The Logistic Equation (Second Order of Nonlinearity)

Write an equation $\boxed{c\dfrac{dx}{dt} = Ax - Bx^2}$

$\Rightarrow$ $\boxed{\dfrac{dx}{dt} = ax - bx^2}$, $\boxed{a, b > 0}$ $\boxed{\begin{array}{c} a = A/c \\ b = B/c \end{array}}$

When $x \to 0$ $\boxed{\dfrac{dx}{dt} \simeq ax}$ (Rate $\propto$ State)

$\Rightarrow$ $\boxed{x \simeq x_0 e^{at}}$ $\Rightarrow$ Early growth is exponential

When x is large, $-bx^2$ inhibits and saturates growth (as in population growth)

Rescaling of variables : $\boxed{\dfrac{dx}{dt} = ax\left(1 - \dfrac{bx}{a}\right)}$

Define $\boxed{K = a/b}$ $\rightarrow$ (Carrying capacity)

$\Rightarrow$ $\boxed{\dfrac{dx}{dt} = ax\left(1 - \dfrac{x}{K}\right)}$

$\Rightarrow$ $\dfrac{d}{d(at)}\left(\dfrac{x}{K}\right) = \left(\dfrac{x}{K}\right)\left(1 - \dfrac{x}{K}\right)$

i) x is scaled by K.
ii) t is scaled by $1/a$.

Define $\boxed{X = x/K}$ and $\boxed{T = at = \dfrac{t}{1/a}}$

$\Rightarrow$ $\boxed{\dfrac{dX}{dT} = X(1-X)}$ The rescaled logistic equation

<u>Integral Solution</u>: (<u>Separation of variables</u>) $\boxed{\int \frac{dx}{x(1-x)} = \int dT}$

Now, by the <u>method of partial fractions</u>,

$$\boxed{\frac{1}{x(1-x)} \equiv \frac{A}{x} + \frac{B}{1-x}} \Rightarrow \boxed{1 \equiv A(1-x) + Bx}$$

$\boxed{\text{i.) } \underline{\text{When } x = 1, B = 1}, \quad \text{ii) } \underline{\text{When } x = 0, A = 1}}$

$$\Rightarrow \boxed{\int \frac{dx}{x(1-x)} = \int \frac{dx}{x} + \int \frac{dx}{1-x} = \int dT}$$

$$\Rightarrow \int \frac{dx}{x} - \int \frac{d(-x)}{1-x} = \int dT$$

| <u>C is an integration constant</u>

$$\Rightarrow \boxed{\ln x - \ln(1-x) = \ln e^T + \ln C}$$

$$\Rightarrow \frac{x}{1-x} = Ce^T \Rightarrow x = Ce^T - xCe^T$$

$$\Rightarrow \boxed{x\left(1 + Ce^T\right) = Ce^T}$$

$$\Rightarrow \boxed{x = \frac{Ce^T}{1+Ce^T} = \frac{1}{1+C^{-1}e^{-T}}}$$

When $\boxed{T = 0 \text{ (i.e. } t = 0)}$, $\boxed{x = x_0 \text{ (or } x = x_0)}$.

(The initial value must <u>NOT</u> be zero)

$$\Rightarrow \boxed{x_0 = \frac{1}{1+C^{-1}}} \boxed{\Rightarrow 1 + C^{-1} = \frac{1}{x_0}}$$

$$\Rightarrow \frac{1}{C} = \frac{1}{X_0} - 1 \Rightarrow \boxed{\frac{1}{C} = \frac{1 - X_0}{X_0}}$$

$$\Rightarrow \boxed{C = \frac{X_0}{1 - X_0} = \frac{x_0/k}{1 - x_0/k} = \frac{x_0}{k - x_0}}.$$

Returning to variables x and t we get,

$$\boxed{X = \frac{x}{k} = \frac{1}{1 + c^{-1} e^{-T}} = \frac{1}{1 + c^{-1} e^{-at}}} \Rightarrow \boxed{x = \frac{k}{1 + c^{-1} e^{-at}}}$$

i.) When $t \rightarrow \infty$, $x \rightarrow k$ (The limiting carrying capacity).
(OR $X \rightarrow 1$) (for ANY initial value)

Further, $\boxed{x = \frac{k e^{at}}{\left(\frac{k - x_0}{x_0}\right) + e^{at}} = \frac{x_0 k e^{at}}{(k - x_0) + x_0 e^{at}}}$

$$\Rightarrow \boxed{x = \frac{x_0 k e^{at}}{k + x_0 (e^{at} - 1)} = \frac{x_0 e^{at}}{1 + \frac{x_0}{k}(e^{at} - 1)}}.$$

ii.) When $\boxed{t \ll a^{-1}}$, $(t \rightarrow 0)$ in the early growth stage.

