Conservative Systems

From Newton's Second Law

$\boxed{F = ma} \Rightarrow \boxed{m\frac{dv}{dt} = F \equiv F(x)}$ (say)

We write F (force) as $\boxed{F = -\frac{d\psi}{dx}}$,
in which $\psi \equiv \psi(x)$ is a potential function.

Multiplying throughout by v we get,

$\boxed{mv\frac{dv}{dt} = -\frac{d\psi}{dx}v}$ Now $\boxed{v = \frac{dx}{dt}}$ (velocity)

$\Rightarrow \quad mv\frac{dv}{dt} + \frac{dx}{dt}\frac{d\psi}{dx} = 0$

$\Rightarrow \quad \boxed{\frac{d}{dt}\left(\frac{1}{2}mv^2 + \psi\right) = 0} \Rightarrow \boxed{\frac{1}{2}mv^2 + \psi = \mathcal{E}}$,

in which $\mathcal{E}$ (total energy) is constant in time.

Reversible Systems

All conservative systems are reversible.

Write $\boxed{m\frac{dv}{dt} = m\frac{d^2x}{dt^2}}$ since $\boxed{v = \frac{dx}{dt}}$.

$\Rightarrow \quad \boxed{m\frac{d^2x}{dt^2} = F(x)}$ This equation is symmetric under $\boxed{t \longrightarrow -t}$.

The time reversal symmetry makes the system $\boxed{m\frac{d^2x}{dt^2} = F(x)}$ reversible in time.

Dissipation and ~~[illegible]~~ Irreversibility

Friction (or viscosity) is effective in opposing motion. This dissipates energy and the conservative condition is lost.

Further, reversibility is also lost.

Since friction (and dissipation) acts only when there is motion, we can write dissipation as a function of velocity, $\boxed{D \equiv D(v)}$. Hence, we get $\boxed{m\frac{d^2x}{dt^2} = F(x) - D(v)}$, or

$\boxed{m\frac{dv}{dt} = F(x) - D(v)}$. The simplest possible way to write this function is by a linear formula $\boxed{D(v) = kv}$, in which k is a proportional constant.

Now since $\boxed{v = dx/dt}$ we can write

$\boxed{m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} + F(x)}$, with the negative sign indicating an opposition to motion. Further the transformation of $\boxed{t \longrightarrow -t}$ is no longer symmetric. The system is IRREVERSIBLE.

The Problem of Atomic Waste Disposal

The <u>v-t equation</u> is $\boxed{v = v_T\left(1 - e^{-t/t_0}\right)}$.

i.) When $t \ll t_0$ $\boxed{v \simeq v_T \frac{t}{t_0}}$ (<u>linear limit</u>).

ii.) When $t \to \infty$ $(t \gg t_0)$ $\boxed{v \simeq v_T}$ (<u>constant</u>).

Since $\boxed{v = dz/dt}$, when <u>$t \ll t_0$</u>, we get $\boxed{\frac{dz}{dt} \simeq \frac{v_T}{t_0} t}$ $\Rightarrow$ $\boxed{z \simeq \left(\frac{v_T}{t_0}\right)\frac{t^2}{2}}$ (<u>parabolic</u>).

And when <u>$t \to \infty$</u>. $\boxed{\frac{dz}{dt} \simeq v_T} \Rightarrow \boxed{z \simeq v_T t}$ (<u>linear</u>).

<u>Fugitive Elasticity</u> (<u>Maxwell</u>)

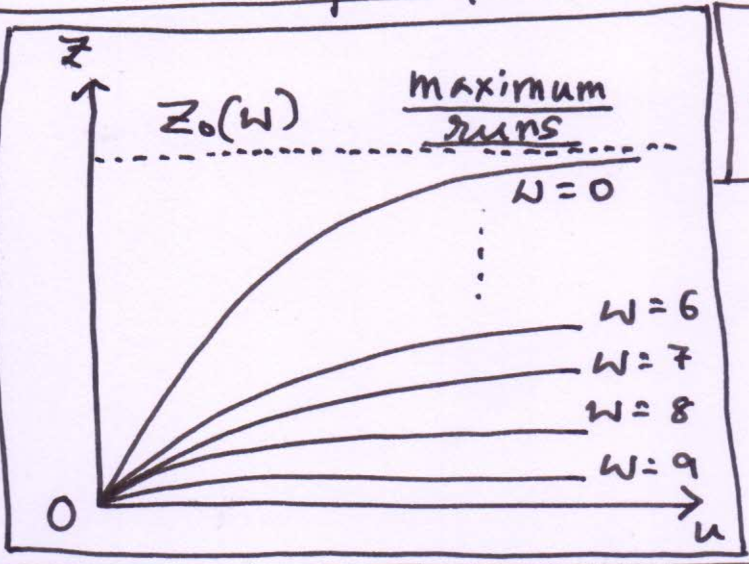
<u>Kelvin's Viscoelastic formula:</u> $\boxed{\frac{d\epsilon}{dt} = \frac{\sigma}{\eta} - \frac{Y}{\eta}\epsilon}$.

