Flows on the Line - (First-Order Systems)

One-Dimensional System: $\rightarrow$ $\boxed{\dot{x} = f(x)}$

$\boxed{\dot{x} = dx/dt}$ $\Rightarrow$ $\boxed{x \equiv x(t)}$ Autonomous System

i.) $x(t)$ is a real valued function of t (time).
ii.) $f(x)$ is a smooth real-valued function of x.

Existence and Uniqueness Theorem:

For the initial-value problem, $\boxed{\dot{x} = f(x)}$,
$\boxed{x \equiv x(t)}$ and $\boxed{x(0) = x_0}$ (initial condition).

i/. $f(x)$ and $\boxed{f'(x) = df/dx}$ are finite-valued and smooth.
ii/. Continuous on an open interval R of the x-axis (in the one-dimensional system).
iii/. x_0 is a point in R (on the x-axis).

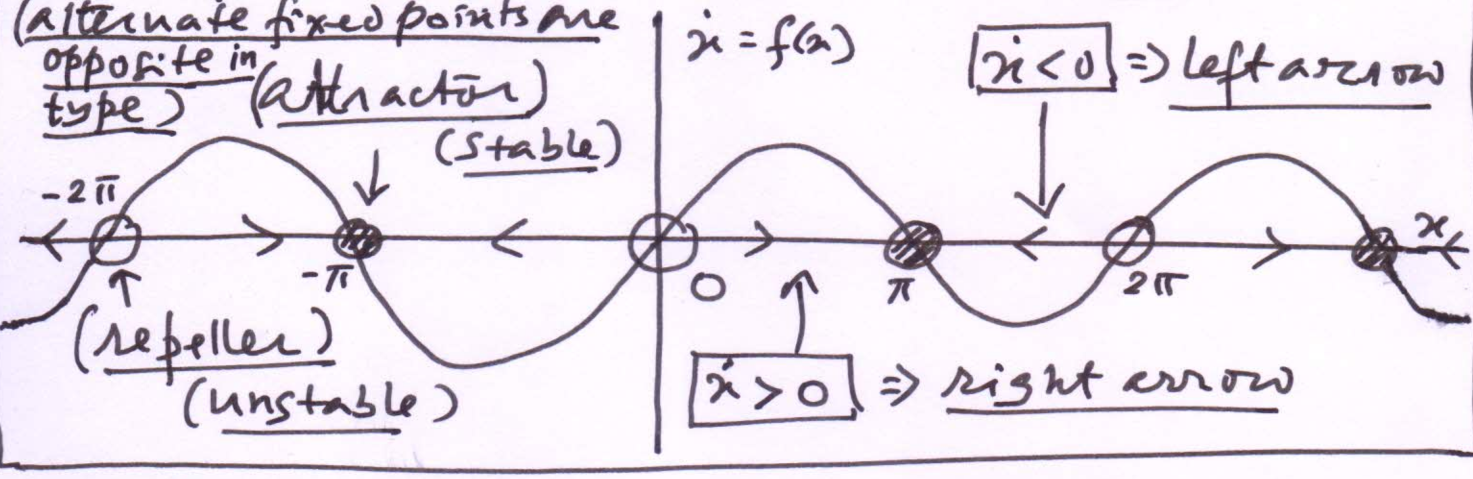
Fulfilling the foregoing conditions, the initial-value problem has a solution on some time interval $(-\tau, \tau)$ about $t = 0$. This solution exists and is unique.

Note: 1/. Effectively the solution is single-valued.
2/. The solution may not exist forever.

$\boxed{\dot{x} = f(x)}$

Phase Portraits: Plotting $\dot{x}$ versus x in,

Example: $\boxed{\dot{x} = f(x) = \sin x}$ Infinite number of roots for $\boxed{\dot{x} = 0}$



~~When~~ For $\boxed{x \equiv x(t)}$, an autonomous first-order system is given by $\boxed{\dot{x} = f(x)}$. The fixed point (or equilibrium point) of such a system is obtained when $\boxed{\dot{x} = f(x) = 0}$. Hence, $\boxed{f(x_c) = 0}$ gives the fixed point (equilibrium point) on the line $\boxed{\dot{x} = 0}$ at $\boxed{x = x_c}$ (in which x_c is the fixed point).

This is not a turning point for $\boxed{x \equiv x(t)}$.

$\boxed{f'(x) = df/dx}$

$$\boxed{\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d\dot{x}}{dt} = \ddot{x} = \frac{df}{dx}\frac{dx}{dt}}$$ (applying chain rule).

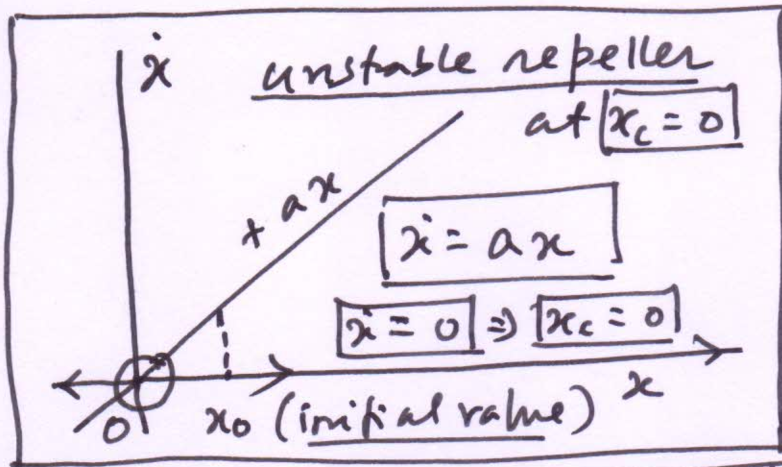
When $\boxed{\dot{x} = 0} \Rightarrow \boxed{\ddot{x} = \frac{df}{dx}\dot{x} = 0}$. Hence both $\boxed{\dot{x} = 0}$ and $\boxed{\ddot{x} = 0}$ at the same time.

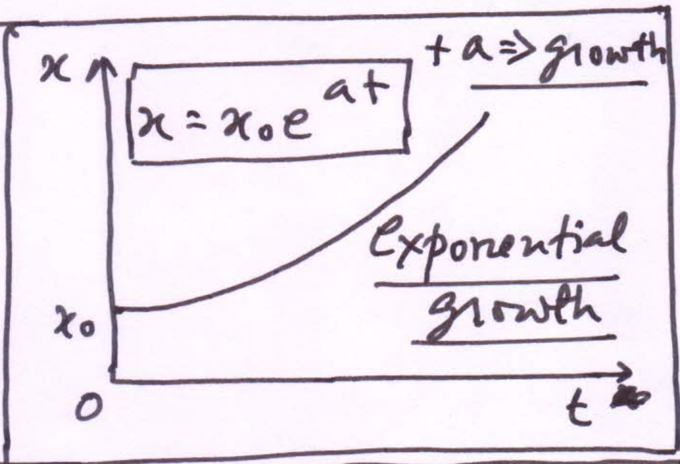
(Turning points usually have non-zero second derivatives.)

