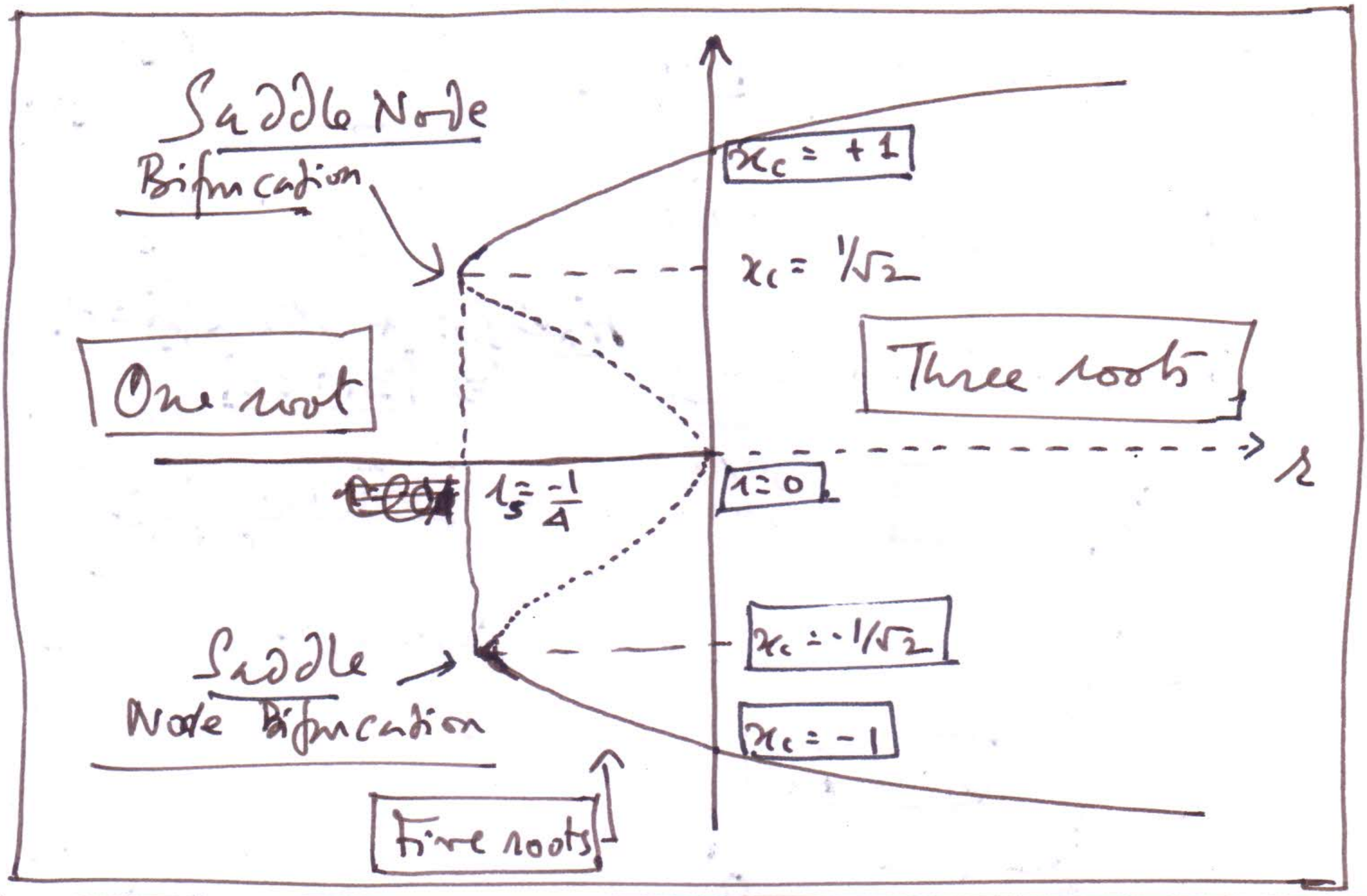
$$\boxed{-1/4 < r < 0}$$

$$\boxed{x_c^2 = \frac{1}{2}\left[1 - \sqrt{1+4r}\right]}$$

are the two inner real roots

Bifurcation Diagram

$\boxed{\dot{x} = 0} \Rightarrow \boxed{f(x_c, \lambda) = 0}$ $\boxed{x_c = 0}$ $\boxed{x_c^2 = \frac{1 \pm \sqrt{1+4\lambda}}{2}}$



1/. Origin stable to small perturbations but not large ones — locally stable

2/. Two saddle-node bifurcations occur when $\boxed{\lambda_s = -1/4}$. $\Rightarrow \boxed{x_c^2 = 1/2}$ $\Rightarrow \boxed{x_c = \pm \frac{1}{\sqrt{2}}}$

3/. Hysteresis and irreversibility as λ is increased and then decreased after starting at $\boxed{x_c = 0}$.