(linear order only)

$$e^{at} - 1 \simeq \cancel{1} + at \cancel{-1} \simeq at \rightarrow 0.$$

Hence, e^{at} in the numerator determines the dynamics, compared to $e^{at} - 1$ in the denominator.

$\Rightarrow \boxed{x \simeq x_0 e^{at}}$ in the early growth, but this also appears as if due to $\boxed{k \rightarrow \infty}$, i.e. an infinite carrying capacity $(b = 0)$.

Going back to ~~[scribble]~~ $\boxed{\dfrac{dx}{dT} = x(1-x) = f(x)}$

we see that starting from $\boxed{x = x_0}$ and tending towards $\boxed{x \rightarrow 1}$ (the upper limit), $\boxed{\dfrac{dx}{dT} > 0}$, i.e. there is always growth.

Now $\boxed{\dfrac{d^2x}{dT^2} = \dfrac{df}{dT} = \dfrac{df}{dx} \cdot \dfrac{dx}{dT}}$ $\rightarrow \dfrac{df}{dx} = 0 \Rightarrow \boxed{x = 1/2}$

$\boxed{f(x) = x(1-x) = x - x^2} \Rightarrow \boxed{\dfrac{df}{dx} = 1 - 2x}$.

i.) when $x < 1/2$, $\dfrac{df}{dx} > 0$,

ii.) when $x > 1/2$, $\dfrac{df}{dx} < 0$.

f(x) has a TURNING POINT at $x = 1/2$

Since $\boxed{\dfrac{dx}{dT} > 0}$ for any FINITE value of T,

we see that $\boxed{\text{for } x < 1/2, \ \dfrac{d^2x}{dT^2} > 0}$, i.e.

growth occurs at an increasing rate. On the other hand $\boxed{\text{for } x > 1/2, \ \dfrac{d^2x}{dT^2} < 0}$,

i.e. growth occurs at a decreasing rate. This means that before $x = 1/2$, the growth is exponential, and beyond $x = 1/2$, the growth starts slowing down towards the carrying capacity.

Hence, $\boxed{X = 1/2}$ is the point where the <u>NONLINEAR</u> <u>effect starts to be functional.</u>

The corresponding <u>time scale</u> T_{nl} (the <u>nonlinear time scale</u>) can be obtained by

$$\boxed{X = \frac{1}{2} = \frac{1}{1 + c^{-1} e^{-T_{nl}}}} \Rightarrow 2 = 1 + c^{-1} e^{-T_{nl}}$$

$$\Rightarrow c^{-1} e^{-T_{nl}} = 1 \Rightarrow \boxed{c\, e^{T_{nl}} = 1}.$$

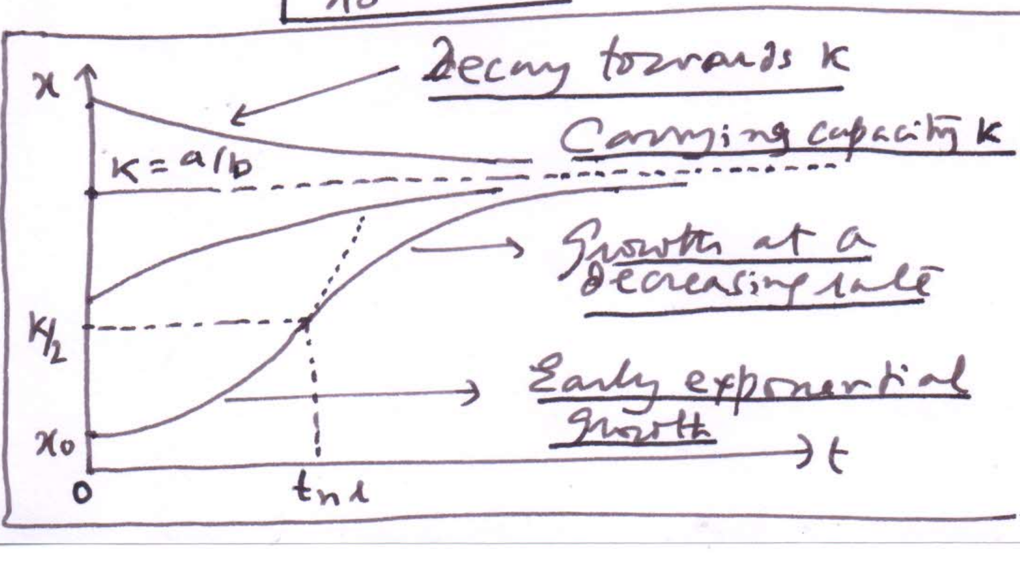
$$\Rightarrow \boxed{T_{nl} = \ln\left(\frac{1}{c}\right) = \ln\left(\frac{1 - X_0}{X_0}\right)}$$

