Population Dynamics

Use a differential equation, i.e, by a Continuum description (differentiable), $x(t)$.

Rate of per capita growth of the population is

$$\boxed{\frac{\Delta x}{x \Delta t} = r(x,t)}$$

$r \rightarrow$ Difference between growth rate and death rate.

By assuming a continuously differentiable function, $x(t)$.

$$\boxed{\frac{1}{x}\frac{dx}{dt} = r(x,t)}$$

Initially (for simplicity), assume that $\boxed{r = a \text{ (constant)}}$. Hence, $\boxed{\frac{dx}{dt} = ax}$

$(a > 0) \Rightarrow$ growth. (autonomous)

$$\Rightarrow \boxed{\int \frac{dx}{x} = \int a\,dt} \Rightarrow \boxed{\ln x = at + \ln A}.$$

when $\boxed{t = t_0,\ x = x_0}$ $\Rightarrow$ $\boxed{\ln A = \ln x_0 - a t_0}$.

$$\Rightarrow \boxed{x = x_0\, e^{a(t-t_0)}}$$

Malthusian Law of Population Growth.

THOMAS ROBERT MALTHUS: An Essay on the Principle of Population.

This law shows an exponential growth.

Between 1700 A.D - 1961 A.D, World population Doubled every 35 years, approximately.

In 1961 A.D., $x_0 = 3.06 \times 10^9$ and $a = 2\% = 0.02$.

i) a was measured from $\boxed{\frac{\Delta x}{x} \cdot \frac{1}{\Delta t} = a}$ which is the percentage increase rate ($t \to$ in years)

ii) For a population size to double, $\boxed{x = 2x_0}$.

Hence, $\boxed{T = t - t_0 = \frac{1}{a} \ln\left(\frac{x}{x_0}\right) = \frac{\ln 2}{a}}$.

$\Rightarrow \boxed{T = \frac{1}{0.02} \ln 2 = 50 \ln 2 \simeq 35 \text{ years}}$ Doubling time

Growth at this rate cannot be sustained in the long run. The Malthusian law fails obviously, when long term growth is considered.

The Logistic Model: (PIERRE FRANÇOIS VERHULST).

(introduce $-bx$ on the R.H.S.)

$$\boxed{\frac{\Delta x}{x \Delta t} = r(x) = a - bx}$$

i) $a, b > 0 \to$ vital coefficients
ii) $r(x)$ becomes small for large x.

$$\Rightarrow \boxed{\frac{dx}{dt} = x(a - bx) = ax\left(1 - \frac{x}{a/b}\right)}$$ The Logistic Equation

Define $\boxed{K = a/b}$ $\to$ The carrying capacity and set $\boxed{x = \frac{K}{1 + c^{-1}e^{-at}}}$. For $t \to \infty$, $x \to K$ (The upper limit).

Practical Examples of Population Dynamics

I) The World Population :
$$\boxed{\frac{1}{x}\frac{dx}{dt} = r = a - bx}$$

Ⓐ Here $r = r(x) = 0.02$ per annum in 1961 A.D.
Ⓑ $a = 0.029$ (ecological estimates). Ⓒ $x = 3.06\times 10^{9}$

Hence $\boxed{r = a - bx}$ $\Rightarrow$ $0.02 = 0.029 - b(3.06\times 10^{9})$

$\Rightarrow$ $\boxed{b = \frac{0.009}{3.06\times 10^{9}} \simeq 3\times 10^{-12}}$. Numerically b is much smaller than a.

Carrying capacity of the world population, ($K = a/b$), is $\boxed{K \simeq \frac{a}{b} \simeq \frac{0.029}{3\times 10^{-12}} \simeq 10^{10} \text{ (10 billion)}}$ Estimate of 1961 A.D.

II.) Population of the U.S.A. :
$$\boxed{x = \frac{K}{1 + c^{-1}e^{-at}}}$$

Write $\boxed{c^{-1} = e^{at_0}}$ (constant) $\Rightarrow$ $\boxed{x = \frac{K}{1 + e^{-a(t-t_0)}}}$ Three unknown parameters, a, K, t_0.

