

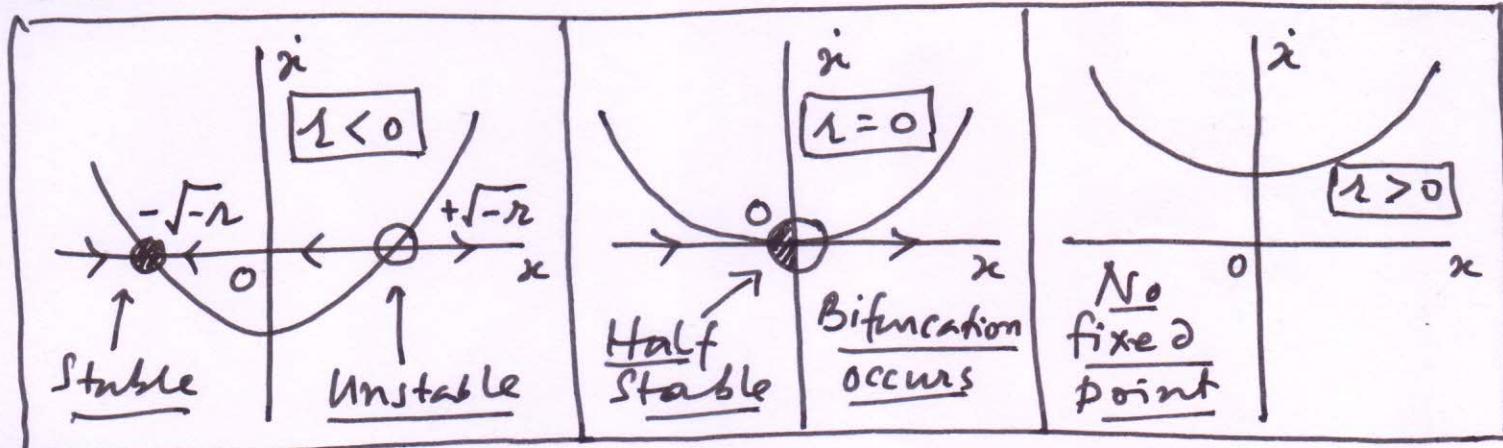
Bifurcations

1. Qualitative changes in the dynamics due to changes in parameters.
2. Result of multiple fixed points (nonlinear).

Saddle-Node Bifurcation

$$\dot{x} = f(x, \alpha) = \alpha + x^2 \quad \alpha \rightarrow \frac{\text{Bifurcation parameter}}{\text{parameter}}$$

Alternative names: 1. Fold Bifurcation,
2. Turning-point bifurcation, 3. Blue-sky Bifurcation.



At $\alpha = 0$, bifurcation occurs. Two fixed points collide and annihilate each other.

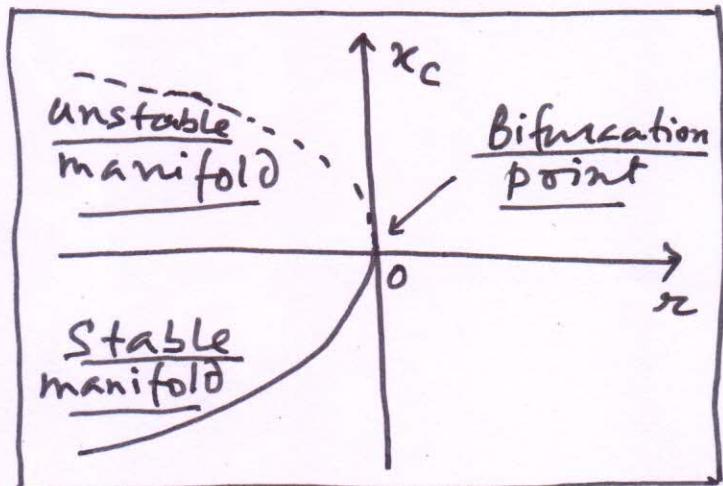
Bifurcation Diagram:

Solve $\dot{x} = f(x_c, \alpha) = 0$

$$\Rightarrow x_c^2 + \alpha = 0 \Rightarrow x_c = \pm \sqrt{-\alpha}$$

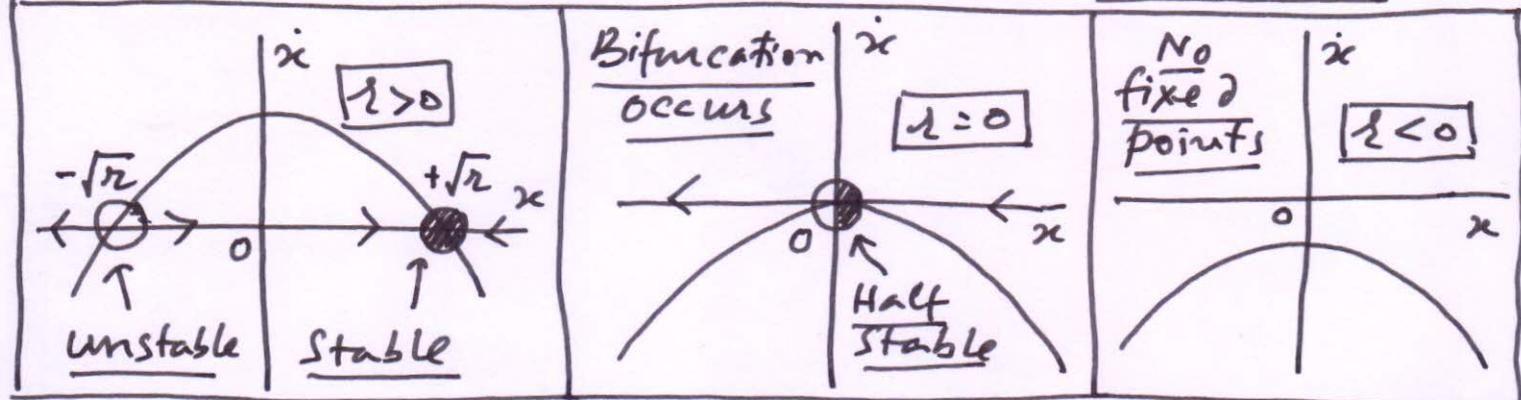
Real x_c only for $\alpha < 0$.

No fixed point for $\alpha > 0$.



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Example: $\dot{x} = f(x, \lambda) = \lambda - x^2$ When $\dot{x} = 0$
 $\Rightarrow x_c = \pm\sqrt{\lambda}$

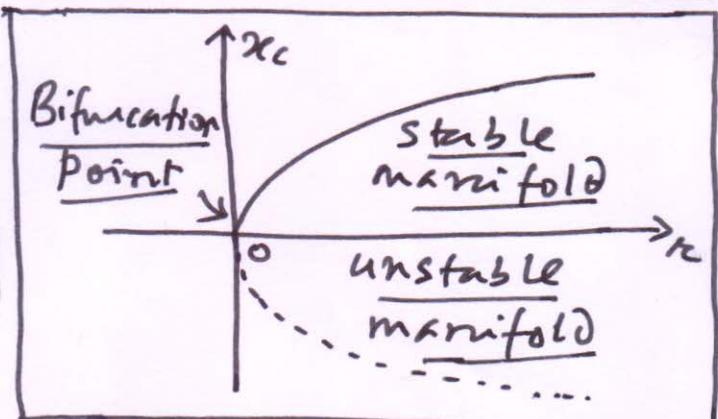


Bifurcation Diagram:

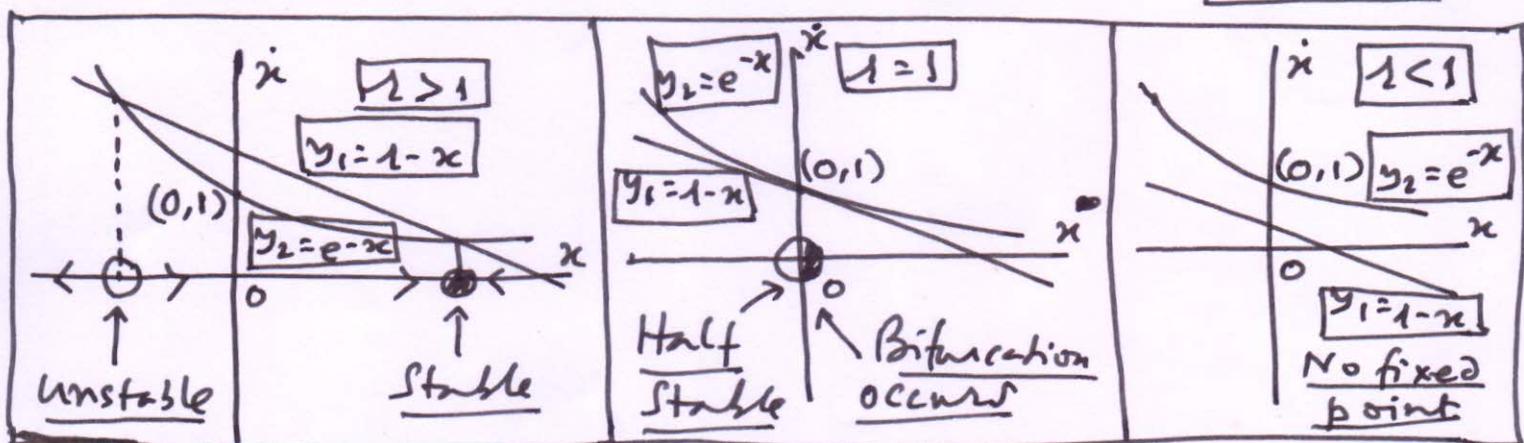
Solve $\dot{x} = f(x_c, \lambda) = 0$

$\Rightarrow \lambda - x_c^2 = 0 \Rightarrow x_c = \pm\sqrt{\lambda}$

Real fixed points only
for $\lambda > 0$.



