The Spread of Technological Innovations

Agricultural Innovation among Farmers

i.) Total number of farmers in a farming community is N (fixed value).
ii.) $x(t) \to$ Number of farmers who have adopted an innovation.
iii.) $N - x(t) \to$ Number of farmers who have not adopted the innovation.

$\boxed{\Delta x \propto \Delta t}$, $\boxed{\Delta x \propto x}$ and $\boxed{\Delta x \propto (N-x)}$

$\therefore \boxed{\Delta x \propto x(N-x)\Delta t}$ (jointly proportional) $\Rightarrow \boxed{\frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt} = c\,x(N-x)}$.

($c > 0$) (proportional constant)

The initial condition is $\boxed{x(0) = 1}$

Rescale: $\boxed{\frac{d(x/N)}{dt} = cN \frac{x}{N}\left(1 - \frac{x}{N}\right)} \Rightarrow$ the logistic equation,

define $\boxed{X = x/N}$ and $\boxed{T = cNt}$ to get

$\boxed{\frac{dX}{dT} = X(1-X)}$ whose solution is $\boxed{X = \frac{1}{1 + A^{-1}e^{-T}}}$.

Hence $\boxed{x = \frac{N}{1 + A^{-1}e^{-cNt}}}$. When $\boxed{t = 0,\ x = 1}$.

$\Rightarrow 1 = \frac{N}{1 + A^{-1}} \Rightarrow \boxed{A^{-1} = N - 1}$.

$$\Rightarrow \boxed{x = \frac{N}{1 + (N-1)e^{-cNt}} = \frac{Ne^{cNt}}{(N-1) + e^{cNt}}}.$$

i.) A discrepancy arises due to not accounting for information obtained through the mass media.
ii.) The ~~[illegible]~~ slowing of the growth rate happens later than expected.

Modification : $\boxed{\Delta x = c'(N-x)\Delta t}$ $(c'>0)$ proportional constant

(Correction due to impersonal communication.)

The total effect is $\boxed{\Delta x = cx(N-x)\Delta t + c'(N-x)\Delta t}$.

$$\Rightarrow \boxed{\frac{\Delta x}{\Delta t} = (cx+c')(N-x)} \Rightarrow \boxed{\frac{dx}{dt} = N(cx+c')\left(1-\frac{x}{N}\right)}.$$

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x+\frac{c'}{c}\right)\left(1-\frac{x}{N}\right)}, \boxed{\frac{c'}{c}>0}.$$

Early growth : When $\boxed{x \ll N}$, $\boxed{\frac{dx}{dt} \simeq Nc\left(x+\frac{c'}{c}\right)}$.

i.) Quicker than exponential, if $c'/c > 0$.
ii.) Slower than exponential, if $c'/c < 0$.

Since $c', c > 0$, the non-human intervention boosts early growth of the function, $x(t)$.

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c} - \frac{x^2}{N} - \frac{c'}{c}\frac{x}{N}\right)}$$

$$\Rightarrow \frac{dx}{dt} = Ncx + Nc' - cx^2 - c'x$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\left[cx^2 - (Nc-c')x - Nc'\right]}$$

Accounting for both human intervention and the mass media

$$\Rightarrow \frac{dx}{dt} = -\left[(\sqrt{c}x)^2 - 2\,\frac{1}{2}\cdot\frac{\sqrt{c}x}{\sqrt{c}}(Nc-c') + \frac{(Nc-c')^2}{4c} - \frac{(Nc-c')^2}{4c} - Nc'\right]$$

$$\Rightarrow \frac{dx}{dt} = -\left[\sqrt{c}\,x - \frac{(Nc-c')}{2\sqrt{c}}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -c\left[x - \frac{(Nc-c')}{2c}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]}$$

Define $\boxed{y = x - \frac{(Nc - c')}{2c}}$ and $\boxed{\alpha^2 = Nc' + \frac{(Nc-c')^2}{4c}}$

Hence $\boxed{\frac{dy}{dt} = \frac{dx}{dt}}$ $\Rightarrow$ $\boxed{\frac{dy}{dt} = \alpha^2 - cy^2}$