Imperfect Bifurcation

$$\boxed{\dot{x} = r x - x^3}$$ Symmetric under transformation $\boxed{x \to -x}$.

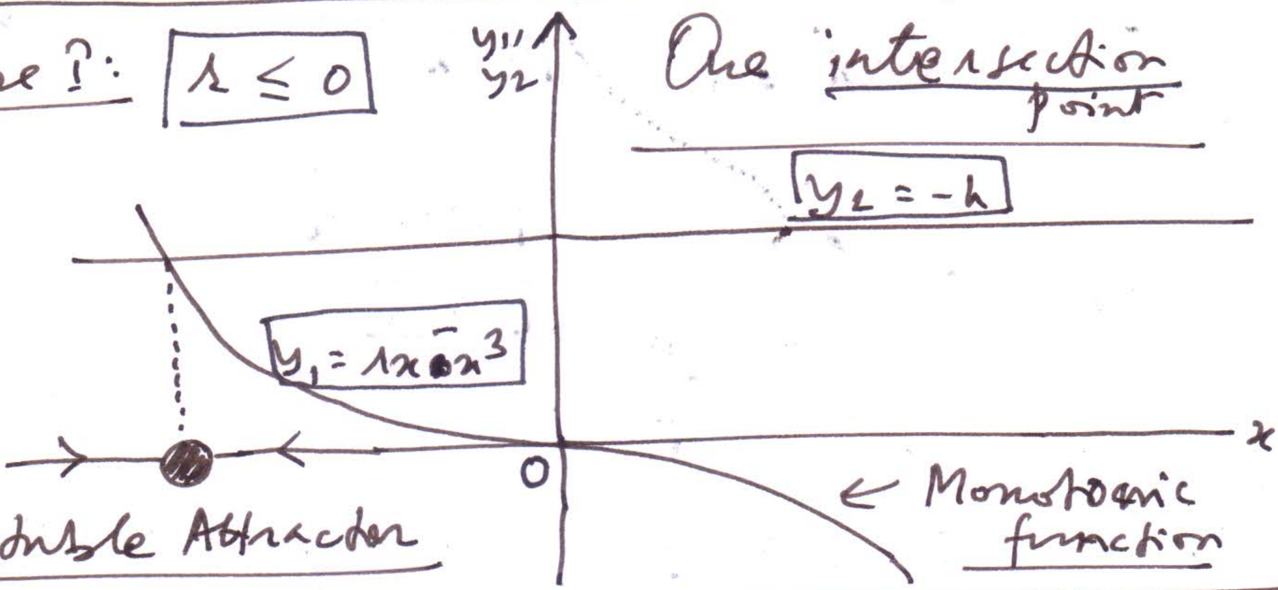
$$\boxed{\dot{x} = h + r x - x^3}$$ $\boxed{h \neq 0} \to$ Imperfection parameter.

Write $\boxed{y_1 = r x - x^3}$ and $\boxed{y_2 = -h}$

$\therefore$ $\boxed{\dot{x} = y_1 - y_2}$ Fixed point when $\boxed{\dot{x} = 0}$. $\Rightarrow \boxed{y_1 = y_2}$