Hence $\boxed{a\, t_{nl} = \ln\left(\frac{1 - x_0/K}{x_0/K}\right) = \ln\left(\frac{K - x_0}{x_0}\right)}$

$$\Rightarrow \boxed{t_{nl} = \frac{1}{a} \ln\left(\frac{K}{x_0} - 1\right)}$$ <u>Realistically $t_{nl} > 0$.</u>

This can <u>only happen</u> if $\boxed{\frac{K}{x_0} - 1 > 1}$

$\Rightarrow \boxed{\frac{K}{x_0} > 2} \Rightarrow \boxed{x_0 < K/2}$ <u>Needed for strong growth</u>



i.) For $\boxed{\frac{K}{2} < x_0 < K}$ there will be <u>ONLY</u> <u>growth at a decreasing rate.</u>

ii.) For $\boxed{x_0 > K}$, there will be <u>ONLY</u> <u>DECAY</u>

Higher Orders of Nonlinearity : Logistic-Type Equation

$$\boxed{\frac{dx}{dt} = ax - bx^{\alpha+1}} \quad \boxed{\alpha \geq 2}, \quad \boxed{\alpha \in \mathbb{Z}}$$

$$\Rightarrow \frac{dx}{dt} = ax\left(1 - \frac{bx^\alpha}{a}\right) = ax\left(1 - \frac{x^\alpha}{a/b}\right)$$

Now transform $\boxed{\xi = x^\alpha}$ $\Rightarrow$ $\boxed{d\xi = \frac{\alpha x^\alpha}{x} dx}$

$$\Rightarrow \boxed{\frac{d\xi}{dt} = \alpha \frac{\xi}{x} \frac{dx}{dt}} \Rightarrow \boxed{\frac{dx}{dt} = \frac{d\xi}{dt} \cdot \frac{x}{\alpha \xi}}$$

Hence $\frac{d\xi}{dt} \frac{\cancel{x}}{\alpha \xi} = a\cancel{x}\left(1 - \frac{\xi}{k}\right)$ $\boxed{k = a/b}$

$$\Rightarrow \boxed{\frac{d\xi}{dt} = a\alpha \xi \left(1 - \frac{\xi}{k}\right)}$$

Now rescale $\boxed{X = \xi/k}$ and $\boxed{T = a\alpha t}$,

$$\Rightarrow \boxed{\frac{d}{d(a\alpha t)}\left(\frac{\xi}{k}\right) = \frac{\xi}{k}\left(1 - \frac{\xi}{k}\right)}$$

$$\Rightarrow \boxed{\frac{dX}{dT} = X(1-X)}$$

is a familiar rescaled form. (logistic equation)

$$\Rightarrow \boxed{X = \frac{\xi}{k} = \frac{1}{1 + C^{-1}e^{-T}} = \frac{1}{1 + C^{-1}e^{-a\alpha t}}}$$

$$\boxed{C = \frac{X_0}{1 - X_0} = \frac{\xi_0/k}{1 - \xi_0/k} = \frac{\xi_0}{k - \xi_0} = \frac{x_0^\alpha}{k - x_0^\alpha}}$$

$$\Rightarrow \boxed{\xi = \frac{k e^{a\alpha t}}{C^{-1} + e^{a\alpha t}}} \quad \boxed{C^{-1} = \frac{k - x_0^\alpha}{x_0^\alpha} = \frac{k - \xi_0}{\xi_0}}$$

$$\Rightarrow \boxed{x^{\alpha} = \frac{k\, e^{a\alpha t}}{\left(\frac{k - x_0^{\alpha}}{x_0^{\alpha}}\right) + e^{a\alpha t}} = \frac{k\, x_0^{\alpha} e^{a\alpha t}}{(k - x_0^{\alpha}) + x_0^{\alpha} e^{a\alpha t}}}$$

$$\Rightarrow \boxed{x^{\alpha} = \frac{x_0^{\alpha} e^{a\alpha t}}{1 + \frac{x_0^{\alpha}}{k}\left(e^{a\alpha t} - 1\right)}}$$

$$\Rightarrow \boxed{x = \frac{x_0\, e^{at}}{\left[1 + \frac{x_0^{\alpha}}{k}\left(e^{a\alpha t} - 1\right)\right]^{1/\alpha}}}$$