$\Rightarrow$ $\boxed{\frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/c}}$. Rescaling now $\boxed{X = \frac{y}{\alpha/\sqrt{c}}}$ and $\boxed{T = \alpha\sqrt{c}\,t}$,

we get $\boxed{\frac{dX}{dT} = 1 - X^2}$, whose solution is known to be $\boxed{\left(\frac{1+X}{1-X}\right) = Ae^{2T}}$. The initial condition is when $\boxed{t = 0,\ x = 0}$. $\Rightarrow$ $\boxed{y_0 = -\frac{Nc - c'}{2c} = \frac{1}{2}\left(\frac{c'}{c} - N\right)}$

Hence, $\boxed{X_0 = \frac{y_0}{\alpha/\sqrt{c}}}$. Making X the subject of T,

we get, $\boxed{X = \frac{Ae^{2T} - 1}{Ae^{2T} + 1}}$ $\Rightarrow$ $\boxed{y = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t} + 1}\right)}$.

Now $4c\alpha^2 = 4Ncc' + (Nc - c')^2$ (from the definition of α)

$\Rightarrow 4c\alpha^2 = 4Ncc' + N^2c^2 - 2Ncc' + c'^2 = (Nc + c')^2$

$\Rightarrow$ $\boxed{2\alpha\sqrt{c} = (Nc + c')}$ & $\frac{\alpha}{\sqrt{c}} = \frac{2\alpha\sqrt{c}}{2c} = \frac{Nc + c'}{2c}$.

$\Rightarrow$ $\boxed{\frac{\alpha}{\sqrt{c}} = \frac{1}{2}\left(N + \frac{c'}{c}\right)}$. Further, when

$t = 0$ ($T = 0$) and $x = 0$ ($X = X_0$), $\boxed{A = \frac{1 + X_0}{1 - X_0}}$.

Therefore, $\boxed{A = \frac{1 + y_0/\alpha\sqrt{c}}{1 - y_0/\alpha\sqrt{c}} = \frac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0}}$

Now, $\boxed{y_0 + \frac{\alpha}{\sqrt{c}} = \frac{1}{2}\frac{c'}{c} - \cancel{\frac{N}{2}} + \cancel{\frac{N}{2}} + \frac{c'}{2c} = \frac{c'}{c}}$.

Also,

$$\boxed{\frac{\alpha}{\sqrt{c}} - y_0 = \frac{N}{2} + \cancel{\frac{c'}{2c}} - \cancel{\frac{1}{2}\frac{c'}{c}} + \frac{N}{2} = N}$$

Hence, $\boxed{A = \frac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0} = \frac{c'}{cN}}$. Hence we write the full integral solutions as $y = x - \frac{(Nc - c')}{2c} = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t} + 1}\right)$

Substituting for A, $2\alpha\sqrt{c}$, and $\alpha/\sqrt{c}$, we get,

$$\boxed{x = \frac{Nc - c'}{2c} + \frac{1}{2}\left(N + \frac{c'}{c}\right) \cdot \frac{(c'/Nc)e^{(Nc+c')t} - 1}{(c'/Nc)e^{(Nc+c')t} + 1}}.$$

$$\Rightarrow x = \frac{Nc - c'}{2c} + \frac{Nc + c'}{2c} \cdot \frac{c'e^{(Nc+c')t} - Nc}{c'e^{(Nc+c')t} + Nc}$$

$$\Rightarrow x = \frac{(Nc - c')\left[c'e^{(Nc+c')t} + Nc\right] + (Nc + c')\left[c'e^{(Nc+c')t} - Nc\right]}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{Ncc'e^{(Nc+c')t} - \cancel{c'^2 e^{(Nc+c')t}} + \cancel{(Nc)^2} - Ncc' + Ncc'e^{(Nc+c')t} + \cancel{c'^2 e^{(Nc+c')t}} - Ncc' - \cancel{(Nc)^2}}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{\cancel{2}N\cancel{c}c'e^{(Nc+c')t} - \cancel{2}N\cancel{c}c'}{\cancel{2c}\left[c'e^{(Nc+c')t} + Nc\right]} = \frac{Nc'e^{(Nc+c')t} - Nc'}{Nc + c'e^{(Nc+c')t}}$$

$$\Rightarrow \boxed{x = \frac{Nc'\left[e^{(Nc+c')t} - 1\right]}{cN + c'e^{(Nc+c')t}}}$$ → The integral solution of $\boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}$.