Or $\boxed{\sigma = Y\epsilon + \eta\frac{d\epsilon}{dt}}$. The <u>right hand side</u> must have <u>dimensional compatibility</u> in a system that shows <u>visco-elastic</u> <u>behaviour</u>.

Hence $\boxed{Y\epsilon \sim \eta\frac{d\epsilon}{dt}}$. We now write $\boxed{t = T t_0}$,

in which <u>T is dimensionless</u> and <u>t_0</u> is <u>a time scale</u>. Hence, $\boxed{Y\epsilon \sim \frac{\eta}{t_0}\frac{d\epsilon}{dT}}$, which gives $\left[\frac{\eta}{Y}\right]$ a <u>time dimension</u>. <u>Viscosity behaves like elasticity</u>. $\boxed{\eta \sim Y t_0}$.

The Graph of the Duckworth-Lewis Equation



$$\boxed{Z(u,w) = Z_0(w)\left[1-e^{-b(w)u}\right]}$$

With larger values of w (wickets lost), values of b increase. Convergence is quicker.

Changes in Population (Discrete/Continuous)

Population changes in discrete step of unity (1). If a population size is x, and it changes (grows) by Δx, then the per capita growth is $\boxed{\frac{\Delta x}{x}}$ and the per capita growth rate is $\boxed{\frac{1}{x}\frac{\Delta x}{\Delta t}}$, in which Δt is the time taken for the growth.

If x is very large and $\boxed{\Delta x \ll x}$, then the discrete quantities can be replaced by continuously changing quantities.

$\Rightarrow \boxed{\frac{1}{x}\frac{\Delta x}{\Delta t} \equiv \frac{1}{x}\frac{dx}{dt}}$ Now x is continuously differentiable with respect to t.

Plotting of Equations like $\boxed{\dfrac{dx}{dT} = -x(1-x)}$

$$\boxed{\frac{d^2x}{dT^2} = \frac{df}{dx}\,\frac{dx}{dT}}$$

in which $\boxed{f(x) = -x + x^2}$

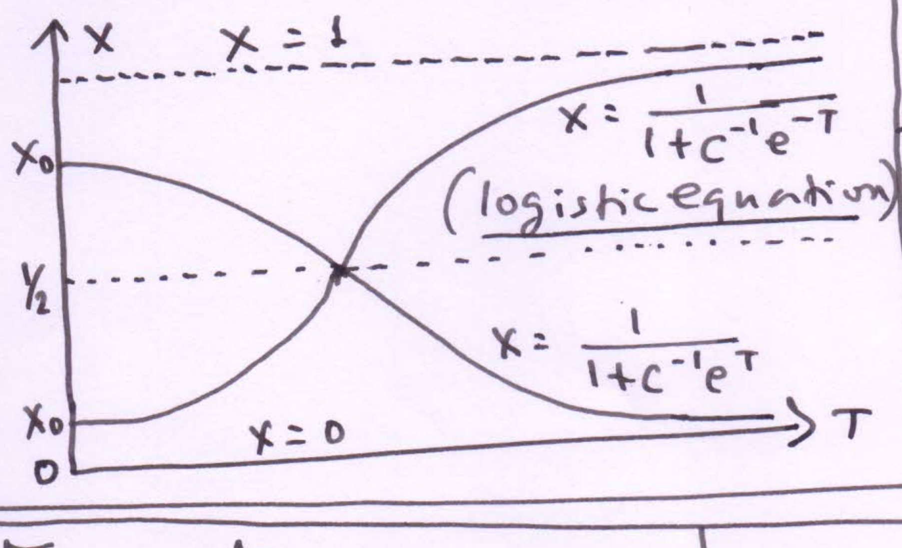
$\Rightarrow \boxed{\dfrac{df}{dx} = f'(x) = -1 + 2x}$

i.) If $\boxed{x < 1/2}$, $\boxed{\dfrac{d^2x}{dT^2} > 0}$ $\left(\because \underline{\dfrac{dx}{dT} < 0} \text{ and } \underline{\dfrac{df}{dx} < 0}\right)$

This means $x(T)$ <u>decreases</u> at an <u>increasing rate</u>.

ii.) If $\boxed{x > 1/2}$, $\boxed{\dfrac{d^2x}{dT^2} < 0}$ $\left(\because \underline{\dfrac{dx}{dT} < 0} \text{ and } \underline{\dfrac{df}{dx} > 0}\right)$,

i.e. <u>$x(T)$ decreases at a decreasing rate</u>.



