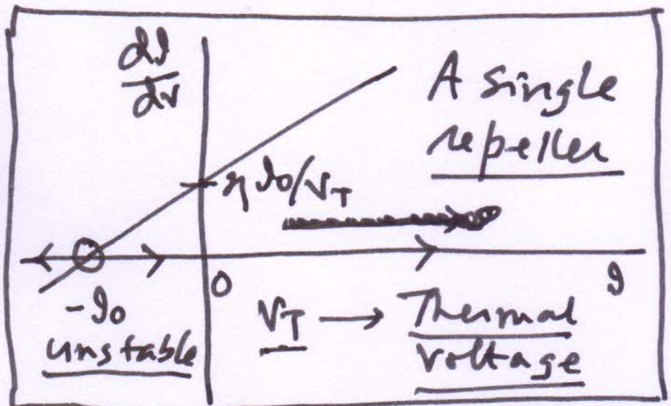
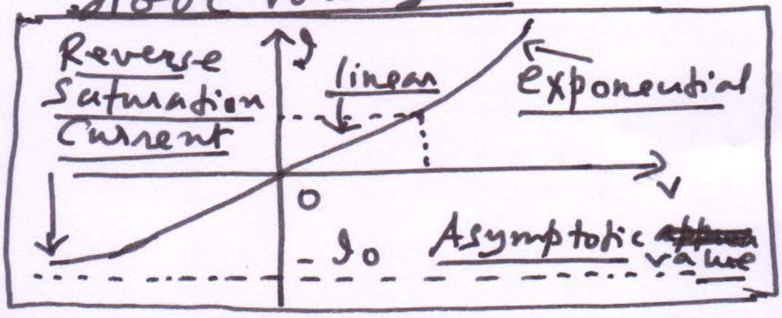
Practical Applications of Phase Portraits

In a p-n junction diode $\boxed{I = I_0\left(e^{\eta V/V_T} - 1\right)}$

$$\Rightarrow \boxed{\frac{dI}{dV} = I_0\, e^{\eta V/V_T} \cdot \frac{\eta}{V_T} = \frac{\eta}{V_T}\left(I + I_0\right)}, \boxed{I \equiv I(V)}$$

in the form of $\boxed{\dot{x} = a + bx}$. I_0, η and V_T are fixed parameters of the system. I is the diode current, V is the diode voltage.





A Few Physical Cases of $\boxed{\dot{x} = a - bx}$ $\boxed{\dot{x} = f(x)}$ (linear functions)

1/. Stokes' Law of Terminal Velocity : $\boxed{\dot{v} = f(v)}$

$$\boxed{\dot{v} = \bar{g} - \frac{k}{m}v}, \boxed{\bar{g} = g\left(1 - \frac{\rho_f}{\rho}\right)} \quad \boxed{k, m > 0} \quad \boxed{g, \rho_f, \rho > 0}$$

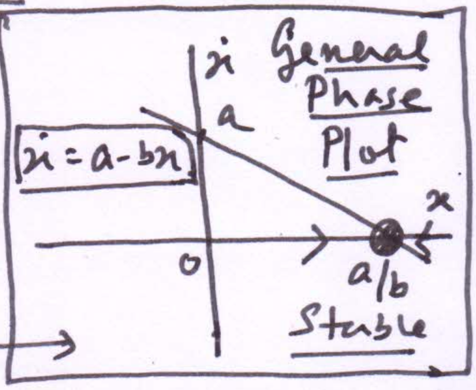
2/. Kelvin's Viscoelastic Formula: $\boxed{\dot{\epsilon} = f(\epsilon)}$

$$\boxed{\dot{\epsilon} = \frac{\sigma}{\eta} - \frac{Y}{\eta}\epsilon} \cdot \boxed{\sigma, \eta, Y > 0}$$ (for highly viscous materials)

3/. Q-R-C Circuit : $\boxed{\dot{Q} = f(Q)}$

$$\boxed{\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}} \quad \boxed{V_0, R, C > 0}$$