The above solution is recast as

$$\boxed{x = \frac{Nc'\left[1 - e^{-(Nc+c')t}\right]}{c' + cNe^{-(Nc+c')t}}}$$ ∴ When $t \to \infty$, $\boxed{x \to \frac{Nc'}{c'} = N}$ The maximum value.

Industrial Innovations: (Edwin Mansfield)

(Study on Coal, iron and steel, brewing and railroads).

$$\boxed{\begin{array}{l} \text{i.) Total number of firms in an industry is } N. \\ \text{ii.) } x(t) \rightarrow \text{Number of firms that have adopted a technological innovation.} \end{array}}$$

$\boxed{\Delta x \propto \Delta t}$ and $\boxed{\Delta x \propto (N-x)}$ $\Rightarrow$ $\boxed{\Delta x \propto (N-x)\Delta t}$

Jointly, we write $\boxed{\Delta x = \lambda (N-x)\Delta t}$,

in which $\lambda \rightarrow$ proportional factor (not constant)

$\boxed{\lambda = \lambda\left(p, s, \frac{x}{N}\right)}$, in which (with N being fixed)

i.) $p \rightarrow$ profitability in investing in an innovation.

ii.) $s \rightarrow$ Investing ability to acquire innovation, as a percentage of the total assets.

iii.) $\frac{x}{N} \rightarrow$ Percentage of firms that have already adopted the innovation.

Edwin Mansfield's Study: (To determine λ).

i.) Carry out a Taylor expansion of λ about some equilibrium values of p, s and x/N, represented with a subscript c $\left(p_c, s_c, \frac{x}{N}\Big|_c\right)$.

ii.) Limit the Taylor expansion only up to the second order, (i.e. orders of p^2, s^2, $\left(\frac{x}{N}\right)^2$.).

iii.) Gather all the coefficients of zeroth, first and second orders.

Accordingly $\boxed{\lambda = f\left(p, s, \frac{x}{N}\right)}$ is Taylor expanded as,

$$\boxed{\begin{aligned}\lambda &= f\left(p_c, s_c, \tfrac{x}{N}\Big|_c\right) \\ &+ \frac{\partial f}{\partial p}\Big|_c (p - p_c) + \frac{\partial f}{\partial s}\Big|_c (s - s_c) + \frac{\partial f}{\partial (x/N)}\Big|_c \left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) \\ &+ \frac{1}{2!}\frac{\partial^2 f}{\partial p^2}\Big|_c (p-p_c)^2 + \frac{1}{2!}\frac{\partial^2 f}{\partial s^2}\Big|_c (s-s_c)^2 + \frac{\partial^2 f}{\partial^2 (x/N)}\Big|_c \left(\frac{x}{N} - \frac{x}{N}\Big|_c\right)^2 \\ &+ \frac{2}{2!}\frac{\partial^2 f}{\partial p \partial s}\Big|_c (p-p_c)(s-s_c) + \frac{2}{2!}\frac{\partial^2 f}{\partial p\, \partial (x/N)}\Big|_c (p-p_c)\left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) \\ &+ \frac{2}{2!}\frac{\partial^2 f}{\partial s\, \partial (x/N)}\Big|_c (s-s_c)\left(\frac{x}{N} - \frac{x}{N}\Big|_c\right) + \cdots\end{aligned}}$$

In deriving the above expression we have used the mathematical principle $\boxed{\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}}$.

We get, on collecting ~~Collecting~~ all the terms of the same order,

$$\boxed{\begin{aligned}\lambda = a_1 + a_2 p + a_3 s + a_4\left(\frac{x}{N}\right) + a_5 p^2 + a_6 s^2 + a_7 ps \\ + a_8 p\left(\frac{x}{N}\right) + a_9 s\left(\frac{x}{N}\right) + a_{10}\left(\frac{x}{N}\right)^2\end{aligned}},$$

in which all ~~a_i~~ a_i are constants, depending on the equilibrium values of p, s and $\frac{x}{N}$, and their derivatives.

Edwin Mansfield's study shows $\boxed{a_{10} = 0}$ and

$$\boxed{a_1 + a_2 p + a_3 s + a_5 p^2 + a_6 s^2 + a_7 ps = 0}.$$

The remaining terms $\boxed{\lambda = (a_4 + a_8 p + a_9 s)\frac{x}{N}}$

Define $\boxed{k = a_4 + a_8 p + a_9 s} \Rightarrow \boxed{\lambda = k\frac{x}{N}}$.