Therefore, Census data were taken for 3 years, 1790 A.D., 1850 A.D. and 1910 A.D. by Pearl and Reed (1920 A.D.).

$\boxed{a \simeq 0.03,\ b \simeq 1.6\times 10^{-10}}$ $\boxed{\text{Carrying capacity } K = a/b \simeq 200 \text{ million.}}$

But the present U.S. population is more than 300 million.

How? Pearl and Reed estimated in 1920. But after World War II, the vital coefficients changed; a increased and b decreased. (Belgium showed similar changes). France, however, gave a good match with predictions.

Policy Implications: $\boxed{\frac{1}{x}\frac{dx}{dt} = r(x) = a\left(1-\frac{x}{k}\right)}$

Percentage growth rate $\boxed{r = a\left(1-\frac{x}{k}\right) = a\left(\frac{k-x}{k}\right)}$

i.) When $x \ll k$, $r \simeq a$, ii.) When $x \to k$, $r \to 0$, i.e. $\boxed{\frac{k-x}{k}}$, the fractional space for growth, is reduced.

Members within the population come in their way.

To maintain a high value of r, either (A) reduce x or (B) increase k (by reducing the value of b).

How? War instincts: Lebensraum, ethnic cleansing, external invasion, increasing national wealth by war and colonisation, preventing immigration.

India is a fertile land, and hence can sustain large populations (in the Ganga Valley)

Criticisms (and Scope for improvement):

i.) Technology, environment and sociological factors are changing rapidly, affecting a and b very rapidly as well. So they need re-calibration more frequently.

ii.) Model by subdividing groups according to age and gender.

iii.) Large populations live in congested conditions and suffer outbreaks of epidemics. Population sizes can fluctuate, not according to the logistic law.

The Laws of Social Dynamics

Elliott W. Montroll

(Analogous to Newton's Laws of Mechanics)

1/. "First Law": In the absence of any social, economic or ecological force, $\boxed{\frac{1}{x}\frac{dx}{dt} = \text{constant}}$ $\boxed{x \equiv x(t) \text{ is the population size.}}$

2/. "Second Law": The constancy of $\boxed{\frac{1}{x}\frac{dx}{dt}}$ is violated when a force (social, economic or ecological) is applied. "Force" causes "replacements". Constancy of $\boxed{\frac{1}{x}\frac{dx}{dt}}$ is the Malthusian law. The simplest form of the replacing "force" is the linear function: $\boxed{a-bx}$.

$\Rightarrow \boxed{\frac{1}{x}\frac{dx}{dt} = a - bx}$ (No longer a constant).

$\Rightarrow \boxed{\frac{dx}{dt} = x(a-bx)} \rightarrow$ The Logistic Equation

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3/. "Third Law": Evolution is the natural response to a replacement. The "force" brings about change. (Eg. Genetic mutation brings about extinction and replacement of species).

 (Example)

Problem of Sharks and Salmon

$\boxed{\dfrac{dx}{dt} = ax - bx^2 - c}$, $\boxed{a, b, c > 0}$
(c is an additive constant)

$\Rightarrow \boxed{\dfrac{dx}{dt} = ax - bx^2 + (-c)}$. Now we already know that a system like $\boxed{\dfrac{dx}{dt} = ax - bx^2 + c}$ can be transformed to a form $\boxed{\dfrac{dy}{dt} = \alpha^2 - by^2}$, in which $\boxed{y = x - \dfrac{a}{2b}}$ and $\boxed{\alpha^2 = \dfrac{a^2}{4b} + c}$. We, thus replace all $\boxed{\text{"c" with "}-c\text{"}}$, i.e. $\boxed{c \longrightarrow -c}$, and rescale further by $\boxed{X = \dfrac{y}{\alpha/\sqrt{b}}}$ and $\boxed{T = \alpha\sqrt{b}\,t}$ to get $\boxed{\dfrac{dX}{dT} = 1 - X^2}$, whose integral solution is $\boxed{X = \dfrac{A - e^{-2T}}{A + e^{-2T}}}$

