The Threshold Theorem of Epidemiology

1. A small group of people introduces an ~~an~~ infectious disease in a large population.
2. The disease has a short incubation period.
3. Recovered individuals gain permanent immunity.

(SIR) There are three classes of population. They are:
i) $x \rightarrow$ The infected class, ii) $y \rightarrow$ The susceptible class.
iii) $z \rightarrow$ The removed class (recovered class).

Rule 1: $\boxed{x(t) + y(t) + z(t) = N}$, where N is the fixed total number of population. (Conserved condition)

Rule 2: $\boxed{\frac{dy}{dt} \propto xy} \Rightarrow \boxed{\frac{dy}{dt} = -Axy}$ $A \rightarrow$ The infection rate.

Rule 3: $\boxed{\frac{dz}{dt} \propto x} \Rightarrow \boxed{\frac{dz}{dt} = Bx}$ $B \rightarrow$ The removal rate. ($A, B \rightarrow$ constants)

Using Rule 1 we can write $\boxed{\frac{dx}{dt} = -\frac{dy}{dt} - \frac{dz}{dt}}$, which gives $\boxed{\frac{dx}{dt} = Axy - Bx}$. In all the three $\boxed{dx/dt}$, $\boxed{dy/dt}$ and $\boxed{dz/dt}$ equations, the right hand side does not depend on z. This a pseudo-third-order system.

I.) The x-y equation :

$$\boxed{\frac{dx/dt}{dy/dt} = \frac{dx}{dy} = \frac{Axy - Bx}{-Axy} = -1 + \frac{B}{Ay}}$$

$$\Rightarrow \boxed{\int dx = \int \left(\frac{B}{Ay} - 1\right) dy} \Rightarrow \boxed{x = \frac{B}{A}\ln y - y + c_1}$$

($c_1 \rightarrow$ Integral constant).

Initial condition: At $\boxed{t=0}$ (initially), $\boxed{x = x_0}$,

$\boxed{y = y_0}$ and $\boxed{z = 0}$. $\Rightarrow$ $\boxed{x_0 + y_0 = N}$.

$$\therefore \boxed{c_1 = x_0 + y_0 - \frac{B}{A}\ln y_0 = N - \frac{B}{A}\ln y_0}.$$

$$\Rightarrow \boxed{x = (x_0 + y_0) - y + \frac{B}{A}\ln(y/y_0)} \quad \boxed{x \equiv x(y)}$$

in closed form.

II.) The y-z equation :

$$\boxed{\frac{dy/dt}{dz/dt} = \frac{dy}{dz} = \frac{-Axy}{Bx} = -\frac{Ay}{B}}$$

$$\Rightarrow \boxed{\int \frac{dy}{y} = -\int \frac{A}{B} dz} \Rightarrow \boxed{\ln y = -\frac{Az}{B} + c_2}$$

($c_2 \rightarrow$ Integral Constant).

At $\boxed{t=0}$, $\boxed{y = y_0}$ and $\boxed{z = 0}$ $\Rightarrow$ $\boxed{c_2 = \ln y_0}$

$\Rightarrow \boxed{\ln\left(\frac{y}{y_0}\right) = -\frac{A}{B} z}$. For $t > 0$, $\boxed{y < y_0}$

$\Rightarrow \boxed{\ln\left(\frac{y}{y_0}\right) < 0}$ for $t > 0$ $\Rightarrow$ $\boxed{z = -\frac{B}{A}\ln\left(\frac{y}{y_0}\right) > 0}$

for $t > 0$.

Also $\boxed{y = y_0 \exp\left(-\frac{A}{B} z\right)}$ $\boxed{y = y(z)}$ in closed form.

III.) The z - x equation :

$$\boxed{\frac{dx/dt}{dz/dt} = \frac{dx}{dz} = \frac{Axy - Bx}{Bx} = \frac{Ay}{B} - 1}.$$

Now $\boxed{y = y_0 e^{-\frac{Az}{B}}} \Rightarrow \boxed{\frac{dx}{dz} = \frac{A}{B} y_0 e^{-\frac{Az}{B}} - 1}$.

$$\Rightarrow x = \int \frac{A}{B} y_0 e^{-\frac{A}{B}z}\, dz - z + C_3$$

$C_3 \rightarrow$ Integral Constant

$$\Rightarrow x = \frac{A}{B} y_0 \frac{e^{-\frac{Az}{B}}}{-A/B} - z + C_3.$$

$$\Rightarrow \boxed{x = -y_0 e^{-Az/B} - z + C_3}.$$

when (at $t=0$), $\boxed{x = x_0}$ and $\boxed{z = 0}$,

$\boxed{C_3 = x_0 + y_0} \Rightarrow \boxed{x = x_0 + y_0\left(1 - e^{-Az/B}\right) - z}$

OR $\boxed{x = (x_0 + y_0) - y_0 \exp\left(-\frac{Az}{B}\right) - z}$ But z is not written in a closed form for x.