For $\dfrac{dx}{dT} = x(1-x)$.

<u>Solution</u>: $\boxed{x = \dfrac{1}{1+c^{-1}e^{-T}}}$

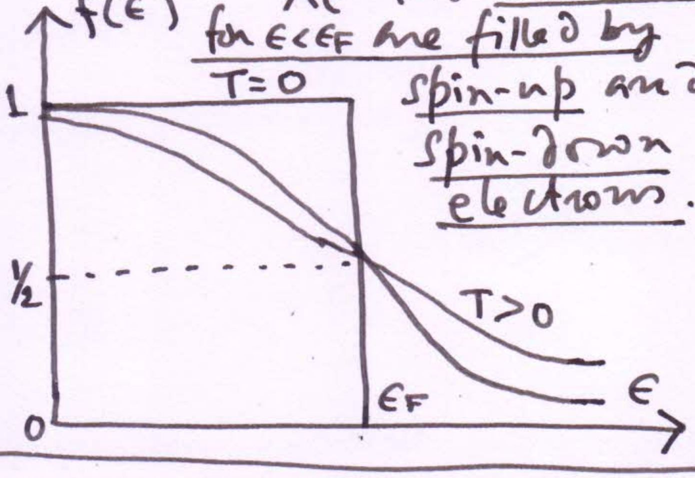
For $\dfrac{dx}{dT} = -x(1-x)$

<u>Solution</u> $\boxed{x = \dfrac{1}{1+c^{-1}e^{T}}}$

<u>Fermi Function</u>:

$$\boxed{f(\epsilon) = \frac{1}{1 + e^{(\epsilon-\epsilon_F)/k_BT}}}$$

Let $T = 0 \Rightarrow$ For $\underline{\epsilon < \epsilon_F}$, $f(\epsilon) = \dfrac{1}{1+e^{-\infty}} = 1$. And for $\epsilon > \epsilon_F$ $f(\epsilon) = \dfrac{1}{1+e^{\infty}} = 0$



At $T=0$ <u>all states for $\epsilon<\epsilon_F$ are filled by spin-up</u> and <u>spin-down electrons</u>.

Power Laws and Their Properties

$\boxed{y = f(x) = Ax^{\gamma}}$ Scale $\boxed{x \longrightarrow \lambda x}$

$\therefore \boxed{f(x) \rightarrow f(\lambda x) = A(\lambda x)^{\gamma} = A\lambda^{\gamma} x^{\gamma} = y\lambda^{\gamma}}$

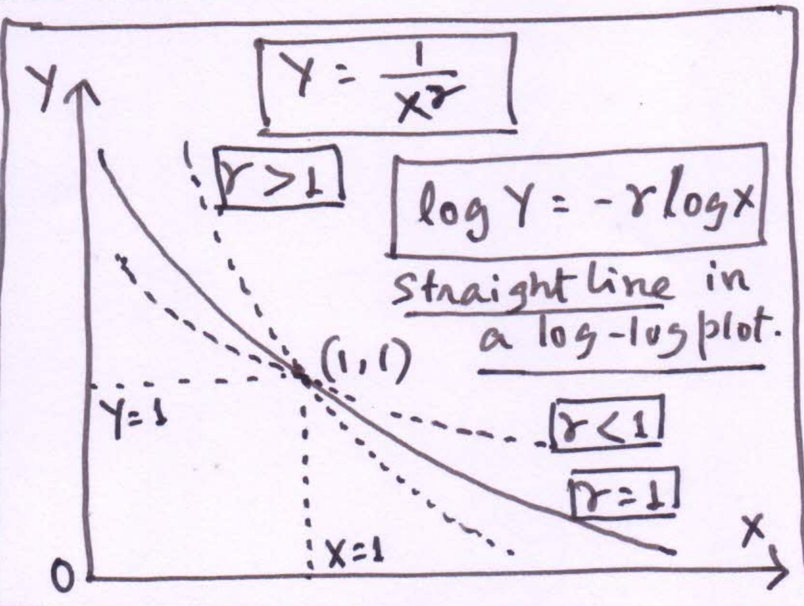
$\Rightarrow$ y is scaled as $\boxed{y\lambda^{\gamma}}$ (Scale invariance)

Inverse Power-Laws

$\boxed{y^{m} x^{n} = c}$ $\Rightarrow \boxed{y x^{n/m} = c^{1/m} = a \text{ (say)}}$

$\Rightarrow \boxed{\dfrac{y}{a} x^{n/m} = 1}$ Rescale $\boxed{Y = y/a}$, $\boxed{X = x}$ and $\boxed{\gamma = n/m}$, $(\gamma > 0)$.