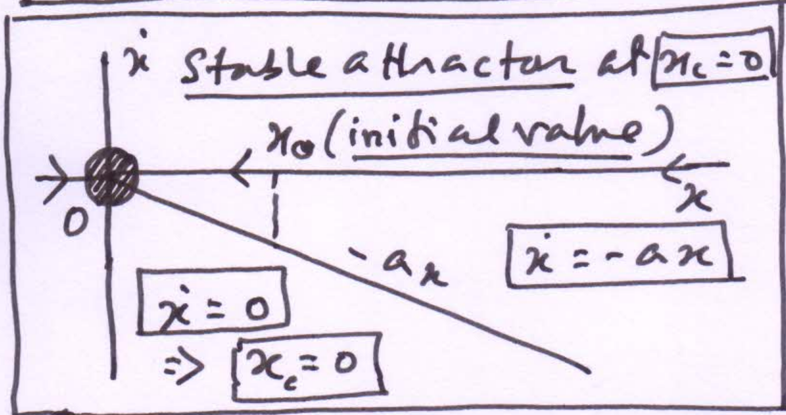
Phase Portraits of Polynomial Forms

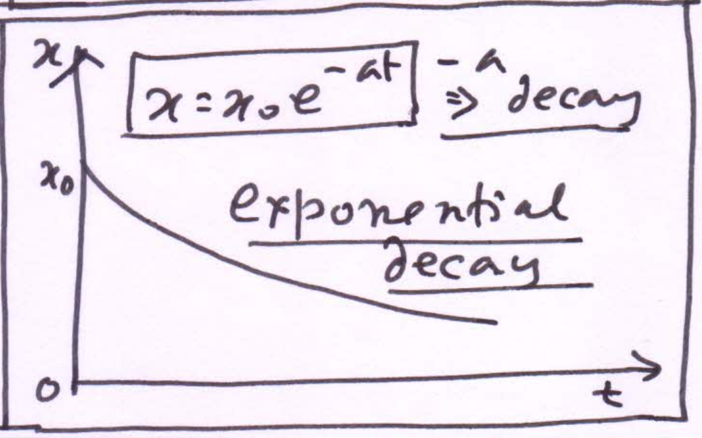
Example:
Consider $\boxed{\dot{x} = \frac{dx}{dt} = f(x) = \pm a x}$ $(a > 0)$.

Integral Solution: $\boxed{x(t) = x_0 e^{\pm a t}}$ x_0 is integration Constant.



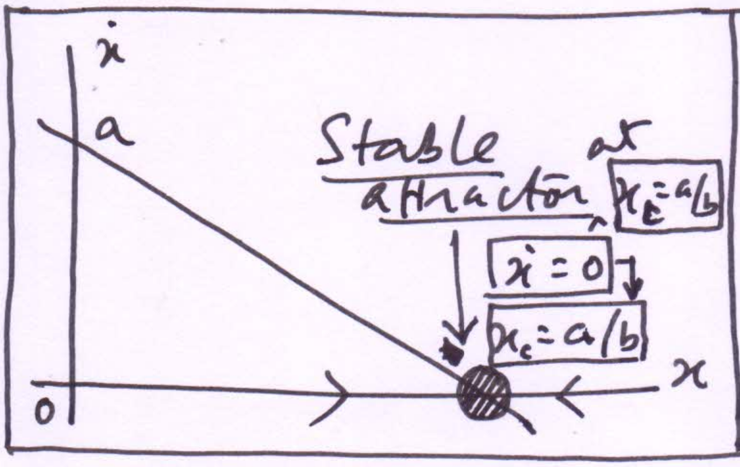


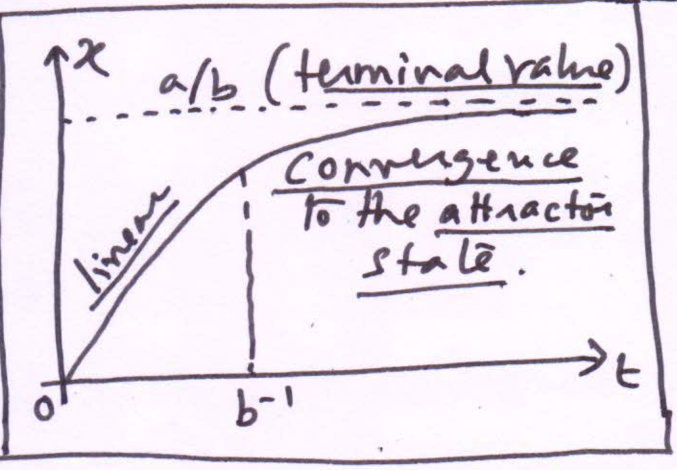




Example: $\boxed{\dot{x} = f(x) = a - b x}$ $(a, b > 0)$.

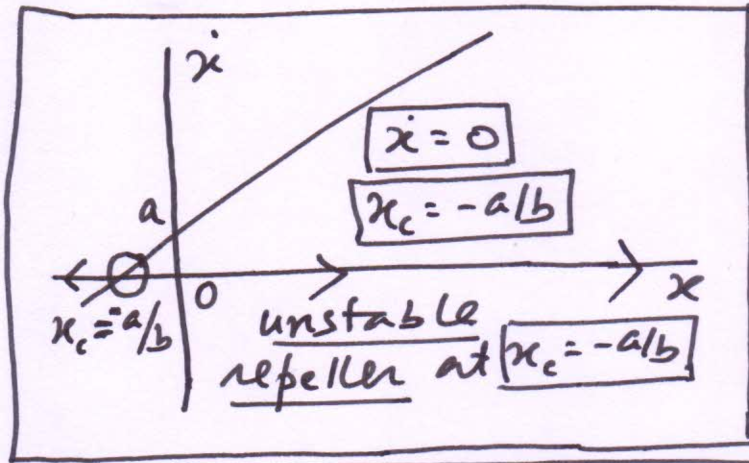
Integral Solution : $\boxed{x(t) = \frac{a}{b}\left(1 - e^{-bt}\right)}$ for $x = 0$ at $t = 0$.

