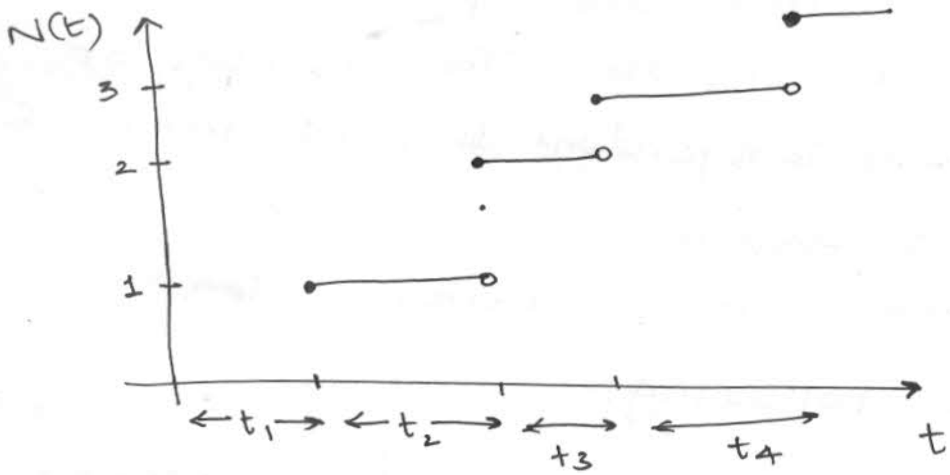
Jump diffusion model

The geometric Brownian motion model of the stock price does not take into account that there is a possibility of sudden price jumps. The Jump diffusion model accounts for such a possibility.

Lets first start with some background that we will need about a random process called the Poisson process $\{N(t) : t \geq 0\}$



N(t) is a random process that represents random arrivals of particles at time t. There are two kinds of randomness inherent in this. If you fix the time t, then the number of particles arriving by time t is a discrete random variable. Also, the inter-arrival times $t_1, t_2, \ldots$ are also continuous random variables. We start with three basic conditions

1. $N(0) = 0$

2. $P\left(N(t+h)=j \mid N(t)=i\right) = \begin{cases} \lambda h + o(h) & \text{if } j = j+1 \\ 1-\lambda h + o(h) & \text{if } j=i \\ o(h) & \text{else} \end{cases}$

3. $N(t-s)$ is independent of $N(s)$ for $0<s<t$

Meaning. The number of particles at time $t=0$ is 0. There is a possibility of either 1 or no particle arriving in a small interval h. The possibility of more than one particle, arriving in interval h is negligible. The number of particles arriving in ~~independent~~ disjoint intervals of time are independent.

Using these three simple criterea one can derive the following:

- $N(t)$ is a Poisson r.v. with parameter λt

  i.e. $P(N(t)=j) = \dfrac{e^{-\lambda t}(\lambda t)^j}{j!}$

- The inter-arrival times t_i are exponential r.v.'s with parameter λ

Jump diffusion model:

~~Let $N(t)$ denote the~~ .

Let us consider a model for price that Superimposes random jumps on a geometric Brownian motion. Let $N(t)$ denote the number of jumps that occur by time t where $N(t)$ is a Poisson process. Also suppose that when the i^{th} jump occurs the stock price is multiplied by J_i where $J_1, J_2, \dots$ are indepent random variables having specified probability distribution. Further let this sequence be independent of the time at which the jump occurs.

Thus the price $S(t)$ is given by

$$S(t) = S^*(t) \prod_{i=1}^{N(t)} J_i , \qquad t \geq 0$$

where $S^*(t)$ is a Brownian motion with volatality σ and drift ~~μ~~. ~~ie~~ to be decided as per risk neutrality.
For risk neutrality, it turns out that

$$S^*(t) = S(0)\, e^{\left(r - \frac{\sigma^2}{2} + \lambda - \lambda \mathbb{E}(J)\right)t + \sigma \cancel{d} W(t)}$$

$\left(r - \frac{\sigma^2}{2} \right.$ is replaced by $\left. r - \frac{\sigma^2}{2} + \lambda - \lambda \mathbb{E}(J) \right)$

$\therefore$ The no-arbitrage price of the option $= \mathbb{E}\left(e^{-rt} \left(S(t) - K \right)^+ \right)$

$$= e^{-rt}\,\mathbb{E}\left[\left(J(t)\,S^*(t)-K\right)^+\right]$$

where $J(t) = \prod_{i=1}^{N(t)} J_i$

and $S^*(t) = S(0)\, e^{\left[\left(r-\frac{\sigma^2}{2}+\lambda-\lambda\mathbb{E}(J)\right)t + \sigma W(t)\right]}$

One can use Monte-Carlo to compute the price of the option. ~~usin~~ that uses the Jump- diffusion model.