$\boxed{K \equiv k(p,s)}$, i.e, k depends on profitability and investing power. (k is not to be confused with the carrying capacity in the logistic equation)

$$\therefore \quad \boxed{\Delta x = k \frac{x}{N}(N-x)\Delta t} \Rightarrow \boxed{\frac{\Delta x}{\Delta t} = k\frac{x}{N}(N-x)}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = k\frac{x}{N}(N-x)} \Rightarrow \boxed{\frac{d(x/N)}{dt} = k\frac{x}{N}\left(1-\frac{x}{N}\right)}$$

Define $\boxed{X = \frac{x}{N}}$ and $\boxed{T = kt}$, to get

$$\boxed{\frac{dX}{dT} = X(1-X)}$$, which is the logistic equation.

The solution is $\boxed{X = \frac{1}{1+A^{-1}e^{-T}}} \Rightarrow \boxed{x = \frac{N}{1+A^{-1}e^{-kt}}}$.

Initial condition: when $\boxed{t = t_0, \ x = 1}$.

$$\Rightarrow 1 = \frac{N}{1+A^{-1}e^{-kt_0}} \Rightarrow \boxed{A^{-1} = (N-1)e^{kt_0}}$$

$$\Rightarrow \boxed{x = \frac{N}{1+(N-1)e^{-k(t-t_0)}}}$$

The integral solution for spread of industrial innovations.

This solution was used to study:

i.) The spread of twelve innovations such as the shuttle car, trackless mobile loaders, mining machines, coke ovens, wide strip mills, etc.

ii.) Across ~~four~~ major industries like coal, iron and steel, brewing and railroads.

The Dynamics of Free-Living Dividing Cell Growth

$\boxed{x \equiv x(t)} \rightarrow$ Volume of dividing cells at time t.

The growth rate equation is $\boxed{\frac{dx}{dt} = \lambda x}$ $(\lambda > 0)$.

At $\boxed{t = t_0 \,,\; x = x_0} \longrightarrow$ Initial condition.

$\Rightarrow \boxed{x(t) = x_0 \exp\left[\lambda(t - t_0)\right]}$. Cell doubling happens when $\boxed{x = 2x_0} \Rightarrow$ Doubling time $\boxed{t - t_0 = \frac{\ln 2}{\lambda}}$

Gompertz Law of Tumour Growth

$\boxed{\frac{dx}{dt} = f(x) = -ax\ln(bx)}$ $\boxed{a, b > 0}$.

$x(t) \rightarrow$ Number of cells in a tumour.

Scale $\boxed{y = x/b^{-1}}$ and $\boxed{T = at}$.

Rescaling:

$\Rightarrow \boxed{\frac{d\left(\frac{x}{b^{-1}}\right)}{d(at)} = -\left(\frac{x}{b^{-1}}\right)\ln\left(\frac{x}{b^{-1}}\right)} \Rightarrow \boxed{\frac{dy}{dT} = -y\ln y}$

Integral Solution: Substitute $\boxed{y = e^X} \Rightarrow \boxed{X = \ln y}$

$\therefore \boxed{\frac{dy}{dT} = e^X\frac{dX}{dT} = y\frac{dX}{dT} = -y\ln y = -yX}$

$\Rightarrow \cancel{y}\frac{dX}{dT} = -\cancel{y}X \quad \Rightarrow \boxed{\frac{dX}{dT} = -X}$.

$\Rightarrow \boxed{\int\frac{dX}{X} = -\int dT} \quad \Rightarrow \boxed{X = X_0\, e^{-T}}$

X_0 is the integration constant

$\Rightarrow$ $\boxed{\ln y = X_0 e^{-T}}$ $\Rightarrow$ $\boxed{\ln(bx) = X_0 e^{-at}}$

$\Rightarrow$ $\boxed{x = \frac{1}{b} \exp\left(X_0 e^{-at}\right)}$. Exponential of an exponential.

i.) When $\boxed{t \to \infty, \ x \to b^{-1}}$ (limiting ~~value~~ value)

ii.) When $\boxed{t = 0, \ x = x_0}$ (initial value).