The foregoing physical systems have this common linear plot. →

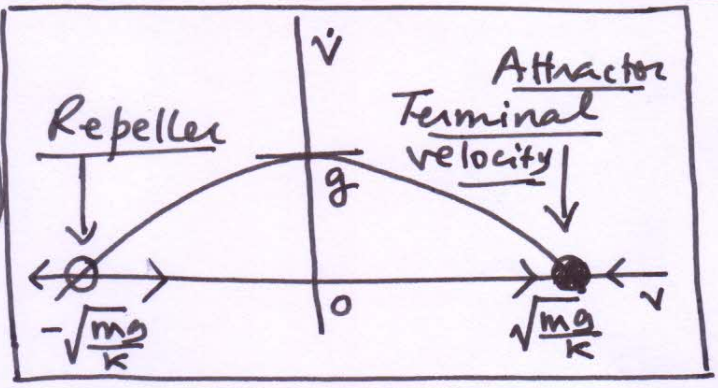


Free Fall of a Parachutist (Terminal Velocity)

$$\boxed{\dot{v} = g - \frac{k}{m}v^2}$$

$\boxed{g, k, m > 0}$ $\boxed{\dot{v} = f(v)}$

$\boxed{\dot{v} = 0} \Rightarrow \boxed{v_c = \pm\sqrt{\frac{mg}{k}}}$



Item Response Theory (Teaching-Learning Process)

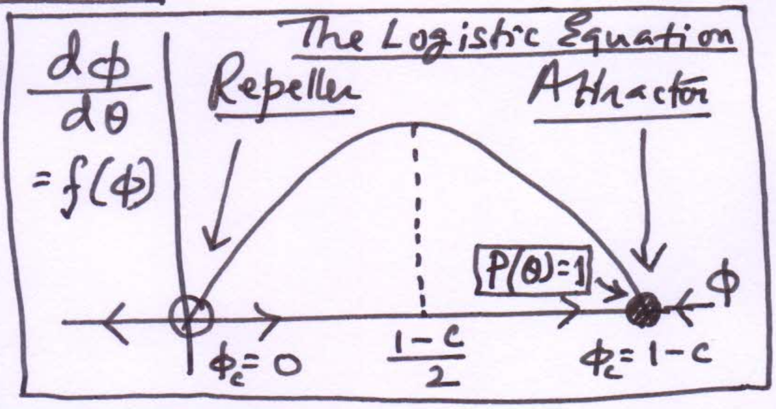
$$\boxed{P(\theta) = c + \frac{1-c}{1+e^{-(\theta-b)/\omega}}}$$

$\boxed{c, b, \omega > 0}$ (fixed parameters)

Define $\boxed{\phi = P(\theta) - c}$

$$\boxed{\frac{d\phi}{d\theta} = \frac{\phi}{\omega}\left[1 - \frac{\phi}{1-c}\right]}$$

$\boxed{\frac{d\phi}{d\theta} = 0} \Rightarrow \boxed{\phi_c = 0}$ $\boxed{\phi_c = 1-c}$



Spread of Agricultural Innovations

$$\boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}$$

$\boxed{N, c, c' > 0}$ (Fixed parameters)

Rescale $\boxed{X = x/N}$, $\boxed{T = cNt}$ and $\boxed{a = c'/cN}$. ($a > 0$)

$\Rightarrow \boxed{\frac{dX}{dT} = (X+a)(1-X)}$ $\boxed{\frac{dX}{dT} = 0} \Rightarrow \boxed{X_c = -a}$ and $\boxed{X_c = 1}$

$\boxed{\frac{dX}{dT} = F(X)} \Rightarrow \boxed{\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT}} \rightarrow$ Chain Rule

$$\boxed{\frac{d(x/N)}{d(Nct)} = \left(\frac{x}{N} + \frac{c'}{cN}\right)\left(1 - \frac{x}{N}\right)} \rightarrow \text{Rescaling}$$

(P.T.O.)

