Monte Carlo simulation

Suppose we want to estimate θ, the expected value of some random variable Y:

$$\theta = \mathbb{E}(Y)$$

Also suppose we are able to generate values of independent r.vs having the same distribution as Y. We perform K simulation runs of the generated values $Y_1, Y_2, \ldots Y_K$

Letting $\bar{Y} = \frac{1}{K} \sum_{i=1}^{K} Y_i$

then by law of large numbers

$$\frac{1}{K} \sum_{i=1}^{K} Y_i \xrightarrow[K \to \infty]{} \theta$$

Hence $\frac{1}{K} \sum_{i=1}^{K} Y_i$ is an estimator of θ

Let $\hat{\theta} = \frac{1}{K} \sum_{i=1}^{K} Y_i$

We have $\mathbb{E}(\hat{\theta}) = \mathbb{E}\left[\frac{1}{K} \sum_{i=1}^{K} Y_i\right] = \theta$ — *

The error in our estimate

$$\mathbb{E}\left[(\hat{\theta} - \theta)^2\right] = \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)^2\right] \quad (\text{from } *)$$

$$= \operatorname{Var}(\hat{\theta})$$

But $\operatorname{Var}(\hat{\theta}) = \operatorname{Var}\left(\frac{\sum_{i=1}^{K} Y_i}{K}\right) = \frac{1}{K^2} \cdot K \operatorname{Var}(Y_i)$

by independence

Note: The mean square error is equal to the variance of estimator.

Suppose $Var(Y) = \sigma^2$

then $\mathbb{E}\left[(\theta - \hat{\theta})^2\right] = \frac{\sigma^2}{K} \xrightarrow[K\to\infty]{} 0$

Suppose one does not know σ^2 then one can estimate it separately

$$\hat{\sigma}^2 = \sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{n-1}$$

<u>Procedure for estimating θ</u> (Monte Carlo)

0. Estimate $\hat{\sigma}$ by generating n values of Y $\hat{\sigma} = \sqrt{\sum_{i=1}^{n} \frac{(Y_i - \bar{Y})^2}{n-1}}$

1. Set error threshold. err. Generate independent values $Y_1, Y_2, \ldots Y_K$ from the probability distribution Y

2. If $K \geq \frac{\hat{\sigma}^2}{err}$ stop

Output $\hat{\theta} = \frac{\sum_{i=1} Y_i}{K}$ as the estimate for θ

More efficient simulation estimators (Variance reduction)

1. Control variate technique

Suppose we are using simulation to estimate

$\theta = \mathbb{E}(Y)$

Also, suppose that in the process of generating the value of the random variable Y we also learn about a r.v. V with mean μ_V, then instead of using Y as the estimator we can use the estimator

$$Z = Y + c(V - \mu_V)$$ (for some constant c)

Now $\mathbb{E}(Z) = \mathbb{E}(Y) = \theta$

It turns out (by some small calculation)

$$\frac{Var(Z)}{Var(Y)} = 1 - Corr^2(Y, V)$$

where $$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}}$$

Hence if $Corr^2(Y, V) < 1$

[Recall $0 \le Corr^2(X, Y) \le 1$]

then we have a reduction of variance of the estimator which will lead to a lesser mean square error

2. Antithetic variables

Let $\theta = \mathbb{E}(Y)$

Again as above consider

$$Z = \frac{Y + X}{2}$$ where $\mathbb{E}(X) = \theta$

then $\mathbb{E}(Z) = \theta$

Now using the fact that

$$Var\left(\frac{Y+X}{2}\right) = \frac{1}{4}\left[Var(Y) + Var(X) + 2\,Cov(X,Y)\right]$$

If X and Y are negatively correlated (antithetic) then this may lead to a reduction in variance of $Z = \dfrac{Y+X}{2}$

3. <u>Condition expectation</u>

Let $\theta = \mathbb{E}(Y)$

Let $Z = \mathbb{E}(Y|X)$ for some r.v. X

then $\mathbb{E}(Z) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(Y)$

It also turns out under certain conditions

$Var(\mathbb{E}(Y|X)) \leq Var(Y)$ so there is variance reduction using a condition expectation estimator.

4. <u>Importance sampling:</u>

$$\theta = \mathbb{E}(Y) = \sum_y y\,f(y)$$

Suppose direct simulation of Y is inefficient.

Let g be another probability density s.t. $f(x) = 0$ whenever $g(x) = 0$ then

$$\theta = \sum_y \frac{y\,f(y)}{g(y)} \cdot g(y) = \mathbb{E}_g\left[\frac{\cancel{Y}\,f(Y)}{g(Y)}\right]$$

Thus θ can be estimated by generating values from the density function g which may be more efficient to simulate.