Example: $\dot{x} = \lambda - x - e^{-x}$ Define $y_1 = 1 - x$
 $\dot{x} = f(x) = y_1 - y_2$ when $\dot{x} = 0 \Rightarrow y_1 = y_2$.



1. Reduce λ so that the fixed points approach each other.
(since $\lambda > 1$)
2. At $\lambda = 1$ the fixed points coalesce and the tangents of y_1 and y_2 are equal.
(P.T.O.)

(continued)

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Bifurcation condition: $\dot{x} = f(x, \lambda) = 0$

$\Rightarrow y_1 = y_2 \Rightarrow \lambda - x_c = e^{-x_c}$. Further,

Since the tangents are the same, $\frac{dy_1}{dx} = \frac{dy_2}{dx}$

$\Rightarrow \frac{d}{dx}(1-x) = \frac{d}{dx}(e^{-x}) \Rightarrow \frac{d}{dx}(1-x-e^{-x}) = 0$.

Since $f(x, \lambda) = 1-x-e^{-x} \Rightarrow \frac{df}{dx} = 0 \Rightarrow f'(x) = 0$

Since the first derivative, $f'(x) = 0$ at bifurcation, \Rightarrow the bifurcation point is neither stable, nor unstable.

$\therefore f'(x) = -1 + e^{-x} = 0 \Rightarrow x_c = 0 \Rightarrow f(x_c) = 0$
 $\Rightarrow 1-x-e^{-x}=0 \Rightarrow 1-0-e^0=0 \Rightarrow \lambda_b = 1$

At the bifurcation point, $\lambda = \lambda_b = 1$.

Normal form of the Saddle-Node Bifurcation

$\dot{x} = 1 \pm x^2 \rightarrow$ The normal form.

Example: $\dot{x} = 1-x-e^{-x}$.

Bifurcation occurs at $x_c = 0$ for $\lambda_b = 1$

Expand about $x_c = 0$. $e^{-x} = 1-x+\frac{x^2}{2!}+\dots$

$\Rightarrow \dot{x} = (1-x) - \left(1-x+\frac{x^2}{2!}+\dots\right)$

$\Rightarrow \dot{x} = 1-x-1+x-\frac{x^2}{2!}+\dots$

Expand and go only up to the second order. (P.T.)

(continued)

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$$\Rightarrow \dot{x} = (\lambda - 1) - \frac{x^2}{2} \quad \text{Up to the second order.}$$

(Parabolic form).

For rescaling we write

$$x = ku$$

k is a constant parameter

$$\Rightarrow \dot{x} = k\dot{u} = \lambda - 1 - \frac{k^2 u^2}{2}$$

Choose $k = 2$ to get

$$\Rightarrow \dot{u} = \left(\frac{\lambda - 1}{k}\right) - \frac{ku^2}{2} \Rightarrow \dot{u} = \frac{\lambda - 1}{2} - u^2$$

$$\Rightarrow \dot{u} = R - u^2 \quad \text{in which } R = \frac{\lambda - 1}{2} \quad (\text{Normal form})$$

When $R = 0 \Rightarrow \lambda = 1$ or $\lambda_b = 1$. At ~~when~~ $u = 0$ bifurcation occurs for $\lambda = \lambda_b = 1$. (also $x = 0$)

Taylor Expansion (about the bifurcation point)

$$\dot{x} = f(x, \lambda) \quad \text{Bifurcation occurs when}$$

$\dot{x} = 0$ at $x = x_c$ for $\lambda = \lambda_b$.

$\therefore f(x_c, \lambda_b) = 0$ and $f'(x_c, \lambda_b) = 0$ are the conditions required for bifurcation.

Expand about x_c as $x = x_c + \epsilon$ and about λ_b as $\lambda = \lambda_b + \rho$. Hence,

$$\begin{aligned} \dot{x} &= \cancel{f(x, \lambda)} f(x, \lambda) = f(x_c, \lambda_b) \\ &\quad + \frac{\partial f}{\partial x} \Big|_{x_c, \lambda_b} \epsilon + \frac{\partial f}{\partial \lambda} \Big|_{x_c, \lambda_b} \rho + \frac{1}{2!} \frac{\partial^2 f}{\partial x^2} \Big|_{x_c, \lambda_b} \epsilon^2 \\ &\quad + \frac{1}{2!} \frac{\partial^2 f}{\partial x \partial \lambda} \Big|_{x_c, \lambda_b} \epsilon \rho + \frac{1}{2!} \frac{\partial^2 f}{\partial \lambda^2} \Big|_{x_c, \lambda_b} \rho^2 + \dots \end{aligned}$$

(P.T.O.)

(continued)

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Since the bifurcation conditions are

$$f(x_c, \lambda_b) = 0 \text{ and } \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b} = 0, \text{ we}$$

retain only the ρ and ϵ^2 terms. Since,

$$\epsilon = x - x_c \text{ and } \rho = \lambda - \lambda_b, \text{ we have}$$

$$\dot{x} = \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b} (\lambda - \lambda_b) + \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} (x - x_c)^2.$$

Writing $a = \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b}$ and $b = \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b}$

gives $\dot{x} = a(\lambda - \lambda_b) + b(x - x_c)^2. (a, b \neq 0)$.

Transform $y = x - x_c \Rightarrow \dot{y} = \dot{x}$.

$$\Rightarrow \dot{y} = a(\lambda - \lambda_b) + b y^2 \text{ Now scale as } y = ku.$$

$$\Rightarrow \dot{y} = ku = a(\lambda - \lambda_b) + b k^2 u^2$$

$$\Rightarrow \dot{u} = \frac{a(\lambda - \lambda_b)}{k} + b k u^2$$

To derive
the
normal
form.

Choose $k = \frac{1}{|b|}$, so that,

$$\dot{u} = a|b|(\lambda - \lambda_b) \pm u^2 \text{ Define } R = a|b|(\lambda - \lambda_b)$$

$$\Rightarrow \dot{u} = R \pm u^2$$

This is the normal form
of the saddle-node bifurcation.

When $R = 0$, $\Rightarrow \lambda = \lambda_b$. Bifurcation occurs
for ~~when~~ $\dot{u} = 0$ at $u = 0$.

Some Examples of Saddle-Node Bifurcation

Example 1: $\dot{x} = f(x, \lambda) = 1 + \lambda x + x^2$

$$\Rightarrow \dot{x} = 1 + 2 \frac{1}{2} x + x^2 + \frac{1^2}{4} - \frac{1^2}{4}$$

$$\Rightarrow \dot{x} = (x + \frac{1}{2})^2 + (1 - \frac{1^2}{4})$$

Set $y = x + \frac{1}{2}$ $\Rightarrow \dot{x} = \dot{y}$

define $R = 1 - \frac{1^2}{4}$

$$\Rightarrow \dot{y} = R + y^2 \cdot \text{Bifurcation occurs}$$

when $R = 0 \Rightarrow 1 - \frac{1^2}{4} = 0 \Rightarrow \lambda_b = \pm 2$

Example 2: $\dot{x} = f(x, \lambda) = 1 + x - \ln(1+x)$

Bifurcation occurs at $x_c = 0$ for $\lambda_b = 0$.

Now $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$ (Expand about $x=0$)

$$\Rightarrow \dot{x} = \lambda + x - \left(x - \frac{x^2}{2} + \dots\right) \quad \text{Going up to the second order.}$$

$$\Rightarrow \dot{x} = \lambda + \frac{x^2}{2} \quad \text{Scale as } x = ku.$$

$$\Rightarrow k\dot{u} = \lambda + \frac{k^2 u^2}{2} \Rightarrow \dot{u} = \frac{1}{k} + \frac{k}{2} u^2$$

Choose $k = 2$ $\Rightarrow \dot{u} = \frac{1}{2} + u^2$, in the form $\dot{u} = R + u^2$. Bifurcation occurs when $R = 0$ or $1 = 0$ at $u = 0$ or $x = 0$

Example 3 : $\dot{x} = f(x, \alpha) = 1 + \frac{x}{2} - \frac{x}{1+x}$

$$\Rightarrow \dot{x} = 1 + \frac{x}{2} - x(1+x)^{-1}$$

$$\Rightarrow \dot{x} = 1 + \frac{x}{2} - x(1-x+\dots)$$

By binomial expansion

$$(1+x)^n = 1 + nx + \dots$$

$$\Rightarrow \dot{x} = 1 + \frac{x}{2} - x + x^2 = 1 - \frac{x}{2} + x^2$$

$$\Rightarrow \dot{x} = 1 - 2x\left(\frac{1}{4}\right) + x^2 + \frac{1}{16} - \frac{1}{16}$$

$$\Rightarrow \dot{x} = \left(2 - \frac{1}{16}\right) + \left(x - \frac{1}{4}\right)^2 \text{ Set } y = x - \frac{1}{4}$$

$$\Rightarrow \dot{y} = R + y^2 \text{ in which } R = 2 - \frac{1}{16}.$$

Bifurcation occurs for $R=0 \Rightarrow u_b = \sqrt{\frac{1}{16}}$.

