Risk-neutral valuation

Computational Finance (DAIICT, 2017-18)

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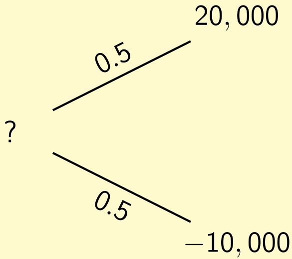
Session Outline

1. Valuation in finance
2. Pricing by no-arbitrage
3. Risk-neutral valuation

Session Outline

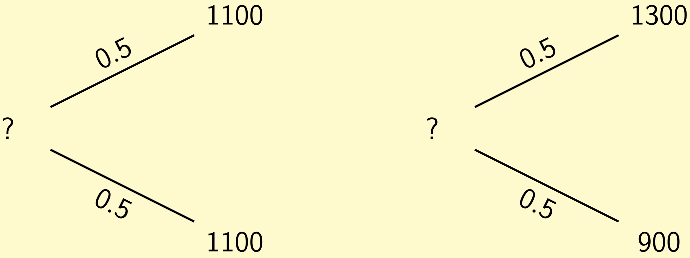
Gambles

How much would you pay to play this gamble?



Gambles

Which one is more attractive?



How much would you pay for each? Let's say a bank is offering you 10% return.

An old puzzle

- Winnings

$$2, 4, 8, \cdots$$

- Associated probabilities

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$$

- Expected value

$$2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \cdots = 1 + 1 + 1 + \cdots = \infty$$

- St Petersburg Paradox: Daniel Bernoulli (1738)

Risk

- Bottom line: People do not like risk and avoid statistically fair gambles, that is value them at less than their expected values
- Meaning: Price is generally less than the expected value of the payoff

$$P_X < \mathbb{E}[X]$$

- For payoffs coming in the future, incorporating time value of money

$$P_X < \frac{\mathbb{E}[X]}{1+r}$$

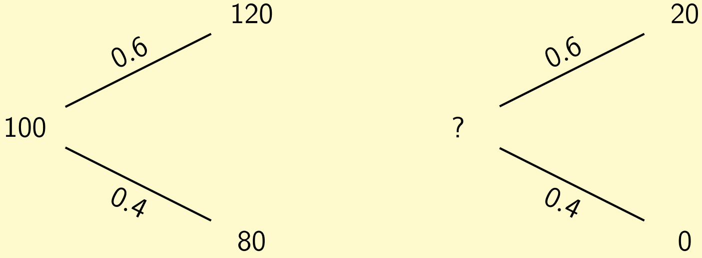
- Much of financial valuation and planning works by changing the 'discount rate', i.e. *riskier stuff is cheaper*

$$P_X = \frac{\mathbb{E}[X]}{1+r_{risky}} \quad \text{with} \quad r_{risky} > r$$

- So, say, one is happy with $r = 7\%$ from an FD but expects a larger return from the stock market. Alternatively, risky stuff is attractive only if it's cheap enough

Risk

So what about this one? A call option



Derivatives are often more conveniently priced relative to the known value of the underlying

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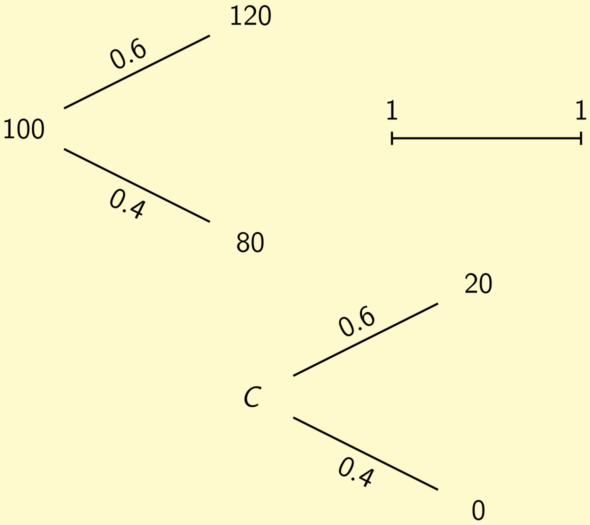
No-arbitrage at work: Replication



No-arbitrage at work: Replication

- For replication to work, we require a portfolio (x, y) which gives the same payoff as the option at maturity:

$$120x + y(1+r) = 20$$
$$80x + y(1+r) = 0$$

- Solution (recall, we are working with $r = 0$):

