Black-Scholes PDE derivation using hedging and the heat equation

Discrete case:

Consider a portfolio of an option and some quantity Δ of stock. Value of the portfolio at $t=0$ is

$$\underbrace{V_0}_{\text{worth of option}} - \underbrace{\Delta}_{\text{quantity of stock}} \underbrace{S_0}_{\text{stock price}}$$

Suppose we carefully choose Δ such that

$$V_1(H) - \Delta S_1(H) = V_1(T) - \Delta S_1(T)$$

so that

$$\Delta = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \quad \text{— } *$$

then we would have eliminated all the risk (since all risk is in the random behaviour of the stock price). The portfolio $V_0 - \Delta S_0$ with the value of delta $(*)$ is a risk-free portfolio. Hence,

$$(V_0 - \Delta S_0)(1+r) = V_1(H) - \Delta S_1(H) = V_1(T) - \Delta S_1(T)$$

Substituting the value of Δ we get the value of the unknown option value V_0. Lets see the same argument in ~~the~~ continuous time.

Let Π be the value of the portfolio at time t $\Pi = V(S,t) - \Delta S$. The

option

option value $V(S,t)$ is a function of the stock price S and time t. We assume that the stock price follows a GBM $dS = \mu S dt + \sigma S dW$. Let $d\Pi$ denote the change in portfolio in a small amount of time dt then

$$d\Pi = dV - \Delta dS$$ (Note: Δ remains constant in time t to t+dt)

By Ito's lemma

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt$$

$$\therefore \quad d\Pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS$$

All the risk in the change in portfolio $d\Pi$ comes from dS term and just like the discrete case we eliminate the risk by choosing $\Delta = \frac{\partial V}{\partial S}$ then

$$\therefore \quad d\Pi = \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt \qquad -(1)$$

But since we have created a risk free portfolio it grows by the risk free rate r

$$d\Pi = r \Pi \, dt$$

$$d\Pi = r\left(V - S\frac{\partial V}{\partial S}\right)dt \quad -(2) \quad \left(\Delta = \frac{\partial V}{\partial S}\right)$$

Equating (1) and (2) we get

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt = \left(rV - rS\frac{\partial V}{\partial S}\right)dt$$

From this we get

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0} \quad -(3)$$

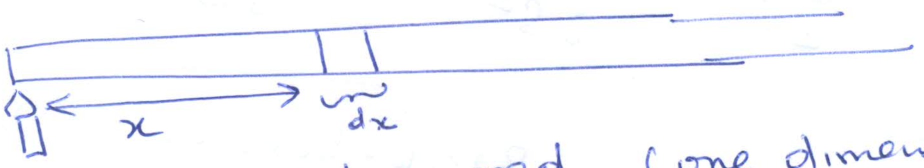
This is the Black-Scholes PDE. It is a linear parabolic PDE. Its solution gives the Black-Scholes option pricing formula. Equation (3) can be converted
transformed
to a famous PDE called the heat equation. This is One of the ways to arrive at the Black-Scholes formula.

One has to be careful because we have made several assumptions and the formula is valid only under these assumptions.

Assumptions of Black-Scholes model

- Stock price follows the log normal distribution
- Interest rates are non-random
- Volatality remains constant
- There are no arbitrage opportunities
- No dividends
- Hedging is done continuously.
- There are no transaction costs.

Heat equation :



Consider an infinite rod (one dimensional) with with some heat ~~Source~~ distribution at one ~~end~~ ~~of the~~ given by the temperature $u(x,t)$ which is a function of the distance x from the beginning of the rod and the time t

Let the initial distribution be

$$\cancel{\forall t} \quad u(x,0) = \psi(x)$$

Consider the heat flow in a small section dx in a time dt

$\frac{\partial u}{\partial x}$ is ~~the~~ proportional to the heat flow across section dx in time dt.

$\frac{\partial^2 u}{\partial x^2}$ is the heat retained by the section in time dt, which ~~w~~ should be proportional to the change in temperature $\frac{\partial u}{\partial t}$. Hence we get the heat equation

$$\begin{cases} \frac{1}{2}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} & \text{--- (4)} \\ u(x,0) = \psi(x) \leftarrow \text{Initial condition} \end{cases}$$

In the next lecture we will see how the Black-Scholes PDE can be transformed to a heat eqn.

Proposition:

The heat equation ~~(##)~~

$$\begin{cases} \frac{\partial g}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y) \\ g(0,y) = \psi(y) \end{cases}$$

has solution $g(t,y) = \int_{-\infty}^{\infty} \psi(z)\, e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}$

cont.

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z)\, e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}$$

Proof:

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \psi(z)\, e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}$$

$$= \int_{-\infty}^{\infty} \psi(z) \frac{\partial}{\partial t} \frac{e^{-\frac{(y-z)^2}{2t}}\, dz}{\sqrt{2\pi t}}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \psi(z) \left(\frac{(y-z)^2}{t^2} - \frac{1}{t} \right) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \psi(z) \frac{\partial^2}{\partial y^2} \left(\frac{e^{-\frac{(y-z)^2}{2t}}}{\sqrt{2\pi t}} \right) dz$$

$$= \frac{1}{2} \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \psi(z)\, e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}$$

$$= \frac{1}{2} \frac{\partial^2 g}{\partial y^2}$$

$$\therefore \frac{\partial g(t,y)}{\partial t} = \frac{1}{2} \frac{\partial^2 g(t,y)}{\partial y^2}$$