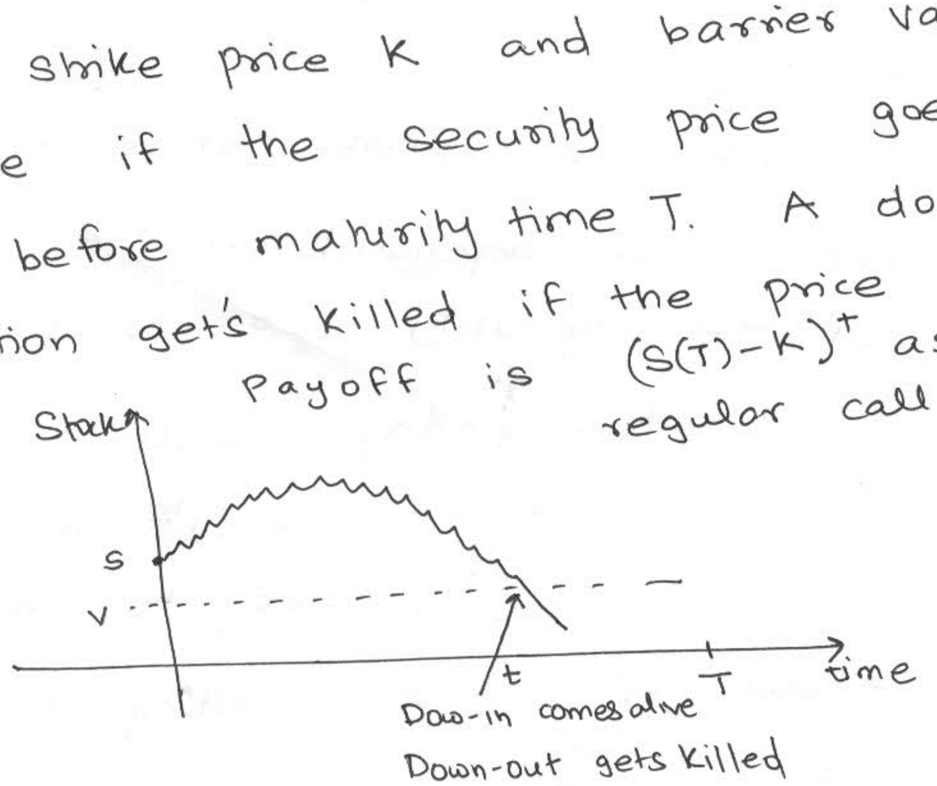
Exotic options and their simulation

There exist several complex products in the markets. Many a time it is not possible to have an analytic expression (Like the Black-Scholes-Merton) or formula for pricing these exotic products. Hence, the main technique here is to use simulation. We look at three different exotic options and give some simulation techniques to value these. 1. Barrier options 2. Asian options 3. Loopback options. A key feature of these options is that the payoff is path dependent.

Barrier options:

There are ~~two~~ different types of barrier options eg: a down-in barrier option and a down-out barrier option. A European down-in barrier option with strike price K and barrier value V comes alive if the security price goes below V before maturity time T. A down-out barrier option get's killed if the price goes below V. Payoff is $(S(T)-K)^+$ as a regular call option



If one owns both a down-in and down-out call option with the same values of K and T then exactly one option will be in play at time $t \leq T$. As a result if $D_i(s, t, K)$ and $D_o(s, t, K)$ represent the value of the down-in and down-out options, then

$$D_i(s, t, K) + D_o(s, t, K) = C(s, t, K)$$

where $C(s, t, K)$ is the Black-Scholes-Merton formula.

There are also up-in and up-out. The up-in barrier option comes alive if the security price exceeds a barrier value v and the up-out option gets killed if security value exceeds barrier v. Arguing similarly

$$U_i(s, t, K) + U_o(s, t, K) = C(s, t, K)$$

Asian option

Asian options are options whose value at time T of exercise is dependent on the average price of the security. Let N denote the number of trading days. $S_d(i)$ is the stock price at the end of the i^{th} trading day i.e. $S_d(i) = S(i/N)$. Let the maturity be at the end of the n^{th} trading day, then the

Asian option is an option whose payoff at exercise time is $\left(\sum_{i=1}^{n} \frac{Sd(i)}{n} - K \right)^{+}$ where K is the strike price.

Loopback option

A loopback option is one whose strike price is the minimum of end-day price upto the options excercise time.

$$Sd(n) - \min_{i=1,\dots n} Sd(i)$$

Monte Carlo simulation of exotic options:

Lets first use regular Monte-Carlo to price the above exotic options. Later we will use more efficient simulators based on variance reduction techniques learnt in the previous lecture.

