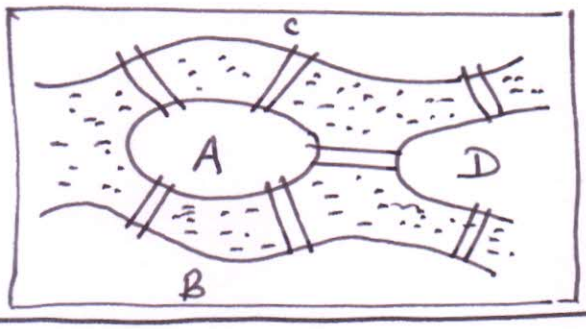
Networks and Complex Systems

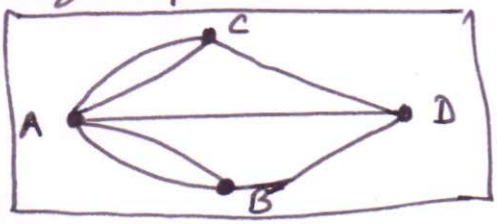
The Königsberg Bridge Problem

1. Seven bridges connecting all of Königsberg City.



2. **Question:** Is it possible to start at any one point in the city, and return to that same point after crossing all the seven bridges ONLY once?

Answer: NO – Leonhard Euler. By introducing the concept of graph structures. Not possible if every node on the graph has an odd number of links.



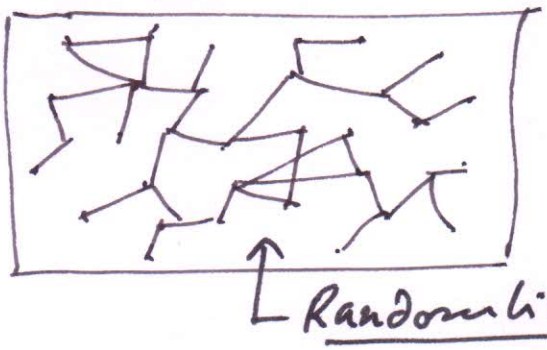
A has 5 links, B has 3, C has 3, D has 3

3. With this work Euler established graph theory as a branch of mathematics.

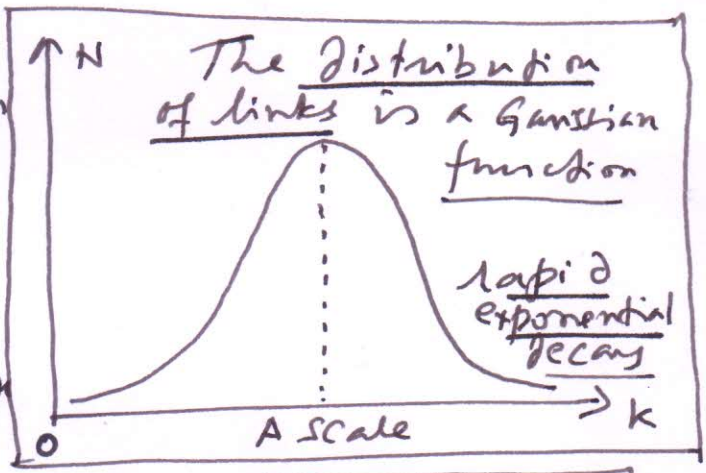
4. Complex systems can be understood by constructing a graph or network structure.

Random Networks : (Paul Erdös, Alfréd Rényi)

1/ A graph in which the nodes are linked in a random way.



2/ The degree distribution of such a graph, in which the frequency of nodes $N(k)$ with a given number of links, k, is a Gaussian function, $N(k) \sim e^{-k^2}$.



Small-World Networks : (Duncan Watts, Steven Strogatz)

1/ In Sociology Mark Granovetter wrote about the strength of weak ties between closely-knit clusters of human social groups with common interests.

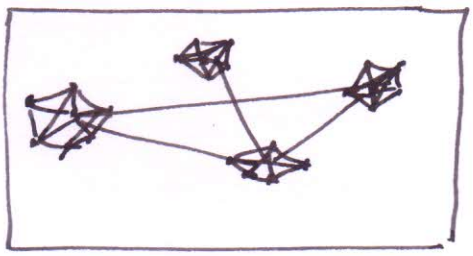
2/ Stanley Milgram found the "six-degrees of separation" between two individuals.