Now, $\boxed{\frac{dX}{dT} = F(X) = (X+a)(1-X) = X + a - aX - X^2}$

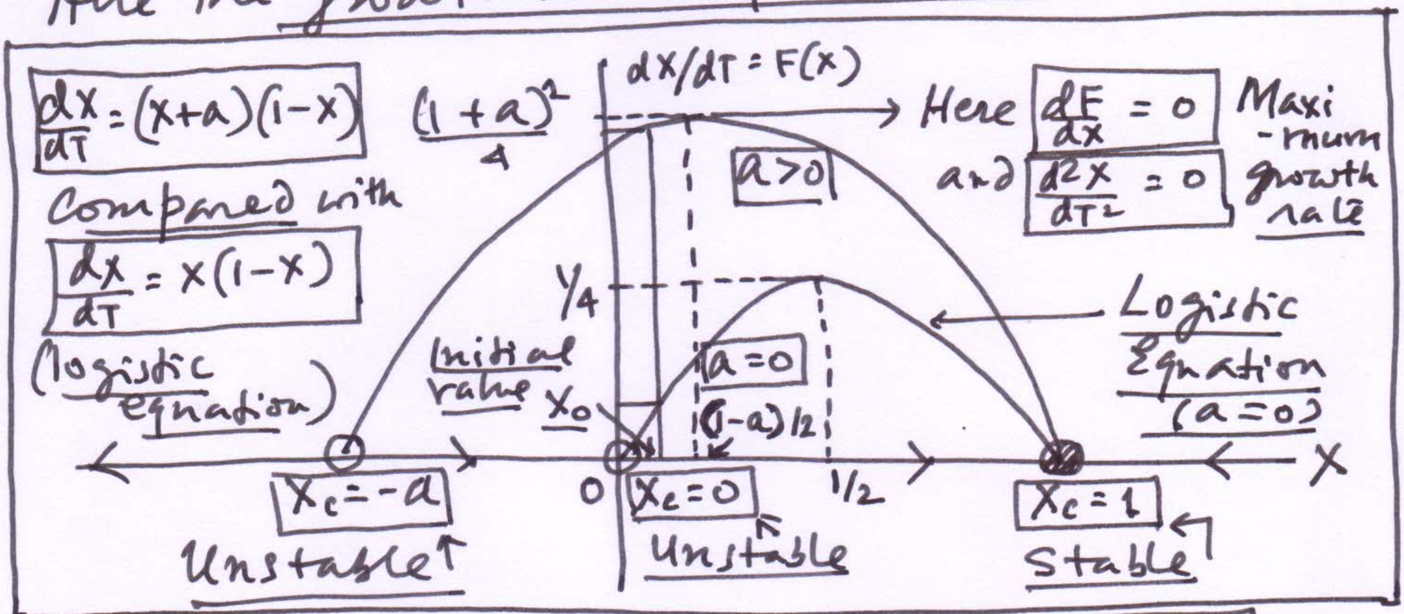
$\Rightarrow \boxed{F(X) = a + (1-a)X - X^2}$. We expect $\boxed{a<1}$, because $\boxed{a = \frac{c'/c}{N}}$ and since N, the number of farmers, has a large ~~number~~ value, $\boxed{a<1}$. $\Rightarrow \boxed{(1-a)>0}$.

$\boxed{\frac{dF}{dX} = 1 - a - 2X}$ When $\boxed{\frac{dF}{dX} = 0} \Rightarrow \boxed{X = \frac{1-a}{2}}$

$$\boxed{F\left(\frac{1-a}{2}\right) = \left(\frac{1-a}{2} + a\right)\left(1 - \frac{1-a}{2}\right) = \frac{(1+a)^2}{4}}$$

At $\boxed{X = \frac{1-a}{2}}$, $\boxed{\frac{d^2X}{dT^2} = \frac{dF}{dX}\frac{dX}{dT} = 0}$ $\left(\because \frac{dF}{dX} = 0\right)$

Here the growth rate of X is highest.



i) For an initial value $\boxed{0 < X_0 < \frac{1-a}{2}}$, a much higher growth rate occurs than it would be for the logistic equation $(a=0)$.

ii) For both cases, $\boxed{a>0}$ and $\boxed{a=0}$, the final attractor state is $\boxed{X_c = 1} \Rightarrow \boxed{x = N}$. But for $\boxed{a>0}$, the early growth is much higher than for $\boxed{a=0}$.

Autocatalysis (In Chemical Reactions)

$$\boxed{A + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 2X}$$

A is a catalyst, that aids chemical X to stimulate its own production — autocatalysis.