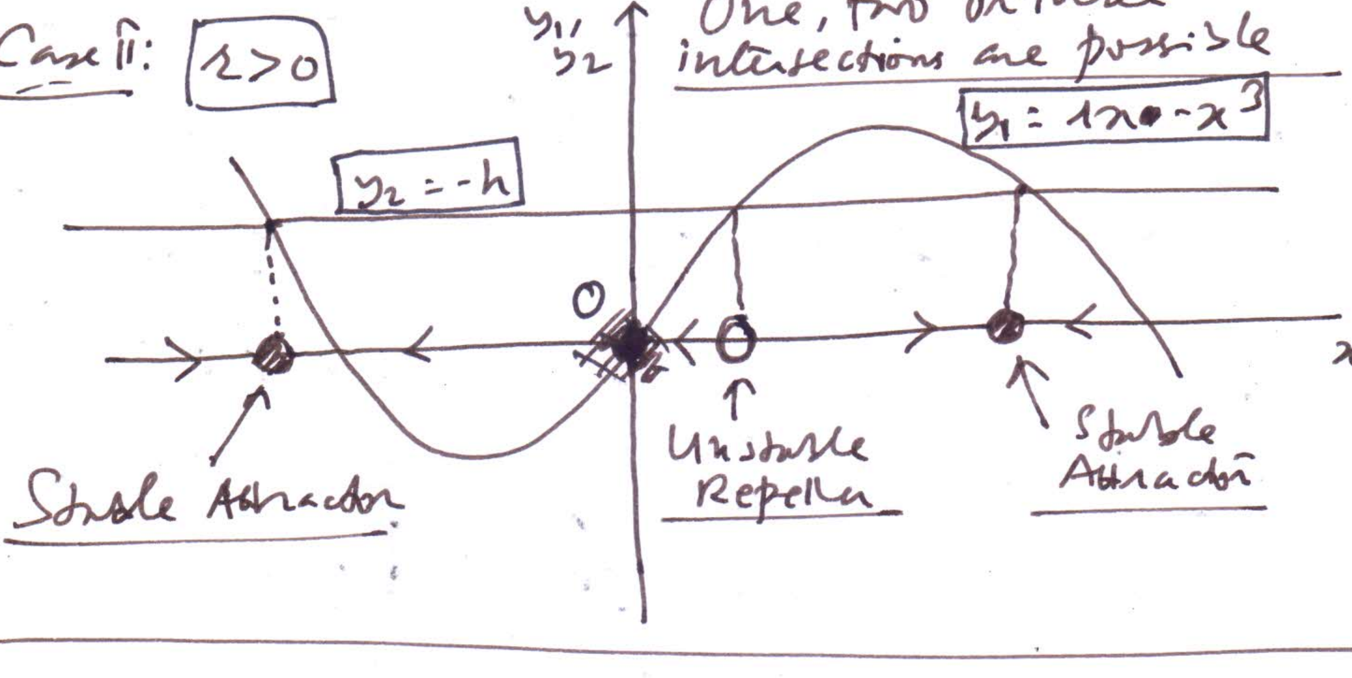
Case I: $\boxed{r \leq 0}$

One intersection point



Case II: $\boxed{r > 0}$

One, two or three intersections are possible



Critical Cases: Saddle-Node Bifurcations

$$\boxed{\frac{dy_1}{dx} = \lambda - 3x^2}$$

$\boxed{\frac{dy_1}{dx} = 0}$ at the turning points

$\Rightarrow \boxed{x^2 = \lambda/3} \Rightarrow \boxed{x = \pm\sqrt{\lambda/3}}$

$\boxed{x_{max} = +\sqrt{\lambda/3}}$ and $\boxed{x_{min} = -\sqrt{\lambda/3}}$

Maximum: $\boxed{y_{1max} = \lambda x_{max} - x_{max}^3}$

$$\Rightarrow y_{1max} = x_{max}\left(\lambda - x_{max}^2\right)$$

$$\Rightarrow y_{1max} = \sqrt{\frac{\lambda}{3}}\left(\lambda - \frac{\lambda}{3}\right) = \frac{2}{3}\lambda\sqrt{\lambda/3}$$

$$\Rightarrow \boxed{y_{1max} = \frac{2}{3\sqrt{3}}\lambda^{3/2}}$$

Minimum: $\boxed{y_{1min} = \lambda x_{min} - x_{min}^3}$

$$\Rightarrow y_{1min} = x_{min}\left(\lambda - x_{min}^2\right)$$

$$\Rightarrow y_{1min} = -\sqrt{\lambda/3} \times \left(\lambda - \lambda/3\right) = -\frac{2}{3}\lambda\sqrt{\lambda/3}$$

