Second-Order Systems of Differential Equations

i.) First-Order Autonomous System: $\boxed{\frac{dx}{dt} = f(x)}$.

ii.) Coupled Second-Order Autonomous System:

$$\boxed{\frac{dx}{dt} = f(x,y)} \quad \text{and} \quad \boxed{\frac{dy}{dt} = g(x,y)}$$

(2-Dimensional System) Dynamical

An Economic Analogy (IBM): Coupled growth of revenue and human resources, given by

$$\boxed{\frac{dR}{dt} = \rho(R,H)} \quad \text{and} \quad \boxed{\frac{dH}{dt} = \eta(R,H)}.$$

R → Revenue
H → Human resources.

iii.) Coupled Third-Order Autonomous System:

$$\boxed{\frac{dx}{dt} = f(x,y,z)}, \quad \boxed{\frac{dy}{dt} = g(x,y,z)}, \quad \boxed{\frac{dz}{dt} = h(x,y,z)}.$$

iv.) Coupled N-Order Autonomous Dynamical System:

$$\boxed{\begin{aligned}
\frac{dx_1}{dt} &= f_1(x_1, x_2, x_3, \ldots, x_N) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, x_3, \ldots, x_N) \\
\frac{dx_3}{dt} &= f_3(x_1, x_2, x_3, \ldots, x_N) \\
&\vdots \\
\frac{dx_N}{dt} &= f_N(x_1, x_2, x_3, \ldots, x_N)
\end{aligned}}$$

N-dimensional System of Coupled N first-order differential equations. All are autonomous.

General Second-Order Autonomous Differential Equation:

$$\boxed{A\left(x, \frac{dx}{dt}\right)\frac{d^2x}{dt^2} + B\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + C\left(x, \frac{dx}{dt}\right)x = 0}.$$

This can then be recast as (dividing by A),

$$\boxed{\frac{d^2x}{dt^2} + F\left(x, \frac{dx}{dt}\right)\frac{dx}{dt} + G\left(x, \frac{dx}{dt}\right)x = 0}.$$

$\boxed{y = \frac{dx}{dt}}$

Writing, $\quad \frac{dx}{dt} = y = 0.x + 1.y$

we get $\rightarrow \quad \frac{dy}{dt} = -F(x,y)y - G(x,y)x$

(Since $y = \frac{dx}{dt}$)

Similarly for a third-order differential equation,

$$\frac{d^3x}{dt^3} + F\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)\frac{d^2x}{dt^2} + G\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)\frac{dx}{dt} + H\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)x = 0$$

we write $\boxed{\dot{x} = y}$, $\boxed{\ddot{x} = \dot{y} = z}$ and $\boxed{\dddot{x} = \dot{z}}$,

in all of which the "dot" implies a time derivative, $\left(\frac{d}{dt}\right)$.

Hence we get a coupled set of three equations.

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \dot{x} = 0.x + 1.y + 0.z \\ \frac{dy}{dt} &= \dot{y} = 0.x + 0.y + 1.z \\ \frac{dz}{dt} &= \dot{z} = -F(x,y,z)z - G(x,y,z)y - H(x,y,z)x \end{aligned}}.$$

An N-order coupled system can always be crafted out of an N-order autonomous differential equation.

Coupled Linear Autonomous Second-Order System:

$$\boxed{\frac{dx}{dt} = Ax + By + C} \quad \text{and} \quad \boxed{\frac{dy}{dt} = Dx + Ey + F}.$$

The most general linear form. Anything else is nonlinear, such as x^3, $\cos y$, xy, e^x, $\ln y$, etc.

Consider a simple system $\boxed{\frac{dx}{dt} = Ax + By}$ and $\boxed{\frac{dy}{dt} = Cx + Dy}$. (without any free constant)

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = A\frac{dx}{dt} + B\frac{dy}{dt} = A\frac{dx}{dt} + B(Cx + Dy)}$$

Substituting $\boxed{By = \frac{dx}{dt} - Ax}$, we get,

$$\boxed{\frac{d^2x}{dt^2} = A\frac{dx}{dt} + BCx + D\left(\frac{dx}{dt} - Ax\right)}.$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = (A+D)\frac{dx}{dt} - (AD - BC)x}$$

y has been eliminated

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} - (A+D)\frac{dx}{dt} + (AD-BC)x = 0}$$

Also $\boxed{\frac{d^2y}{dt^2} - (A+D)\frac{dy}{dt} + (AD-BC)y = 0}$ (in y).

Writing in a matrix form

$$\boxed{\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} \Rightarrow \boxed{\frac{d\vec{X}}{dt} = \tilde{A}\vec{X}}$$

$\vec{X} \equiv \{x_i\}$

We see that $\boxed{A + D = \tau}$, the trace of matrix $\tilde{A}$, and $\boxed{AD - BC = \Delta}$, the determinant of matrix $\tilde{A}$.