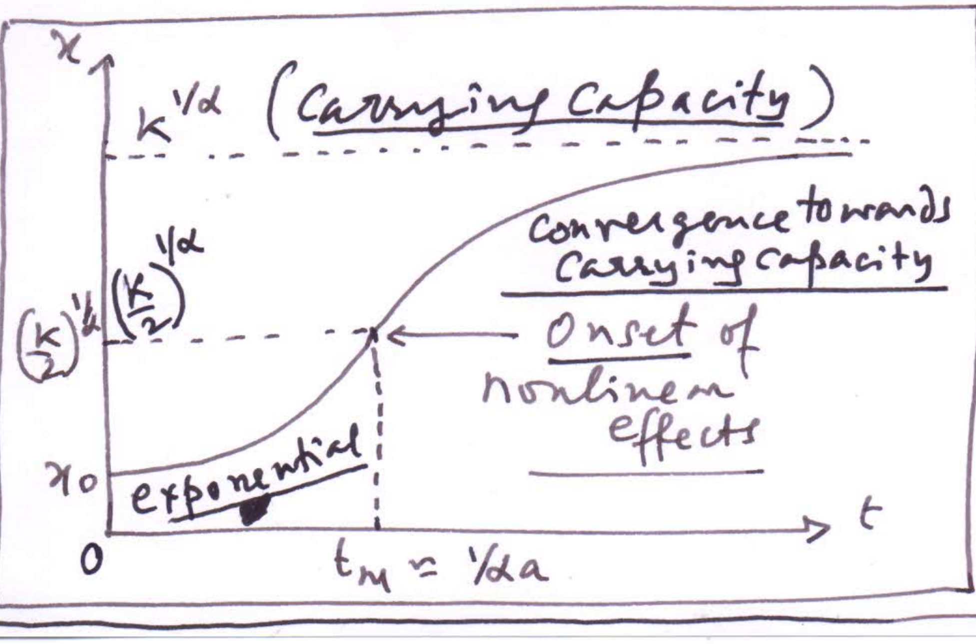
In $\boxed{t \to 0}$ Exponential Early Growth as $\boxed{x \simeq x_0 e^{at}}$

From $\boxed{x^{\alpha} = \frac{k}{1 + c^{-1} e^{-a\alpha t}}}$, we see that for $\boxed{t \to \infty}$, $\boxed{x \to k^{1/\alpha}}$, i.e. the carrying capacity has been reduced to $k^{1/\alpha}$.

Nonlinear Time Scale: $\boxed{T_{nl} = \ln\left(\frac{1}{c}\right)}$.

$$\Rightarrow \boxed{t_{nl} = \frac{1}{\alpha a} \ln\left(\frac{k - x_0^{\alpha}}{x_0^{\alpha}}\right) = \frac{1}{a\alpha} \ln\left(\frac{k}{x_0^{\alpha}} - 1\right)}$$

Realistically for $t_{nl} > 0$, $\boxed{\frac{k}{x_0^{\alpha}} - 1 > 1} \Rightarrow \boxed{x_0 < \left(\frac{k}{2}\right)^{1/\alpha}}$



For $\alpha \geq 2$, the Carrying Capacity is $k^{1/\alpha}$ in x. In ξ it is k, and in X it is 1.

i/. The Carrying Capacity is reduced.

ii/. The nonlinear time is also reduced.

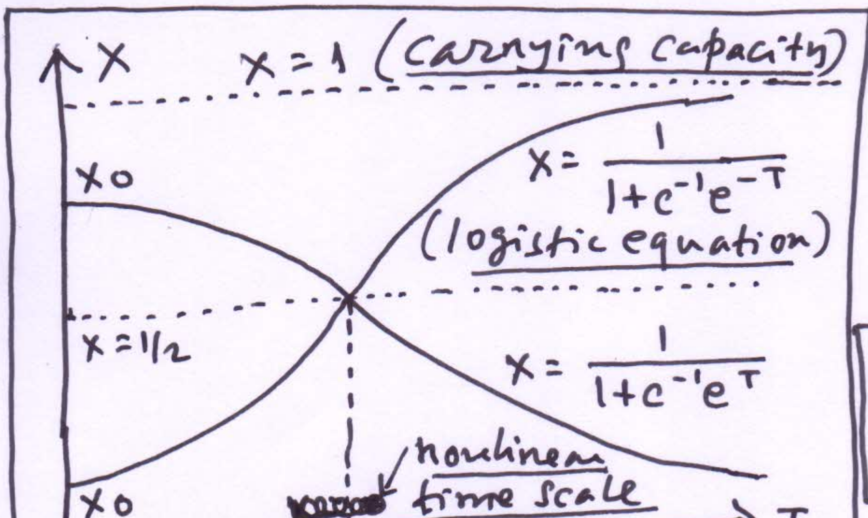
An Equation of the form $\boxed{\frac{dx}{dT} = -x(1-x)}$