$\boxed{c = 0 \Rightarrow \text{logistic equation}}$

A is an integra-tion constant

This is then written as,

$$\boxed{x = \frac{a}{2b} + \frac{\alpha}{\sqrt{b}}\left[\frac{A - e^{-2\alpha\sqrt{b}t}}{A + e^{-2\alpha\sqrt{b}t}}\right]}$$

When $t \to \infty$,

$$\boxed{x \longrightarrow \frac{a}{2b} + \frac{1}{\sqrt{b}} \cdot \sqrt{\frac{a^2}{4b} - c}} \Rightarrow \boxed{x \longrightarrow \frac{a}{2b}\left(1 + \sqrt{1 - \frac{4bc}{a^2}}\right)}$$

This limiting value of the population does not depend on the value of A.

(Example) Population of New York City: A Critical Case

$$\boxed{\frac{dx}{dt} = ax - bx^2 - c} \quad \boxed{a, b, c > 0}.$$

a → growth parameter, b, c → decline parameters

$$a = \frac{1}{25} = 4\times10^{-2},\ b = \frac{1}{25\times10^6} = 4\times10^{-8},\ c = 10^4$$

$$\therefore \boxed{\frac{4bc}{a^2} = \frac{4\times4\times10^{-8}\times10^4}{4\times4\times10^{-4}} = 1} \Rightarrow \boxed{a^2 = 4bc} \Rightarrow \boxed{\frac{a^2}{4b} - c = 0}$$

$$\Rightarrow \boxed{\alpha^2 = 0} \Rightarrow \boxed{\frac{dy}{dt} = -by^2} \Rightarrow \boxed{\int y^{-2}\,dy = -b\int dt}$$

$$\Rightarrow \boxed{-y^{-1} = -bt + \text{Constant}}. \Rightarrow \boxed{\frac{1}{y} = bt + A}$$ A is the integration constant.

$$\Rightarrow \boxed{y = \frac{1}{bt+A}} \Rightarrow \boxed{x = \frac{a}{2b} + \frac{1}{bt+A}}$$ Power-law convergence.

When $t \to \infty$, $y \to 0$, and $\boxed{x \to \frac{a}{2b}}$.

This is the limiting value of the population, $x \to 0.5$ million. $\Rightarrow \boxed{x \to \frac{4\times10^{-2}}{2\times4\times10^{-8}}}$

This convergence is slow as in a power-law. This happens in all critical phenomena, such as phase transitions. Power laws are also seen in gas laws, Zipf's law (GEORGE KINGSLEY ZIPF) and Pareto's law in income and wealth distributions (VILFREDO PARETO). They are SCALE-FREE.

(Parachute is not opened)

Free Fall of a Parachutist

Also $\boxed{Re = \frac{\rho l v}{\eta}}$

Reynold's Number : $\rightarrow$ $\boxed{Re = \frac{l v}{\nu}}$

$l \rightarrow$ characteristic length, $v \rightarrow$ characteristic velocity.

$\boxed{\nu = \eta / \rho}$ $\rightarrow$ Kinematic viscosity, in which ρ is the density, η is the dynamic viscosity.

$\boxed{\nu_{water} \sim 10^{-2} \text{ S.I. units}}$. $\boxed{\nu_{air} \sim 10^{-4} \text{ S.I. units}}$.

$\boxed{\text{Drag force}, \quad D \propto v^{\gamma}}$, where v is the velocity.

i/. When $\boxed{R_e \sim 10}$ (low v, high ν), $\boxed{\gamma = 1}$.
ii/. When $\boxed{Re \sim 10^3}$ (high v, low ν), $\boxed{\gamma = 2}$.
(This is due to turbulence in air).
iii/. $10 < R < 10^3$, γ is uncertain. $\gamma = ?$

For a freely falling parachutist, $\boxed{\gamma = 2}$. $\boxed{D = kv^2}$

$\Rightarrow$ $\boxed{m\frac{dv}{dt} = mg - kv^2}$, $\boxed{k > 0}$. We now get, $\boxed{\frac{dv}{dt} = g - \frac{k}{m}v^2}$.