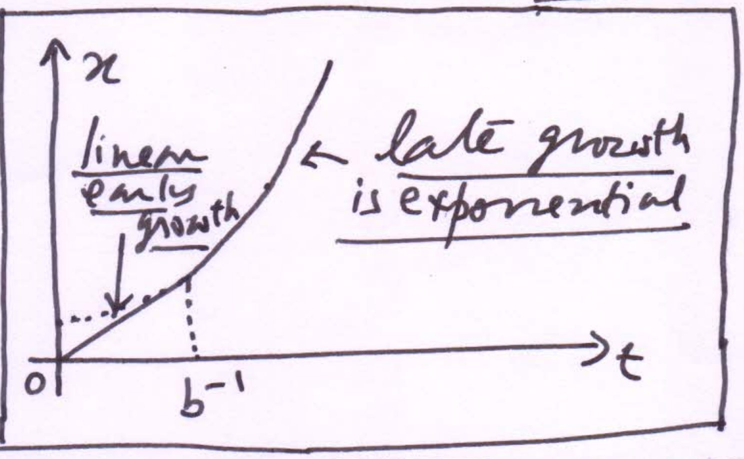




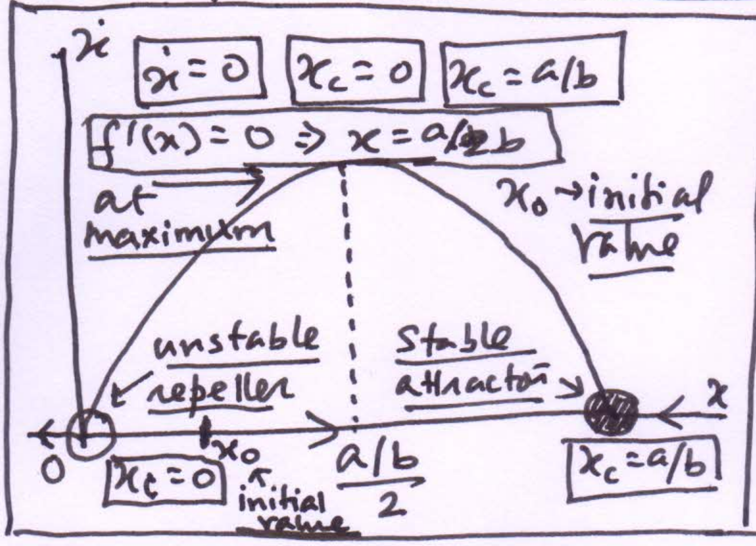
Example: $\boxed{\dot{x} = f(x) = a + bx}$ $(a, b > 0)$

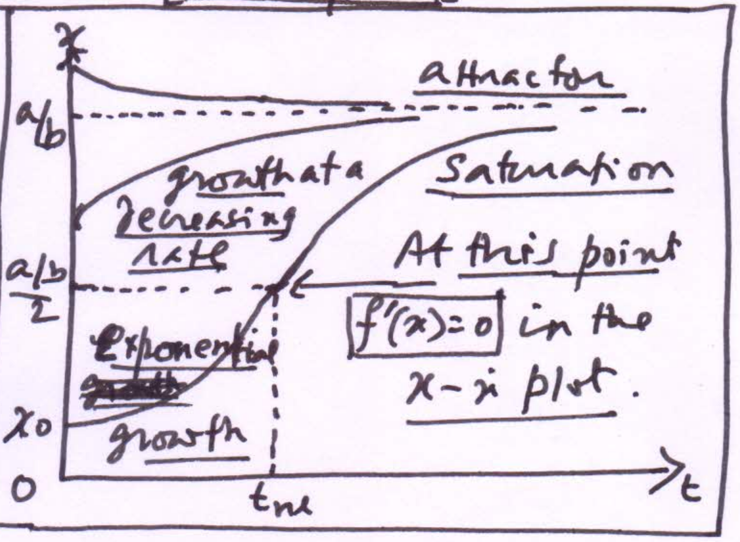
Integral solution: $\boxed{x(t) = \frac{a}{b}\left(e^{bt} - 1\right)}$ for $x = 0$ at $t = 0$





Example: $\boxed{\dot{x} = f(x) = ax - bx^2}$ $(a, b > 0)$ Nonlinear function [logistic equation]





Rescaling $\boxed{X = \frac{x}{a/b}}$ and $\boxed{T = at}$, we get a parameter-free equation $\boxed{\frac{dX}{dT} = X - X^2} \Rightarrow \boxed{\frac{dX}{dT} = F(X)}$

i) When $\boxed{\frac{dX}{dT} = F(X) = 0}$, $\boxed{X = 0}$ and $\boxed{X = 1}$ (fixed points)

ii) $\boxed{\frac{dF}{dX} = 1 - 2X} \Rightarrow \boxed{\frac{dF}{dX} = 0}$ is at $\boxed{X = \frac{1}{2}}$ (turning point for $F(X)$)

iii) $\boxed{\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT}}$ ← chain rule $\Rightarrow$ $\boxed{\frac{d^2X}{dT^2} = (1 - 2X)\frac{dX}{dT}}$

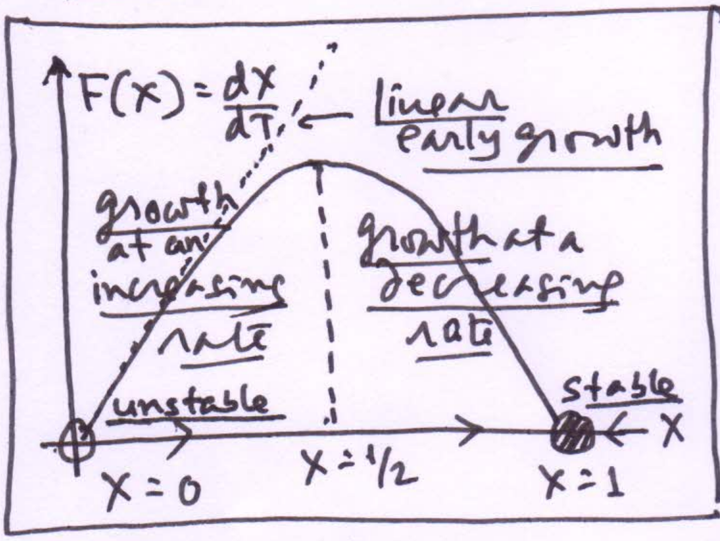
(P.T.O.)

(continued)

Now for $\boxed{0 < X \leq 1}$, $\boxed{\frac{dX}{dT} > 0}$ $\Rightarrow$ X always grows with T.

But F(X) has a turning point at $\boxed{X = 1/2}$.