$\Rightarrow \boxed{\frac{dx}{d(-T)} = x(1-x)}$ → This equation is the equivalent of $\boxed{T \longrightarrow -T}$ in the logistic equation, $\boxed{\frac{dx}{dT} = x(1-x)}$. Its solution is → $\boxed{x = \frac{1}{1+c^{-1}e^{-T}}}$

$\therefore$ Transforming the rescaled time $\boxed{T \longrightarrow -T}$.

gives $\boxed{x = \frac{1}{1+c^{-1}e^{T}}}$ $\Rightarrow$ when $\boxed{T \longrightarrow \infty}$, $\boxed{x \longrightarrow 0}$.

At $\boxed{T = 0}$, $\boxed{x = x_0}$ (the initial condition).



This solution can be compared with the Fermi-Dirac Distribution function.

$$\boxed{f(\epsilon) = \frac{1}{1 + e^{(\epsilon - \epsilon_F)/k_B T}}}$$

$T \rightarrow$ temperature (constant) (not to be confused with rescaled time)

$$\therefore \boxed{f(\epsilon) = \frac{1}{1 + e^{-\epsilon_F/k_B T}\, e^{\epsilon/k_B T}}}$$

$\epsilon_F \rightarrow$ Parameter (constant) (Fermi Energy)

Inverting the solution of the logistic equation gives the Fermi-Dirac type solution.

An Equation of the form $\boxed{\dfrac{dx}{dt} = a - bx^2}$

We write $\boxed{\dfrac{1}{a}\dfrac{dx}{dt} = 1 - \dfrac{x^2}{a/b}}$ and

define $\boxed{X = \dfrac{x}{\sqrt{a/b}}}$ $\Rightarrow$ $\dfrac{\sqrt{a/b}}{a}\dfrac{dX}{dt} = 1 - X^2$

Now also define $\boxed{T = \sqrt{ab}\, t}$, to get

$$\boxed{\frac{dX}{dT} = 1 - X^2} \Rightarrow \boxed{\int \frac{dX}{(1-X)(1+X)} = \int dT}$$

Using the method of partial fractions,

$$\boxed{\frac{1}{(1-X)(1+X)} = \frac{A}{1-X} + \frac{B}{1+X}} \Rightarrow 1 = A(1+X) + B(1-X)$$

i.) When $X = 1$. $\Rightarrow 1 = A.2 \Rightarrow \boxed{A = 1/2}$.

ii.) When $X = -1$. $\Rightarrow 1 = B.2 \Rightarrow \boxed{B = 1/2}$.

$$\Rightarrow \int \frac{dX}{(1-X)(1+X)} = \frac{1}{2}\int \frac{dX}{1-X} + \frac{1}{2}\int \frac{dX}{1+X} = \int dT$$

$$\Rightarrow \int \frac{dX}{1+X} - \int \frac{d(-X)}{1+(-X)} = 2\int dT$$

$$\Rightarrow \boxed{\ln(1+X) - \ln(1-X) = 2T + c}$$

The initial condition CAN be $x = 0$ at $t = 0$.

When $\boxed{t = 0,\ \text{i.e, } T = 0}$ and $\boxed{x = 0,\ \text{i.e. } X = 0}$,

$\boxed{c = 0}$ under this initial condition.

$$\Rightarrow \boxed{\ln\left(\frac{1+x}{1-x}\right) = 2T = \ln e^{2T}}.$$

$$\Rightarrow \boxed{\frac{1+x}{1-x} = e^{2T}} \Rightarrow 1+x = e^{2T} - x e^{2T}$$

$$\Rightarrow x(1+e^{2T}) = e^{2T} - 1 \Rightarrow \boxed{x = \frac{e^{2T}-1}{e^{2T}+1}}$$

$$\Rightarrow \boxed{x = \frac{e^{2T}-1}{e^{2T}+1} = \frac{(e^{T}-e^{-T})/2}{(e^{T}+e^{-T})/2}}$$

Now $\boxed{\sinh(T) = \frac{e^{T}-e^{-T}}{2}}$, $\boxed{\cosh(T) = \frac{e^{T}+e^{-T}}{2}}$

Hence $\boxed{x = \tanh(T)} \Rightarrow \boxed{x = \sqrt{\frac{a}{b}}\tanh(\sqrt{ab}\,t)}$.

