Differential Equations

Contain Derivatives

Consider the equation of a straight line $\boxed{y = mx + c}$, with m and c being fixed parameters.

Taking the first derivative we get, $\boxed{\frac{dy}{dx} = m}$ and the second derivative gives us $\boxed{\frac{d^2y}{dx^2} = 0}$.

i/. Successive derivatives reduce the number of fixed parameters. This implies greater generalisation and more universal relevance.

ii/. Derivatives capture changes, and are relevant for evolving systems.

These are the two advantages of working with differential equations.

Changes in an independent variable, t, ("time", but it can be anything else).

We use a differential equation to express changes of a variable, x, in time, t.

dependent variable, x $\longrightarrow$ Population, Capital, height, position, etc.

$\boxed{\frac{dx}{dt}}$ $\longrightarrow$ Rate at which x changes with t.

Since $\boxed{x \equiv x(t)}$, i.e. x depends on ONLY one variable, we get a full derivative (or ordinary derivative) in t. This requires an ordinary differential equation.

Orders of a differential equation:

A.) First-order: Highest derivative is $\boxed{\frac{dx}{dt}}$.

B.) Second-order: Highest derivative is $\boxed{\frac{d^2x}{dt^2}}$.

Examples:

A) First-order ordinary differential equation

$$\boxed{\frac{dx}{dt} = x}$$ Eg. Compound interest.

B) Second-order ordinary differential equation.

$$\boxed{\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0}$$ Eg. Damped oscillator.

Order of the differential equation ~~derivatives~~ = The number of initial (or boundary) conditions required in an integral solution.

If there are more than one independent variables, as in $\boxed{\psi(x,t)}$, then we have a partial differential equation, such as The Diffusion (or Heat) Equation:

$$\boxed{\frac{\partial\psi}{\partial t} = \nu\frac{\partial^2\psi}{\partial x^2}}$$ which requires one initial condition (first order in t) and two boundary conditions (second order in space).

The Wave Equation: $\boxed{\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2}}$

Requires two initial conditions and two boundary conditions, because it is second order in t and also second order in x.

Second-order Differential Equations

Consider Newton's Second Law.

$\boxed{F = kma} \Rightarrow \boxed{F = m\frac{d^2x}{dt^2}}$ $(k=1)$

Now we write $\boxed{\frac{d^2x}{dt^2} = \frac{F(x,t)}{m}}$,

in which we substitute,

$\boxed{\frac{dx}{dt} = v} \Rightarrow \boxed{x = x(t)}$

and $\boxed{\frac{dv}{dt} = \frac{F(x,t)}{m}} \Rightarrow \boxed{v = v(t)}$

At ~~at~~ a given time, $t = t_0$, two initial conditions are required, $x(t_0)$ (an initial position) and $v(t_0)$ (an initial velocity). The former specifies the state and the latter the rate at which the state is changing (velocity).

Rate $\propto$ State : $\boxed{\frac{dx}{dt} \propto x}$

First - order system

We consider a system $\boxed{\frac{dx}{dt} = \pm a x}$ in which $\boxed{a > 0}$. (geometric growth) for $+a$

$\boxed{+ \text{ sign} \Rightarrow \text{growth} \mid - \text{ sign} \Rightarrow \text{decay}}$

Rescaling : $\boxed{\frac{dx}{d(at)} = \pm x}$

~~Now~~ Now we rescale $\boxed{T = at}$, and get

$$\boxed{\frac{dx}{dT} = \pm x}$$

$\boxed{x = 0}$ is a trivial solution.

Separation of Variables : $\boxed{\int \frac{dx}{x} = \pm \int dT}$

$$\Rightarrow \boxed{\ln x = \ln A \pm \ln e^T}$$

A $\rightarrow$ integral constant

$$\Rightarrow \boxed{x = A e^{\pm T}} \Rightarrow \boxed{x = A e^{\pm at}}$$

A Linear First - Order Autonomous Ordinary Differential Equation : $\boxed{\frac{dx}{dt} = f(x)}$

$$\boxed{\frac{dx}{dt} = f(x) = a \pm bx}$$

$\boxed{a, b > 0}$

(An autonomous form)

$\boxed{\frac{dx}{dt} = f(x,t)} \rightarrow$ is in a NON - AUTONOMOUS form.