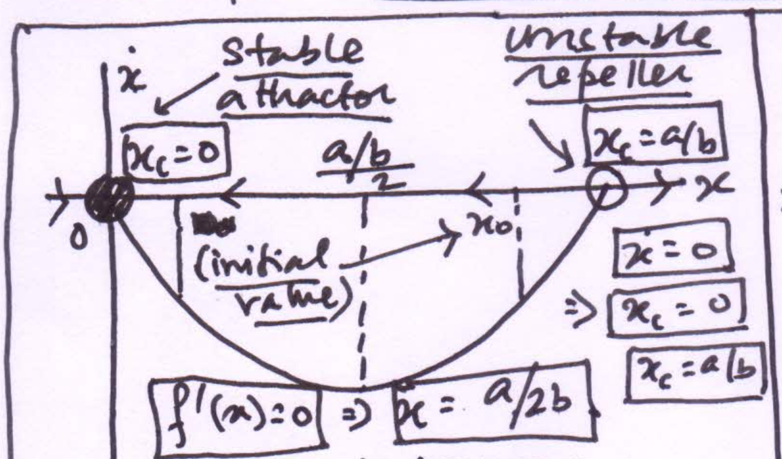
$\boxed{\frac{dF}{dX} = 1 - 2X}$ $\Rightarrow$ $\boxed{\frac{d^2F}{dX^2} = -2 < 0}$ $\therefore$ The turning point at $X = 1/2$ is a maximum.



iv.) When $\boxed{X < 1/2}$, $\boxed{\frac{dF}{dX} > 0}$ $\Rightarrow$ $\boxed{\frac{d^2X}{dT^2} > 0}$. Hence, for $\boxed{X < \frac{1}{2}}$, growth occurs at an increasing rate.

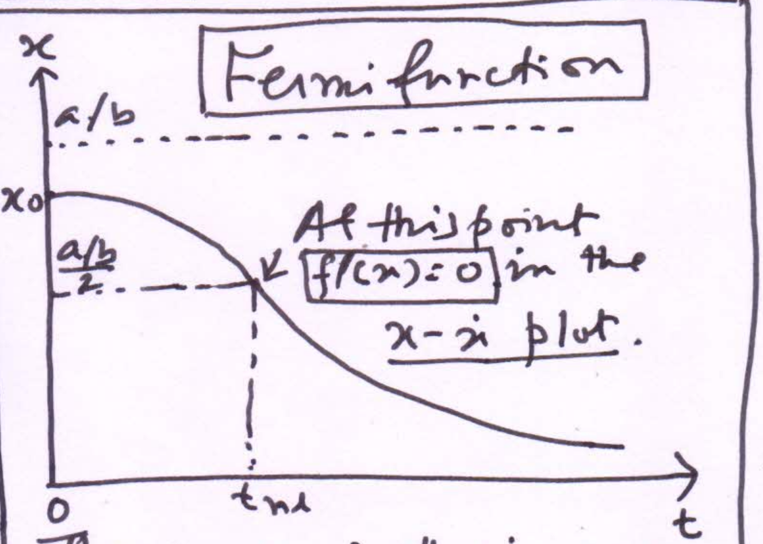
v.) When $\boxed{X > 1/2}$, $\boxed{\frac{dF}{dX} < 0}$ $\Rightarrow$ $\boxed{\frac{d^2X}{dT^2} < 0}$. Hence, for $\boxed{X > 1/2}$, growth occurs at a decreasing rate. The slowing down of growth due to the nonlinear (X^2) term starts at $\boxed{X = 1/2}$.

Example: $\boxed{\dot{x} = f(x) = -ax + bx^2}$ $(a, b > 0)$ (Inverse of the logistic equation)



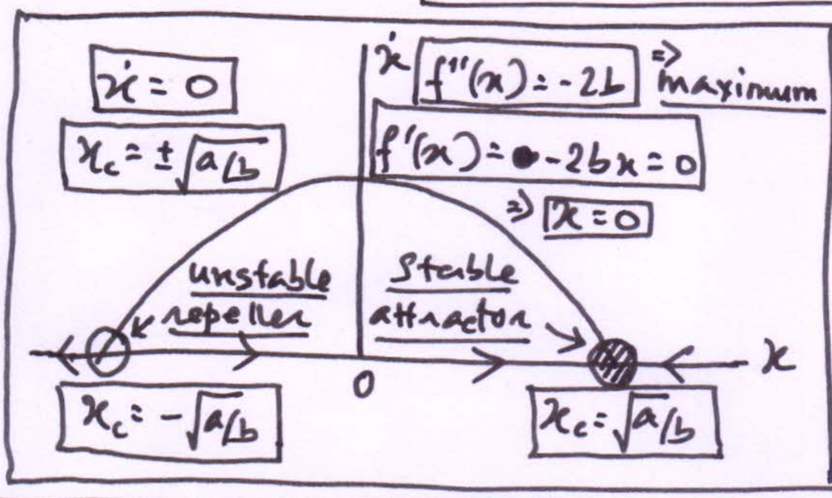
$\boxed{f'(x) = -a + 2bx}$

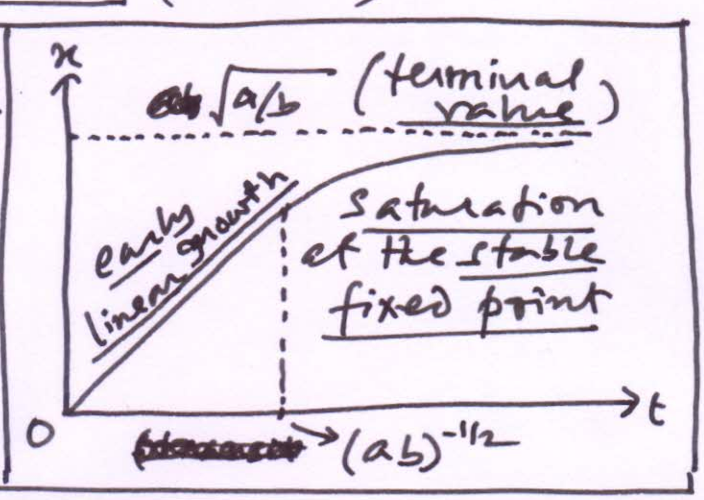
$\boxed{f''(x) = 2b > 0}$ $\Rightarrow$ minimum



This graph is the inversion of the logistic equation.

