CS-401 Insem 1 Solns

1.

a) Put call parity states that if call and put European options of the same strike price K are available then

$$C + K e^{-rT} = P + S_0 \quad ②$$

$C \sim$ cost of call option
$P \sim$ " " put option
$S_0 \sim$ spot price of underlying stock
$T \sim$ time to expiry
$r \sim$ risk free interest rate

Consider the following two portfolio's

③

Portfolio A: one European + one Zero coupon bond with payoff K

Portfolio B: One European put + one share of stock

At time T:

		$S_T > K$		$S_T < T$
Worth of A	call	$S_T - K$		0
	ZC bond	K		K
	Total	S_T		K
Worth of B	put	0	put	$K - S_T$
	share	S_T		S_T
	Total	S_T		K

i. Portfolio A = Portfolio B at $t = T$

$\therefore$ They must be worth the same today.

$$\therefore \quad c + \underbrace{K e^{-rT}}_{\text{discounting to present}} = p + S_0$$

b. $c = 3$, $p = 2.25$ $r = 10\%$ $T = \frac{3}{12} = \frac{1}{4}$

$K = 30$, $S_0 = 31$

$$c + K e^{-rT} = 3 + 30 e^{-0.01 \cdot \frac{1}{4}} = 32.26$$

$$p + S_0 = 2.25 + 31 = 33.25$$

$$\therefore \quad c + K e^{-rT} < p + S_0$$

Arbitrage possibility.

Today	In 3mths if $S_T > 30$
1. Buy Call \$3	1. Receive $30.25 \times e^{0.01 \cdot \frac{1}{4}} =$ 31.02 from investment
2. Sell put to realize 2.25	2. Ex. call to buy stock for 30
3. Short Stock to realize 31	3. Settle the short position in stock from 2.
4. Invest $31 + 2.25 - 3 = 30.25$ in bank for 3mths	Net Profit = 1.02 (2)
(3)	In 3mths if $S_T < 30$
	1. Receive 31.02 from inv
	2. Buy stock for 30 from Put option (Exercised by buyer)
	3. Settle short on stock
	Net Profit = 1.02

2. (Two Marks each)

a) $P_k(A) \sim$ Probability that Gambler is ruined starting with k dollars

$$P_k(A) = P_k(A \mid B)P(B) + P_k(A \mid B^c)\,P(B^c)$$

where B is the event that the first toss is head.

$$P_k(A) = \frac{1}{2} P_k(A \mid B) + \frac{1}{2} P(A \mid B^c)$$

But $P_k(A \mid B) = P_{k+1}(A)$ and

$P_k(A \mid B^c) = P_{k-1}(A)$ so

$$P_k(A) = \frac{1}{2} P_{k+1}(A) + \frac{1}{2} P_{k-1}(A)$$

b) Denote $P_k(A) = P_k$

then $P_k = \frac{1}{2} P_{k+1} + \frac{1}{2} P_{k-1}$

$\therefore \quad P_{k+1} - P_k = P_k - P_{k-1}$

i.e $\quad P_2 - P_1 = P_1 - P_0$

$\quad P_3 - P_2 = P_2 - P_1$

$\quad \vdots$

$\quad P_N - P_{N-1} = P_{N-\cancel{2}} - P_{N-2}$

Now $\quad P_N = 0, \quad P_0 = 1$

$\therefore \quad P_N - P_0 = -1 = \sum_{i=0}^{N-1} (P_{i+1} - P_i)$

$\quad P_N - P_0 = N(P_1 - P_0)$

$\therefore$

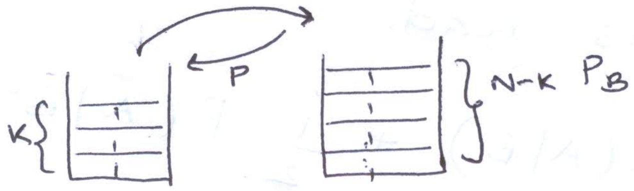
$$\therefore \; P_1 - P_0 = \frac{-1}{N}$$

Similarly

$$P_K - P_0 = \sum_{i=0}^{K-1} (P_{i+1} - P_i) = K\,(P_1 - P_0)$$

$$\therefore \; P_K - P_0 = \frac{-K}{N} \Rightarrow \boxed{P_K = 1 - \frac{K}{N}}$$

c) P_A



A and B play a game where counters of denomination 1 are arranged. At each step depending on head or tails counters from one pile are placed to the other. A wins if all counters are in his pile and B wins if all counters are on his pile.