Example 4 : $\dot{x} = f(x, \alpha) = 1 - \cosh x$ Bifurcation at $x_c = 0$

Now $\cosh x = \frac{e^x + e^{-x}}{2}$. Series expansion

$$\Rightarrow \dot{x} = 1 - \frac{1}{2}(e^x - e^{-x}) = 1 - \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} + \dots\right)\right]$$

$$\Rightarrow \dot{x} = 1 - \frac{1}{2}(2 + x^2) = (1-1)\bar{x} \frac{x^2}{2} \quad \text{Scale } \bar{x} = Ku$$

$$\Rightarrow Ku = 1 - 1 - \frac{k^2 u^2}{2} \Rightarrow \dot{u} = \frac{1-1}{K} - \frac{ku^2}{2}$$

Choose $k=2$ $\Rightarrow \dot{u} = \left(\frac{1-1}{2}\right) - u^2$ Define $R = \frac{1-1}{2}$

$$\Rightarrow \dot{u} = R - u^2 \quad \text{Bifurcation occurs for}$$

$$R=0, \Rightarrow \left[\frac{1-1}{2} = 0\right] \Rightarrow [u_b = 1] \text{ at } u=0 \Rightarrow x=0.$$

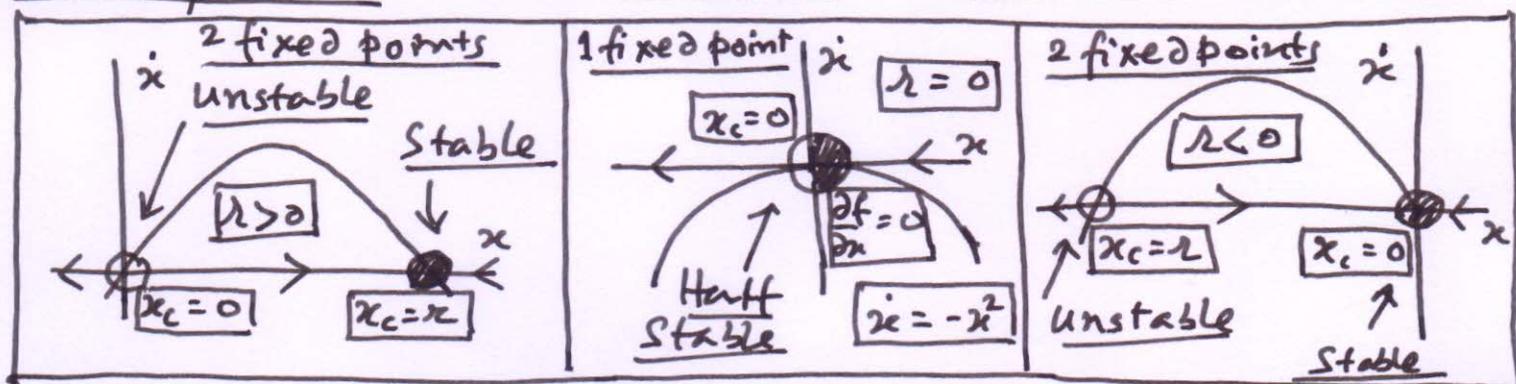
Transcritical Bifurcation

1. Two fixed points ALWAYS exist.
2. They exchange their stability properties.

$$\dot{x} = f(x, r) = rx - x^2$$

Normal form of
transcritical bifurcation.

Fixed points: When $\dot{x} = 0 \Rightarrow x_c = 0, x_c = r$.



1. Bifurcation occurs when $r = 0$ at $x_c = 0$.

2. The two fixed points exchange their stabilities.

3. At the bifurcation point, $\frac{df}{dx} = 0$

4. No fixed point is eliminated (unlike saddle-node)

5. Resembles the logistic equation: $\dot{x} = ax - bx^2$

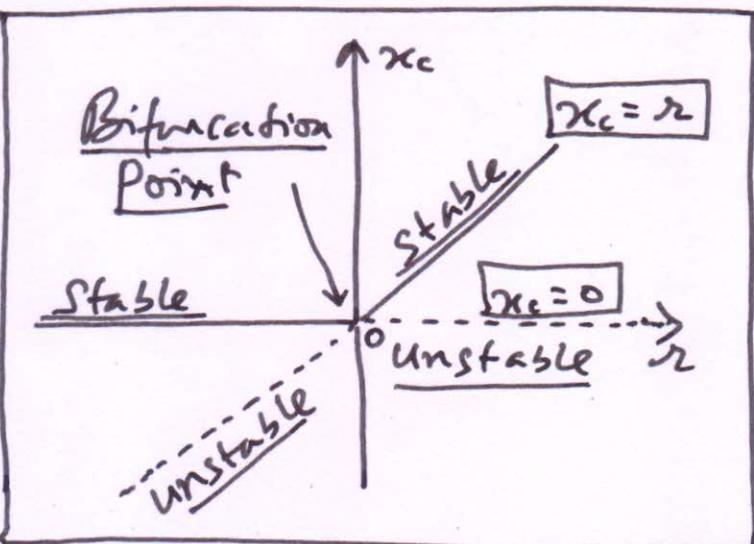
Bifurcation Diagram:

Solve $\dot{x} = f(x_c, r) = 0$

$$\Rightarrow rx_c - x_c^2 = 0 \Rightarrow x_c = 0$$

and $x_c = r$ (straight lines).

Both fixed points occur for all real values of r



Some Examples of Transcritical Bifurcation

Example 1: $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

$\dot{x} = f(x, a, b)$ for all a, b , $x_c = 0$ is a fixed point.

Expand $e^{-bx} = 1 - bx + \frac{b^2x^2}{2!} + \dots$ about $x=0$.

$$\Rightarrow \dot{x} = x - x^3 - a\left(bx - \frac{b^2x^2}{2}\right) \quad \begin{matrix} \uparrow \\ \text{up to the} \\ \text{second order.} \end{matrix}$$

$$\Rightarrow \dot{x} = x(1-ab) + \frac{ab^2x^2}{2} - x^3 \quad \begin{matrix} \text{The cubic order} \\ \text{is to be neglected.} \end{matrix}$$

$$\Rightarrow \dot{x} = x(1-ab) + \frac{ab^2}{2}x^2 \quad \text{when } \dot{x} = 0 \quad x_c = 0 \quad \text{and} \\ x_c = \frac{2(ab-1)}{ab^2}.$$

We scale $x = ku$

$$\Rightarrow k\dot{u} = ku(1-ab) + \frac{ab^2k^2u^2}{2} \Rightarrow \dot{u} = u(1-ab) + \frac{ab^2ku^2}{2}$$

Choose $k = -2/ab^2$ and define $R = 1-ab$.

$$\Rightarrow \dot{u} = Ru - u^2 \quad \begin{matrix} \text{Transcritical bifurcation occurs} \\ \text{when } R = 0 \Rightarrow ab = 1 \quad (\text{at } u = 0 \Rightarrow x = 0). \end{matrix}$$

The derivatives at the bifurcation: $\dot{x} = f(x, a, b)$

$$f(x, a, b) = x(1-x^2) - a(1-e^{-bx})$$

All the

i.) $\frac{\partial f}{\partial a} = -\left(1-e^{-bx}\right) \therefore \text{When } x=0, \frac{\partial f}{\partial a} = 0$ first derivatives

ii.) $\frac{\partial f}{\partial b} = ae^{-bx} \cdot (-x) \therefore \text{When } x=0, \frac{\partial f}{\partial b} = 0$ vanish

iii.) $\frac{\partial f}{\partial x} = f'(x) = 1-3x^2 + a e^{-bx}(-b) \therefore \text{When } x=0, \frac{\partial f}{\partial x} = 1-ab$

Since bifurcation occurs when $ab = 1 \Rightarrow \frac{\partial f}{\partial x} = 0$..

Example 2:

$$\dot{x} = f(x, \lambda) = \lambda \ln x + x - 1$$

when $\dot{x} = 0$, $x_c = 1 \rightarrow$ the fixed point.

Transform $y = x - 1 \Rightarrow x = 1 + y \& \dot{x} = y$.

$$\therefore \dot{y} = \lambda \ln(1+y) + y \quad \text{Now } \ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} \dots$$

$$\Rightarrow \dot{y} = \lambda \left(y - \frac{y^2}{2} \right) + y \quad \begin{matrix} \text{(series expansion)} \\ \text{up to the second order} \end{matrix}$$

$$\Rightarrow \dot{y} = (\lambda + 1)y - \lambda \frac{y^2}{2} \quad \text{Scaling } y = ku,$$

$$\Rightarrow k\dot{u} = ku(\lambda + 1) - \lambda \frac{k^2 u^2}{2} \Rightarrow \dot{u} = (\lambda + 1)u - \lambda \frac{ku^2}{2}$$

Choose $k = 2/\lambda$ and $R = \lambda + 1$

$$\Rightarrow \dot{u} = Ru - u^2 \quad (\text{Normal form of the transcritical bifurcation.})$$

for bifurcation, $R = 0 \Rightarrow \dot{x}_b = -1$ at $u = y = 0$
or $x = 1$.