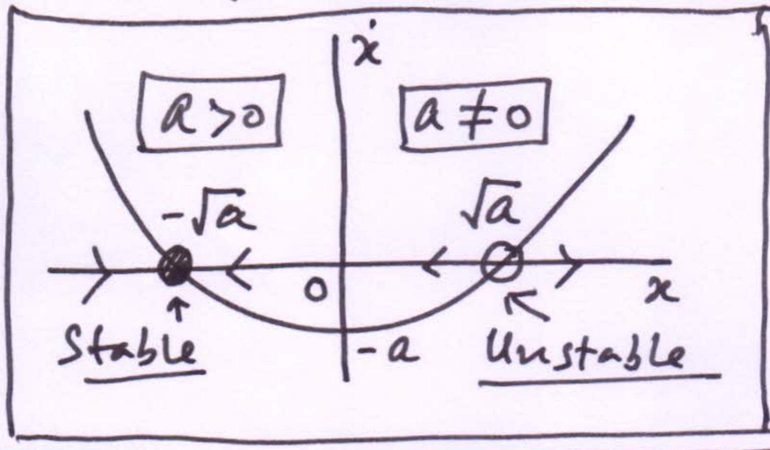
Example: $\boxed{\dot{x} = f(x) = a - bx^2}$ $(a, b > 0)$

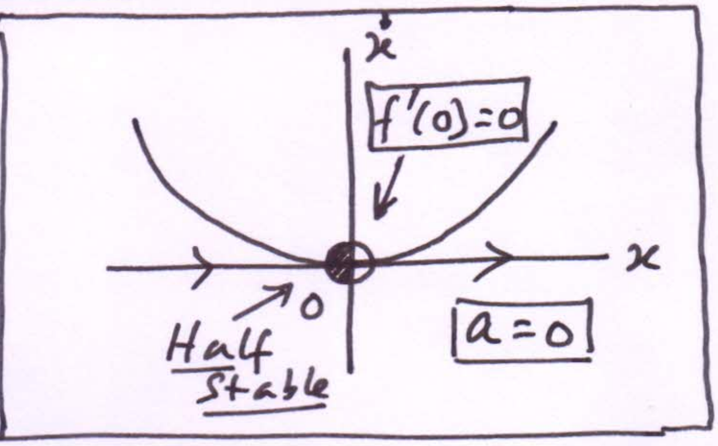




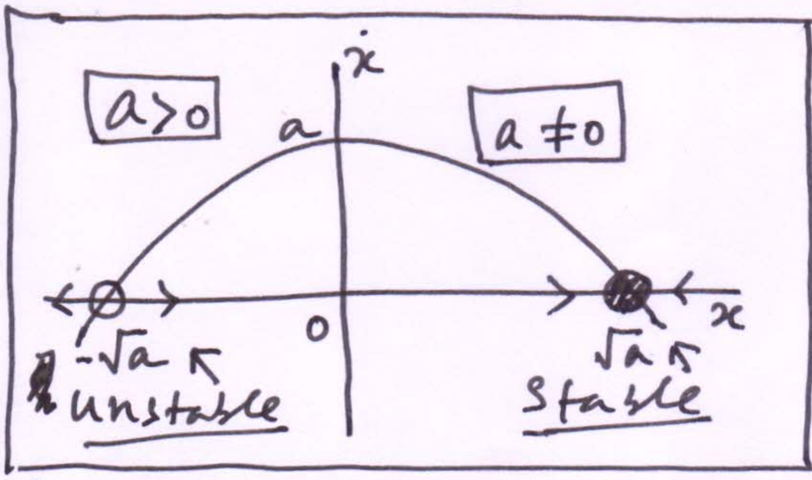
Critical Cases (Zero First Derivative)

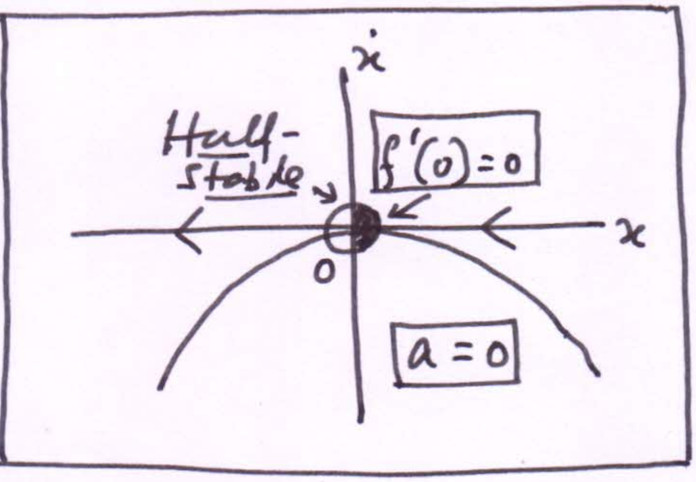
Example! $\boxed{\dot{x} = f(x) = x^2 - a}$ $\Rightarrow \boxed{f'(x) = 2x}$ $(a > 0)$





Example: $\boxed{\dot{x} = f(x) = a - x^2}$ $\Rightarrow \boxed{f'(x) = -2x}$ $(a > 0)$

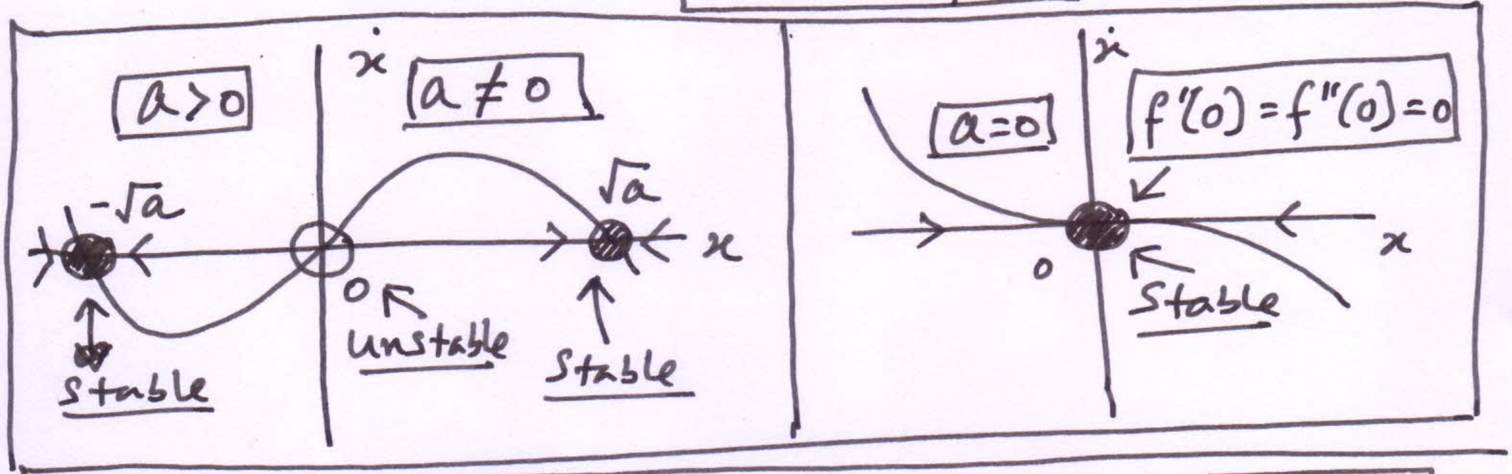




In the two cases above, the stable attractor and the unstable repeller exchange positions.