Transformation of variables:

Write $\boxed{y = a \pm bx}$. $\Rightarrow$ $\boxed{\dfrac{dy}{dt} = \pm b \dfrac{dx}{dt}}$

But $\boxed{\dfrac{dx}{dt} = a \pm bx = y}$.

Hence, $\boxed{\dfrac{dy}{dt} = \pm by}$, which we

<u>rescale</u> to get $\boxed{\dfrac{dy}{d(bt)} = \pm y}$ $\boxed{T = bt}$.

and, therefore, $\boxed{\dfrac{dy}{dT} = \pm y}$. This

equation is in the <u>rate & state form</u>.

Its solution is $\boxed{y = c\, e^{\pm T}}$, as before.

$$\Rightarrow \boxed{a \pm bx = c\, e^{\pm bt}}$$

$c \rightarrow$ <u>Integration Constant</u>

$$\Rightarrow \mp bx = a - c e^{\pm bt}$$

$$\Rightarrow \mp x = \frac{a}{b} - \frac{c}{b} e^{\pm bt}$$

$$\Rightarrow \boxed{x = \mp\left(\frac{a}{b} - \frac{c}{b} e^{\pm bt}\right)}.$$

The <u>choice</u> of the <u>lower</u> (~~[illegible]~~) <u>sign</u> gives $\boxed{x = \dfrac{a}{b} - \dfrac{c}{b} e^{-bt}}$ from $\boxed{\dfrac{dx}{dt} = a - bx}$

Solving $\boxed{\dfrac{dx}{dt} = a - bx}$ where $\boxed{a, b > 0}$

Separation of variables: $\boxed{\dfrac{dx}{f(x)} = dt}$

$$\Rightarrow \boxed{\frac{dx}{a-bx} = dt} \Rightarrow \boxed{\int \frac{d(-bx)}{a-bx} = \int d(-bt)}$$

$$\Rightarrow \boxed{\ln(a-bx) = \ln c - bt = \ln c + \ln e^{-bt}}$$

$$\Rightarrow \boxed{a - bx = c e^{-bt}} \Rightarrow bx = a - ce^{-bt}$$

$$\Rightarrow \boxed{x = \frac{a}{b} - \frac{c}{b} e^{-bt}}$$

$c \to$ Integration Constant

Since we started with a first-order differential equation in t, we require ONE INITIAL condition, which is when $\boxed{t = 0, \quad x = 0}$. $\Rightarrow \boxed{0 = \frac{a}{b} - \frac{c}{b} e^{-b0}}$

$\Rightarrow \boxed{c = a}$, by which we get.

$$\boxed{x = \frac{a}{b}\left(1 - e^{-bt}\right)}$$

. We now define a scale for x as $\boxed{x_0 = a/b}$ and a scale for t as $\boxed{\tau = 1/b}$. Using these scales we can write $\boxed{x = x_0\left(1 - e^{-t/\tau}\right)}$

(Alternative approach)

Rescaling $\boxed{X = \frac{x}{x_0}}$ and $\boxed{T = \frac{t}{\tau}}$,

we get $\boxed{X = 1 - e^{-T}}$. We can also perform a rescaling on $\boxed{\frac{dx}{dt} = a - bx}$ to obtain $\boxed{X = 1 - e^{-T}}$. This can be done as $\boxed{\frac{1}{b}\frac{dx}{dt} = \frac{a}{b} - x}$.

$$\Rightarrow \boxed{\frac{dx}{d(bt)} = \frac{a}{b} - x}.$$

Since $\boxed{T = bt}$ and $\boxed{x_0 = a/b}$ (x_0 and τ are NATURAL scales)

we write $\boxed{\frac{dx}{dT} = x_0 - x}$.

$$\Rightarrow \boxed{\frac{d(x/x_0)}{dT} = 1 - (x/x_0)}.$$

Since $\boxed{X = \frac{x}{x_0}}$

we finally get $\boxed{\frac{dX}{dT} = 1 - X}$, a rescaled (parameter free) differential equation whose solution is as before, $\boxed{X = 1 - e^{-T}}$. The limiting cases of this solution are when $\boxed{T = 0, \quad X = 0}$ and when $T \longrightarrow \infty$, $X \longrightarrow 1$, which is a convergence to a finite value.

