In-Sem 2 Solns

1. We have for GBM $dS = \mu S dt + \sigma S dW$

$$\ln\left(\frac{S_T}{S_0}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right)$$

$$\therefore\ P(S_T > 80) = P\left(\ln\left(\frac{S_T}{S_0}\right) > \ln\left(\frac{80}{50}\right)\right)$$

$$= P\left(X > \ln\left(\frac{80}{50}\right)\right)$$

where $X \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right) = N(\mu', \sigma'^2)$

$\mu' = \left(\mu - \frac{\sigma^2}{2}\right)T, \quad \sigma'^2 = \sigma^2 T$

$$\therefore\ P(S_T > 80) = P\left(\frac{X - \mu'}{\sigma'} > \frac{\ln\left(\frac{80}{50}\right) - \mu'}{\sigma'}\right)$$

$$= P\left(Z > \frac{\ln\left(\frac{80}{50}\right) - \mu'}{\sigma'}\right)$$

where Z is now $N(0,1)$

$$P(S_T > 80) = 1 - P\left(Z \le \frac{\ln\left(\frac{80}{50}\right) - \mu'}{\sigma'}\right)$$

$$= 1 - N\left(\frac{\ln\left(\frac{80}{50}\right) - \mu'}{\sigma'}\right)$$

Substituting $\mu' = \left(\mu - \frac{\sigma^2}{2}\right)T$

$$\left\{\begin{array}{l} \mu = 0.12 \text{ p.a} \\ \sigma = 0.3 \text{ p.a} \\ T = 2 \end{array}\right.$$

$\sigma' = \sigma\sqrt{T}$

$\mu' = 0.15 \qquad \sigma' = 0.3\sqrt{2}$

$$= 1 - N(0.7542)$$

$$P(S_T > 80) = 1 - 0.7734)$$

$$P(S_T > 80) = 0.2266$$

2. We apply Ito's lemma.

$$dF = \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW$$

$$F = S e^{r(T-t)}$$

Then,

$$\frac{\partial F}{\partial t} = -rS e^{r(T-t)} = -rF$$

$$\frac{\partial F}{\partial S} = e^{r(T-t)} = \frac{F}{S}$$

$$\frac{\partial^2 F}{\partial S^2} = 0$$

$$\therefore dF = \left(-rF + \mu S \cdot \frac{F}{S} + \frac{\sigma^2}{2} S^2 (0) \right) dt + \sigma S \frac{F}{S} dW$$

$$\therefore dF = \underbrace{(\mu - r)}_{\mu'} F\, dt + \underbrace{\sigma}_{\sigma'} F\, dW$$

Let T be the time to safety for the miner

$$\mathbb{E}(T) = \mathbb{E}(T \mid \text{Door 1}) . P(\text{Door 1}) + \mathbb{E}(T \mid \text{Door 2}) P(\text{Door 2}) + \mathbb{E}(T \mid \text{Door 3}) P(\text{Door 3})$$

$$= \frac{1}{3} \mathbb{E}(T \mid \text{Door 1}) + \frac{1}{3} \mathbb{E}(T \mid \text{Door 2}) + \frac{1}{3} \mathbb{E}(T \mid \text{Door 3})$$

$$\mathbb{E}(T) = \frac{1}{3} . 3 + \frac{1}{3}(5 + \mathbb{E}(T)) + \frac{1}{3}(7 + \mathbb{E}(T))$$

$$\therefore \mathbb{E}(T) = 1 + \frac{5}{3} + \frac{7}{3} + \frac{2}{3} \mathbb{E}(T)$$

$$\frac{\mathbb{E}(T)}{3} = \frac{15}{3}$$

$$\therefore \mathbb{E}(T) = 15$$

4. $$S_n = \sum_{i=1}^{n} X_i$$

$$S_{n+1} = S_n + X_{n+1}$$

$$S_{n+1}^2 = S_n^2 + X_{n+1}^2 + 2 S_n X_{n+1}$$

$$\mathbb{E}(S_{n+1}^2 \mid \mathcal{F}_n) = \mathbb{E}\left(S_n^2 + X_{n+1}^2 + 2 S_n X_{n+1} \mid \mathcal{F}_n\right)$$

Using linearity of conditional expectation

$$\mathbb{E}(S_{n+1}^2 | \mathcal{F}_n) = \mathbb{E}(S_n^2 | \mathcal{F}_n) + \mathbb{E}(X_{n+1}^2 | \mathcal{F}_n) + \mathbb{E}(2 X_{n+1} S_n | \mathcal{F}_n)$$

$$= S_n^2 + \mathbb{E}(X_{n+1}^2) + 2 S_n \mathbb{E}(X_{n+1} | \mathcal{F}_n)$$

S_n^2 is completely determined by $\mathcal{F}_n$

→ independence

$$= S_n^2 + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 + 2(0)$$

$$\mathbb{E}(S_{n+1}^2 | \mathcal{F}_n) = S_n^2 + 1$$

$$\mathbb{E}(S_{n+1}^2 - (n+1) | \mathcal{F}_n) = \mathbb{E}(S_{n+1}^2 | \mathcal{F}_n) - n - 1$$

$$= S_n^2 + 1 - n - 1$$

$$= S_n^2 - n$$

$$\therefore \mathbb{E}(Y_{n+1} | \mathcal{F}_n) = Y_n$$

$\therefore Y_n = S_n^2 - n$ is a Martingale.

a. $P(S_T > K) \equiv$ Probability that option will be excercised.

$$= P\left(\ln\left(\frac{S_T}{S_0}\right) > \ln\left(\frac{K}{S_0}\right)\right)$$

$$\ln\left(\frac{S_T}{S_0}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\right)$$

$$\therefore P(S_T > K) = P\left(X > \ln\left(\frac{K}{S_0}\right)\right)$$

$$\text{wher } X \sim N\left(\overbrace{\left(\mu - \frac{\sigma^2}{2}\right)T}^{\mu'},\ \overbrace{\sigma^2 T}^{\sigma^{2\prime}}\right)$$

$$\therefore P(S_T > K) = P\left(\frac{X - \mu'}{\sigma'} > \frac{\ln\left(\frac{K}{S_0}\right) - \mu'}{\sigma'}\right)$$

$$= P\left(Z > \frac{\ln\left(\frac{K}{S_0}\right) - \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= 1 - N\left(-\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right)$$

$$= 1 - N(-d_2)$$

$$P(S_T > K) = N(d_2)$$

b. $F = 100$

$K = 40$

$S_0 = 38$

$\sigma = 0.32$ p.a

$r = 0.06$ p.a

$T = 6$ mths $= 0.5$ years

From a. Payoff at time T = F if $S_T > K$

So Expected payoff $F\,N(d_2)$

to find the option price at $t=0$

we have to discount this by e^{-rT}.

$$\therefore \quad c = F\,N(d_2)\,e^{-rT}$$

$$= 100\,N(d_2) \cdot e^{-0.06 \times 0.5}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

Substituting values we get

$$d_2 = -0.2$$

$$\therefore \quad c = 100 \times N(-0.2)\,e^{-0.06 \times 0.5}$$

$$c = 100 \times 0.9704 \times (1 - N(0.20))$$

$$c = 97.04\,(1 - 0.58)$$

$$c = 40.5$$