Plot of $x-y$: $\boxed{x = (x_0 + y_0) - y + \frac{B}{A} \ln\left(\frac{y}{y_0}\right)}$.

i.) when $\boxed{y \longrightarrow 0}$, $\boxed{x \longrightarrow -\infty}$.

ii.) $\boxed{\frac{dx}{dt} = x(Ay - B)} \Rightarrow \boxed{\frac{dx}{dt} = 0}$, when either $\boxed{x = 0}$ OR $\boxed{y = B/A}$.

iii.) $\boxed{\frac{dx}{dy} = \frac{B}{Ay} - 1} \Rightarrow \boxed{\frac{dx}{dy} = 0}$, when $\boxed{y = \frac{B}{A} = \rho \text{ (say)}}$

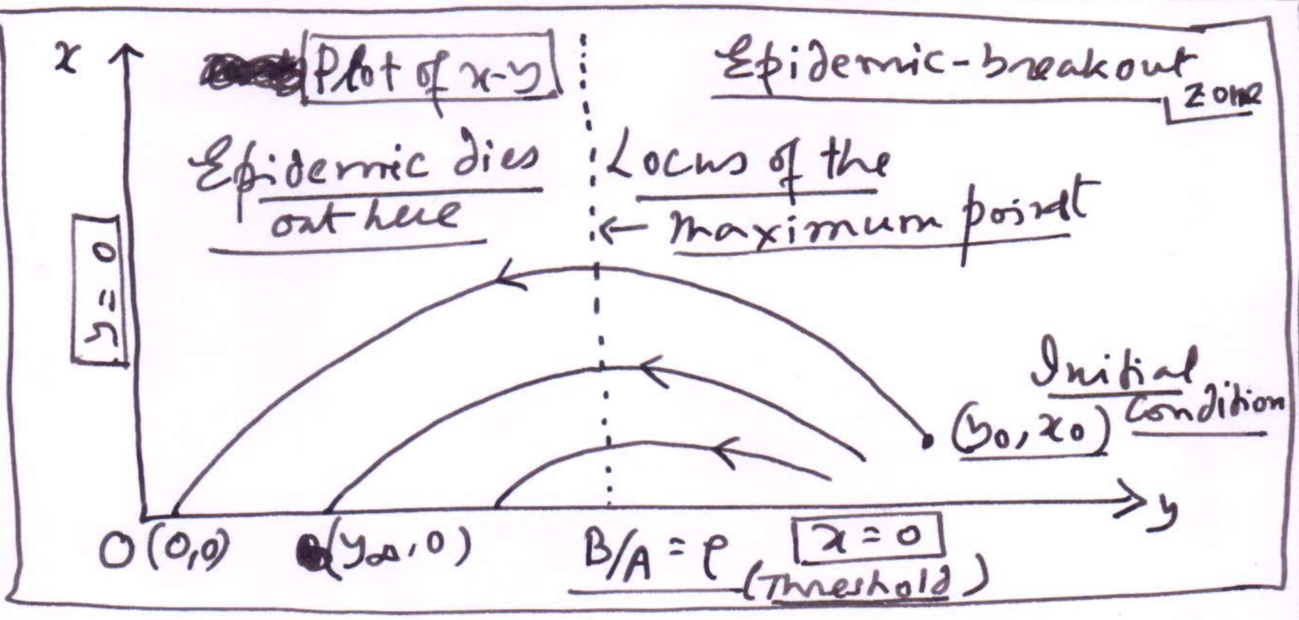
iv.) $\boxed{\frac{d^2x}{dy^2} = -\frac{B}{Ay^2}}$ At $\boxed{y = \frac{B}{A}}$, $\boxed{\frac{d^2x}{dy^2} = -\frac{A}{B} < 0}$. Hence, $y = B/A$ is a maximum.

(P.T.O.)

v.) When $\boxed{x = 0}$, write $\boxed{y = y_\infty}$ (say).

With $\boxed{x=0}$, the point becomes an equilibrium point, since both $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$ (in the phase plot)

Note: x is plotted along the vertical axis, and y along the horizontal.