$\therefore$ $\boxed{x_0 = \frac{1}{b} \exp(X_0)}$ $\Rightarrow$ $\boxed{e^{X_0} = x_0 b}$ ~~[illegible]~~

Fixing the unknown X_0 $\Rightarrow$ $\boxed{X_0 = \ln(x_0 b)}$

Since $\boxed{x_0 < b^{-1}}$ (the tumour GROWS from x_0 to b^{-1}).

$\Rightarrow$ $\boxed{\frac{x_0}{b^{-1}} < 1}$ $\Rightarrow$ $\boxed{X_0 = \ln\left(\frac{x_0}{b^{-1}}\right) < 0}$

Hence, $\boxed{x = \frac{1}{b} \exp\left[\ln\left(\frac{x_0}{b^{-1}}\right) e^{-at}\right]}$,

the Gompertz formula for tumour growth, which satisfies over 1000-fold growth.

We differentiate the $x \equiv x(t)$ equation to get.

$$\boxed{\frac{dx}{dt} = \frac{1}{b} \exp\left[\ln\left(\frac{x_0}{b^{-1}}\right) e^{-at}\right] \cdot \ln\left(\frac{x_0}{b^{-1}}\right) e^{-at} \cdot (-a)}$$

Now $\boxed{\ln\left(\frac{x_0}{b^{-1}}\right) < 0}$ $\therefore$ $\boxed{-a \ln\left(\frac{x_0}{b^{-1}}\right) = -aX_0 > 0}$

We write $\boxed{\lambda = -aX_0 > 0}$ to get $\boxed{\frac{dx}{dt} = x \lambda e^{-at}}$.

This equation is in the form $\boxed{\frac{dx}{dt} = f(x,t)}$

The non-autonomous form $\boxed{\frac{dx}{dt} = f(x,t)}$, can be ~~[illegible]~~ cast in two ways. They are:

i.) $\boxed{\frac{dx}{dt} = \left(\lambda e^{-at}\right) x = \bar{\lambda}(t)\, x} \rightarrow \bar{\lambda}$ depends on t.

ii.) OR $\boxed{\frac{dx}{dt} = \lambda\left(x e^{-at}\right)} \rightarrow \lambda$ is a constant

First form: $\boxed{\frac{dx}{dt} = \bar{\lambda}(t)\, x}$ (Rate $\propto$ State)

The time scale for tumour generation is $\boxed{\bar{t} \sim \frac{1}{\bar{\lambda}}}$ (On comparing with $\boxed{t - t_0 = \frac{\ln 2}{\lambda}}$ in free-living and dividing cells)

$\Rightarrow \boxed{\bar{t} \sim \lambda^{-1} e^{at}}$ $\Rightarrow$ As t increases, longer time is taken for the same amount of growth. The cells mature and divide more slowly.

Second form: $\boxed{\frac{dx}{dt} = \lambda\left(x e^{-at}\right)}$. λ is constant and now rate is proportional to a state, ~~[illegible]~~ $\boxed{x e^{-at}}$.

This effective state, contributing to the growth of the tumour, decreases due to necrosis at the core of the tumour, with lower number of living cells.

First form: Growth process slows down. [SUMMARY]

Second form: Number of cells in the growth is lower.

Bacteria versus Toxin: (A non-autonomous system)

$$\boxed{\begin{array}{l} x(t) \rightarrow \text{Number of bacteria at time, } t. \\ T(t) \rightarrow \text{Amount of toxin at time, } t. \end{array}}$$

i.) In the ~~presence~~ absence of toxin bacteria grow, $\boxed{\frac{dx}{dt} = bx}$. $(b > 0)$.

ii.) In the presence of toxin, bacteria die out. $\boxed{\frac{dx}{dt} = -axT}$ $(a > 0)$.

iii.) Growth rate of toxin. $\boxed{\frac{dT}{dt} = c}$. $(c > 0)$.

$\Rightarrow \boxed{T = ct + k}$ (linear function) ← Initial condition: When $t = 0$, $T = 0$ $\Rightarrow \boxed{k = 0}$ $\Rightarrow \boxed{T = ct}$.

$\therefore \boxed{\frac{dx}{dt} = -axct}$ in the presence of toxin.

Combined Equation: $\boxed{\frac{dx}{dt} = bx - axct}$

$\Rightarrow \boxed{\frac{dx}{dt} = f(x,t) = x(b - act)}$ Non-autonomous equation

Integral Solution: $\boxed{\int \frac{dx}{x} = \int (b - act)\, dt}$

$\Rightarrow \boxed{\ln x = \ln x_0 + bt - \frac{act^2}{2}}$ | x_0 is an integration constant.