gives $P_A + P_B = 1$

$$P_A(N-K) - P_B K = 0 \Rightarrow P_B = 1 - \frac{K}{N}$$

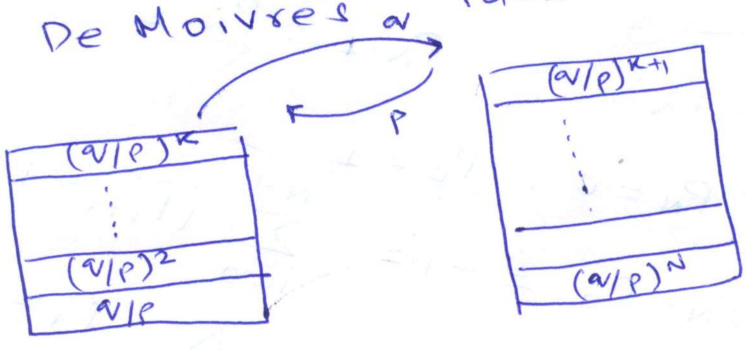
d) $\cancel{\lim_{K\to\infty}} P_K = \cancel{\lim}$ The game stops when either the gambler is ruined or the gambler wins N dollars. Since

$K << N$ $\quad P_K \sim 1$ at some time

(known as the stopping time)

e) Use the reccurence idea or

use De Moivre's idea



A B

A and B play a game with counters with denominalions $(q/p)^i$ as shown. At each iteration either a counter is transferred from A's pile to B or vice versa depending on a tail or a head ocurng

Let $P_A \sim$ Probability of A winning (if all the counters are on A's side)

$P_B \sim$ Probability of B winning.

At each iteration the expected earning of (A or B) is

$$q \cdot \left(\frac{q}{p}\right)^{k+1} - p\left(\frac{q}{p}\right)^{k} = 0$$

Hence at stop time.

$$P_A\left[\left(\frac{q}{p}\right)^{k+1} \cdots + \left(\frac{q}{p}\right)^{N}\right] - P_B\left[\left(\frac{q}{p}\right) + \cdots + \left(\frac{q}{p}\right)^{k}\right] = 0 \quad -(1)$$

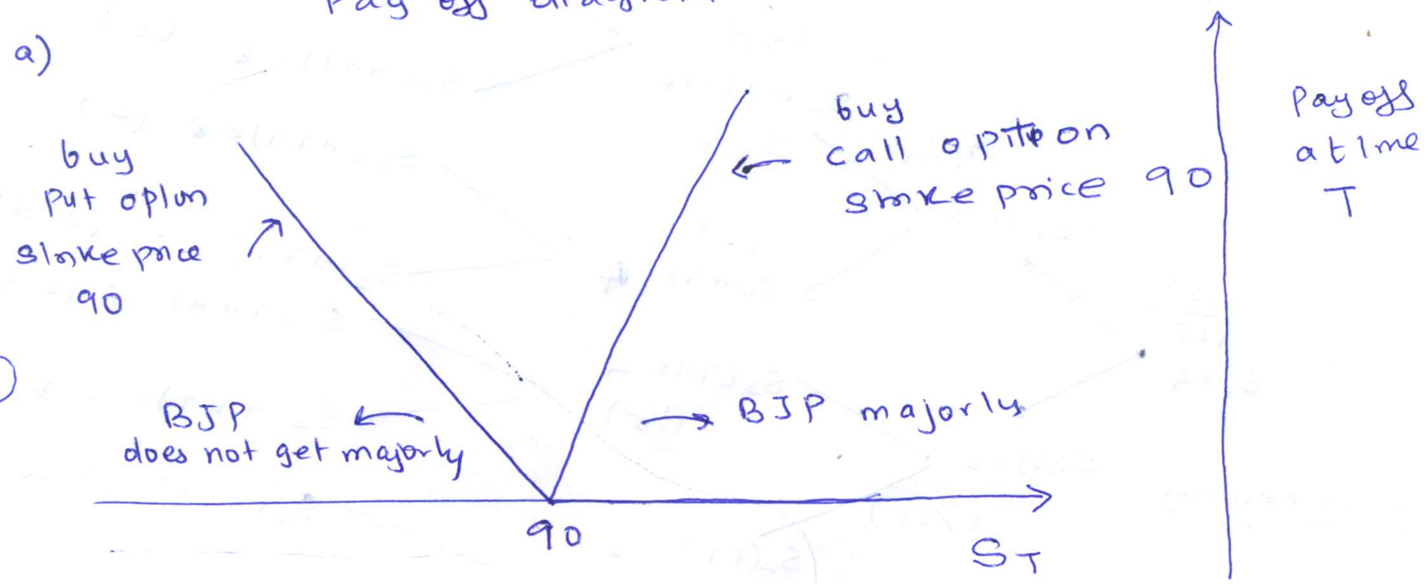
Also $\quad P_A + P_B = 0 \quad -(2)$

Solving the two eqns. we get

$$P_B = \frac{\left(\frac{q}{p}\right)^{N} - (q/p)^{k}}{(q/p)^{N} - 1} \sim \text{Probability of Ruin}$$