Conclusion 1: An epidemic will breakout, if ($y_0 > \rho$), i.e. the initial number of susceptibles are above the threshold ρ. This is the Threshold Theorem of Epidemiology (Kermack & McKendrick)

Conclusion 2: The spread of the disease stops ($x=0$) because the infective population is reduced to zero, even though there may be some susceptibles left (at y_∞).

Practically speaking epidemics breakout due to overcrowding of a susceptible population in an unhygienic environment.

Case of Initial Number of Susceptibles Slightly Higher than the Threshold

$\boxed{\frac{dx}{dt} = x(Ay - B)}$ and $\boxed{x = (x_0 + y_0) - y + \frac{B}{A}\ln\left(\frac{y}{y_0}\right)}$.

When $\boxed{\frac{dx}{dt} = 0}$ Either $\boxed{y = B/A = \rho}$ or $\boxed{x = 0}$.

i.) The former case is a turning point in which $\boxed{y = \rho}$ is a maximum $\Rightarrow$ $\boxed{d^2x/dt^2 < 0}$.

ii.) $\boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt}(Ay - B) + x\frac{d}{dt}(Ay - B)}$. When $\boxed{x = 0}$ at $\boxed{\frac{dx}{dt} = 0}$ $\Rightarrow$ $\boxed{\frac{d^2x}{dt^2} = 0}$. This is an equilibrium point. (at $x = 0$).

When $\boxed{x = 0}$, we write $\boxed{y = y_\infty}$. With these, we get $\boxed{0 = x_0 + y_0 - y_\infty + \frac{B}{A}\ln\left(\frac{y_\infty}{y_0}\right)}$. Usually $\boxed{x_0 \ll y_0}$, and so we neglect x_0 above.

Now y_0 is the initial value of the susceptibles, and y_∞ is the final value of the susceptibles.

$\therefore$ $\boxed{y_0 - y_\infty}$ is the number of people who have contracted the infectious disease. Further, $\boxed{y_\infty < y_0}$, which implies that $\boxed{y_0 - y_\infty > 0}$.

Hence we can write $\boxed{(y_0 - y_\infty) + \frac{B}{A}\ln\left(\frac{y_0 - y_0 + y_\infty}{y_0}\right) \simeq 0}$

$\Rightarrow$ $\boxed{(y_0 - y_\infty) + \rho\ln\left(1 - \frac{y_0 - y_\infty}{y_0}\right) \simeq 0}$ in which we have neglected x_0.

(P.T.O.)

We now consider <u>the case where the initial number of susceptibles is slightly greater than the threshold</u>, i.e. $\boxed{y_0 = \rho + \epsilon}$, in which $\boxed{\epsilon \ll \rho}$. $\therefore \boxed{\frac{(y_0 - \rho)}{\rho} = \frac{\epsilon}{\rho} \ll 1}$.

From the <u>plot of x-y</u> we can see that in this case $\boxed{(y_0 - y_\infty) \ll y_0}$. Hence using the <u>formula</u> $\boxed{\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \dots}$, we can write $\boxed{\ln\left[1 - \left(\frac{y_0 - y_\infty}{y_0}\right)\right] = -\left(\frac{y_0 - y_\infty}{y_0}\right) - \frac{1}{2}\left(\frac{y_0 - y_\infty}{y_0}\right)^2}$

going <u>only up to the second order term.</u>

Hence, $\boxed{(y_0 - y_\infty) + \rho\left[-\frac{(y_0 - y_\infty)}{y_0} - \frac{1}{2}\left(\frac{y_0 - y_\infty}{y_0}\right)^2\right] \simeq 0}$

$\Rightarrow \boxed{(y_0 - y_\infty)\left[1 - \frac{\rho}{y_0} - \frac{\rho}{2}\frac{(y_0 - y_\infty)}{y_0^2}\right] \simeq 0}$ $\left(\text{Now, } \underline{y_0 - y_\infty \neq 0}\right)$

$\Rightarrow \boxed{y_0 - y_\infty \simeq \frac{2y_0^2}{\rho}\left(1 - \frac{\rho}{y_0}\right) \simeq 2y_0\left(\frac{y_0}{\rho} - 1\right)}$

But $\boxed{y_0 = \rho + \epsilon}$ and $\boxed{\left(\frac{y_0}{\rho} - 1\right) = \frac{\epsilon}{\rho}}$. Using these

$(\epsilon \ll \rho)$

we get, $\boxed{y_0 - y_\infty \simeq 2y_0\epsilon/\rho = 2(\rho + \epsilon)\epsilon/\rho}$

<u>Neglecting ϵ</u> in $\boxed{\rho + \epsilon \simeq \rho}$, we finally get.