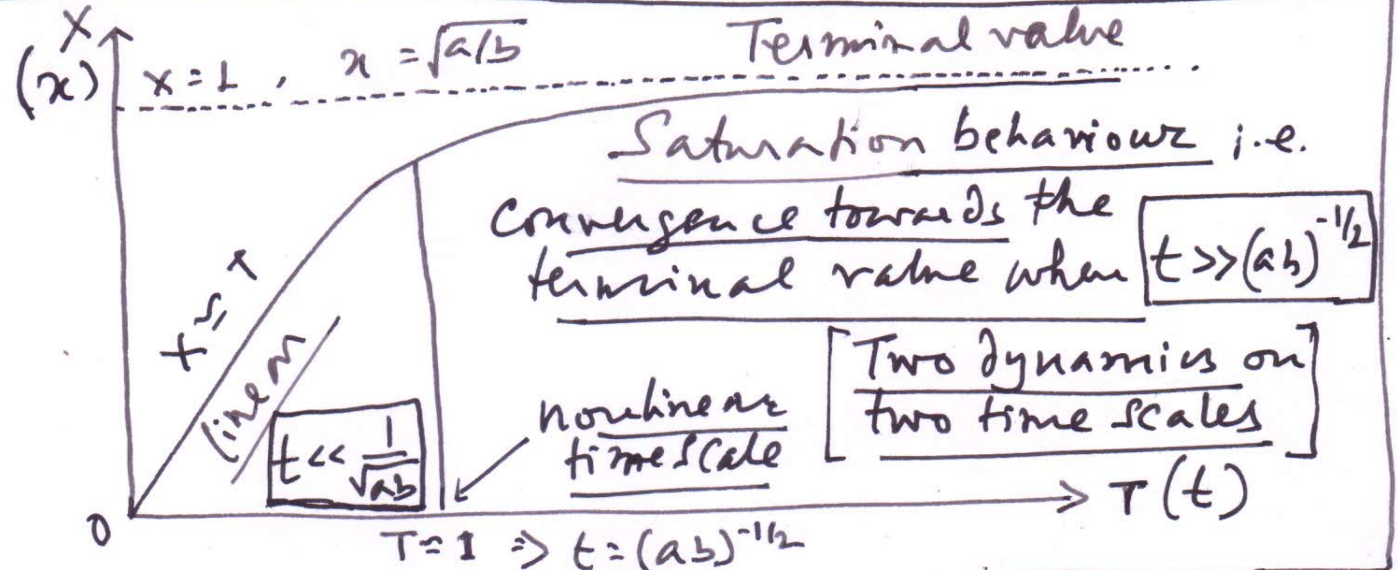
i.) When $\underline{T \ll 1}$, $\boxed{e^{T} \simeq 1+T}$ and $\boxed{e^{-T} \simeq 1-T}$.

$$\therefore\ x \simeq \frac{(1+T)-(1-T)}{(1+T)+(1-T)} \simeq \frac{2T}{2} \simeq T \quad \text{(linear)}$$

(on small time)

ii.) When $\underline{T \to \infty}$, $x = \frac{1-e^{-2T}}{1+e^{-2T}} \longrightarrow 1$ (on long time)

i.e $\boxed{x \longrightarrow \sqrt{a/b}}$ (approach towards this terminal value)



Modifications of the Logistic Equation

$$\boxed{\frac{dx}{dt} = ax - bx^2 + c} \text{ where } \boxed{a, b, c > 0}$$

(adding a constant to the right hand side)

$$\Rightarrow \frac{dx}{dt} = -(\sqrt{b}\,x)^2 + 2\sqrt{b}\,x\,\frac{a}{2\sqrt{b}} + c + \frac{a^2}{4b} - \frac{a^2}{4b}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\left[(\sqrt{b}\,x)^2 - 2(\sqrt{b}\,x)\left(\frac{a}{2\sqrt{b}}\right) + \frac{a^2}{4b}\right] + \left(\frac{a^2}{4b} + c\right)}$$

$$\Rightarrow \boxed{\frac{dx}{dt} = \left(\frac{a^2}{4b} + c\right) - \left(\sqrt{b}\,x - \frac{a}{2\sqrt{b}}\right)^2}$$