$\Rightarrow \boxed{YX^{\gamma} = 1}$ (as in $\boxed{PV^{\gamma} = \text{constant}}$).



1/. All the curves pass through (1,1).

2/. As $x \rightarrow \infty$, the decay is faster for higher values of γ.

3/. For finite values of X and Y, no curve touches $\boxed{X = 0}$ or $\boxed{Y = 0}$.

4/. Any part of a curve is self-similar to any other part — scale-invariant.

Fall of a Parachutist | Free Fall

• The equation $\boxed{m\frac{dv}{dt} = mg - kv^2}$ is used to describe the free-fall of a parachutist from a height of about 30,000 ft to about 2,000 ft. After that the parachute is opened (no longer in free fall).

Bernoulli Equation

$$\boxed{\frac{v^2}{2} + \frac{P}{\rho} + gz = \text{Constant}}$$

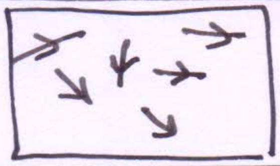
$z \to$ height
$v \to$ velocity
$P \to$ Pressure

$\rho \to$ Density, $g \to$ acceleration due to gravity.

i.) Streamline Motion:

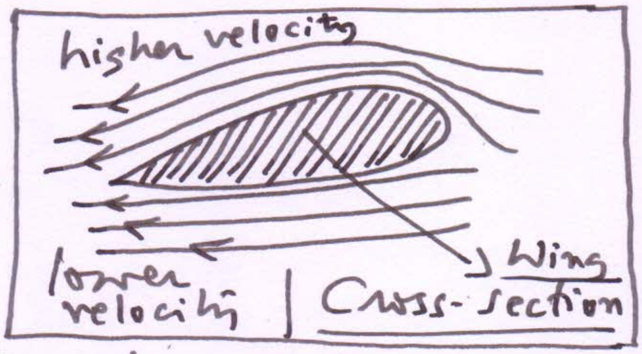
Smooth and laminar

ii.) Turbulent Motion:



Random and chaotic

Lift of an Aircraft



i.) Above the wing closer stream-lines have higher velocity. Hence pressure is lower.

ii.) Below the wing the streamlines have lower velocity. Hence at nearly the same height the pressure is higher. This gives the lift.

Item Response Theory: Additional Points

i.) Item discrimination: $\boxed{P = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$.

When $\boxed{\omega = 0}$, for $\boxed{\theta > b, \quad P = c+1-c = 1}$, and for $\boxed{\theta < b}$, $\boxed{P = c}$ (probability that a candidate with low ability responds correctly).

$\Rightarrow$ P varies between c (non-zero lowerbound) and unity (completely perfect response).

ii.) Item difficulty: The parameter b sets a scale for ability (θ). Higher ability to respond to an item is $\boxed{\theta > b}$, and low ability is $\boxed{\theta < b}$.

Sigmoid Activation Function

Biological neurons have a floor and ceiling of activity. This is expressed by the logistic function $\boxed{y_i = \frac{1}{1+e^{-x_j}} - a}$.

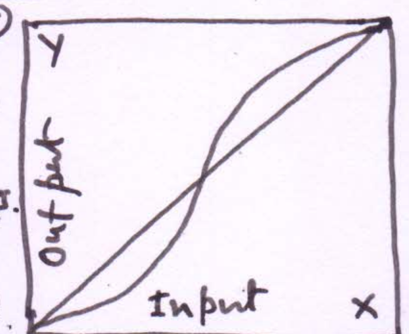
$x_j \rightarrow$ input to j-th unit. $y_i \rightarrow$ Activation response.

The Hill Function

$$\boxed{Y = \frac{1}{1+(X/\theta)^{-N}}}$$ Power law

$N \rightarrow$ Hill coefficient, $\theta \rightarrow$ Threshold (constant)

Used for Positive Cooperativity, in haemoglobin, which has four monomers. Binding of one monomer with oxygen increases the affinity for binding in the other three. After that it saturates.