Lets assume that the price of the security follows the risk neutral geometric Brownian motion.

$$S(t) = S(0)\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma dW}$$

For risk-neutrality μ is replaced by the risk free interest rate r. Let $Sd(i)$ denote the price of the security at the end of day i

Let $X(i) = \log\left(\frac{Sd(i)}{Sd(i-1)} \right)$. One of properties of GBM is that successive daily price ratio changes are independent. Thus, it follows that $X(1), X(2), \ldots, X(n)$ are independent normal random variables each with mean $\frac{1}{N}\left(r - \frac{\sigma^2}{2}\right)$ and variance $\frac{\sigma^2}{N}$ (Here N is the total number of trading days and n is the expiry day of the option). We begin our simulation by generating n independent normal r.v.'s each with mean $\frac{1}{N}\left(r - \frac{\sigma^2}{2}\right)$ and variance $\frac{\sigma^2}{N}$. Set them equal to $X(1), X(2), \ldots, X(n)$, then the day prices are

$$
\begin{aligned}
Sd(0) &= S(0) \\
Sd(1) &= Sd(0)\, e^{X(1)} \\
Sd(2) &= Sd(1)\, e^{X(2)} \\
&\vdots \\
Sd(i) &= Sd(i-1)\, e^{X(i)} \\
&\vdots \\
Sd(n) &= Sd(n-1)\, e^{X(n)}
\end{aligned}
$$

Let $I = \begin{cases} 1 & \text{if } Sd(i) < v \text{ for some } i = 1, \ldots n) \\ 0 & \text{if } Sd(i) \geq v \text{ for all } i = 1, \ldots n \end{cases}$

[This is a down-in call option and will be alive only if $I = 1$]

(Pay off) $Y = e^{-rn/N}\, I\, (Sd(n) - K)^+$

Call this payoff Y_1. Repeat this procedure $k-1$ times to get a set of $Y_1, Y_2, \ldots Y_k$ realizations. The estimate of the price of

Barrier option at $t=0$ is

$$\frac{\sum_{i=1}^{k} Y_i}{K}$$

Risk neutral valuations of Asian and loopback options are obtained by takin

$$Y = e^{-rn/N}\left(\sum_{i=1}^{n}\frac{S_d(i)}{n} - k\right)^{+} \quad \text{— (Asian)}$$

$$Y = e^{-rn/N}\left(S_d(n) - \min_i S_d(i)\right) \quad \text{— (Loopback)}$$

<u>More efficient simulation estimators based on variance reduction techniques</u>

a.) For the Asian option (<u>Control variate technique</u>)

$$Y = e^{-rn/N}\left(\frac{\sum_{i=1}^{n} S_d(i)}{n} - k\right)^{+}$$

It is observed that Y is positively correlated with $V = \sum_{i=0}^{n} S_d(i)$

We use the simulator

$$Z = Y + c\,(V - \mu_v) \quad \text{where } \mu_v = \mathbb{E}(V)$$

$$\mu_v = \mathbb{E}(V) = \mathbb{E}\left[\sum_{i=0}^{n} S_d(i)\right] = \sum_{i=0}^{n} \mathbb{E}\left(S_d(i)\right)$$

$$= S(0)\sum_{i=0}^{n}\left(e^{r/N}\right)^{i} = S(0)\,\frac{1-e^{r(n+1)/N}}{1-e^{r/N}}$$

b. Antithetic variables.

This method generates $X(1), \ldots X(n)$ and uses ~~then~~ it to compute Y. It then reuses the same data with $X(i) \Longrightarrow \frac{2(r - \sigma^2/2)}{N} - X_i$

The value of Y based on these new values is then computed. The estimate from that simulation run is the average of the two values. It can be shown that reusing the data in this manner would result in a smaller variance.

c. Conditional expectation technique

Consider a down-in barrier option. Suppose that the first time the stock price falls below the barrier v occurs on the j^{th} day with the price at the end of day being $S_d(j) < v$. At this moment the option comes alive and ~~is~~ worth exactly equal to a regular call option. ~~If the~~ This means that the options worth is now $C\left(S_d(j), \frac{j}{N}, K\right)$

where $C(x, t, K)$ is the Black-Scholes-Merton price. Consequently we follow the following strategy → End the simulation when the end of day price falls below barrier v and use the resulting Black-Scholes-Merton as the estimator from this run. The resulting estimator is called conditional exp. estimator (conditioning on price falling below barrier)