$\Rightarrow \boxed{x = x_0 \exp\left[bt - \frac{act^2}{2}\right]}$. From this solution we see that when $\boxed{t = 0,\ x = x_0}$ (initial condition). Further when $t \rightarrow \infty$, the square power dominates and $x \rightarrow 0$ (the limiting condition).

Now we write $\boxed{bt - \frac{act^2}{2} = \frac{2bt - act^2}{2}}$.

This is $-\frac{1}{2}\left[(\sqrt{ac}\,t)^2 - 2\frac{b}{\sqrt{ac}}\sqrt{ac}\,t + \frac{b^2}{ac} - \frac{b^2}{ac}\right]$

which can be written as a square,

$$\boxed{-\frac{1}{2}\left[\left(\sqrt{ac}\,t - \frac{b}{\sqrt{ac}}\right)^2 - \frac{b^2}{ac}\right] = \frac{b^2}{2ac} - \frac{ac}{2}\left(t - \frac{b}{ac}\right)^2}$$

Hence $\boxed{x = x_0\, e^{b^2/2ac} \times \exp\left[-\frac{ac}{2}\left(t - \frac{b}{ac}\right)^2\right]}$

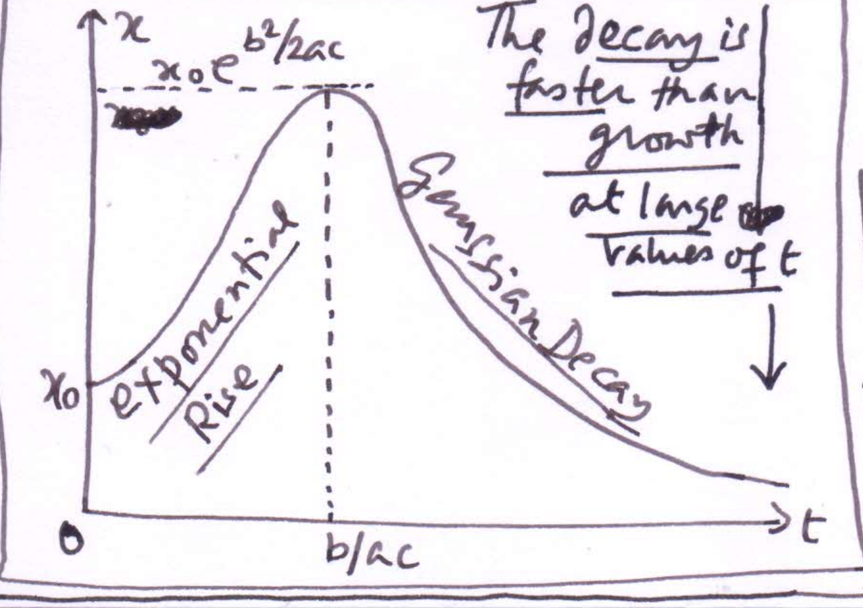
Clearly, i.) When $\boxed{t = 0,\ x = x_0}$, ii.) When $\boxed{t \to \infty,\ x \to 0}$, and iii.) When $\boxed{t = \frac{b}{ac}}$, $\boxed{x = x_0 e^{b^2/2ac} > x_0}$

Looking at $\boxed{\frac{dx}{dt} = x(b - act)}$, we see that when $\boxed{t = b/ac}$, $\boxed{\frac{dx}{dt} = 0}$.

Hence, $\boxed{t = b/ac}$ gives a turning point for $x(t)$.

The Second Derivative: $\boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt}(b - act) + x(-ac)}$

When $\boxed{t = b/ac}$, $\boxed{\frac{d^2x}{dt^2} = -acx < 0}$. This is the condition for a maximum value of $x(t)$.



Rescale: $\boxed{X = x/x_0}$ and $\boxed{T = t/(b/ac)}$. This gives

$$\boxed{X = e^{b^2/2ac} \times \exp\left[-\frac{b^2}{2ac} \times (T-1)^2\right]}.$$

i.) For $\boxed{T < 1}$, early growth is exponential.

ii.) For $\boxed{T > 1}$, the decay is Gaussian.