Plotting $\boxed{X = 1 - e^{-T}}$ (Plotting by hand)

i.) We know that when $\boxed{T = 0}$, $\boxed{X = 0}$.

Now, when $\boxed{0 < T \ll 1}$, we expand

$$\boxed{e^{-T} = 1 - T + \frac{T^2}{2!} - \frac{T^3}{3!} + \cdots}$$ Infinite Series

Successive terms in this series diminish very rapidly since $T \ll 1$. Hence,

$$\boxed{e^{-T} \simeq 1 - T} \text{ when } T \ll 1$$ (Small time limit)

$$\therefore \quad \boxed{X = 1 - e^{-T} \simeq \cancel{1} - (\cancel{1} - T)}$$

$$\Rightarrow \quad \boxed{X \simeq T} \quad \Rightarrow \quad \boxed{\frac{x}{x_0} \simeq \frac{t}{\tau}}.$$

$$\therefore \quad x \simeq x_0 \frac{t}{\tau} = \frac{a}{\cancel{b}} \cancel{b} t \simeq at$$

Hence, for (small time) $t \ll \tau$ (or $T \ll 1$), $\boxed{x \simeq at}$.

ii.) In the opposite limit when $T \to \infty$.
(long time limit)

$$\boxed{X = 1 - e^{-\infty} = 1} \Rightarrow X \to 1, \text{ when } T \to \infty.$$

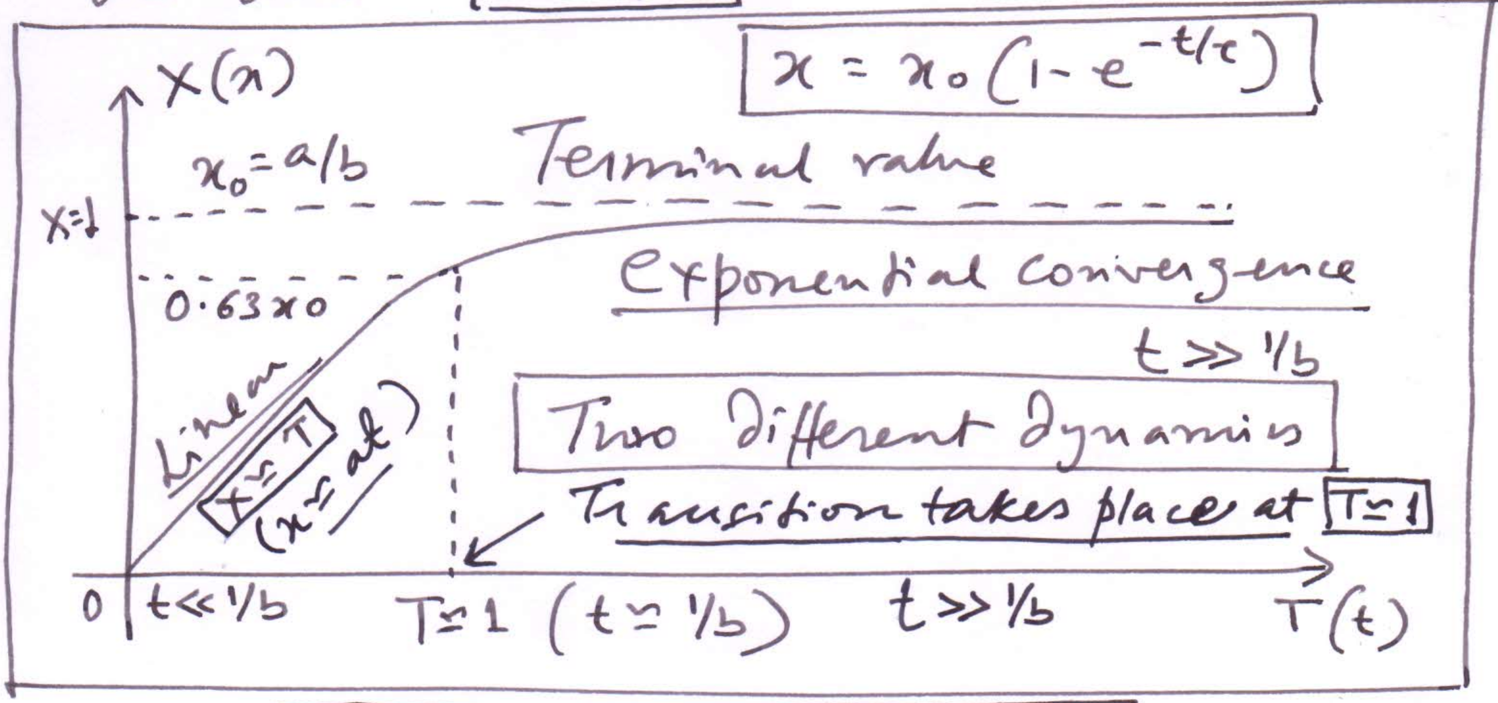
$\Rightarrow \boxed{x \to x_0 = a/b}$, when $\boxed{t \to \infty}$. So for long time scale, x converges towards a limiting value of a/b.