Taylor Expansion in Multiple Variables

I/. **One variable:** $\boxed{f \equiv f(x)}$ expanded about $\boxed{x = x_c}$.

$$\Rightarrow \boxed{f = f(x_c) + \left.\frac{df}{dx}\right|_{x_c}(x - x_c) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x_c}(x - x_c)^2 + \cdots}$$

II/. **Two Variables:** $\boxed{f \equiv f(x,y)}$ about $\boxed{(x_c, y_c)}$.

$$\Rightarrow \boxed{\begin{aligned} f = {} & f(x_c, y_c) \longrightarrow \text{1 zero-order term } (2^0) \\ & + \left.\frac{\partial f}{\partial x}\right|_{x_c,y_c}(x - x_c) + \left.\frac{\partial f}{\partial y}\right|_{x_c,y_c}(y - y_c) \longrightarrow \text{2 first-order terms } (2^1) \\ & + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,y_c}(x - x_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{x_c,y_c}(x - x_c)(y - y_c) \\ & + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_c,y_c}(y - y_c)(x - x_c) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_c,y_c}(y - y_c)^2 + \cdots \\ & \longrightarrow \text{4 second-order terms } (2^2) \end{aligned}}$$

III/. **Three Variables:** $\boxed{f \equiv f(x,y,z)}$ about $\boxed{(x_c, y_c, z_c)}$.

$$\Rightarrow \boxed{\begin{aligned} f = {} & f(x_c, y_c, z_c) \longrightarrow \text{1 zero-order term } (3^0) \\ & + \left.\frac{\partial f}{\partial x}\right|_{x_c,y_c,z_c}(x - x_c) + \left.\frac{\partial f}{\partial y}\right|_{x_c,y_c,z_c}(y - y_c) + \left.\frac{\partial f}{\partial z}\right|_{x_c,y_c,z_c}(z - z_c) \longrightarrow \text{3 first-order terms } (3^1) \\ & + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_c,y_c,z_c}(x - x_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_c,y_c,z_c}(y - y_c)^2 + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial z^2}\right|_{x_c,y_c,z_c}(z - z_c)^2 \\ & + \frac{2}{2!}\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{x_c,y_c,z_c}(x - x_c)(y - y_c) + \frac{2}{2!}\left.\frac{\partial^2 f}{\partial y \partial z}\right|_{x_c,y_c,z_c}(y - y_c)(z - z_c) \\ & + \frac{2}{2!}\left.\frac{\partial^2 f}{\partial z \partial x}\right|_{x_c,y_c,z_c}(z - z_c)(x - x_c) + \cdots \longrightarrow \text{9 second-order terms } (3^2), \\ & \text{with 6 mixed terms.} \end{aligned}}$$

Additional Discussions on the Spread of Industrial Innovations (E. Mansfield)

$\boxed{\lambda = f\left(p, s, \frac{x}{N}\right)}$. Following a Taylor expansion we are able to write $\boxed{\lambda = (a_4 + a_8 p + a_9 s)\frac{x}{N}}$.

In λ, we have p and s as variables.

Writing $\boxed{\lambda = k(x/N)}$, where $\boxed{k = a_4 + a_8 p + a_9 s}$,

we use it in $\boxed{\frac{dx}{dt} = k\frac{x}{N}(N-x)}$. In this equation, $\boxed{k \equiv k(p,s)}$ has p and s as free parameters, with their values fixed at the beginning.

Nonlinear Time Scale in Mansfield's Equation

Given $\boxed{x = \frac{N}{1+(N-1)e^{-k(t-t_0)}}}$, which is the solution of the logistic equation, we set $\boxed{x = N/2}$, the scale of nonlinearity in time, $\boxed{(t-t_0)|_{nl}}$.

$\therefore \boxed{\frac{\cancel{N}}{2} = \frac{\cancel{N}}{1+(N-1)e^{-k(t-t_0)|_{nl}}}} \Rightarrow 2 = 1 + (N-1)e^{-k(t-t_0)|_{nl}}$

$\Rightarrow \boxed{(N-1)\,e^{-k(t-t_0)|_{nl}} = 1} \Rightarrow \boxed{(N-1) = e^{k(t-t_0)|_{nl}}}$.

$\therefore k(t-t_0)|_{nl} = \ln(N-1) \Rightarrow \boxed{(t-t_0)|_{nl} = \frac{1}{k}\ln(N-1)}$.

The nonlinear time.