Computational Finance - In Sem 2 Solns:

1.

$$\mathbb{E}^{Q}(S_3|S_1) = q_u^2 u^2 S_1 + 2 q_u q_d u d S_1 + q_d^2 d^2 S_1$$

$$= \frac{1}{4} \cdot 4 S_1 + \frac{1}{4} \cdot 2 \cdot \frac{1}{2} S_1 + \frac{1}{4} \cdot \frac{1}{4} S_1$$

(5)

$$= \frac{25 S_1}{16}$$

$$\mathbb{E}^{Q}(S_3|S_2) = q_u u S_2 + q_d d S_2$$

$$= \frac{1}{2} \cdot 2 \cdot S_2 + \frac{1}{2} \cdot \frac{1}{2} S_2$$

(2)

$$= \frac{5}{4} S_2$$

$$\mathbb{E}^{Q}(\mathbb{E}^{Q}(S_3|S_2)|S_1)$$

$$= \mathbb{E}^{Q}\left(\frac{5}{4} S_2 | S_1\right)$$

$$= \frac{5}{4} \mathbb{E}^{Q}(S_2|S_1)$$

(3)

$$= \frac{5}{4}(q_u u S_1 + q_d d S_1)$$

$$= \frac{5}{4}\left(S_1 + \frac{S_1}{4}\right) = \frac{25}{16} S_1$$

$$\therefore \mathbb{E}^{Q}(S_3|S_1) = \mathbb{E}^{Q}(\mathbb{E}^{Q}(S_3|S_2)|S_1)$$

2)

$$F = S^n$$

$$dF = \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S} dW$$

3 $$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial S} = n S^{n-1}, \quad \frac{\partial^2 F}{\partial S^2} = n(n-1) S^{n-2}$$

$$dF = \left(\mu S . n S^{n-1} + \frac{\sigma^2}{2} S^2 n(n-1) S^{n-2} \right) dt + \sigma S n S^{n-1} dW$$

5 $$= \left(\mu n S^n + \frac{\sigma^2}{2} n(n-1) S^n \right) dt + \sigma n S^n dt$$

$$= \left(\mu n + \frac{\sigma^2}{2} n(n-1) \right) S^n dt + \sigma n S^n dW$$

$$dF = \mu' F dt + \sigma' F dW$$

where $\boxed{\mu' = \mu n + \frac{\sigma^2}{2} n(n-1), \quad \sigma' = \sigma n}$

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a) $\mathbb{E}(X|Y=y) = \sum_x x \, P(X=x|Y=y)$

$$\mathbb{E}(\mathbb{E}(X|Y)) = \sum_y \sum_x x \, P(X=x|Y=y) \, P(Y=y)$$

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$$= \sum_y \sum_x x \, \frac{P(X=x, Y=y)}{P(Y=y)} \, P(Y=y)$$

$$= \sum_x x \sum_y P(X=x, Y=y)$$

$$= \sum_x x \, P(X=x) = \mathbb{E}(X)$$

b) $\mathbb{E}(Y_{n+1}|X_1, \ldots X_n) = Y_n$

Taking expectations on both sides.

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$$\mathbb{E}(\mathbb{E}(Y_{n+1}|X_1, \ldots X_n)) = \mathbb{E}(Y_n)$$

From a) $LHS = \mathbb{E}(Y_{n+1})$

It follows that

$$\mathbb{E}(Y_{n+1}) = \mathbb{E}(Y_n) = \ldots = \mathbb{E}(Y_0)$$

c) $Y_n = \left(\frac{q}{p}\right)^{S_n}$, $S_n = \sum_{i=1}^{n} X_i$

$$\mathbb{E}(Y_{n+1}|X_1 \ldots X_n)$$

$$= \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_{n+1}} \middle| X_1 \ldots X_n\right)$$

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$$= \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_n + X_{n+1}} \middle| X_1 \ldots X_n\right)$$

$$= \mathbb{E}\left(\left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p}\right)^{X_{n+1}} \middle| X_1 \ldots X_n\right)$$

$$= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{X_{n+1}} \middle| X_1 \dots X_n\right]$$

Since $\left(\frac{q}{p}\right)^{S_n}$ is a function of $X_1, X_2 \dots X_n$

$$= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left(\frac{q}{p}\right)^{X_{n+1}}$$