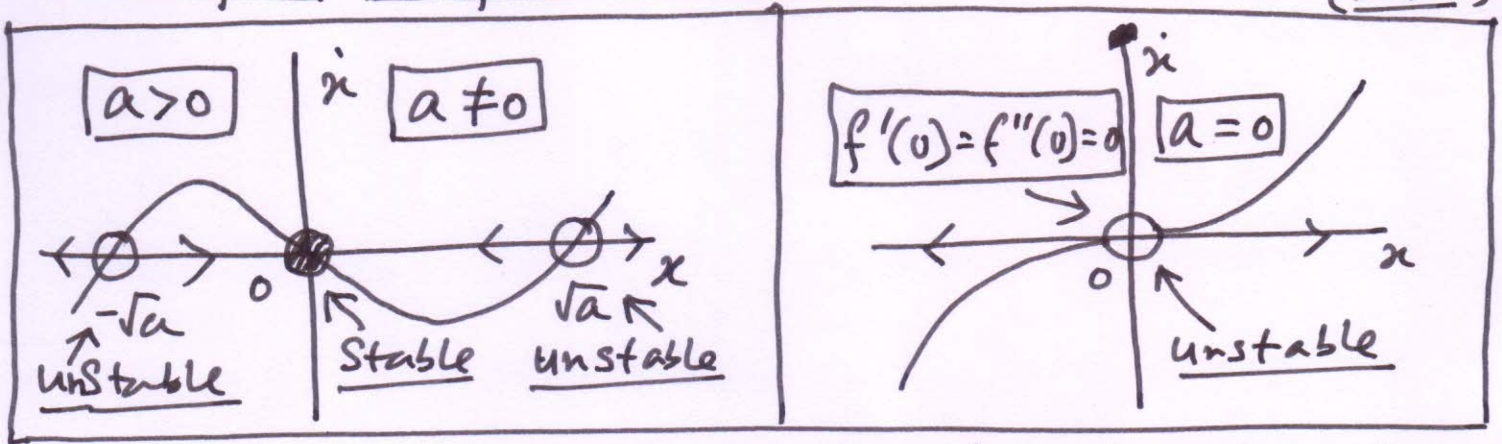
Example: $\boxed{\dot{x} = f(x) = ax - x^3}$ $\boxed{f'(x) = -3x^2}$ $(a>0)$

$\boxed{f''(x) = d^2f/dx^2}$ $\boxed{f''(x) = -6x}$



Example: $\boxed{\dot{x} = f(x) = -ax + x^3}$ $\boxed{f'(x) = 3x^2}$ $\boxed{f''(x) = 6x}$

$(a>0)$



In the two cases above, the fixed points exchange their stability properties.

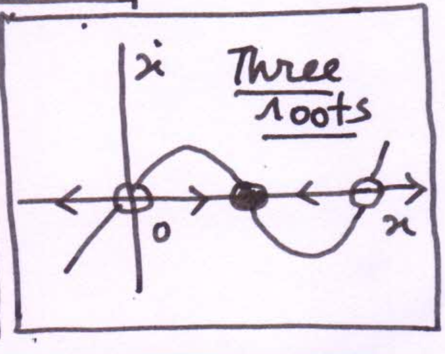
Plotting of a Polynomial Series

$$\boxed{\dot{x} = f(x) = a_0 x^0 \pm a_1 x \pm a_2 x^2 \pm a_3 x^3 \pm \cdots} \rightarrow \text{Polynomial Power Series.}$$

i.) Small powers dominate for small x.
ii.) Large powers dominate for large x.

Example: $\boxed{\dot{x} = f(x) = a_1 x - a_2 x^2 + a_3 x^3}$

i) Negative sign $\Rightarrow$ Downward direction
ii) Positive sign $\Rightarrow$ Upward direction



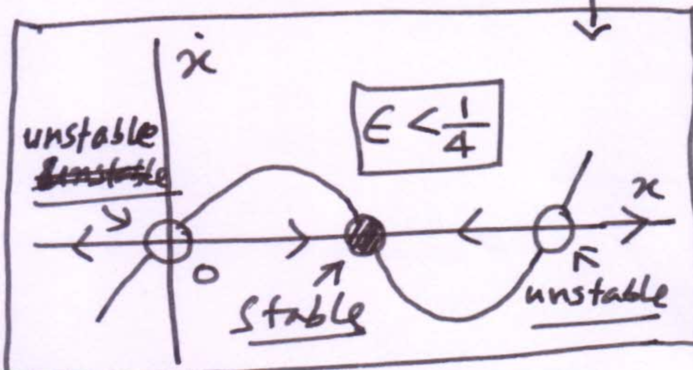
Plotting Cubic Polynomials

$\boxed{\dot{x} = f(x) = x - x^2 + \epsilon x^3}$ $(\epsilon > 0)$

Second-order inhibits, third-order enhances.

When $\boxed{\dot{x} = 0}$, $\boxed{x_c = 0}$ & $\boxed{1 - x_c + \epsilon x_c^2 = 0}$.

$\Rightarrow \boxed{x_c = \dfrac{1 \pm \sqrt{1-4\epsilon}}{2\epsilon}}$. If $\boxed{\epsilon < \frac{1}{4}}$, there are three real fixed points.



$\boxed{x_c = \dfrac{1}{2\epsilon} \pm \dfrac{1}{2\epsilon}(1-4\epsilon)^{1/2}}$

When $\epsilon \to 0$ $\boxed{(1-4\epsilon)^{1/2} \simeq 1 - 2\epsilon}$

[Binomial theorem] → (approximation). → $\boxed{(1+z^n) \simeq 1+nz}$

$\Rightarrow \boxed{x_c \simeq \dfrac{1}{2\epsilon} \pm \dfrac{1}{2\epsilon}(1-2\epsilon)} \Rightarrow \boxed{x_c \simeq \dfrac{1}{2\epsilon} \pm \dfrac{1}{2\epsilon} \mp 1}$

When $\boxed{\epsilon \to 0}$, choose lower sign $\Rightarrow \boxed{x_c \to 1}$.