The derivatives at the bifurcation: $\dot{x} = f(x, \lambda)$

$$f(x, \lambda) = \lambda \ln x + x - 1 \quad [x \rightarrow \text{variable}, \lambda \rightarrow \text{parameter}]$$

i.) $\frac{\partial f}{\partial x} = f'(x) = \frac{1}{x} + 1 \quad \therefore \text{When } \lambda = 0-1, x = 1$
 ~~$\frac{\partial f}{\partial x} = 0$~~

ii.) $\frac{\partial f}{\partial \lambda} = \ln x \quad \therefore \text{When } \lambda = 1, \frac{\partial f}{\partial \lambda} = 0$ first derivatives vanish.

iii.) $\frac{\partial^2 f}{\partial x \partial \lambda} = \frac{1}{x} = \frac{\partial^2 f}{\partial \lambda \partial x}$ At $x = 1, \frac{\partial^2 f}{\partial x \partial \lambda} = 1 = \frac{\partial^2 f}{\partial \lambda \partial x}$

The mixed second derivative is not zero.

Example 3: $\dot{x} = f(x, \lambda) = \lambda x + x^2$

$$\text{Scale } x = ku \Rightarrow k\dot{u} = \lambda ku + k^2 u^2$$

$$\Rightarrow \dot{u} = \lambda u + k u^2 \quad \text{Choose } k = -1 \text{ to get}$$

$$\dot{u} = \lambda u - u^2. \text{ Bifurcation occurs for } \lambda_b = 0 \text{ at } u = x = 0.$$

The derivatives at the bifurcation: $\dot{x} = f(x, \lambda)$

$$f(x, \lambda) = \lambda x + x^2 \Rightarrow \frac{\partial f}{\partial x} = \lambda + 2x, \quad \frac{\partial f}{\partial \lambda} = x$$

$$\text{when } \lambda = 0 \text{ and } x = 0 \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial \lambda} = 0.$$

$$\frac{\partial^2 f}{\partial x^2} = 2 = \frac{\partial^2 f}{\partial \lambda \partial x} \quad \begin{array}{l} \text{first derivatives vanish.} \\ \text{The mixed second derivative survives.} \end{array}$$

Example 4: $\dot{x} = f(x, \lambda) = \lambda x - \ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \dots \quad \text{up to the second order.}$$

$$\Rightarrow \dot{x} = \lambda x - \left(x - \frac{x^2}{2} \right) \Rightarrow \dot{x} = (\lambda - 1)x + \frac{x^2}{2}$$

$$\text{Scale } x = ku, \Rightarrow k\dot{u} = (\lambda - 1)ku + \frac{k^2 u^2}{2}.$$

$$\Rightarrow \dot{u} = (\lambda - 1)u + \frac{ku^2}{2} \quad \begin{array}{l} \text{choose } k = -2 \text{ to get} \\ \text{and define } R = \lambda - 1. \end{array}$$

$$\Rightarrow \dot{u} = Ru - u^2 \rightarrow \text{The normal form of transcritical bifurcation}$$

$$\text{when } R = 0, \lambda_b = 1 \text{ at } u = x = 0$$

The derivatives at the bifurcation: $\dot{x} = f(x, \lambda)$

~~$$f(x, \lambda) = \lambda x - \ln(1+x) \Rightarrow \frac{\partial f}{\partial x} = 1 - \frac{1}{1+x}$$~~

$$\text{and } \frac{\partial f}{\partial \lambda} = x. \quad \frac{\partial^2 f}{\partial x \partial \lambda} = 1 = \frac{\partial^2 f}{\partial \lambda \partial x}. \quad \text{when } x = 0, \lambda_b = 1.$$

Both first derivatives vanish.

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial \lambda} = 0$$

Example 5: $\dot{x} = f(x, r) = x(r - e^x)$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \Rightarrow \dot{x} = rx - x\left(1 + x + \frac{x^2}{2!} + \dots\right)$$

$$\Rightarrow \dot{x} = (r-1)x - x^2 \text{ up to the second order.}$$

Define $R = r-1 \Rightarrow \dot{x} = Rx - x^2$ (~~normal form~~)

Bifurcation occurs at $x_c = 0$ for $R = 0 \Rightarrow \lambda_b = 1$

The derivatives at the bifurcation: $\dot{x} = f(x, r)$

$$f(x, 1) = rx - xe^x \Rightarrow \frac{\partial f}{\partial r} = x \quad \frac{\partial f}{\partial x} = 1 - xe^x - e^x$$

When $x=0, r=1 \quad \frac{\partial f}{\partial r} = 0 = \frac{\partial f}{\partial x}$ First derivatives vanish.

$$\frac{\partial^2 f}{\partial r \partial x} = 1 = \frac{\partial^2 f}{\partial x \partial r} \text{. The mixed second derivative is non-zero.}$$

Example 6: $\dot{x} = f(x, r) = x - rx(1-x)$

$$\Rightarrow \dot{x} = (1-r)x + rx^2 \text{ Scale } x = ku.$$

$$\Rightarrow k\dot{u} = (1-r)ku + rk^2u^2 \Rightarrow \dot{u} = (1-r)u + rk^2u^2$$

Choose $k = -1/r$, Define $R = 1-r \Rightarrow \dot{u} = Ru - u^2$ (Normal form)

Bifurcation occurs at $u=0$

$$\Rightarrow x=0 \text{ for } R=0 \Rightarrow \lambda_b=1$$

The derivatives at the bifurcation: $\dot{x} = f(x, r)$

$$\frac{\partial f}{\partial r} = 1-r+2rx, \quad \frac{\partial f}{\partial x} = -r+x^2. \text{ When } x=0, r=1 \quad \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial r}$$

$$\frac{\partial^2 f}{\partial r \partial x} = -1+2x = \frac{\partial^2 f}{\partial x \partial r} \quad \text{At the bifurcation the mixed second derivative is non-zero}$$

Both the first derivatives vanish at bifurcation.

Taylor Expansion (about the bifurcation point)

$$\dot{x} = f(x, \lambda)$$

Bifurcation occurs when
 $x = 0$ at $x = x_c$ for $\lambda = \lambda_b$.

The conditions at the bifurcation are

$$f(x_c, \lambda_b) = 0, \quad \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b} = f'(x_c, \lambda_b) = 0, \quad \left. \frac{\partial f}{\partial \lambda} \right|_{x_c, \lambda_b} = 0.$$

All first derivatives vanish for transcritical bifurcations.

Expand about x_c as $x = x_c + \epsilon$ and about λ_b as $\lambda = \lambda_b + \rho$ $\Rightarrow \begin{cases} \epsilon = x - x_c \\ \rho = \lambda - \lambda_b \end{cases}$

$$\begin{aligned} \dot{x} = f(x, \lambda) &= f(x_c, \lambda_b) + \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b} \epsilon + \left. \frac{\partial f}{\partial \lambda} \right|_{x_c, \lambda_b} \rho \\ &+ \frac{1}{2!} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} \epsilon^2 + \frac{1}{2!} \left. \frac{\partial^2 f}{\partial \lambda \partial x} \right|_{x_c, \lambda_b} \epsilon \rho + \frac{1}{2!} \left. \frac{\partial^2 f}{\partial \lambda^2} \right|_{x_c, \lambda_b} \rho^2 + \dots \end{aligned}$$

$$\text{Now } f(x_c, \lambda_b) = \left. \frac{\partial f}{\partial \lambda} \right|_{x_c, \lambda_b} = \left. \frac{\partial f}{\partial \lambda} \right|_{x_c, \lambda_b} = 0 \quad \text{which}$$

are the conditions for the transcritical bifurcation.

Retain only the ϵ^2 term or the $(x - x_c)^2$ term and the $\rho \epsilon$ terms or the $(\lambda - \lambda_b)(x - x_c)$ terms.

$$\therefore \dot{x} = \left[\left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} (\lambda - \lambda_b) \right] (x - x_c) + \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} (x - x_c)^2$$

Writing $a = \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b}$ and $b = \frac{1}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b}$

gives $\dot{x} = a(\lambda - \lambda_b)^{(x-x_c)} + b(x - x_c)^2 \quad (a, b \neq 0)$

Transform $[y = x - x_c] \Rightarrow \dot{x} = y \quad (\text{P.T.O.})$

$$\therefore \boxed{y = a(\lambda - \lambda_b)y + by^2} \quad \text{Scaling as } \boxed{y = ku}$$

$$\Rightarrow ku = a(\lambda - \lambda_b)ku + bk^2u^2$$

$$\Rightarrow u = a(\lambda - \lambda_b)u + bkku^2$$

Choose $K = -\frac{1}{b}$
and define

$$\therefore \boxed{u = Ru - u^2} \quad \begin{matrix} \text{Normal} \\ \text{form of} \end{matrix} \quad \text{the transcritical bifurcation.}$$