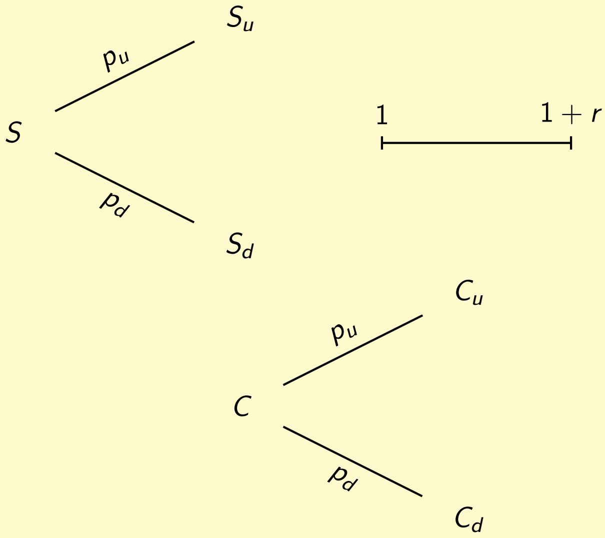
$$x = \frac{20 - 0}{120 - 80} = 0.5$$
$$y = -80x = -80 \times 0.5 = -40$$

- And if no-arbitrage, or the law of one price works, then we must have:

$$C = xS + y = 0.5 \times 100 + (-40) = 10$$

$$\Rightarrow \boxed{C = xS + y = 10}$$

The Formal Set-up



Replication

- Portfolio of stock and money-in-the-bank $\equiv (x, y)$
- Replication to work requires:

$$xS_u + y(1+r) = C_u$$
$$xS_d + y(1+r) = C_d$$

- Solution:

$$x = \frac{C_u - C_d}{S_u - S_d} = \frac{\Delta C}{\Delta S}$$
$$y = \frac{1}{1+r}\frac{C_d S_u - C_u S_d}{S_u - S_d}$$

- No-arbitrage implies: $C = xS + y$:

$$C = xS + y = \frac{C_u - C_d}{S_u - S_d}S + \frac{1}{1+r}\frac{C_d S_u - C_u S_d}{S_u - S_d}$$

Replication

- Writing $S_u = S \times u$, and $S_d = S \times d$; $u > d$ implies:

$$\begin{aligned}
C &= \frac{C_u - C_d}{S \times u - S \times d} S + \frac{1}{1+r} \frac{C_d S \times u - C_u S \times d}{S \times u - S \times d} \\
&= \frac{C_u - C_d}{u - d} + \frac{1}{1+r} \frac{u C_d - d C_u}{u - d} \\
&= \frac{1}{1+r} \Big[\frac{(1+r) - d}{u - d} C_u + \frac{u - (1+r)}{u - d} C_d \Big]
\end{aligned}$$

- Useful for later to call:

$$\boxed{q_u = \frac{(1+r) - d}{u - d}} \quad \text{and} \quad \boxed{q_d = \frac{u - (1+r)}{u - d}} = 1 - q_u$$

And then $C = \dfrac{1}{1+r} [q_u C_u + q_d C_d]$

Session Outline

Risk-neutral valuation

- For payoffs coming in the future, incorporating time value of money

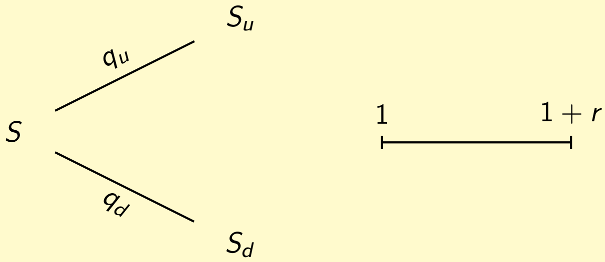
$$P_X < \frac{\mathbb{E}[X]}{1+r}$$

- Risk-neutral valuation is about an alternative way of dealing with the inequality
- And much of derivatives pricing works by doing just that - changing the numerator, i.e. by changing the expected value and writing it as

$$P_X = \frac{\mathbb{E}^{\mathbb{Q}}[X]}{1+r} \quad \text{with} \quad \mathbb{E}^{\mathbb{Q}}[X] < \mathbb{E}[X]$$