This term is a perfect square

$$\Rightarrow \boxed{\frac{dx}{dt} = \left(c + \frac{a^2}{4b}\right) - b\left(x - \frac{a}{2b}\right)^2}$$

define $\boxed{\alpha^2 = \frac{a^2}{4b} + c}$ and $\boxed{y = x - \frac{a}{2b}}$,

to get, $\boxed{\frac{dy}{dt} = \alpha^2 - by^2}$ $\boxed{\text{Since, } \frac{dx}{dt} = \frac{dy}{dt}}$

$$\Rightarrow \boxed{\frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/b}}$$, Now define $\boxed{X = \frac{y}{\alpha/\sqrt{b}}}$

$$\Rightarrow \boxed{\frac{1}{\alpha^2}\cdot\frac{\alpha}{\sqrt{b}}\frac{dX}{dt} = 1 - X^2} \Rightarrow \boxed{\frac{dX}{dT} = 1 - X^2},$$

when $\boxed{T = \alpha\sqrt{b}\,t}$. The solution of this equation, as earlier is

$$\boxed{\frac{1+X}{1-X} = Ae^{2T}} \Rightarrow \boxed{X = \frac{Ae^{2T} - 1}{Ae^{2T} + 1}}$$

A is an integration constant

Power Laws in Non-Autonomous Systems

Consider a non-autonomous equation $\boxed{\frac{dx}{dt} = \alpha \frac{x}{t}}$.

Integral Solution: $\boxed{\int \frac{dx}{x} = \alpha \int \frac{dt}{t}} \Rightarrow \boxed{\ln x = \alpha \ln t - \alpha \ln c}$

$\therefore \boxed{x = \left(\frac{t}{c}\right)^{\alpha}}$ when $\alpha < 0$, for $t \to \infty$, $x \to 0$ and for $t \to 0$, $x \to \infty$.

To prevent this divergence we translate $\boxed{t \to t + t_0}$.
Hence $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$. We write an equation as $\boxed{\frac{dx}{dt} = \alpha \frac{x}{t + t_0}}$, which we transform as $\boxed{\frac{dx}{dT} = \alpha \frac{x}{T}}$. The integral solution of this equation is $\boxed{x = \left(\frac{t + t_0}{c}\right)^{\alpha}}$, in which when $t \to 0$ (for $\alpha < 0$), the divergence on x is contained by $\boxed{x \to (t_0/c)^{\alpha}}$.

A Nonlinear Generalisation: Consider now $\boxed{(t + t_0)\frac{dx}{dt} = \alpha x - b x^{\mu + 1}}$, which is a nonlinear, non-autonomous equation.

Substitute $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$, and $\boxed{\xi = x^{\mu}}$.

$\therefore$ We get, $\boxed{T \frac{dx}{dT} = \alpha x \left(1 - \frac{x^{\mu}}{\alpha / b}\right)}$. $\boxed{k = \frac{\alpha}{b}}$

Now $\frac{d\xi}{dT} = \mu \frac{x^{\mu}}{x} \frac{dx}{dT} \Rightarrow \boxed{\frac{dx}{dT} = \frac{x}{\mu \xi} \frac{d\xi}{dT}}$

$$T\frac{dx}{dT} = \frac{T\cancel{x}}{\mu\,\xi_s}\frac{d\xi_s}{dT} = \alpha\cancel{x}\left(1-\frac{\xi_s}{k}\right)$$

$$\Rightarrow \boxed{\frac{d\xi_s}{dT} = \alpha\mu\,\frac{\xi_s}{T}\left(1-\frac{\xi_s}{k}\right)}$$

. Now rescale $\boxed{X = \xi_s/K}$.

$$\Rightarrow \frac{d(\xi_s/k)}{dT} = \alpha\mu\,\frac{(\xi_s/k)}{T}\left(1-\frac{\xi_s}{k}\right)$$

$$\Rightarrow \boxed{\frac{dX}{dT} = \alpha\mu\,\frac{X}{T}(1-X)}$$

. We integrate this equation by the method of separation of variables and partial fractions.

$$\Rightarrow \boxed{\int\frac{dX}{X(1-X)} = \alpha\mu\int\frac{dT}{T}}$$

. Now $\boxed{\frac{1}{X(1-X)} \equiv \frac{A}{X}+\frac{B}{1-X}}$

$$\Rightarrow \boxed{1 \equiv A(1-X)+BX}$$

Now when $X=0,\ A=1$ and when $X=1,\ B=1$.

$$\therefore \int\frac{dX}{X(1-X)} = \int\frac{dX}{X} - \int\frac{d(-X)}{1-X} = \alpha\mu\int\frac{dT}{T}$$