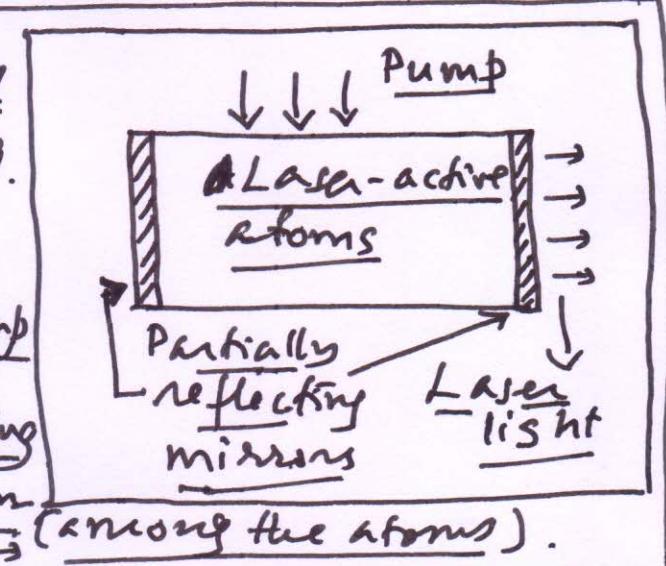
Bifurcation occurs when $R = 0 \Rightarrow \lambda = \lambda_b$ at $\boxed{u=0}$

Laser Threshold

1. Laser action happens only above a threshold pumping.

2. Before the threshold the laser is like an ordinary lamp

3. The process is self-organising due to cooperative interaction



i.) $n(t)$ is the photon number in the laser.

ii.) $N(t)$ is the number of stimulated atoms.

iii.) $\frac{dn}{dt} = \text{gain} - \text{loss}$. Gain is due to stimulated emission. Loss is due to leakage.

iv.) Gain happens after the random encounter between photons and stimulated atoms.

v.) $\text{Gain} \propto nN$, \rightarrow (Jointly proportional).

vi.) $\text{Gain} = G n N$ $\boxed{G > 0} \rightarrow$ Gain Coefficient.
(P.T.O.)

(continued)

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vii.) $\boxed{\text{Loss} \propto n} \Rightarrow \boxed{\text{Loss} = kn}$ $\boxed{K > 0}$ \rightarrow Loss coefficient.

viii.) $\frac{dn}{dt} = r = GnN - kn$

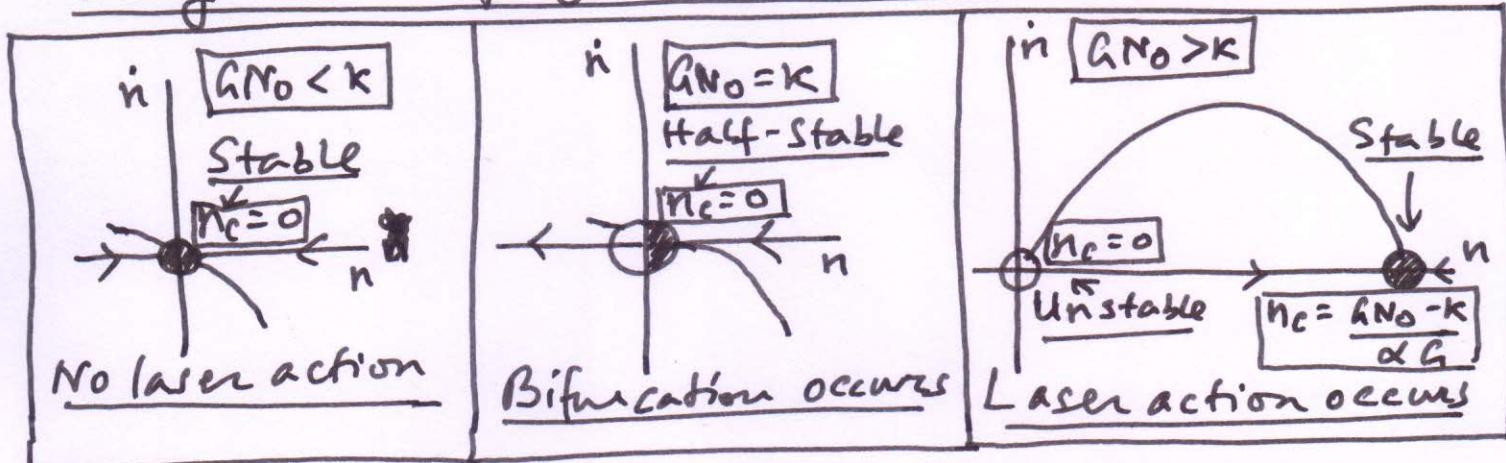
ix.) $N(t) = N_0 - \alpha n$ No is the fixed number of atoms in the absence of lasing action. α is the rate at which atoms return to the ground state. ($\alpha > 0$)

$$\therefore r = Gn(N_0 - \alpha n) - kn = (G N_0 - K)n - \alpha G n^2$$

(Like the logistic equation leading to transcritical bifurcation)

Fixed points: $n=0 \Rightarrow n_c = 0$ and $n_c = \frac{G N_0 - K}{\alpha G}$

Only $n > 0$ is physically valid. (Two fixed points)



Bifurcation Diagram:

Solve $\boxed{r = f(n_c, N_0) = 0}$

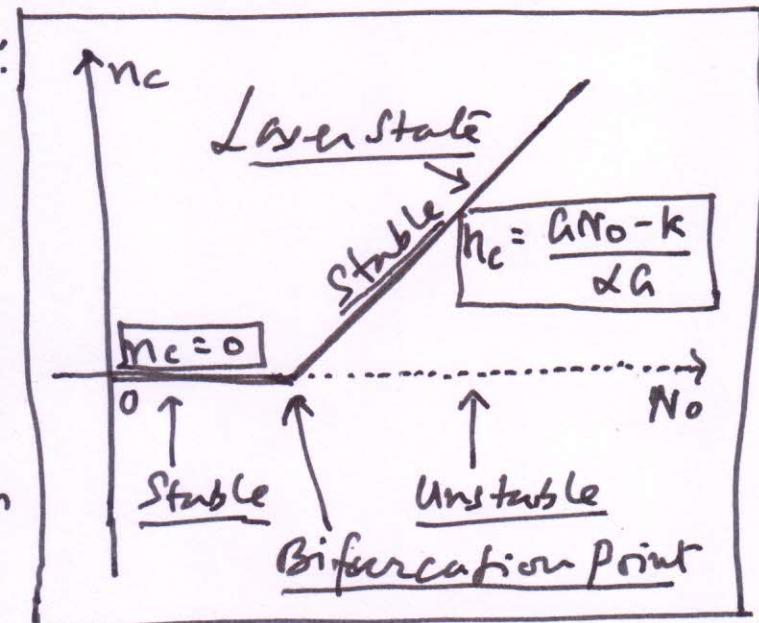
(N_0 is the parameter)

$\Rightarrow \boxed{n_c = 0}$ and $\boxed{n_c = \frac{G N_0 - K}{\alpha G}}$

1. Only $\boxed{n_c > 0}$ gives laser action

2. Laser action begins at the transcritical

bifurcation point. For $\boxed{G N_0 > K}$ the laser state is an attractor.



Pitchfork Bifurcation

1. Either one fixed point or three fixed points.
2. At the bifurcation, two fixed points annihilate each other (saddle-node bifurcation).
3. Symmetric (invariant) for $[x \rightarrow -x]$.

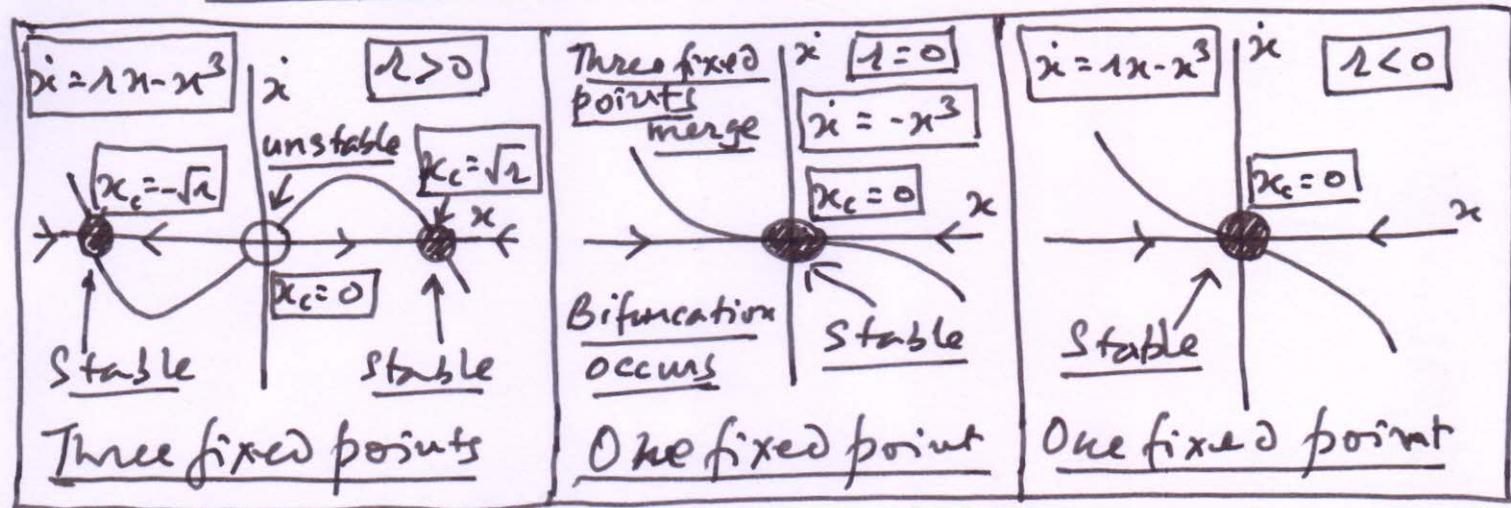
Supercritical Pitchfork Bifurcation

$$\dot{x} = f(x, \lambda) = \lambda x - x^3$$

Normal form of the supercritical pitchfork bifurcation.

fixed points : $\dot{x} = 0 \Rightarrow x_c = 0$ and $x_c = \pm\sqrt{\lambda}$

- i.) Three fixed points when $\lambda > 0$. Bifurcation condition is $\lambda = 0$.
- ii.) One fixed point when $\lambda < 0$.