$\boxed{y_0 - y_\infty \simeq 2\cancel{\rho}\,\epsilon/\cancel{\rho}} \Rightarrow \boxed{y_0 - y_\infty = 2\epsilon}$.

$\therefore \boxed{y_0 - y_\infty \simeq 2(y_0 - \rho)}$, an <u>approximate result</u> that is <u>valid only</u> when <u>y_0 is slightly greater than ρ.</u>

$z(t)$ as a Function of Time : $z \equiv z(t)$.

$$\boxed{\frac{dz}{dt} = Bx}, \quad \boxed{x = (x_0 + y_0) - y + \frac{B}{A}\ln\left(\frac{y}{y_0}\right)} \text{ and}$$

$$\boxed{y = y_0 \exp\left(-\frac{Az}{B}\right)}. \Rightarrow \boxed{z = -\frac{B}{A}\ln\left(y/y_0\right)}.$$

Hence, $\boxed{\dfrac{dz}{dt} = B\left[x_0 + y_0 - y_0\exp\left(-\dfrac{A}{B}z\right) - z\right]}$,

an equation in an autonomous form $\boxed{\dfrac{dz}{dt} = f(z)}$.

Noting $\boxed{\dfrac{B}{A} = \rho}$, we consider the case of $\boxed{z \ll \rho}$.

$\therefore \boxed{e^{-z/\rho} \simeq 1 - \dfrac{z}{\rho} + \dfrac{1}{2}\left(\dfrac{z}{\rho}\right)^2 + \cdots}$, going up to the second order.

$$\Rightarrow \frac{dz}{dt} \simeq B\left[x_0 + \cancel{y_0} - \cancel{y_0} + y_0\frac{z}{\rho} - \frac{y_0}{2}\left(\frac{z}{\rho}\right)^2 - z\right]$$

$$\Rightarrow \boxed{\frac{dz}{dt} = B\left[x_0 + \left(\frac{y_0}{\rho} - 1\right)z - \frac{y_0}{2}\left(\frac{z}{\rho}\right)^2\right]} \quad \boxed{\frac{1}{B}\frac{dz}{dt} = x} \Rightarrow \boxed{x \equiv x(z)}$$

We define $\boxed{\xi = z/\rho}$, by which we get,

$$\boxed{\frac{\rho}{B}\frac{d\xi}{dt} = x_0 + (y_0 - \rho)\xi - \frac{y_0}{2}\xi^2}.$$

Multiplying throughout by $2/y_0$ we get.

$$\boxed{\frac{2}{y_0}\frac{\rho}{B}\frac{d\xi}{dt} = \frac{2x_0}{y_0} + 2\left(1 - \frac{\rho}{y_0}\right)\xi - \xi^2}.$$

We further define $\boxed{\alpha^2 = 2x_0/y_0}$ and $\boxed{\beta = 1 - \frac{\rho}{y_0}}$.

(P.T.O.)

Hence $\boxed{\frac{2}{y_0}\frac{\ell}{B}\frac{d\mathcal{S}}{dt} = -\left(\mathcal{S}^2 - 2\beta\mathcal{S} + \beta^2 - \beta^2\right) + \alpha^2}$

$\Rightarrow \boxed{\frac{2}{y_0 A}\frac{d\mathcal{S}}{dt} = (\beta^2+\alpha^2) - (\mathcal{S}-\beta)^2} \because \left[\frac{\ell}{B} = \frac{B}{A}\cdot\frac{1}{B}\right]$

Further define $\boxed{k^2 = \alpha^2+\beta^2}$ and $\boxed{\xi_s = \mathcal{S}-\beta}$, which gives $\boxed{\frac{d\mathcal{S}}{dt} = \frac{d\xi_s}{dt}}$. Using this we get,

$\boxed{\frac{2}{A y_0}\frac{d\xi_s}{dt} = k^2 - \xi_s^2} \Rightarrow \boxed{\frac{2}{A y_0 k}\frac{d}{dt}\left(\frac{\xi_s}{k}\right) = 1 - \left(\frac{\xi_s}{k}\right)^2}$

Again, defining $\boxed{\psi = \xi_s/k}$ and $\boxed{\tau = \frac{A y_0 k\, t}{2}}$, we obtain finally, $\boxed{\frac{d\psi}{d\tau} = 1-\psi^2}$. This equation can be integrated to obtain

$\boxed{\ln\left(\frac{1+\psi}{1-\psi}\right) = 2\tau + 2c}$, where c is an integration constant.