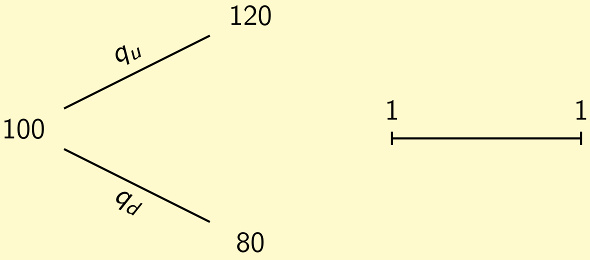
- So one works with synthetic probabilities called the risk-neutral probabilities, because one continues to discount by the risk-free rate

Risk-neutral valuation



For risk-neutral valuation to work we should be able to find q_u and q_d

Risk-neutral valuation



For risk-neutral valuation to work we should be able to find q_u and q_d

Risk-neutral valuation

- In our example (with $r = 0$), this is akin to saying:

$$100 = \mathbb{E}^{\mathbb{Q}}[S_1] = 120q_u + 80q_d$$
$$1 = 1.2q_u + 0.8q_d$$

- Since probabilities must add up to 1, i.e. $q_u + q_d = 1$

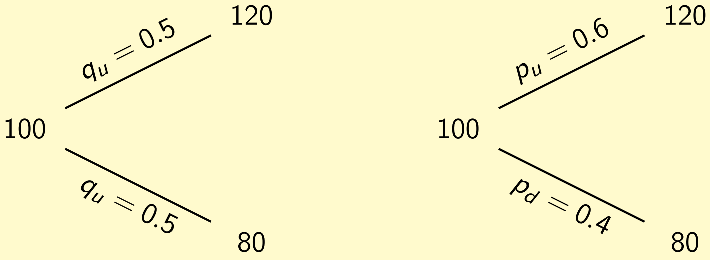
$$\Rightarrow q_u = \frac{1 - 0.8}{1.2 - 0.8} = 0.5, \quad q_d = \frac{1.2 - 1}{1.2 - 0.8} = 0.5$$

- Once $\mathbb{Q}$ probabilities exist, we can price all risky assets within that model using such probabilities as:

$$C = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[C_1] = \frac{1}{1+r}\left[q_u C_u + q_d C_d\right] = 0.5 \times 20 + 0.5 \times 0 = 10$$

- There is no need for 'physical/actual' probabilities p_u and p_d: No-arbitrage explains this

Risk-neutral valuation



Notice anything?

Risk-neutral valuation

- In general, we require (working with $S_u/S = u$, $S_d/S = d$):

$$S = \frac{1}{1+r}\left[q_u S_u + q_d S_d\right], \text{ and } q_u + q_d = 1$$
$$\Rightarrow q_u = \frac{(1+r)-d}{u-d}, \quad q_d = 1 - q_u = \frac{u-(1+r)}{u-d}$$

- And:

$$C = \frac{1}{1+r}\big[q_u C_u + q_d C_d\big]$$
$$= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[C_1] = \frac{1}{1+r}\Big[\frac{(1+r)-d}{u-d}C_u + \frac{u-(1+r)}{u-d}C_d\Big]$$

- In general, whether or not $\mathbb{Q}$ exists is a matter of no-arbitrage

Compare with answer from replication

- Assuming $S_u = S \times u$, and $S_d = S \times d$; $u > d$ implies:

$$\begin{aligned} C &= \frac{C_u - C_d}{S \times u - S \times d} S + \frac{1}{1+r} \frac{C_d S \times u - C_u S \times d}{S \times u - S \times d} \\ &= \frac{C_u - C_d}{u - d} + \frac{1}{1+r} \frac{u C_d - d C_u}{u - d} \\ &= \frac{1}{1+r} \Big[\frac{(1+r) - d}{u - d} C_u + \frac{u - (1+r)}{u - d} C_d \Big] \end{aligned}$$

- Useful for later to call:

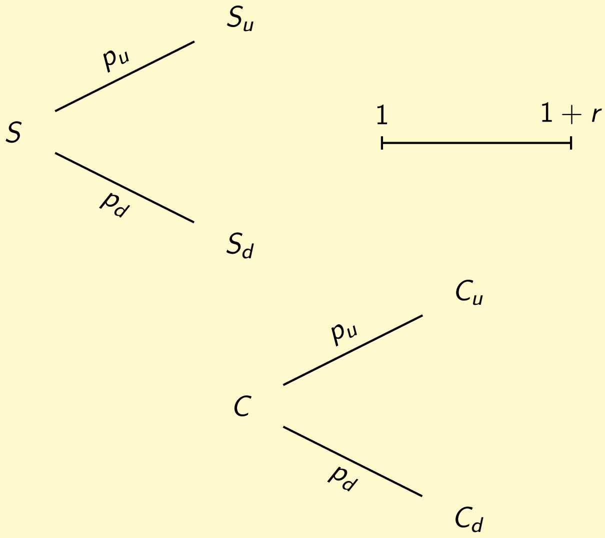
$$\boxed{q_u = \frac{(1+r) - d}{u - d}} \quad \text{and} \quad \boxed{q_d = \frac{u - (1+r)}{u - d}} = 1 - q_u$$

And then $C = \dfrac{1}{1+r} \left[q_u C_u + q_d C_d \right]$

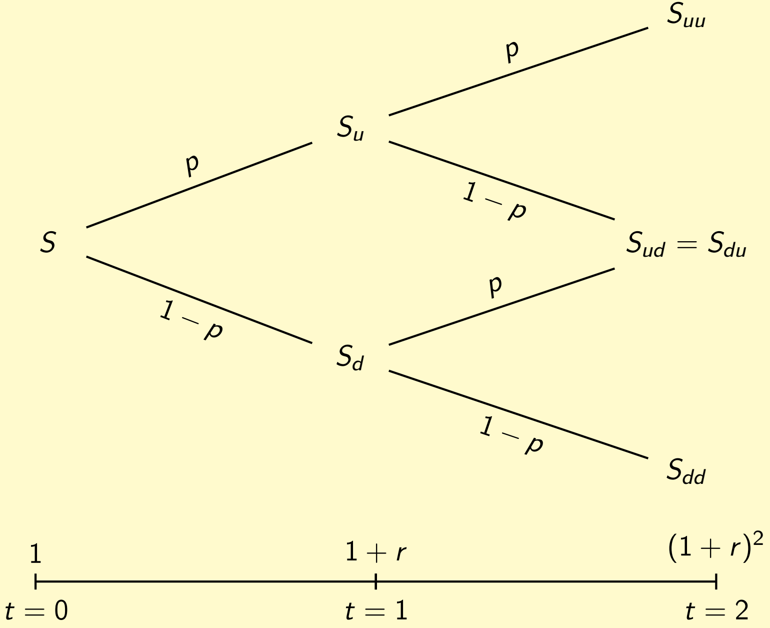
Arbitrage in the Binomial model

- Consider $S_u = 120, S_d = 105, r = 0$; $x = 1$, $y = -100$
- And now, $S_u = 95, S_d = 90, r = 0$; $x = -1$, $y = 100$
- Both $u > d > 1 + r$ and $u < d < 1 + r$ implies free money on the table (and unreasonable probabilities)
- **First fundamental theorem of asset pricing**: No-arbitrage is equivalent to existence of a risk-neutral measure

The Binomial model



The Binomial model: 2-period



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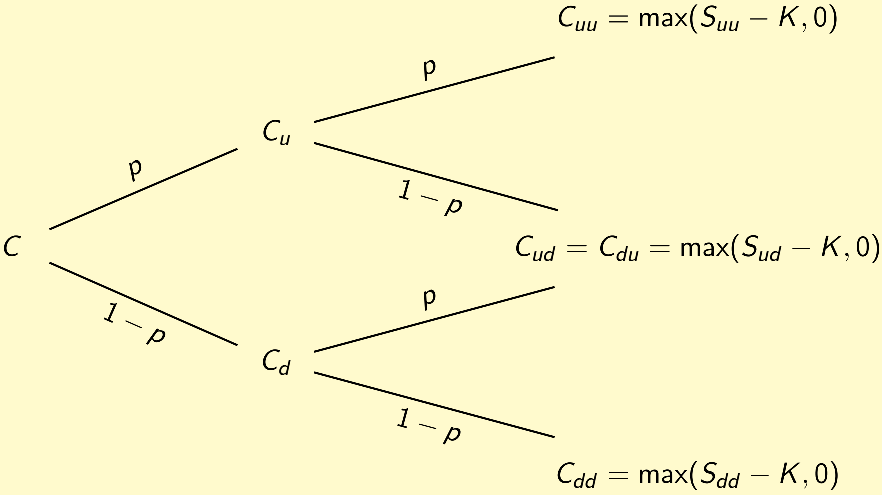