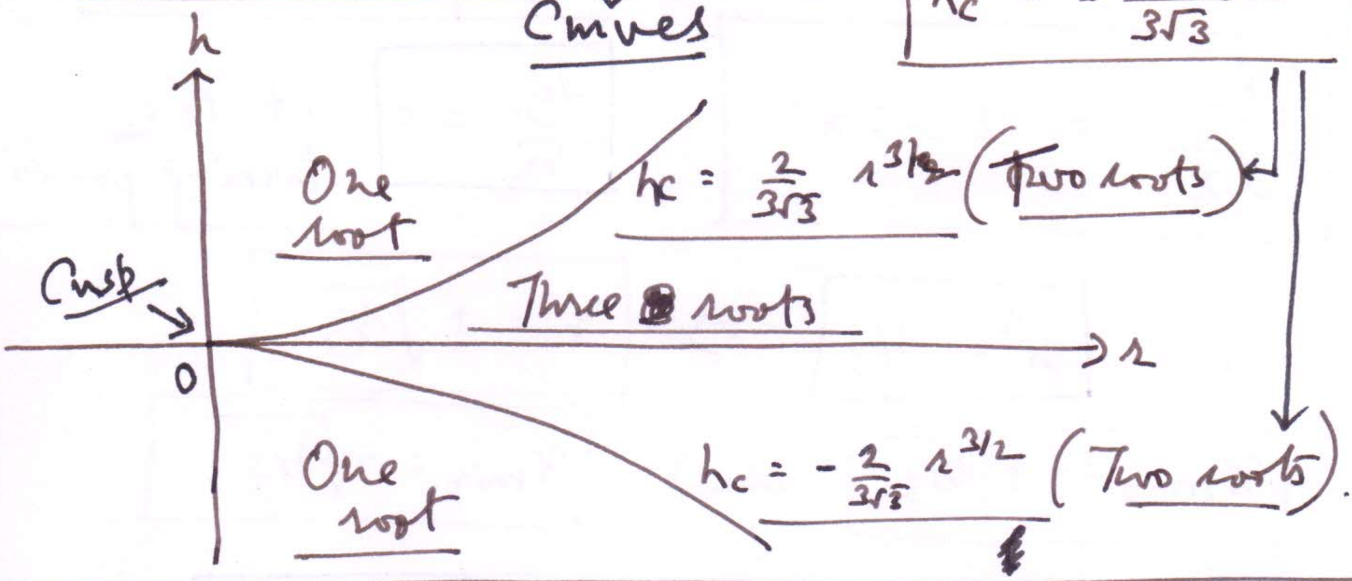
$$\Rightarrow \boxed{y_{1min} = -\frac{2}{3\sqrt{3}}\lambda^{3/2}}$$

Define $\boxed{h_c = \frac{2}{3\sqrt{3}}\lambda^{3/2}}$

1/. If $h = \pm h_c$, then we have criticality (2 roots).
2/. If $|h| < h_c$, then 3 roots. 3/ If $|h| > h_c$, then 1 root.

Bifurcation ~~Diagram~~ Curves:

$$\boxed{h_c = \pm \frac{2}{3\sqrt{3}} \lambda^{3/2}}$$



Cubic Equation: Solution by Cardan's Method.

$$\boxed{a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0}$$

Transform $\boxed{z = x + k}$ and make the second-degree terms vanish.

$\therefore$ $\boxed{x^3 + px + q = 0}$ Discriminant $\Delta = 0$,

$$\boxed{\Delta = \frac{q^2}{4} + \frac{p^3}{27}}$$

Criticality Condition

1 real root to 3 real roots.

Now $\boxed{\dot{x} = \lambda x - x^3 + h = 0}$ → for fixed points.

$\Rightarrow$ $\boxed{x^3 - \lambda x - h = 0}$ where $\boxed{p \equiv -\lambda}$, $\boxed{q = -h}$.

$\therefore$ $\boxed{\Delta = \frac{h^2}{4} - \frac{\lambda^3}{27} = 0}$ for criticality.

$\Rightarrow$ $\boxed{h^2 = \frac{4}{27} \lambda^3}$ $\Rightarrow$ $\boxed{h = \pm \frac{2}{3\sqrt{3}} \lambda^{3/2}}$ → Bifurcation Condition

$\therefore$ Bifurcation condition is $\boxed{\Delta = 0}$ in Cardan's method.