3. Pay off diagram

a)



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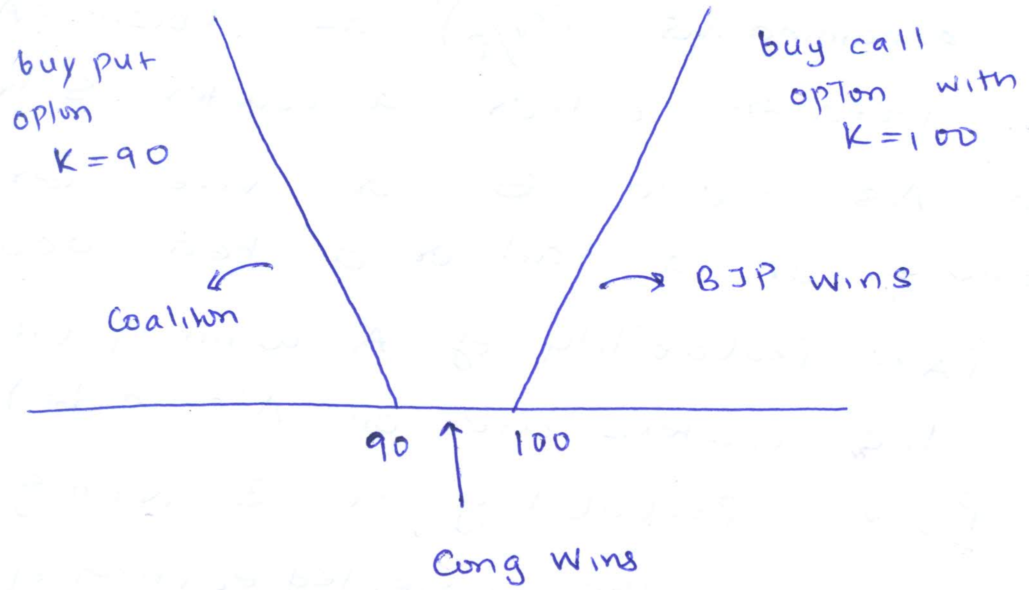
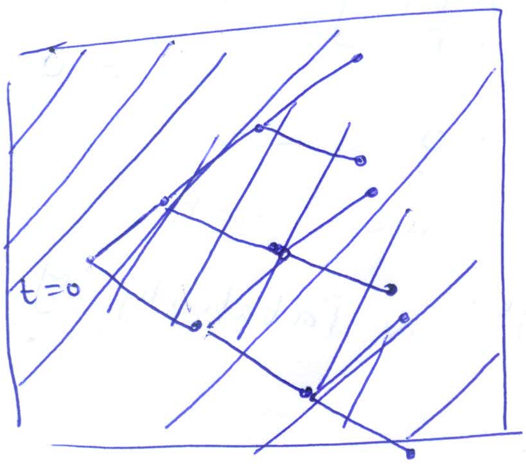
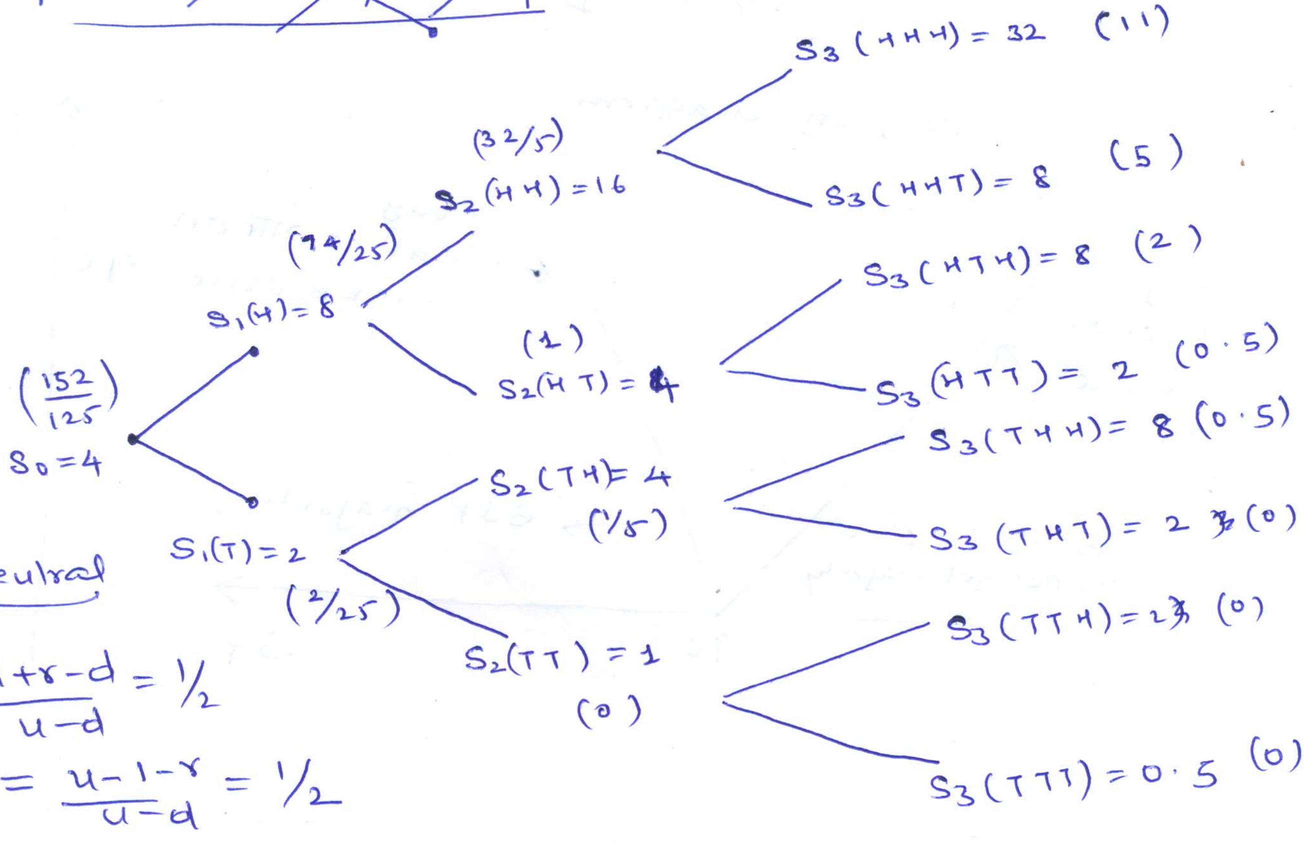
b)