4. Bifurcation occurs when $\lambda = 0$ at $x_c = 0$.
2. At the bifurcation, two fixed points disappear (as in a saddle-node bifurcation, they merge).
3. After the bifurcation only one fixed point remains.
4. At the bifurcation $\frac{\partial f}{\partial u} = 0$ and $\frac{\partial^2 f}{\partial u^2} = 0$.

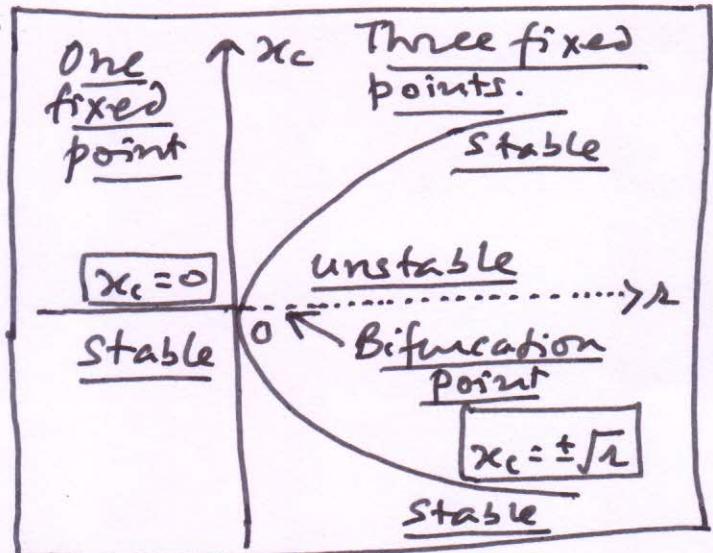
Bifurcation Diagram:

Solve $\dot{x} = f(x_c, \lambda) = 0$

$$\Rightarrow \lambda x_c - x_c^3 = 0 \Rightarrow x_c = 0$$

and $x_c = \pm \sqrt{\lambda}$ if Two real roots for $\lambda > 0$. The

bifurcation diagram looks like a pitchfork.



The bifurcation is called "supercritical" because the non-zero fixed points are above ("super") ($\lambda > 0$) the bifurcation point at $x_c = 0$ for $\lambda = \lambda_b = 0$.

Example: $\dot{x} = -x + \beta \tanh x$ (β is a parameter).

Define $y_1 = x$ and $y_2 = \beta \tanh x$ $\Rightarrow \dot{x} = y_2 - y_1$

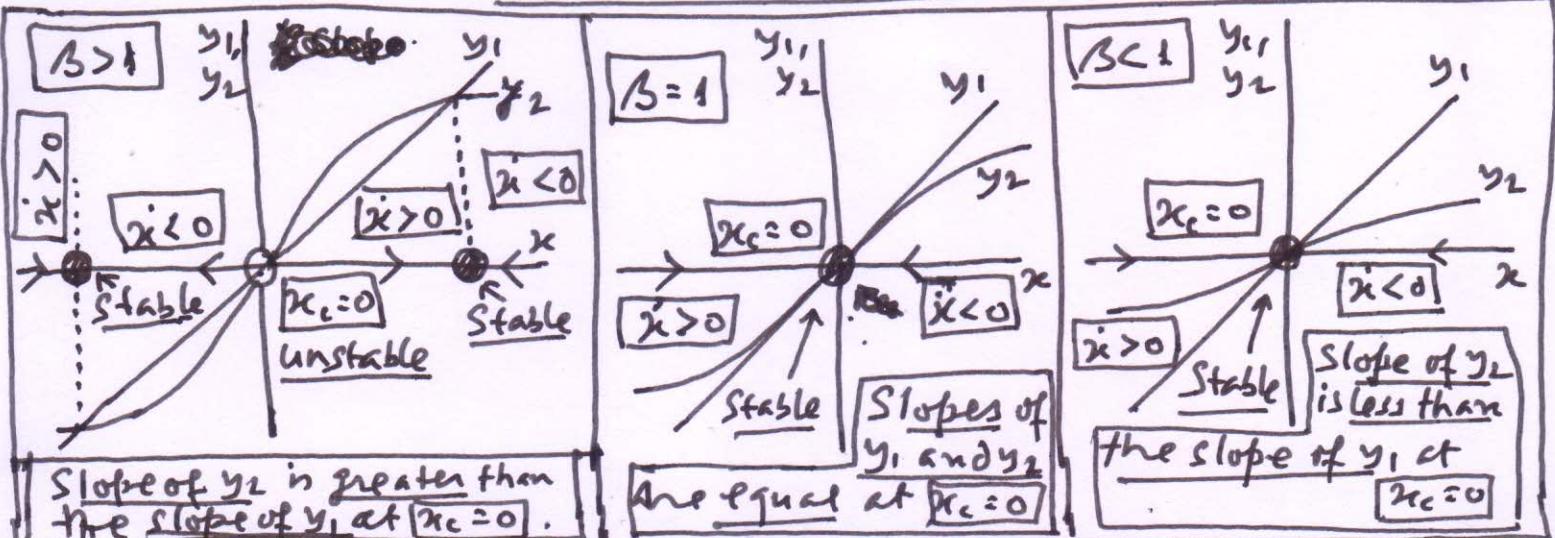
Now $\tanh x \approx \frac{e^x - e^{-x}}{e^x + e^{-x}}$

i) When $x \rightarrow 0$
 $e^x \approx 1+x$ and $e^{-x} \approx 1-x$

ii) $\Rightarrow \tanh x = \frac{x+x-(1-x)}{1+x+1-x} \Rightarrow \tanh x \approx \frac{2x}{2} \approx x \Rightarrow y_2 \approx \beta x$

iii) When $x \rightarrow \infty$, $\tanh x = \frac{1+e^{-2x}}{1+e^{-2x}} \rightarrow 1 \Rightarrow y_2 \rightarrow \beta$

When $x=0$
 $y_2=0$



(continued)

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The normal form of

$$\dot{x} = -x + \beta \tanh x$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$\Rightarrow \tanh x = \frac{(1+x+\cancel{x^2/2!}+x^3/3!+\dots)-(1-x+\cancel{x^2/2!}-x^3/3!+\dots)}{(1+x+\cancel{x^2/2!}+x^3/3!+\dots)+(1-x+\cancel{x^2/2!}-x^3/3!+\dots)}$$

$$\Rightarrow \tanh x \approx \frac{2x + 2x^3/6}{2 + x^2} = \frac{x + x^3/6}{1 + x^2/2} \quad \text{(up to the third order)}$$

$$\Rightarrow \tanh x \approx \left(x + \frac{x^3}{6}\right) \left(1 + \frac{x^2}{2}\right)^{-1} \approx \left(x + \frac{x^3}{6}\right) \left(1 - \frac{x^2}{2}\right)$$

$$\Rightarrow \tanh x \approx x + \frac{x^3}{6} - \frac{x^3}{2} = x + \frac{x^3}{6} - \frac{3x^3}{6} \approx x - \frac{x^3}{3}$$

$$\Rightarrow \dot{x} = -x + \beta \left(x - \frac{x^3}{3}\right) = (\beta - 1)x - \frac{\beta}{3}x^3$$

$$\text{Recall } x = ku \Rightarrow k\dot{u} = (\beta - 1)ku - \frac{\beta}{3}k^3u^3$$

$$\Rightarrow \dot{u} = (\beta - 1)u - \frac{\beta}{3}k^2u^3 \quad \text{Choose } k^2 = \frac{3}{\beta} \quad \text{Define } R = \beta - 1.$$

$$\Rightarrow \dot{u} = Ru - u^3 \quad \text{Normal form of the supercritical pitchfork bifurcation for } \beta = 1.$$

Bifurcation Diagram :