$\Rightarrow$ $\boxed{\frac{1}{g}\frac{dv}{dt} = 1 - \frac{v^2}{(\sqrt{mg/k})^2}}$ We rescale, $\boxed{X = \frac{v}{\sqrt{mg/k}}}$.

$\Rightarrow$ $\frac{\sqrt{mg/k}}{g}\frac{dX}{dt} = 1 - X^2$. Now rescale $\boxed{T = \sqrt{kg/m}\, t}$.

$\Rightarrow$ $\boxed{\frac{dX}{dT} = 1 - X^2}$ The initial condition is $\boxed{t = 0, \; v = 0}$.

$\Rightarrow$ $\boxed{T = 0}$ and $\boxed{X = 0}$ are the rescaled initial conditions.

The integral solution is $\boxed{X = \tanh(T)}$.

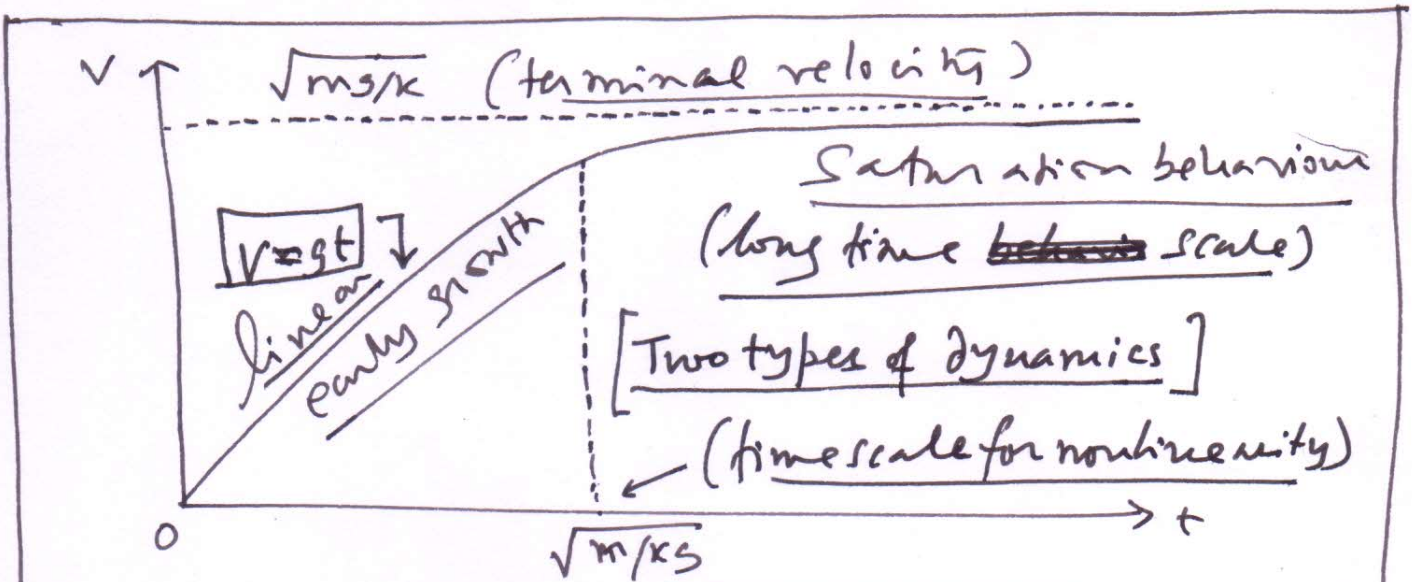
$$\Rightarrow \boxed{\frac{V}{\sqrt{mg/k}} = \tanh\left(\sqrt{kg/m}\, t\right)}$$

$$\Rightarrow \boxed{V = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}\, t\right)}$$

When $t \to \infty$, $V \to \sqrt{\frac{mg}{k}}$.