3/ Human social networks are NOT formed randomly. There is order.

(P.T.O.)

4/. Small-world clusters form in networks, in which all nodes are connected to one another (Watts & Strogatz)

5/. The clusters are connected to other clusters through



one or two weak links spanning across clusters. Without these weak linkages, the clusters will become isolated.

6/. Used to explain phenomena like synchrony and self-organisation.

Clustering Coefficient: Consider a cluster in which there are K_i nodes. Then the maximum number of links that can form, connecting any two nodes, is

$$K_i C_2 \Rightarrow \frac{K_i!}{2!(K_i-2)!} = \frac{K_i(K_i-1)}{2}.$$

If the actual number of links is T_i, then the clustering coefficient is

$$C = \frac{T_i}{K_i(K_i-1)/2}.$$

For a tightly-knit cluster, $C \to 1$. For a random graph $C \to 0$.

Basic Definitions :

1.) Nodes (vertices) → Points (or vertices) in the network.

2.) Edge → An undirected link joining two nodes.

3.) Arc → A directed link joining two nodes.

4.) Neighbours → Two directly connected nodes.

5.) Neighbourhood → A node and all its neighbours.

6.) Path → Sequence of edges (or arcs) between any two nodes.

7.) Path length → Number of edges or arcs between two nodes.

8.) Distance → Shortest path length.

9.) Diameter → The maximum distance between any two nodes.

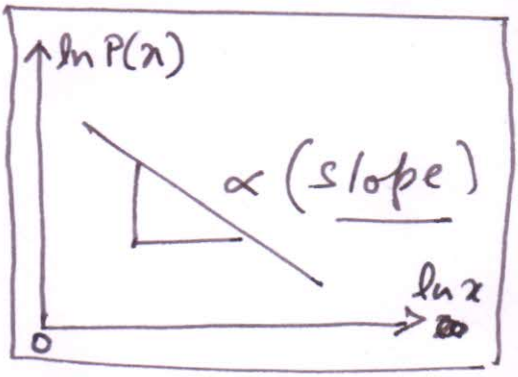
10.) Clustering → Formation of closely-knit and tight cliques.

11.) Degree Distribution → Distribution plot of the number of nodes with a given number of links.

Power Laws: $P(x) = C x^{-\alpha}$ $(\alpha > 0)$

$\therefore \ln P(x) = \ln C - \alpha \ln x$

Equation of a straight line



Has no scale in this distribution function. Hence scale-free.

1/ Events with a power-law distribution:

i) Earthquakes, ii) Extinction events,
iii) War/Terrorist fatalities, iv) Stock fluctuations (Crashes)
v) Traffic jams.

2/ Resource distributions with a power law:

i) Income distribution, ii) Wealth distribution,
iii) City-size distributions, iv) Word frequency in languages.
(Both of these are Zipf's law).

Networks with Power-Law Distributions:

1/ The World Wide Web, 2/ The Internet,
3/ Movie actors and science collaboration networks.
4/ The cellular network, 5/ Ecological network,
6/ Phone call networks, 7/ Citation networks,
8/ Networks in linguistics, 9/ Power and neural networks.
10/ Protein folding, 11/ The web of human sexual contact.

Scale-free Networks: (Albert-László Barabási)

1/ The degree distribution is like a power law.

2/ The distribution, therefore, has no characteristic scale — scale-free.

3/ The network is dynamically growing.

4/ Links are made preferentially with the heavily-linked nodes.

5/ The most heavily-linked nodes are the ones that bind separate clusters through weak links in a small-world network.

6/ Since the heavily-linked nodes are disproportionately small in number, and yet they are dominant in the network, they may avoid being adversely affected in a random attack or a random failure. The network is robust against random attacks.

7/ Scale-free networks with small-world structures, are vulnerable to targetted attacks (e.g. spread of AIDS, World Trade Centre, British Raj in India).