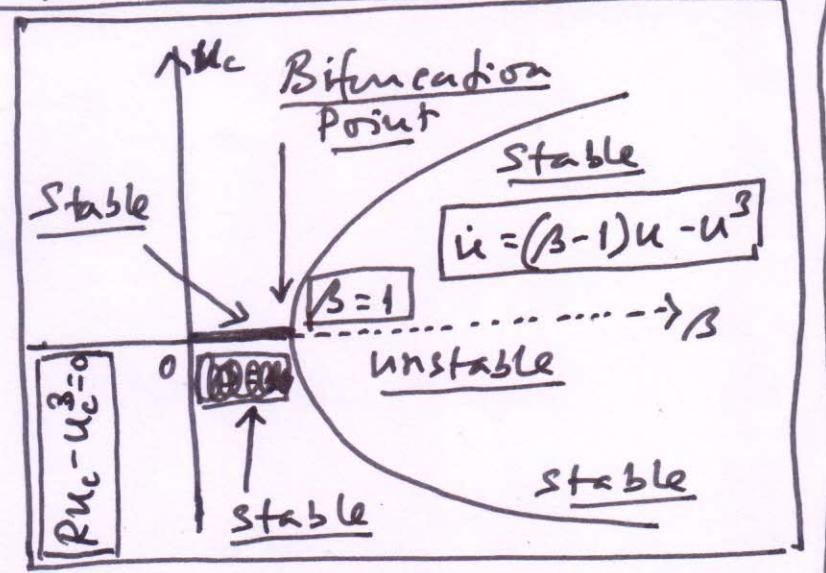
$$\text{Solve } \dot{x} = f(x_c, \beta) = 0 \\ \text{or } \dot{u} = f(u_c, R) = 0 \quad \text{(in this problem)}$$

$$\Rightarrow u_c = 0, \quad u_c = \pm \sqrt{R}$$

$$\Rightarrow u_c = \pm \sqrt{\beta - 1}$$

i.) One fixed point for $\beta < 1$.

ii.) Three fixed points for $\beta > 1$.



Subcritical Pitchfork Bifurcation

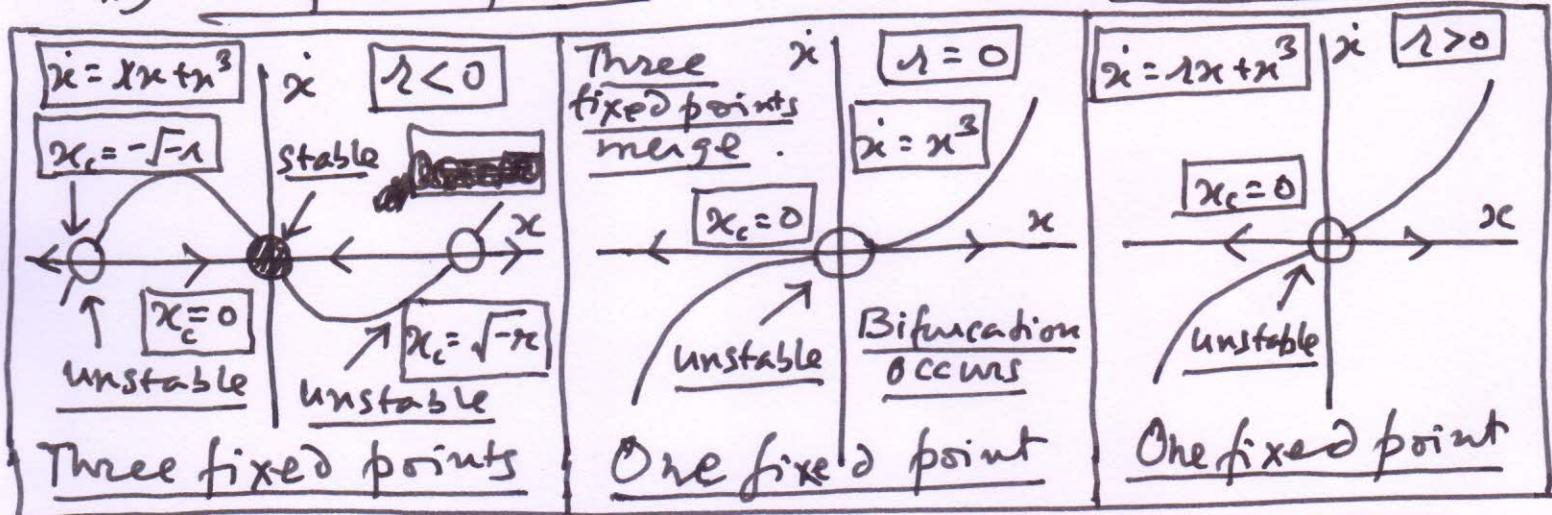
$$\dot{x} = f(x, \lambda) = \lambda x + x^3$$

Normal form of the
subcritical pitchfork
bifurcation

The cubic term destabilizes for $\lambda > 0$.

Fixed points: $\dot{x} = 0 \Rightarrow x_c = 0$ and $x_c = \pm\sqrt{-\lambda}$.

- i.) Three fixed points when $\lambda < 0$. Bifurcation condition is $\lambda = 0$.
- ii.) One fixed point when $\lambda > 0$.



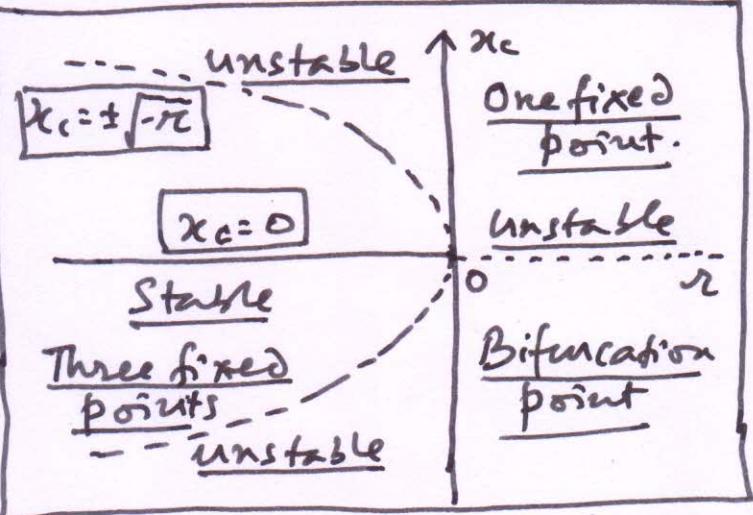
All conditions for the supercritical pitchfork bifurcation apply in the subcritical case.

Bifurcation Diagram:

Solve $\dot{x} = f(x_c, \lambda) = 0$

$\Rightarrow \lambda x_c + x_c^3 = 0 \Rightarrow x_c = 0$

and $x_c = \pm\sqrt{-\lambda}$. Two real roots for $\lambda < 0$. The



bifurcation is called "subcritical" because the non-zero fixed points are below ("sous") ($\lambda < 0$) the bifurcation point at $|x_c = 0|$ for $\lambda = \lambda_b = 0$.

Some Examples of Pitchfork Bifurcation

Example 1: $\boxed{\dot{x} = f(x, \lambda) = \lambda x + 4x^3}$ Fixed point at $x_c = 0$

Scale $x = ku$ $\Rightarrow \boxed{ku = \lambda ku + 4k^3 u^3}$.

$\Rightarrow \boxed{\dot{u} = \lambda u + 4k^2 u^3}$. Choose $\boxed{k^2 = 1/4}$ pitchfork bifurcation.

$\Rightarrow \boxed{\dot{u} = \lambda u + u^3}$, Normal form of subcritical

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{\lambda_b = 0}$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x, \lambda)}$

$$f(x, \lambda) = \lambda x + 4x^3 \Rightarrow \boxed{\frac{\partial f}{\partial x} = \lambda}, \boxed{\frac{\partial^2 f}{\partial x^2} = 0}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 1 + 12x^2}, \boxed{\frac{\partial^2 f}{\partial x^2} = 24x}, \boxed{\frac{\partial^2 f}{\partial x \partial \lambda} = \frac{\partial^2 f}{\partial \lambda \partial x} = 1}$$

When $\boxed{x=0, \lambda=0}$, $\boxed{\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \lambda} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \lambda^2} = 0}$

$$\boxed{\frac{\partial^3 f}{\partial x^3} = 24}. \text{ Only the mixed and the third derivatives are non-zero.}$$

Example 2: $\boxed{\dot{x} = f(x, \lambda) = \lambda x - 4x^3}$ Fixed point at $\boxed{x_c = 0}$.

Scale $x = ku$ $\Rightarrow \boxed{ku = \lambda ku - 4k^3 u^3}$.

$\Rightarrow \boxed{\dot{u} = \lambda u - 4k^2 u^3}$. Choose $\boxed{k^2 = 1/4} \Rightarrow \boxed{\dot{u} = \lambda u - u^3}$

This is the normal form of supercritical pitchfork bifurcation.

Bifurcation occurs at $\boxed{x_c = 0}$ for $\boxed{\lambda_b = 0}$.

The derivatives at the bifurcation: $\boxed{\dot{x} = f(x, \lambda)}$

$$f(x, \lambda) = \lambda x - 4x^3 \Rightarrow \boxed{\frac{\partial f}{\partial x} = \lambda}, \boxed{\frac{\partial^2 f}{\partial x^2} = 0}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = 1 - 12x^2}, \boxed{\frac{\partial^2 f}{\partial x^2} = -24x}, \boxed{\frac{\partial^2 f}{\partial x \partial \lambda} = \frac{\partial^2 f}{\partial \lambda \partial x} = 1}$$

Example 2 (continued): When $x=0, \lambda=0$,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \lambda} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial \lambda^2} = 0, \quad \frac{\partial^3 f}{\partial x^3} = -24.$$

Only the mixed and the third derivatives are non-zero.