When $t \to 0$, $\boxed{\tanh\left(\sqrt{\frac{kg}{m}}\, t\right) \simeq \sqrt{\frac{kg}{m}}\, t}$

$$\Rightarrow \boxed{V \simeq \sqrt{\frac{\cancel{m}}{\cancel{k}}}\sqrt{g} \cdot \sqrt{\frac{\cancel{k}}{\cancel{m}}}\sqrt{g}\, t \simeq gt} \quad \text{(linear)}$$



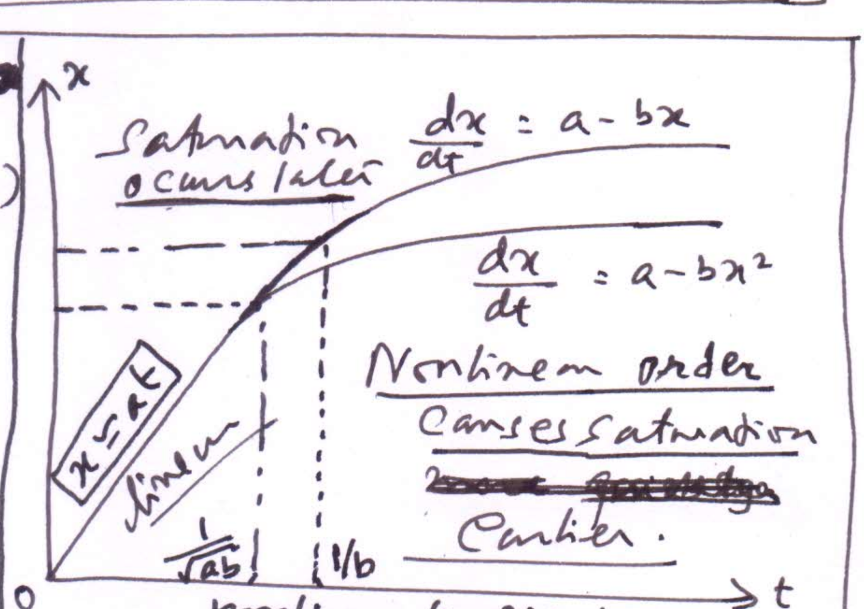
Comparison of $\boxed{\frac{dx}{dt} = a - bx}$ (linear)

and $\boxed{\frac{dx}{dt} = a - bx^2}$

(Nonlinear).

For $t \to 0$,

$\boxed{x \simeq at}$ for both (linear).



Item Response Theory

Singh, Pathak & Pandey

$$\boxed{P(\theta) = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$$ Item Response Function

$\theta \rightarrow$ Ability , $P(\theta) \rightarrow$ Performance Index.
$c \rightarrow$ Probability that a candidate with low ability will respond correctly to an item.
$\omega \rightarrow$ Item discrimination parameter.
$b \rightarrow$ Item difficulty parameter.

Define $\boxed{\phi = P(\theta) - c} \Rightarrow \boxed{\phi = (1-c)\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-1}}$

$$\Rightarrow \frac{d\phi}{d\theta} = (1-c) . \times \cancel{-}\left[1 + e^{-\frac{\theta-b}{\omega}}\right]^{-2} \times e^{-\left(\frac{\theta-b}{\omega}\right)} \times \cancel{-}\frac{1}{\omega}$$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{(1-c)}{\omega} \cdot \frac{\phi^2}{(1-c)^2} \cdot e^{-\left(\frac{\theta-b}{\omega}\right)}.$$

Now $\boxed{\left[1+e^{-\frac{\theta-b}{\omega}}\right]^{-1} = \frac{\phi}{1-c}} \Rightarrow \boxed{e^{-\frac{\theta-b}{\omega}} = \frac{1-c}{\phi} - 1}$

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{(1-c)}{\omega} \cdot \frac{\phi^2}{(1-c)^2} \cdot \left[-1 + \frac{1-c}{\phi}\right]$$ ← Autonomous form ↓

$$\Rightarrow \frac{d\phi}{d\theta} = \frac{\cancel{(1-c)}}{\omega} \cdot \frac{\phi}{\cancel{1-c}}\left[1 - \frac{\phi}{1-c}\right] \quad \boxed{\frac{d\phi}{d\theta} = f(\phi)}$$

$$\Rightarrow \boxed{\frac{d\phi}{d\theta} = \frac{\phi}{\omega}\left[1 - \frac{\phi}{1-c}\right]}$$ → The logistic equation.