K_1 and k_{-1} are rate constants for the forward and backward reactions, respectively.

$\boxed{x = [X] \rightarrow \text{Concentration of } X, \quad a = [A] \rightarrow \text{Concentration of } A}$

For an unlimited amount ~~amount~~ of A, $\boxed{a = \text{constant}}$.

Law of mass action of chemical Kinetics:
Rate of a chemical reaction is proportional to the product of the concentration of the reactants.

$A + X \xrightarrow{k_1} X + X$ (Depletion of X)

(Creation of X) $X + X \xrightarrow{K_{-1}} A + X$

(Rate balance)

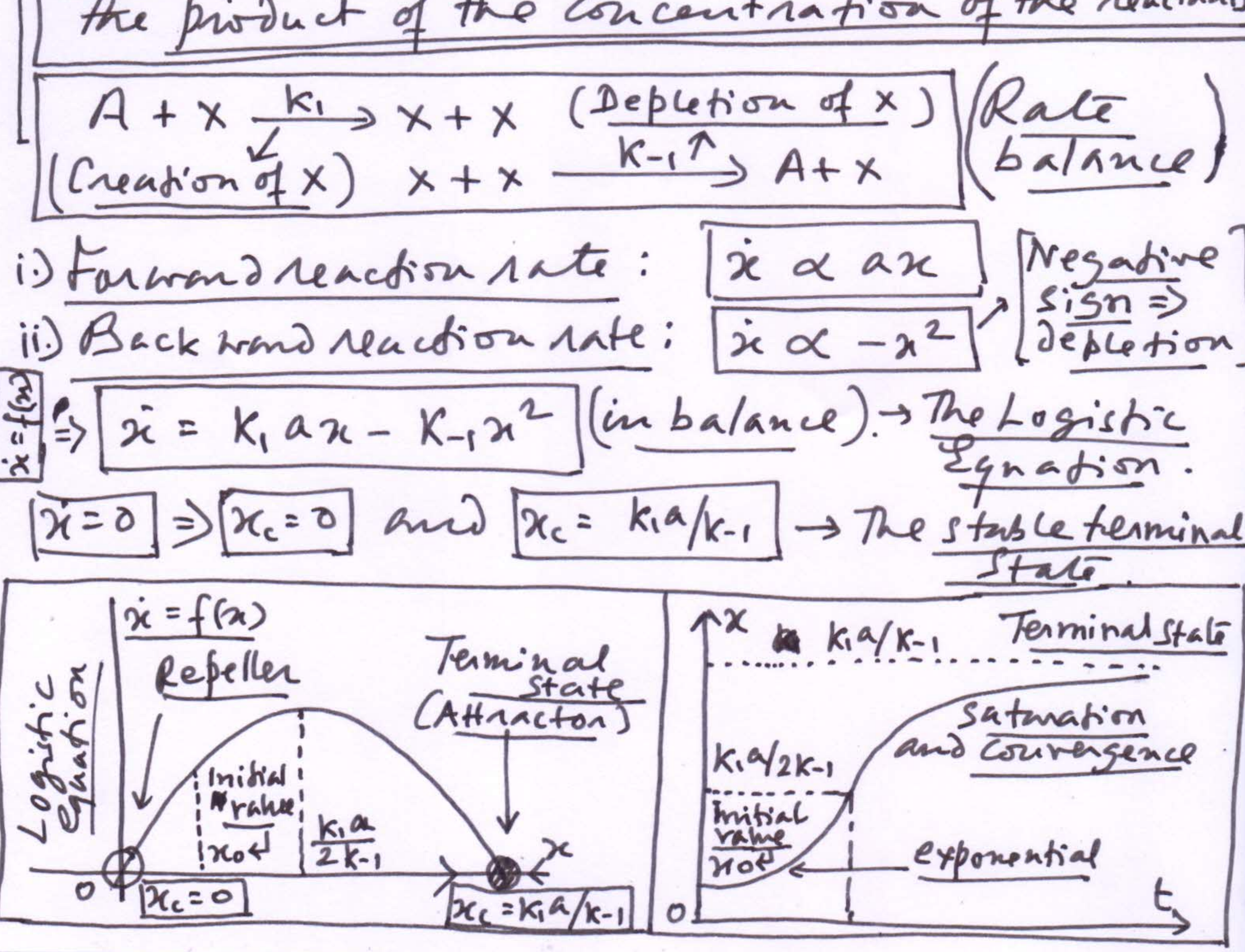
i) Forward reaction rate: $\boxed{\dot{x} \propto ax}$

ii) Backward reaction rate: $\boxed{\dot{x} \propto -x^2}$

[Negative sign ⇒ depletion]

