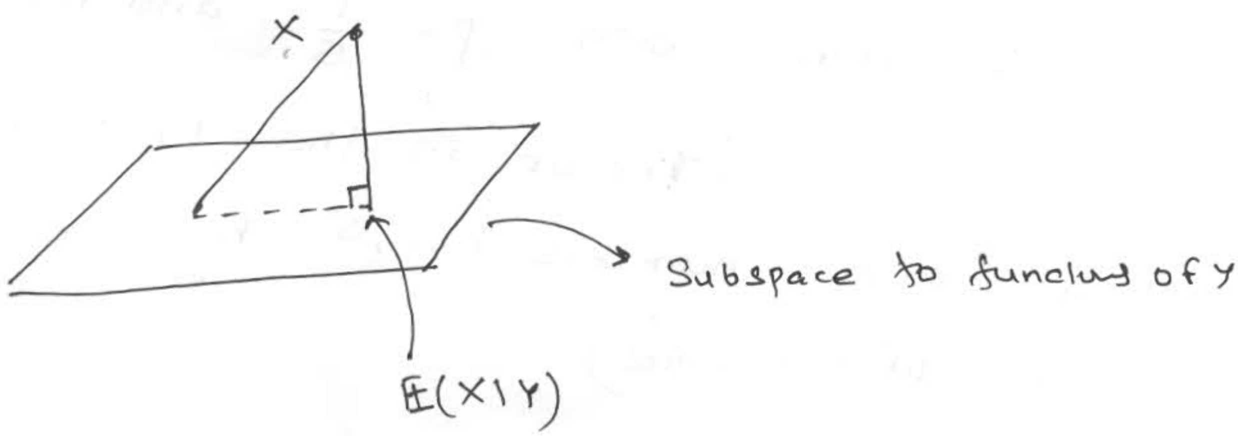
Conditional expectation of a random variable X given Y denoted by $\mathbb{E}(X|Y)$ is a random variable that represents the best estimate of X given the observation Y. Here, we can only observe the random variable Y and our estimation of X is based on a reconstruction using some function $f(Y)$ such that

$$\mathbb{E}\left(X - \mathbb{E}(X|Y)\right)^2 \leq \mathbb{E}\left(X - f(Y)\right)^2 \quad (*)$$

Points to remember about $\mathbb{E}(X|Y)$

- It is a random variable
- It is some function of Y
- It is that function of Y that is closest of X in the sense $(*)$



Mathematically for a discrete random variable we can define

$$\mathbb{E}(X \mid Y=y) = \sum_{x} x \, P(X=x \mid Y=y)$$

As we vary y we get a different number.

Hence $\mathbb{E}(X \mid Y) = f(Y)$ for some funcion f.

Example 1 :

Suppose ~~one~~ you ~~casts~~ cast a die till you get the number six. What is the best estimate of the number of one's?

Let $Y =$ number of sixes, (Note: this is Geometric R.V.)

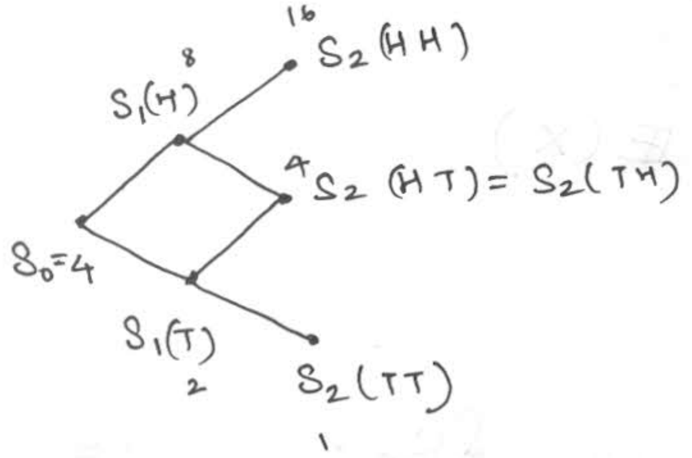
$X =$ number of ones

The best estimate of X given Y is $\mathbb{E}(X \mid Y)$. Which function of Y should it be?

Well, the number of ones is a binomial random variable with $p = \frac{1}{5}$ and $n = Y-1$ (Since six cannot occur in the first $Y-1$ trials and all other numbers $1, \ldots, 5$ have equal probability of occurence)

$$\therefore \mathbb{E}(X \mid Y) = \frac{Y-1}{5}$$

Example 2: Best estimate of stock price.



Find $\mathbb{E}^{Q}(S_2 | S_1)$ $\xrightarrow{}$ risk neutral measure.