Example 3: $\dot{x} = f(x, \lambda) = x + \frac{\lambda x}{1+x^2}$ at $x_c=0$.

$$\Rightarrow \dot{x} = x + \lambda x(1+x^2)^{-1} = x + \lambda x(1-x^2) \quad (\text{Binomial expansion})$$

$$\Rightarrow \dot{x} = x + \lambda x - \lambda x^3 = (1+\lambda)x - \lambda x^3 \quad [(1+x)^n = 1+nx]$$

$$\text{Scale } x = ku \Rightarrow \dot{x} = (1+\lambda)ku - \lambda k^3 u^3$$

$$\Rightarrow \dot{u} = (k+1)u - \lambda k^2 u^3. \text{ Choose } k^2 = 1/\lambda.$$

$$\Rightarrow \dot{u} = Ru - u^3 \quad \begin{array}{l} \text{Normal} \\ \text{form of the supercritical} \\ \text{pitchfork bifurcation.} \end{array}$$

Bifurcation occurs at $x_c=0$ for $\lambda_b=-1$, ($R=0$).

The derivatives at the bifurcation: $\dot{x} = f(x, \lambda)$.

$$f(x, \lambda) = x + \lambda x(1+x^2)^{-1} \Rightarrow \frac{\partial f}{\partial x} = \frac{x}{1+x^2}, \quad \frac{\partial^2 f}{\partial x^2} = 0.$$

$$\frac{\partial f}{\partial \lambda} = 1 + \frac{1}{1+x^2} + \frac{1x + 1x^2x}{(1+x^2)^2} = 1 + \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1 \cdot 2x}{(1+x^2)^2} - 2x \left[\frac{2x}{(1+x^2)^2} + \frac{x^2 - 2x^2x}{(1+x^2)^3} \right]$$

$$\Rightarrow \frac{\partial^2 f}{\partial \lambda^2} = -\frac{2x}{(1+x^2)^2} - 2x \left[\frac{2x}{(1+x^2)^2} - \frac{4x^3}{(1+x^2)^3} \right]$$

When $x_c=0, \lambda_b=-1$

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial \lambda} = \frac{\partial^2 f}{\partial \lambda^2} = 0 \quad (\text{P.I.O.})$$

$\Rightarrow (x=0, \lambda=-1)$

Example 3 (Continued): - 22 -

(continued)

$$\frac{\partial^3 f}{\partial x^3} = \frac{-2x}{(1+x^2)^2} - \frac{2x^2 \times -2 \times 2x - 2x}{(1+x^2)^3} \left[\frac{2}{(1+x^2)^2} \right] \\ + \frac{2x \times -2 \times 2x}{(1+x^2)^3} - \frac{4x^3}{(1+x^2)^3} - \frac{4x^3 \times -3 \times 2x}{(1+x^2)^4}$$

When $x = -1, x = 0$ $\frac{\partial^3 f}{\partial x^3} = 2 + 4 = 6$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} = \frac{\partial^2 f}{\partial x^2} \quad \text{When } x = 0 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 1$$

Only the mixed and the third derivatives are non-zero.

Example 4: $x_i = f(x, r) = rx - \sinh x$ fixed point at $x_c = 0$.

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \quad \begin{matrix} \text{Retaining only} \\ \text{up to the cubic order.} \end{matrix}$$

$$\Rightarrow x_i = rx - \frac{1}{2} \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \right) \right]$$

$$\Rightarrow x_i = rx - \frac{1}{2} \left(2x + \frac{x^3}{3} \right) = (r-1)x + \frac{x^3}{6} \quad \begin{matrix} \text{Scale} \\ x = ku \end{matrix}$$

$$\Rightarrow k u = (r-1)ku + \frac{k^3 u^3}{6} \quad \begin{matrix} \text{Choose} \\ k^2 = 6 \end{matrix} \quad \text{and define} \quad R = r-1$$

$$\Rightarrow u = Ru - u^3 \quad \text{Normal form of the supercritical pitchfork bifurcation.}$$

Bifurcation occurs at $x_c = 0$ for $R = 0 \Rightarrow k_b = 1$.

The derivatives at the bifurcation: $x_i = f(x, r)$

$$f(x, r) = rx - \left(\frac{e^x - e^{-x}}{2} \right), \quad \frac{\partial f}{\partial x} = r, \quad \frac{\partial^2 f}{\partial x^2} = 0$$

(P.T.O.)

Example 4 (continued): - 23 - (continued)

$$\frac{\partial f}{\partial x} = \lambda - \left(e^x + e^{-x} \right) / 2$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{2} (e^x - e^{-x})$$

When $x=0$, $\lambda=1$

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial \lambda} = \frac{\partial^2 f}{\partial \lambda^2} = 0$$

$$\frac{\partial^3 f}{\partial x^3} = -\frac{1}{2} (e^x + e^{-x}) = -1 \quad \left(\begin{array}{l} x=0 \\ \lambda=1 \end{array} \right) \quad \frac{\partial^4 f}{\partial x^2 \partial \lambda} = 1 = \frac{\partial^2 f}{\partial \lambda \partial x}$$

Only the mixed and the third derivatives are non-zero.

Taylor Expansion (about the bifurcation point)

$$x = f(x, \lambda) \quad \text{Bifurcation occurs when } x=0 \text{ at } x=x_c \text{ for } \lambda=\lambda_b$$

The conditions at the bifurcation are

$$f(x_c, \lambda_b) = 0, \quad \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b} = f'(x_c, \lambda_b) = 0, \quad \left. \frac{\partial f}{\partial \lambda} \right|_{x_c, \lambda_b} = 0,$$

$$\left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} = 0, \quad \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} = 0 \quad \text{All first and second derivatives vanish in a pitchfork bifurcation.}$$

$$\text{Expand about } x_c \text{ as } x = x_c + \epsilon \quad \Rightarrow \quad \epsilon = x - x_c$$

and about λ_b as $\lambda = \lambda_b + \rho \quad \Rightarrow \quad \rho = \lambda - \lambda_b$

$$x = \cancel{\text{linear terms}} + \cancel{\text{quadratic terms}} + \dots + f(x, \lambda) = f(x_c, \lambda_b)$$

$$+ \left. \frac{\partial f}{\partial x} \right|_{x_c, \lambda_b} \epsilon + \left. \frac{\partial f}{\partial \lambda} \right|_{x_c, \lambda_b} \rho + \frac{1}{2!} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x_c, \lambda_b} \epsilon^2 + \frac{1}{2!} \left. \frac{\partial^2 f}{\partial x \partial \lambda} \right|_{x_c, \lambda_b} \rho \epsilon$$

$$+ \frac{1}{2!} \left. \frac{\partial^4 f}{\partial x^2 \partial \lambda} \right|_{x_c, \lambda_b} \epsilon \rho + \frac{1}{2!} \left. \frac{\partial^4 f}{\partial \lambda^2} \right|_{x_c, \lambda_b} \rho^2 + \frac{1}{3!} \left. \frac{\partial^3 f}{\partial x^3} \right|_{x_c, \lambda_b} \epsilon^3 + \dots$$

Retain only the $\epsilon \rho$ or the $(x-x_c)(\lambda-\lambda_b)$ and the $(P.T.O.) \epsilon^3$ or $(x-x_c)^3$ terms in the Taylor expansion.

(Continued)

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$$\therefore \dot{x} = \left[\frac{\partial^2 f}{\partial x \partial n} \Big|_{x_c, 1_b} (1 - 1_b) \right] (x - x_c) + \frac{1}{3!} \frac{\partial^3 f}{\partial x^3} \Big|_{x_c, 1_b} (x - x_c)^3$$

Writing $a = \frac{\partial^2 f}{\partial x \partial n} \Big|_{x_c, 1_b}$ and $b = \frac{1}{6} \frac{\partial^3 f}{\partial x^3} \Big|_{x_c, 1_b}$,

gives $\dot{x} = a(1 - 1_b)(x - x_c) + b(x - x_c)^3$ ($a, b \neq 0$)

Transform $y = x - x_c \Rightarrow \dot{x} = \dot{y}$

$$\Rightarrow \dot{y} = a(1 - 1_b)y + by^3 \quad \text{scale } y = ku$$

$$\Rightarrow k\dot{u} = a(1 - 1_b)ku + bk^3u^3 \quad R = a(1 - 1_b)$$

$$\Rightarrow \dot{u} = a(1 - 1_b)u + bk^2u^3 \quad \text{Define } u = \frac{v}{k} \uparrow$$

Choose $k^2 = \frac{1}{|b|}$ (only the absolute value of b).

$$\Rightarrow \dot{u} = Ru \pm u^3 \quad \text{Normal form of the pitchfork bifurcation.}$$

i.) If the sign of b is positive (+), the bifurcation is subcritical.

ii.) If the sign of b is negative (-), the bifurcation is supercritical.

Bifurcation occurs for $\dot{u} = 0$ at $u = 0$ (or $x = 0$), when $|R| = 0$ i.e. $|R| = 1_b$.

NOTE: In all the cases of bifurcation, retain only the specifically relevant terms in the Taylor expansion about the bifurcation point. This will give the appropriate normal form.