Compare with $\boxed{\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right)}$. $\Rightarrow$ The limiting value of ϕ is $1-c$ (like carrying capacity).

By definition

$\Rightarrow \boxed{P(\theta) - c = \phi \rightarrow 1 - c} \rightarrow$ when $\theta \rightarrow \infty$.

$\because \boxed{P(\theta) - c = 1 - c} \Rightarrow \boxed{P(\theta) \rightarrow 1}$ when $\theta \rightarrow \infty$.

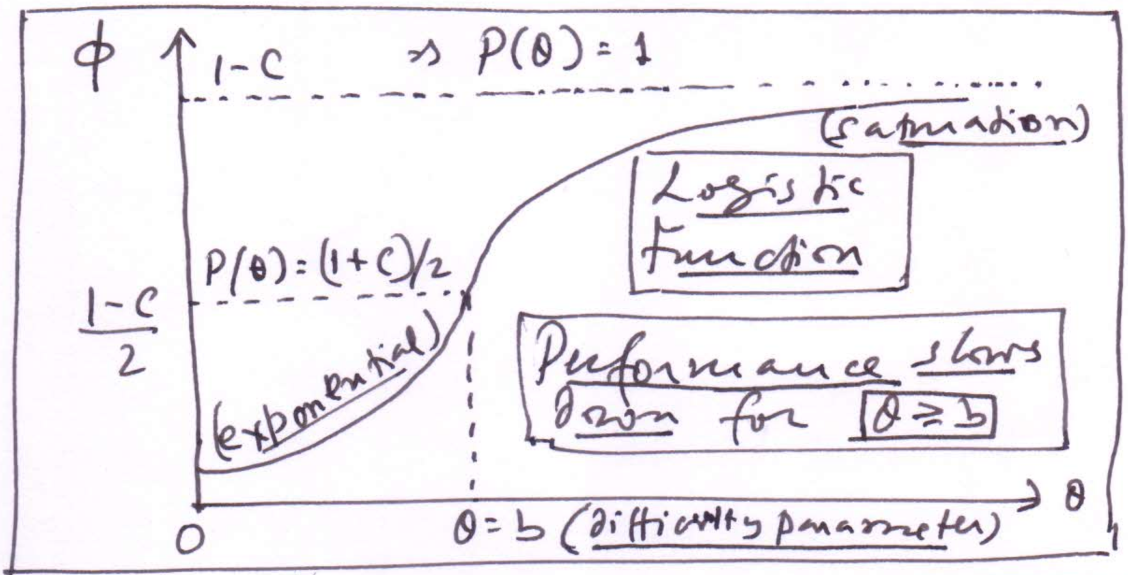
(Absolutely perfect performance when ability is infinite).

When $\boxed{\phi = \frac{1-c}{2}}$,

$\Rightarrow P - c = \frac{1-c}{2}$

$\Rightarrow \boxed{P = \frac{1+c}{2}}$

Performance starts to decelerate after this value.



Using the $P(\theta)$ function, we write $\boxed{\frac{1+c}{2} = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$

$\Rightarrow \boxed{\frac{\cancel{1-c}}{2} = \frac{\cancel{1-c}}{1+e^{-(\theta-b)/\omega}}} \Rightarrow \boxed{e^{-(\theta-b)/\omega} = 1}$

$\Rightarrow \boxed{\theta = b}$ when $\boxed{P = \frac{1+c}{2}}$.

$\Rightarrow$ Beyond $\boxed{\theta = b}$ (item difficulty parameter), performance increases at a decreasing rate (i.e. slows down)

Item discrimination parameter, ω: When ω is small the logistic function is step-like and steep.

Let $\boxed{\omega = 0}$, in $\boxed{P = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$

i/. when $\theta > b$, $e^{-(\theta-b)/\omega} = e^{-\infty} = 0$

and $\boxed{P = 1, \phi = 1-c}$

ii/. when $\theta < b$,

$e^{-(\theta-b)/\omega} = e^{\infty} \rightarrow \infty \Rightarrow$ and $\boxed{P = c, \phi = 0}$