$u = 2$, $d = \frac{1}{2}$

$r = \frac{1}{4}$

$q_u = \frac{1}{2} = q_d$

$$\mathbb{E}^{Q}(S_2 | S_1 = H) = \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 4 = 8 + 2 = 10$$

$$\mathbb{E}^{Q}(S_2 | S_1 = T) = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 1 = \frac{5}{2}$$

OR

$$\mathbb{E}^{Q}(S_2 | S_1) = q_u u \cdot S_1 + q_d d \cdot S_1$$

$$= \frac{1}{2} \cdot 2 \cdot S_1 + \frac{1}{2} \cdot \frac{1}{2} \cdot S_1$$

$$\mathbb{E}^{Q}(S_2 | S_1) = S_1 + \frac{S_1}{4} = \frac{5}{4} S_1$$

Ex: What is $\mathbb{E}(S_3 | S_1)$?

Properties of conditional expectation:

1. $\mathbb{E}(X + Y | Z) = \mathbb{E}(X | Z) + \mathbb{E}(Y | Z)$
2. $\mathbb{E}(X | Y) = \mathbb{E}(X)$ if X and Y are ind.
3. $\mathbb{E}(X | Y) = X$ if $X = f(Y)$

Iterated condition

4. $$\mathbb{E}\left(\mathbb{E}(X \mid Y) \mid Z\right) = \mathbb{E}(X \mid Z)$$

if $Z = h(Y)$

In particular

$$\mathbb{E}\left(\mathbb{E}(X \mid Y)\right) = \mathbb{E}(X)$$

Martingales:

Defn: A Stochastic process $\{S_n\}_{n \geq 1}$ is a martingale w.r.t. $\{X_n\}_{n \geq 1}$ if

$$\mathbb{E}(S_{n+1} \mid X_1, X_2, \ldots X_n) = S_n.$$

Martingale idea is related to ~~a~~ the idea of a "fair game". If S_n represents the gamblers (gain/loss) at time n then the best estimate of the gambler's gain/loss at time n+1 given the information uptil time n i.e. $\mathbb{E}(S_{n+1} \mid X_1, X_2, \ldots X_n)$ is exactly equal to the gamblers gain/loss at time n.

Example 1. (Random walk - Symmetric)

Let S_n be the position of the random walker at time n. then $S_n = \sum_{i=1}^{n} X_i$

where $X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ toss in H} \\ -1 & \text{if } i^{\text{th}} \text{ toss is T} \end{cases}$

then S_n is a martingale w.r.t. X_n

$$\mathbb{E}(S_{n+1} \mid X_1, \dots X_n)$$

$$= \mathbb{E}(S_n + X_{n+1} \mid X_1 \dots X_n)$$

$$= \mathbb{E}(S_n \mid X_1 \dots X_n) + \mathbb{E}(X_{n+1} \mid X_1 \dots X_n)$$

(Due to linearity)

$$= S_n + \mathbb{E}(X_{n+1})$$

By property (3) $S_n = f(X_1, \dots X_n)$

By property (2) X_{n+1} is ind of $X_1, \dots X_n$

$$= S_n + \frac{1}{2}1 - \frac{1}{2}1$$

$$= S_n$$

$$\therefore \mathbb{E}(S_{n+1} \mid X_1 \dots X_n) = S_n$$

Example 2: (De Moivre's martingale)

Let S_n be the position of a asymmetric random walker then

$\left(\frac{q}{p}\right)^{S_n}$ is a martingale w.r.t. X_n

Note: In this case you can check that S_n is not a martingale.

$$\mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}} \middle| X_1, \dots X_n\right]$$

$$= \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n + X_{n+1}} \middle| X_1 \dots X_n\right]$$

$$= \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_n} \cdot \left(\frac{q}{p}\right)^{X_{n+1}} \middle| X_1 \dots X_n\right]$$

$$= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}} \middle| X_1 \dots X_n\right]$$

(from property 3)

$$= \left(\frac{q}{p}\right)^{S_n} \cdot \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right]$$ (from property 2)

$$= \left(\frac{q}{p}\right)^{S_n} \cdot \left[p\left(\frac{q}{p}\right)^{1} + q\left(\frac{q}{p}\right)^{-1} \right]$$

$$= \left(\frac{q}{p}\right)^{S_n} \left[q + p \right] = \left(\frac{q}{p}\right)^{S_n} 1.$$

$$\therefore \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}} \middle| X_1, \ldots X_n \right] = \left(\frac{q}{p}\right)^{S_n}$$

<u>Example 3:</u> (Polya's urn.)