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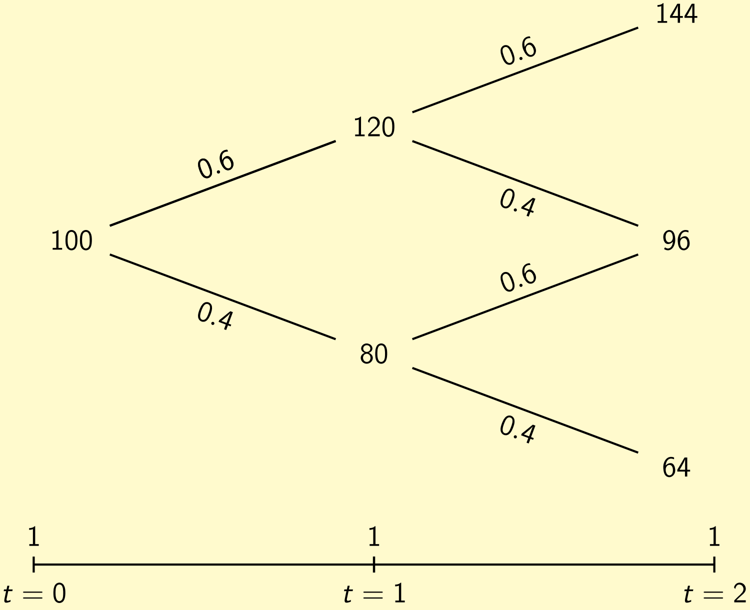
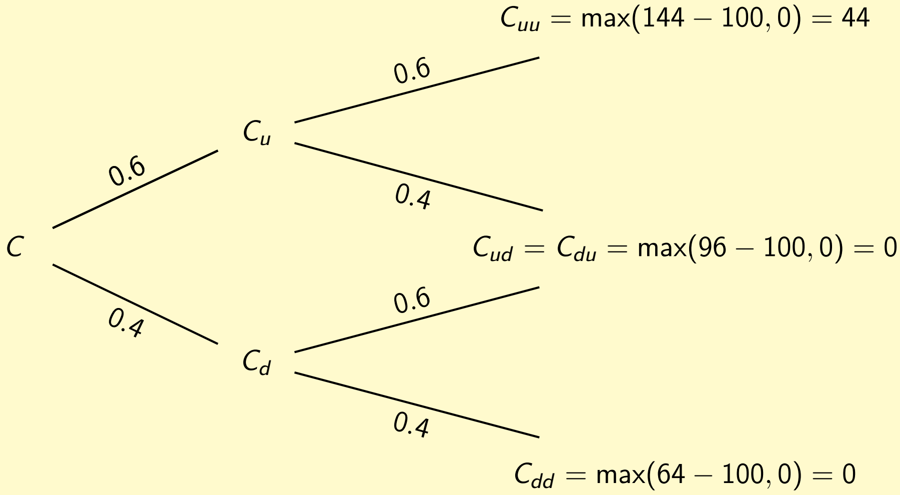


The Binomial model: 2-period



Solving the 2-period model: Risk-neutral valuation

- Any of the three methods - replication, hedging or risk-neutral valuation may be used here, but risk-neutral valuation is the simplest
- The idea is to approach each sub-tree as a single-period model starting at different time points, starting S
- In this example, $S_{uu}/S_u = S_u/S = 1.2$, $S_{ud}/S_u = S_d/S = 0.8$, i.e. all ratios are the same and $r = 0$
- So all up-moves and down-moves probability are the same at each time point:

$$q_{uu} = \frac{(1+r)-d}{u-d} = \frac{1-0.8}{1.2-0.8} = 0.5 = q_{du}$$
$$q_{ud} = \frac{u-(1+r)}{u-d} = \frac{1.2-1}{1.2-0.8} = 0.5 = q_{dd}$$