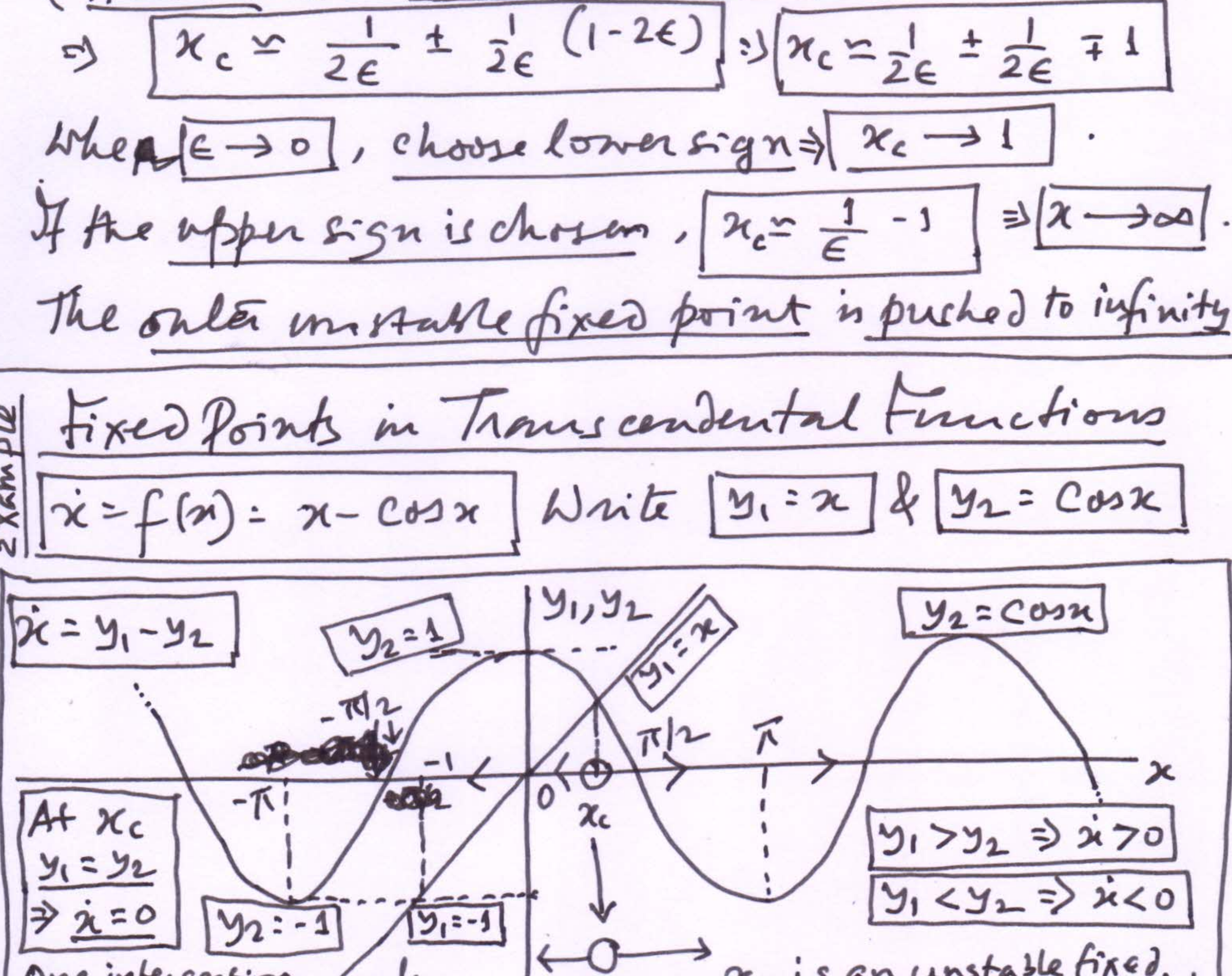
If the upper sign is chosen, $\boxed{x_c \simeq \dfrac{1}{\epsilon} - 1} \Rightarrow \boxed{x \to \infty}$.

The outer unstable fixed point is pushed to infinity.

Example

Fixed Points in Transcendental Functions

$\boxed{\dot{x} = f(x) = x - \cos x}$ Write $\boxed{y_1 = x}$ & $\boxed{y_2 = \cos x}$



Stability Analysis of Fixed Points

For a first-order autonomous dynamical system, $\boxed{\dot{x} = f(x)}$, the fixed point condition is $\boxed{\dot{x} = 0}$ at $\boxed{x = x_c}$ $\Rightarrow$ $\boxed{f(x_c) = 0}$ at $x = x_c$.

The fixed point coordinate x_c is perturbed by a small amount ϵ, i.e. $\epsilon \ll x_c$. Hence, we write $\boxed{x = x_c + \epsilon}$. Using this in $\boxed{\dot{x} = f(x)}$,

$$\boxed{\dot{x} = \dot{x}_c + \dot{\epsilon} = f(x) = f(x_c + \epsilon)}$$ Now $\boxed{\dot{x}_c = 0}$.

$$\therefore \boxed{\dot{\epsilon} = f(x_c + \epsilon) = f(x_c) + f'(x_c)\epsilon + \frac{f''(x_c)}{2!}\epsilon^2 + \cdots}$$

by a Taylor expansion. Now $\boxed{f(x_c) = 0}$, and truncating the Taylor expansion at the linear order (i.e. ignoring ϵ^2 and all higher order terms, due to the smallness of ϵ), we get $\boxed{\dot{\epsilon} \simeq f'(x_c)\epsilon}$, a linear differential equation in ϵ.

$$\Rightarrow \boxed{\frac{d\epsilon}{dt} = f'(x_c)\,\epsilon} \;\Rightarrow\; \boxed{\int \frac{d\epsilon}{\epsilon} = f'(x_c)\int dt} \quad \boxed{f'(x_c) \text{ is constant}}$$

$$\Rightarrow \boxed{\ln \epsilon = \ln A + f'(x_c)\,t} \Rightarrow \boxed{\epsilon = A e^{f'(x_c)t}}$$

Hence $\boxed{x \simeq x_c + A e^{f'(x_c)t}}$ (A is an integration constant).

The foregoing result is due to a LINEAR STABILITY ANALYSIS when $\boxed{f(x_c) = 0}$, $\boxed{f'(x_c) \neq 0}$

(P.T.O.)

(continued)

1/. For a stable fixed point, as $\boxed{t \to \infty}$, $\boxed{\epsilon \to 0}$.

$\therefore$ $\boxed{x \longrightarrow x_c}$, i.e. there is a convergence towards x_c.

This happens only when $\boxed{f'(x_c) < 0}$.

2/. For an unstable fixed point, as $\boxed{t \to \infty}$.

$\boxed{\epsilon \to \infty}$, i.e. a divergence away from x_c occurs.

This happens only when $\boxed{f'(x) > 0}$.

$\Rightarrow$ i.) If $\boxed{f'(x_c) < 0}$, the fixed point is stable.

ii.) If $\boxed{f'(x_c) > 0}$, the fixed point is unstable.