5



4.





Risk neutral

$$q_u = \frac{1+r-d}{u-d} = \frac{1}{2}$$

$$q_d = \frac{u-1-r}{u-d} = \frac{1}{2}$$

Payoff

$$V_3 = \left(\frac{1}{4}\sum_{k=0}^{3} S_k - K\right)^+$$

$$\therefore\ V_3(HHH) = \left(\frac{4+8+16+32}{4} - 4\right)^+ = 11$$

$$V_3(HHT) = \left(\frac{4+8+16+8}{4} - 4\right)^+ = 5$$

$$V_3(HTH) = \left(\frac{4+8+4+8}{4} - 4\right)^+ = 2$$

(4) $$V_3(HTT) = \left(\frac{4+8+4+2}{4} - 4\right)^+ = 0.5$$

$$V_3(THH) = \left(\frac{4+2+4+8}{4} - 4\right)^+ = 0.5$$

$$V_3(THT) = \left(\frac{4+2+4+2}{4} - 4\right)^+ = 0$$

$$V_3(TTH) = \left(\frac{4+2+1+2}{4} - 4\right)^+ = 0$$

$$V_3(TTT) = \left(\frac{4+2+1+0.5}{4} - 4\right) = 0$$

$$V_n(\omega_1\omega \ldots \omega_n H) = \frac{1}{1+r}\left[q_u V_{n+1}(\omega_1 \ldots \omega_n H) + q_d V_{n+1}(\omega_1 \ldots \omega T)\right]$$

$$\therefore\ V_2(HH) = \frac{4}{5}\left(\frac{1}{2}\cdot 11 + \frac{1}{2}\cdot 5\right) = \frac{32}{5}$$

(4)

$$V_2(HT) = \frac{4}{5}\left(\frac{1}{2}\cdot 2 + \frac{1}{2}\,0.5\right) = 1$$

$$V_2(TH) = \frac{4}{5}\left(\frac{1}{2}\,0.5 + \frac{1}{2}\cdot 0\right) = \frac{1}{5}$$

$$V_2(TT) = 0$$

$$V_1(H) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{32}{5} + \frac{1}{2}\cdot 1\right) = \frac{74}{25}$$

$$V_1(T) = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{1}{5} + \frac{1}{2}\cdot 0\right) = \frac{2}{25}$$

$$V_0 = \frac{4}{5}\left(\frac{1}{2}\cdot\frac{74}{25} + \frac{1}{2}\cdot\frac{2}{25}\right)$$

2 $$\boxed{V_0 = \frac{152}{125} = 1.216}$$