Solving the 2-period model: Risk-neutral valuation

- Given q_{uu}, q_{ud}, q_{du} and q_{dd}, we can find C_u and C_d:

$$C_u = \frac{1}{1+r}\left[q_u C_{uu} + q_d C_{ud}\right] = 0.5 \times 44 = 22$$
$$C_d = \frac{1}{1+r}\left[q_u C_{du} + q_d C_{dd}\right] = 0$$

- Having found C_u and C_d we are back to the single-period model at $t = 0$, with:

$$q_u = \frac{(1+r) - d}{u - d} = \frac{1 - 0.8}{1.2 - 0.8} = 0.5$$
$$q_d = \frac{u - (1+r)}{u - d} = \frac{1.2 - 1}{1.2 - 0.8} = 0.5$$

- And then:

$$C = \frac{1}{1+r}\left[q_u C_u + q_d C_d\right] = 0.5 \times 22 = 11$$

Real world applications

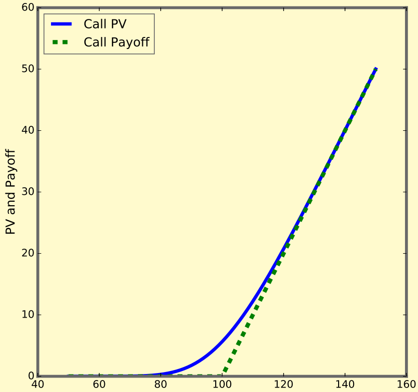
- In more realistic models, binomial model becomes Normal distribution, and tree becomes the Brownian motion
- Risk-neutral valuation is then even easier: With change in probability measure Brownian motion remains a Brownian motion
- Since Brownian motion paths are not difficult to simulate, in real world application valuing complex products becomes a simple application of Law of Large Numbers

$$P_X = \frac{\mathbb{E}^{\mathbb{Q}}[X]}{1+r} \;\equiv\; \frac{1}{1+r} \times \frac{1}{N}\sum_{i=1}^{N} X_i$$

- Much of use of computational science in finance is about doing efficient simulation of X_i, and it helps that the science of Monte Carlo simulation is quite well-developed by now and computing comes cheap
- In practice one does not do it 'by hand', but uses a library like QuantLib: `http://www.quantlib.org`

Option Price

- $\mathbb{E}^{\mathbb{Q}}[\max(S_1 - K, 0)] = 10$ vs. $\max(\mathbb{E}^{\mathbb{Q}}[S_1] - K, 0) = 0$? What's wrong with $\max(\mathbb{E}^{\mathbb{Q}}[S_1] - K, 0)$?



Jensen's Inequality

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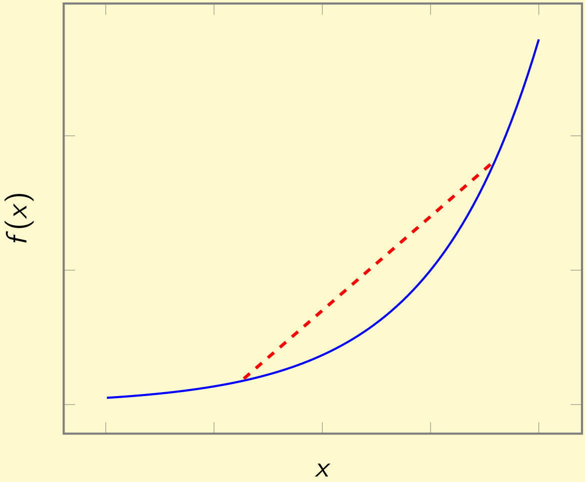

Jensen's Inequality

- $\mathbb{E}^{\mathbb{Q}}[\max(S_1 - K, 0)]$ vs. $\max(\mathbb{E}^{\mathbb{Q}}[S_1] - K, 0)$?
- Mathematical answer: For convex functions $\mathbb{E}^{\mathbb{Q}}[f(X)] \geq f(\mathbb{E}^{\mathbb{Q}}[X])$
- Economic answer: Insurance premium depends not on your 'average future health' but average future insurance claim
- Option value does not depend on the average future stock price but the average future payoff
- Jensen's inequality for convex functions gives us the mathematical answer, and the insurance intuition gives us an economic explanation of why $\mathbb{E}^{\mathbb{Q}}[\max(S_1 - K, 0)] \geq \max(\mathbb{E}^{\mathbb{Q}}[S_1] - K, 0) = 0$