3/. Now $\boxed{x = x_c + A\, e^{f'(x_c)t}} \Rightarrow \boxed{\epsilon = A\, e^{f'(x_c)t}}$

$$\Rightarrow \boxed{t = \frac{1}{f'(x_c)} \ln\left(\frac{\epsilon}{A}\right) = \frac{1}{f'(x_c)} \ln\left(\frac{x - x_c}{A}\right)}$$

For a stable fixed point, $\boxed{f'(x_c) < 0}$ and $\boxed{x \to x_c}$ or $\boxed{\epsilon \to 0}$. Hence $\boxed{t \to \infty}$, for $\boxed{x \to x_c}$

The convergence to x_c takes infinitely long.

Hence, for a first-order system, there is No overshoot of the stable fixed point, and no oscillation about the fixed point is possible. Oscillations are only possible when $\boxed{f'(x_c)}$ is imaginary, but since $\boxed{\dot{x} = f(x)}$ is real, this is not allowed in a first-order system.

Critical Condition in the Stability Analysis

Given $\boxed{\dot{x} = f(x)}$, the fixed point is at $\boxed{\dot{x} = 0}$, i.e. $\boxed{f(x_c) = 0}$. Perturbing $\boxed{x = x_c + \epsilon}$, we get

$$\boxed{\dot{x} = \dot{\epsilon} = f(x_c) + f'(x_c)\,\epsilon + \frac{1}{2!} f''(x_c)\,\epsilon^2 + \cdots} \quad (\dot{x}_c = 0).$$

At the fixed point $\boxed{f(x_c) = 0}$. In addition if $\boxed{f'(x_c) = 0}$, then we have a critical condition (no longer linear).

$$\Rightarrow \boxed{\dot{\epsilon} \simeq \frac{f''(x_c)}{2!}\,\epsilon^2}$$ (ϵ^2 can no longer be neglected, but higher powers of ϵ^2 are neglected).

$$\Rightarrow \boxed{\frac{d\epsilon}{dt} \simeq \frac{f''(x_c)}{2}\,\epsilon^2} \Rightarrow \boxed{\int \frac{d\epsilon}{\epsilon^2} = \frac{f''(x_c)}{2} \int dt}.$$

$$\Rightarrow \boxed{\frac{\epsilon^{-1}}{-1} \simeq \frac{f''(x_c)}{2}\,(t - A)} \rightarrow$$ A is an integration constant [$f''(x_c)$ is constant]

$$\Rightarrow \boxed{\epsilon \simeq \frac{-2}{f''(x_c)}\,\frac{1}{t - A}} \Rightarrow \boxed{x \simeq x_c - \frac{2}{f''(x_c)}\,\frac{1}{t - A}}$$

When $\boxed{t \to \infty}$, $\boxed{x \to x_c}$ (slow power-law convergence).

1/. When $\boxed{f'(x_c) < 0}$, $\boxed{x \simeq x_c + e^{f'(x_c)t}}$. As $t \to \infty$, the convergence towards x_c is a rapid exponential convergence.

2/. When $\boxed{f'(x_c) = 0}$ (critical condition),
$\boxed{x \simeq x_c + \frac{B}{t - A}}$ $\left(B = \frac{-2}{f''(x_c)}\right)$, the convergence ~~towards~~ towards x_c as $t \to \infty$, is a slow power-law convergence.

Testing Stability of Fixed Points $\boxed{\dot{x} = f(x)}$

1/. $\boxed{\dot{x} = f(x) = a - x^2}$ $(\underline{a > 0})$ $\boxed{f'(x) = -2x}$ $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = \pm\sqrt{a}}$

$\therefore$ $\boxed{f'(\sqrt{a}) = -2\sqrt{a} < 0 \text{ (Stable)}}$, $\boxed{f'(-\sqrt{a}) = 2\sqrt{a} > 0 \text{ (unstable)}}$

2/. $\boxed{\dot{x} = f(x) = x^2 - a}$ $(\underline{a > 0})$ $\boxed{f'(x) = 2x}$ $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = \pm\sqrt{a}}$

$\therefore$ $\boxed{f'(\sqrt{a}) = 2\sqrt{a} > 0 \text{ (unstable)}}$, $\boxed{f'(-\sqrt{a}) = -2\sqrt{a} < 0 \text{ (Stable)}}$

3/. $\boxed{\dot{x} = f(x) = a - bx}$ $\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = a/b}$ $\boxed{f'(x) = -b}$

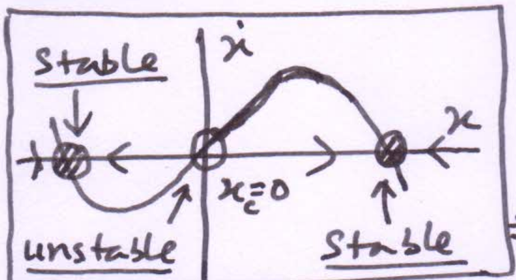
<u>If $b > 0$, x_c is stable</u>. <u>If $b < 0$, x_c is unstable</u>.

4/. $\boxed{\dot{x} = f(x) = ax - bx^2}$, $(a, b > 0)$. $\boxed{f'(x) = a - 2bx}$

$\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = 0,\ a/b}$. $\boxed{f'(0) = a > 0 \text{ (unstable)}}$ $\boxed{f'(a/b) = -a < 0}$ (Stable)

5/. $\boxed{\dot{x} = f(x) = x + x^2 - x^3}$ (<u>Second order enhances, third order inhibits</u>).

$\boxed{\dot{x} = 0}$ $\Rightarrow$ $\boxed{x_c = 0}$ and $\boxed{x_c^2 - x_c - 1 = 0}$ $\Rightarrow$ $\boxed{x_c = \frac{1 \pm \sqrt{5}}{2}}$ (3 roots)



$\boxed{f'(x) = 1 + 2x - 3x^2}$ $\boxed{f'(0) = 1 > 0 \text{ (unstable)}}$

$$f'\left(\frac{1 \pm \sqrt{5}}{2}\right) = 1 + (1 \pm \sqrt{5}) - \frac{3}{4}(1 \pm \sqrt{5})^2$$

$\Rightarrow$ $\boxed{f'\left(\frac{1 \pm \sqrt{5}}{2}\right) = -\frac{\sqrt{5}}{2}(\sqrt{5} \pm 1) < 0}$ (Both outer roots are Stable).

6/. $\boxed{\dot{x} = f(x) = e^x - \cos x}$ Write $\boxed{y_1 = e^x}$ and $\boxed{y_2 = \cos x}$

At $\boxed{x = 0}$, $\boxed{y_1 = 1 = y_2}$ $\Rightarrow$ $\boxed{\dot{x} = 0}$ Hence $\boxed{x_c = 0}$ is a <u>fixed point</u>.

$\boxed{f'(x) = e^x + \sin x}$ $\Rightarrow$ $\boxed{f'(0) = 1 > 0}$ $\boxed{x_c = 0}$ is an <u>unstable fixed point</u>.