Solution to the Black-Scholes PDE using the heat equation

From the previous lecture we have

Proposition 1: The heat equation

$$\begin{cases} \dfrac{\partial g(t,y)}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 g(t,y)}{\partial y^2} \\ g(0,y) = \psi(y) \end{cases}$$

has solution $g(t,y) = \displaystyle\int_{-\infty}^{\infty} \psi(y)\, e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}$.

Proposition 2: Assume that $f(t,x)$ solves the Black-Scholes PDE

$$\begin{cases} \dfrac{\partial f}{\partial t} + rx\dfrac{\partial f}{\partial x} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 f}{\partial x^2} - rf = 0 \\ f(T,x) = (x-K)^+ \end{cases}$$

with terminal condition $h(x) = (x-K)^+$ then the function

$$g(t,y) = e^{rt} f\left(T-t,\ e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}\right)$$

solves the heat eqn

$$\frac{\partial g(t,y)}{\partial t} = \frac{1}{2}\frac{\partial^2 g(t,y)}{\partial y^2}$$

with initial condition $g(0,y) = h(e^{\sigma y})$

Proof:

Let $s = T-t$ and $x = e^{\sigma y - \left(\frac{\sigma^2}{2} - r\right)t}$

Now
$$\frac{\partial g}{\partial t}(t,y) = \frac{\partial}{\partial t} e^{rt} f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$= f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right) \frac{\partial}{\partial t} e^{rt}$$

$$+ e^{rt} \frac{\partial}{\partial t} f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$= r e^{rt} f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$+ e^{rt}\left(\frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t}\right)$$

$$= r e^{rt} f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$- e^{rt} \frac{\partial f}{\partial s}\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$+ \left(\frac{\sigma^2}{2} - r\right) e^{rt} e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t} \frac{\partial f}{\partial x}\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$= r e^{rt} f(T-t, x) - e^{rt} \frac{\partial f}{\partial s}(T-t, x)$$

$$+ \left(\frac{\sigma^2}{2} - r\right) e^{rt} x \frac{\partial f}{\partial x}(T-t, x)$$

$$= \frac{1}{2} e^{rt} x^2 \sigma^2 \frac{\partial^2 f}{\partial x^2}(T-t, x) + \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T-t, x)$$

where in the last step we have used that f satisfies the Black-Scholes PDE

$$\frac{\partial g}{\partial y}(t,y) = \frac{\partial}{\partial y} e^{rt} f\left(T-t, e^{\sigma y+\left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$= e^{rt} \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial y}$$

$$= \sigma e^{rt} e^{\sigma y\left(\frac{\sigma^2}{2}-r\right)t} \frac{\partial f}{\partial x}\left(T-t, e^{\sigma y+\left(\frac{\sigma^2}{2}-r\right)t}\right)$$

and

$$\frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y) = \frac{\sigma^2}{2} e^{rt} e^{\sigma y+\frac{1}{2}\left(\frac{\sigma^2}{2}-r\right)t} \frac{\partial f}{\partial x}\left(T-t, e^{\sigma y+\left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$+ \frac{\sigma^2}{2} e^{rt} e^{2\sigma y+2\left(\frac{\sigma^2}{2}-r\right)t} \frac{\partial^2 f}{\partial x^2}\left(T-t, e^{\sigma y+\left(\frac{\sigma^2}{2}-r\right)t}\right)$$

$$= \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T-t, x) + \frac{\sigma^2}{2} e^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T-t, x)$$

lso,

$$g(0,y) = f(T, e^{\sigma y}) = h(e^{\sigma y})$$

inally we have the following ~~proposition~~ theorem.

Theorem (Black-Scholes)

Vhen $h(x) = (x-K)^+$ the solution to he Black-Scholes PDE is given by

$$(t,x) = x\,\Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$

wher $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$

$$= \frac{\log\left(\frac{x}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma(T-t)}, \quad d_- = d_+ - \sigma\sqrt{T-t}$$

We have $g(t,y) = e^{rt} f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}\right)$

Inversion leads to

$$f(t,x) = e^{-r(T-t)} g\left(T-t, \frac{-\left(\frac{\sigma^2}{2} - r\right)(T-t) + \log x}{\sigma}\right)$$

Since $g(t,y)$ solves the heat equalu

$$f(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \psi\left(\underbrace{\frac{-\left(\frac{\sigma^2}{2} - r\right)(T-t) + \log x}{\sigma} + z}_{y}\right) e^{\frac{-z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} h\left(e^{\sigma y}\right) e^{\frac{-z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} h\left(x e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)}\right) \cdot e^{\frac{-z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} \left(x e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)} - K\right)^+ e^{\frac{-z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$= e^{-r(T-t)} \int_{\frac{\left(-r + \frac{\sigma^2}{2}\right)(T-t + \log\left(\frac{K}{x}\right)}{\sigma}}^{\infty} \left(x e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)} - K\right) \frac{e^{\frac{-z^2}{2(T-t)}} dz}{\sqrt{2\pi(T-t)}}$$

$$= x\, e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)}\, e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$- K e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$= x \int_{-d_-\sqrt{T-t}}^{\infty} e^{\sigma z - \frac{\sigma^2}{2}(T-t) - \frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}$$

subst.. $y = \frac{z}{\sqrt{T-t}}$

$$- K e^{-r(T-t)} \int_{-d_-}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}$$

$$= x \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{1}{2(T-t)}(z - \sigma(T-t))^2} \frac{dz}{\sqrt{2\pi(T-t)}}$$

$$- K e^{-r(T-t)} \left(1 - \Phi(-d_-)\right)$$

$$= x \int_{-d_-\sqrt{T-t} - \sigma(T-t)}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}} - K e^{-r(T-t)} \Phi(d_-)$$

subs $z - \sigma(T-t) = y$

$1 - \Phi(-a) = \Phi(a)$

$$= x \int_{-d_- - \sigma\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{2\pi} - K e^{-r(T-t)} \Phi(d_-)$$

$$= x \int_{-d_+}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} - K e^{-r(T-t)} \Phi(d_-)$$

$$= x \left(1 - \Phi(-d_+)\right) - K e^{-r(T-t)} \Phi(d_-)$$

$$= x \, \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$