Integral constant $C>0$

$$\Rightarrow \boxed{\ln X - \ln(1-X) = \alpha\mu\ln T - \alpha\mu\ln c}$$

$$\Rightarrow \boxed{\ln\left(\frac{X}{1-X}\right) = \ln\left(\frac{T}{c}\right)^{\alpha\mu}} \Rightarrow \boxed{\frac{X}{1-X} = \left(\frac{T}{c}\right)^{\alpha\mu}}.$$

$$\Rightarrow X = \left(\frac{T}{c}\right)^{\alpha\mu} - X\left(\frac{T}{c}\right)^{\alpha\mu} \Rightarrow \boxed{X\left[1+\left(\frac{T}{c}\right)^{\alpha\mu}\right] = \left(\frac{T}{c}\right)^{\alpha\mu}}$$

$$\Rightarrow \boxed{X = \frac{(T/c)^{\alpha\mu}}{1+(T/c)^{\alpha\mu}}} \Rightarrow \boxed{X = \frac{1}{1+(T/c)^{-\alpha\mu}}}$$

$$\boxed{X = \frac{x^\mu}{k}} \Rightarrow \boxed{x^\mu = \frac{k\,(T/c)^{\alpha\mu}}{1+(T/c)^{\alpha\mu}}}$$

in which $\boxed{T = t+t_0}$.

Case I: $\boxed{\mu = 1}$ and $\boxed{\alpha > 0}$ and $\boxed{t_0 = 0}$.

$$\therefore \boxed{x = \frac{k\,(t/c)^{\alpha}}{1+(t/c)^{\alpha}}}$$

i) When $\boxed{t \to 0}$, $\boxed{1 + \left(\frac{t}{c}\right)^{\alpha} \simeq 1}$

$\Rightarrow \boxed{x \simeq k\,(t/c)^{\alpha}}$ for small values of t.

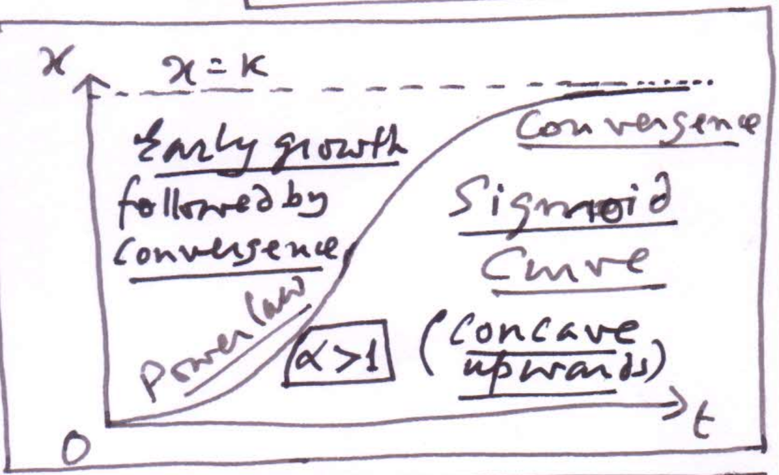
$\Rightarrow$ When $\boxed{t = 0, \; x = 0}$.

ii) When $\boxed{t \to \infty}$,

$$\boxed{x = \frac{k}{1+(t/c)^{-\alpha}}}.$$

$\Rightarrow \boxed{x \longrightarrow k}$ (limiting value)

(x starts at $x = 0$)



Case II: $\boxed{\mu = -1}$ and $\boxed{\alpha < 0}$ and $\boxed{t_0 \neq 0}$.

We write $\boxed{k^{-1} = \eta}$ in $\boxed{x^{\mu} = \dfrac{1}{k^{-1} + k^{-1}(T/c)^{-\alpha\mu}}}$

and $\boxed{\dfrac{1}{k} \cdot \dfrac{1}{c^{-\alpha\mu}} = \dfrac{1}{c_1^{-\alpha\mu}}}$, to get,

$$\boxed{x = \left[\frac{1}{\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}}\right]^{1/\mu}} \Rightarrow \boxed{x = \left[\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}\right]^{-1/\mu}}$$

when $\boxed{\mu = -1}$, $\boxed{x = \eta + \left(\frac{t+t_0}{c_1}\right)^{\alpha}}$

We know $\boxed{\alpha < 0}$. For the special case of $\boxed{\alpha = -2}$ (Zipf's law), (GEORGE KINGSLEY ZIPF)

$$\boxed{x = \eta + \left(\frac{c_1}{t+t_0}\right)^{2}}$$

When $\boxed{t \to \infty}$ $\boxed{x \longrightarrow \eta}$.