iii.) We can obtain the derivative of $\boxed{X = 1 - e^{-T}}$, as $\boxed{\frac{dX}{dT} = e^{-T}}$. If $\boxed{\frac{dX}{dT} = 0}$ $\Rightarrow$ $\boxed{T \to \infty}$.

The second derivative is $\boxed{\frac{d^2X}{dT^2} = -e^{-T}}$

when $\boxed{T \to \infty, \ \frac{d^2X}{dT^2} = 0}$. Hence this is not a turning point $\boxed{\text{At } X = 1}$.

iv.) The transition from the linear behaviour $\boxed{X \simeq T}$ to an exponential convergence of $\boxed{X = 1 - e^{-T}}$ takes place when $\boxed{T \simeq 1}$ or when $\boxed{t \simeq 1/b}$ (natural time scale).



When $\boxed{t = \tau}$, $\boxed{x = x_0(1 - e^{-1})}$ $\Rightarrow$ $\boxed{x \simeq 0.63x_0}$

There are two different dynamics on two different time scales. Eg. Growth of humans or the inflationary Universe.

Systems of the form: $\boxed{\dfrac{dx}{dt} = a + bx}$

$\boxed{a, b > 0}$

We know when $\boxed{\dfrac{dx}{dt} = a - bx}$ (with $a,b>0$) the solution is $\boxed{x = \frac{a}{b}\left(1 - e^{-bt}\right)}$. When $\boxed{\dfrac{dx}{dt} = a + bx = a - (-b)x}$, we make the transformation $\boxed{b \to -b}$

Hence, $\boxed{x = \dfrac{a}{-b}\left(1 - e^{bt}\right)}$

$\Rightarrow \boxed{x = \frac{a}{b}\left(e^{bt} - 1\right)}$ is the solution of $\boxed{\dfrac{dx}{dt} = a + bx}$.

Writing $\boxed{x_0 = a/b}$ and $\boxed{\tau = 1/b}$, we get

$\boxed{x = x_0\left(e^{t/\tau} - 1\right)}$ OR $\boxed{X = e^{T} - 1}$ $\left(\begin{array}{c} X = x/x_0 \\ \text{and } T = t/\tau \end{array}\right)$

$\tau = 1/b$ is the natural time scale.

Limiting behaviour:

$\left(e^{t/\tau} = 1 + t/\tau + t^2/2!\tau^2 + \ldots\right)$

i.) when $\boxed{t \ll \tau}$, $\boxed{e^{t/\tau} \simeq 1 + t/\tau}$ (linear order only)
(small time)

$\therefore x \simeq x_0\left(\cancel{1} + \frac{t}{\tau} - \cancel{1}\right) = x_0 \frac{t}{\tau} = at$.

$\Rightarrow \boxed{x = at}$ (early growth is linear).

ii.) when $t \gg \tau$, $e^{t/\tau} - 1 \simeq e^{t/\tau}$ (for long time).