(Check for the solution of $\boxed{\frac{dx}{dT} = 1-x^2}$ equation)

Writing $\boxed{T = \tau + c}$, we get, $\boxed{\frac{1+\psi}{1-\psi} = e^{2T}}$, from which we get $\boxed{1+\psi = (1-\psi)e^{2T}} \Rightarrow \boxed{\psi(1+e^{2T}) = e^{2T}-1}$

$\Rightarrow \boxed{\psi = \frac{e^{2T}-1}{e^{2T}+1}} \Rightarrow \boxed{\psi = \frac{e^{T}-e^{-T}}{e^{T}+e^{-T}} = \tanh(T)}$.

$\therefore \boxed{\frac{\xi_s}{k} = \tanh(\tau+c)} \Rightarrow \boxed{\frac{\mathcal{S}-\beta}{k} = \tanh(\tau+c)}$.

(P.T.O.)

$$\Rightarrow \boxed{\zeta = \beta + k \tanh(\tau + c)}$$

$$\Rightarrow \boxed{\frac{Z}{\rho} = \beta + k \tanh\left(\frac{A y_0 k}{2} t + c\right)}$$

$$\Rightarrow \boxed{Z = \beta\rho + k\rho \tanh\left(\frac{A y_0 k}{2} t + c\right)} \quad \boxed{Z \equiv Z(t)}$$

Hence, $$\boxed{\frac{dZ}{dt} = k\rho \cdot \left(\frac{A y_0 k}{2}\right) \operatorname{sech}^2\left(\frac{A y_0 k}{2} t + c\right)}$$

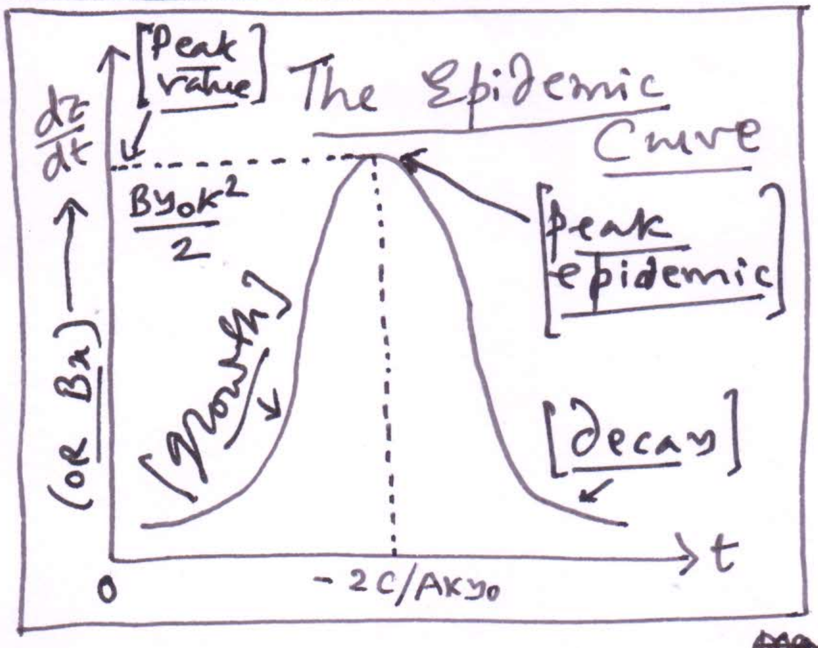
$$\Rightarrow \boxed{\frac{dZ}{dt} = \frac{B y_0 k^2}{2} \operatorname{sech}^2\left(\frac{A y_0 k}{2} t + c\right)} \quad \boxed{\frac{1}{B}\frac{dZ}{dt} = x} \Rightarrow \boxed{x \equiv x(t)}$$

$(\rho = B/A)$

The integration constant c can be fixed from the initial condition, $\boxed{Z = 0}$ at $\boxed{t = 0}$.

$$\Rightarrow \boxed{0 = \beta\rho + k\rho \tanh(c)} \Rightarrow \boxed{\tanh(c) = -\frac{\beta}{k}}$$

Now $\boxed{\beta = 1 - \frac{\rho}{y_0}}$ $\therefore$ $\boxed{\tanh(c) = \frac{1}{k}\left(\frac{\rho}{y_0} - 1\right)}$.



i) The epidemic curve reaches a maximum when $\boxed{\frac{A y_0 k}{2} t + c = 0}$.

$\Rightarrow \boxed{t = -2c/Aky_0}$ $(c < 0)$

ii) The removal rate $\boxed{dZ/dt}$ climbs to a peak value and dies out.

iii) Data matched by the Bombay plague of 1905.