Since $\left(\frac{q}{p}\right)^{X_{n+1}}$ is independent of $X_1, X_2, \dots, X_n$

$$= \left(\frac{q}{p}\right)^{S_n} \left[p\left(\frac{q}{p}\right)^{1} + q.\left(\frac{q}{p}\right)^{-1}\right]$$

$$= \left(\frac{q}{p}\right)^{S_n} (q + p) = \left(\frac{q}{p}\right)^{S_n} = Y_n$$

$\therefore \mathbb{E}(Y_{n+1}^{*} \mid X_1 \dots X_n) = Y_n \quad \therefore Y_n$ is a Martingale

d) Let T be the random time at which the game ends

$$\mathbb{E}(Y_T) = P_K \left(\frac{q}{p}\right)^{0} + (1-P_K)\left(\frac{q}{p}\right)^{N}$$

$$= P_K + (1-P_K)\left(\frac{q}{p}\right)^{N}$$

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$$\mathbb{E}(Y_0) = \left(\frac{q}{p}\right)^{S_0} = \left(\frac{q}{p}\right)^{K}$$

Since Y_n is a Martingale. $\mathbb{E}(Y_0) = \mathbb{E}(Y_T)$

$$\therefore \left(\frac{q}{p}\right)^{K} = P_K + (1-P_K)\left(\frac{q}{p}\right)^{N}$$

which gives

$$\boxed{P_K = \frac{\left(\frac{q}{p}\right)^{N} - \left(\frac{q}{p}\right)^{K}}{\left(\frac{q}{p}\right)^{N} - 1}}$$

4) Binary Call

$$\text{Payoff} = \cancel{e^{-rT}}\, I_{(S(T)-K)^+}$$

$$V_B^{call}(x,t) = \mathbb{E}^{Q}\left[e^{-r(T-t)} I_{(S(T)-K)^+} \,\middle|\, \mathcal{F}_t\right]$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} I_{(S(T)-K)^+} \, dW(t)$$

$$S(T) > K \Rightarrow x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} Z} > K \quad (\tau = T-t)$$

$$\text{i.e.} \quad \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau} Z > \log\left(\frac{K}{x}\right)$$

$K >$

$$\therefore \quad Z > \frac{-\log\left(\frac{x}{K}\right) - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

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$Z > -d_2$ where $K > \cancel{x}e$

$$d_2 = \frac{\log\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$\therefore \quad V_B^{call}(x,t) = e^{-r(T-t)} \int_{-d_2}^{\infty} \frac{e^{-z^2}}{\sqrt{2\pi}} \, dz$$

$$= e^{-r\tau} N(d_2)$$

$$\therefore \quad V_B(x,0) = e^{-rT} N(d_2)$$

Binary put payoff = $\mathbb{I}(K - S(T))^{+}$

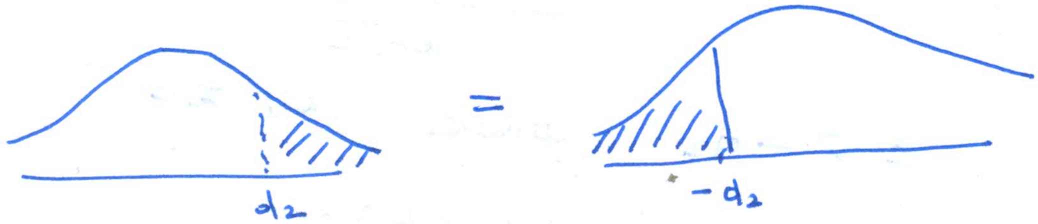
Similarly we compute the value of the binary put

$$V_B^{put}(x,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} \mathbb{I}_{(K-S(T))^{+}} \, dW(t)$$

(3) Proceeding similarly we get

$$V_B^{put}(x,t) = e^{-r(T-t)} \int_{d_2}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz$$

$$V_B^{put}(x,t) = e^{-r(T-t)} \, N(-d_2)$$



b. At maturity

Binary Call Binary Put Portfolio of binary call + put

(4) Since at expiry time T the combined portfolio of binary call and put is the same as a zero coupon bond. Therefore a ZC bond is worth the same as a sum of a Binary call and Put

at time t ZC bond is worth

$$e^{-r(T-t)} N(d_2) + e^{-r(T-t)} N(-d_2) = e^{-r(T-t)}$$