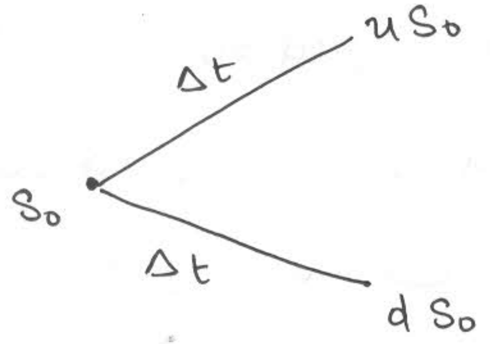
Pricing using Binomial model



The risk neutral formula is

$$\boxed{S_0 e^{\Delta t r} = q_u u S_0 + q_d d S_0} \quad — \ (1)$$

How to incorporate volatality in the binomial model?

We know that for a GBM the variance of the stock in time period Δt is $\sigma^2 \Delta t$.

Let X represent the percentage change in stock price in time period Δt, then

$\mathbb{E}(1+X) = e^{r\Delta t}$ (under the risk neutral measure)

Also $\mathbb{E}\left[(1+X)^2\right] = u^2 q_u + d^2 q_d$

Hence $Var(1+X) = u^2 q_u + d^2 q_d - e^{2r\Delta t}$

Since $Var(X) = Var(1+X)$ we see that the variation of the stock mapping GBM back to the binomial model is

$$\boxed{u^2 q_u + d^2 q_d - e^{2r\Delta t} = \sigma^2 \Delta t} \quad — \ (2)$$

Now multipling ① by $u+d$ we get

$$S_0 e^{\Delta r t}(u+d) = q_u u^2 S_0 + q_u u d S_0 + q_d d^2 S_0 + q_d u d S_0$$

Substituting $\sigma^2 \Delta t$ from ② we get

$$\boxed{e^{\Delta r t}(u+d) - ud - e^{2r\Delta t} = \sigma^2 \Delta t} \quad — ③$$

Further we assume

$$\boxed{u = \frac{1}{d}} \quad — ④$$

From ① we get

$$\boxed{q_u = \frac{e^{r\Delta t} - d}{u-d}, \quad q_d = 1 - q_u} \quad — ⑤$$

It turns out upto terms of first order in Δt

$$\boxed{u = e^{\sigma\sqrt{\Delta t}}} \quad — ⑥$$

$$\boxed{d = e^{-\sigma\sqrt{\Delta t}}} \quad — ⑦$$

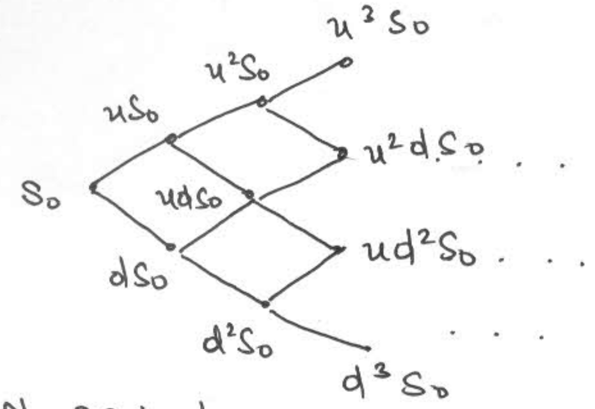
$u \approx 1 + \sigma\sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2} + (\text{ignore})$

$d = 1 - \sigma\sqrt{\Delta t} + \frac{\sigma^2 \Delta t}{2}$

satisfy eqn ③.

⑤, ⑥, ⑦ can be mapped to the binomial model for pricing call & put options.

Thus, we can price European and American call & put options



For N-period

Let $f_{N,j} = \max\left(S_0 u^j d^{N-j} - K, 0 \right)$

$j = 0, 1, 2, \ldots N$

and $f_{i,j} = e^{-r\Delta t}\left[q_u f_{i+1,j+1} + q_d f_{i+1,j} \right]$

and work backword to compute the price of option $f_{0,0}$

Example:

Consider a 5-month ~~Am~~ European call option on a non-dividend paying stock. The current stock price is 50, Strike price $K = 50$, $r = 10\%$ p.a and $\sigma = 40\%$ p.a.

From the above discussion.

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.4\sqrt{0.0833}} = 1.2224$

$d = e^{-\sigma\sqrt{\Delta t}} = 0.8903$

$q_u = \dfrac{e^{r\Delta t} - d}{u - d} = 0.5073$, $q_d = 1 - q_u = 0.4927$

We take $\Delta t = 1$ mth $= \frac{1}{12}$ year $= 0.0833$ y

Construct a 5-period binomial model and compute the option price at $t=0$.