$\therefore \boxed{x = x_0 e^{t/\tau}}$ (late growth is exponential)

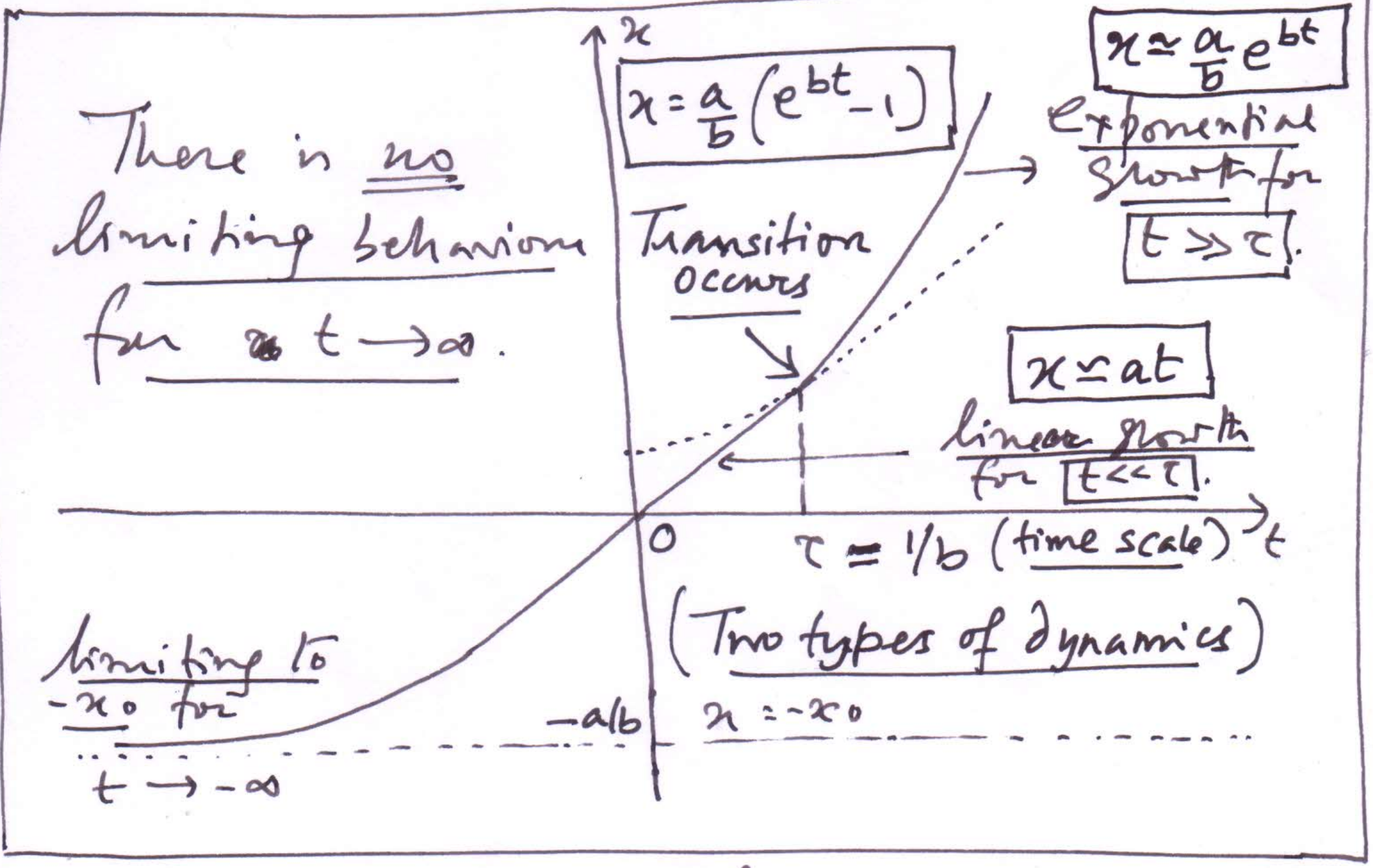
Consider a hypothetical case when $\boxed{t<0}$.

iii) For $t \to -\infty$, $\boxed{x \to -x_0}$ (limiting terminal value)

iv) For $|t| \ll \tau$, $\boxed{e^{t/\tau} \simeq 1 + t/\tau}$ (linear order)

$\Rightarrow \quad x \simeq x_0 t/\tau \quad \Rightarrow \quad \boxed{x \simeq at}$ (linear)

Plotting: $\boxed{x = x_0\left(e^{t/\tau} - 1\right)}$ $\quad \boxed{x_0 = a/b}$ $\quad \boxed{\tau = 1/b}$



There is an exchange of the functional behaviour from the first to the third quadrant as $\boxed{\frac{dx}{dt} = a - bx}$ goes to $\boxed{\frac{dx}{dt} = a + bx}$.