An urn originally contains a white balls and b black balls. An experimenter draws one ball at random from the urn, examines the color, then puts the ball back along with one more ball of the same colour. Let M_n denote the proportion of white balls in the urn, then M_n is a martingale w.r.t. to itself

We have

$$M_{n+1}\Big|_{M_n = \frac{m}{n+a+b}} = \begin{cases} \dfrac{m+1}{n+a+b+1} & \text{with prob. } \dfrac{m}{n+a+b} \\[2ex] \dfrac{m}{n+a+b+1} & \text{with prob } \dfrac{n+a+b-m}{n+a+b} \end{cases}$$

$$\mathbb{E}\left(M_{n+1} \middle| M_1, M_2 \ldots M_n \right) = \frac{m}{n+a+b} \cdot \frac{m+1}{n+a+b+1} + \frac{n+a+b-m}{n+a+b} \cdot \frac{m}{n+a+b+1}$$

$$\mathbb{E}\left(M_{n+1} \middle| M_1 \dots M_n\right) = \frac{m}{n+a+b} = M_n$$

Example 4: (Discounted Stock price is a martingale under the risk neutral measure)

$$\mathbb{E}^{Q}\left(\frac{S_{n+1}}{(1+r)^{n+1}} \middle| \mathcal{F}_n \right) \longrightarrow \text{Information till time } n$$

$$= \frac{q_u S_{n+1}(\omega_1 \dots \omega_n H) + q_d S_{n+1}(\omega_1 \dots \omega_n T)}{(1+r)^{n+1}}$$

$$= \frac{u q_u S_n + d q_d S_n}{(1+r)^{n+1}}$$

$$= \frac{S_n}{(1+r)^n} \left(\frac{u q_u + d q_d}{1+r} \right)$$

$$= \frac{S_n}{(1+r)^n} \left[\frac{(1+r-d)u}{(u-d)(1+r)} + \frac{(u-1-r)d}{(u-d)(1+r)} \right]$$

$$= \frac{S_n}{(1+r)^n} \cdot \frac{(u-d)(1+r)}{(u-d)(1+r)} = \frac{S_n}{(1+r)^n}$$

Example 5: (Discounted option price is a Martingale under risk neutral measure)

We shall look at the discounted portfolio process $\frac{X_{n+1}}{(1+r)^{n+1}}$

We have

$$X_{n+1} = \underbrace{\Delta_n}_{\text{stock qty.}} \underbrace{S_{n+1}}_{\text{stock price at time } n+1} + \underbrace{(1+r)}_{\text{growth in 1 time period.}} \underbrace{(X_n - \Delta_n S_n)}_{\text{amt in bank or money mkt}}$$

$$\mathbb{E}^Q\left(\frac{X_{n+1}}{(1+r)^{n+1}} \,\middle|\, \mathcal{F}_n\right)$$

$$= \mathbb{E}^Q\left(\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} + \frac{(X_n - \Delta_n S_n)}{(1+r)^n} \,\middle|\, \mathcal{F}_n\right)$$

$$= \mathbb{E}^Q\left(\frac{\Delta_n S_{n+1}}{(1+r)^{n+1}} \,\middle|\, \mathcal{F}_n\right) + \mathbb{E}\left(\frac{X_n - \Delta_n S_n}{(1+r)^n} \,\middle|\, \mathcal{F}_n\right)$$

$$= \Delta_n \frac{S_n}{(1+r)^n} + \frac{X_n}{(1+r)^n} - \frac{\Delta_n S_n}{(1+r)^n}$$

discounted Stock price is a Martingal

(property 3)

$$= \frac{X_n}{(1+r)^n}$$

$$\frac{C_n}{(1+r)^n} = \mathbb{E}^Q\left(\frac{C_{n+1}}{(1+r)^{n+1}} \,\middle|\, \mathcal{F}_n \right)$$

option price at time n

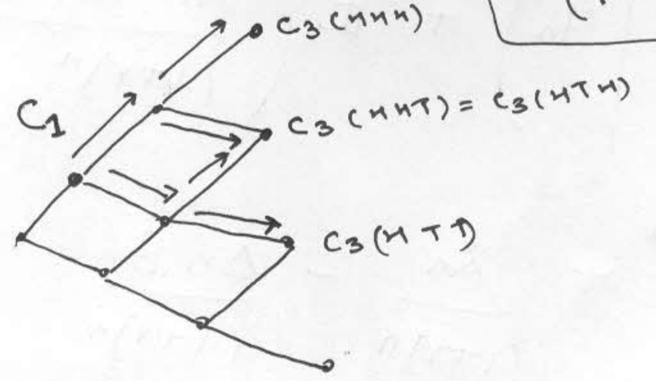
By property 4. of conditional expectation we can derive.

$$\frac{C_n}{(1+r)^n} = \mathbb{E}^Q\left[\frac{C_N}{(1+r)^N} \,\middle|\, \mathcal{F}_n \right]$$

$$\boxed{\text{ie} \quad C_n = \mathbb{E}^Q\left[\frac{C_N}{(1+r)^{N-n}} \,\middle|\, \mathcal{F}_n \right]}$$

option pricing formula.

$$C_1^{(H)} = \frac{(q_u)^2 C_3^{(HHH)}}{(1+r)^{3-1}} + \frac{q_u q_d C_3(HHT)}{(1+r)^{3-1}} + \frac{q_d q_u C_3(HTH)}{(1+r)^{3-1}} + \frac{(q_d)^2 C_3(HTT)}{(1+r)^{3-1}}$$



3-period model