$\dot{x} = f(x)$ ⇒ $\boxed{\dot{x} = K_1 a x - K_{-1} x^2}$ (in balance) → The Logistic Equation.

$\boxed{\dot{x} = 0} \Rightarrow \boxed{x_c = 0}$ and $\boxed{x_c = k_1 a / k_{-1}}$ → The stable terminal state.



Gompertz Law of Tumour Growth

$$\boxed{\dot{x} = \frac{dx}{dt} = f(x) = -ax\ln(bx)} \quad (\underline{a, b > 0})$$

(fixed point)

$\boxed{\dot{x} = 0} \Rightarrow$ i.) $\ln(bx) = 0 \Rightarrow \boxed{bx_c = 1} \Rightarrow \boxed{x_c = b^{-1}}$

ii.) $\boxed{\dot{x} = -\frac{a\ln(bx)}{x^{-1}}}$, when $\underline{x \to 0}$, $\boxed{\ln(bx) \to -\infty}$ and $\boxed{x^{-1} \to \infty}$.

Applying $\underline{\text{L'Hospital Rule}}$, when $\boxed{x \to 0}$,

$$\Rightarrow \boxed{\dot{x} = \frac{-ab(1/bx)}{-x^{-2}} = -\frac{ax^{-1}|_{x=0}}{-x^{-2}|_{x=0}} \to \frac{a\infty}{\infty^2} = 0}$$

$\Rightarrow$ when $\boxed{x \to 0}$, $\boxed{\dot{x} \to 0} \Rightarrow \boxed{x_c = 0}$ is a $\underline{\text{fixed point}}$.

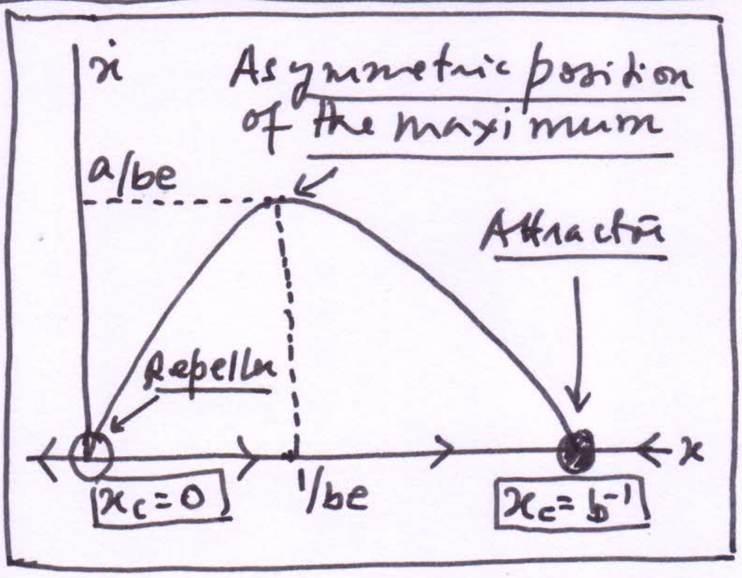
$\underline{\text{Turning point of } f(x)}$: $\boxed{f(x) = -ax\ln(bx)}$.

$\Rightarrow f'(x) = -a\left[\ln(bx) + x \cdot \frac{1}{bx} \cdot b\right] = -a\left[1 + \ln(bx)\right]$

$\Rightarrow f'(x) = 0 \Rightarrow \boxed{\ln(bx) = -1} \Rightarrow bx = \frac{1}{e} \Rightarrow \boxed{x = \frac{1}{be}}$

$f''(x) = -a \cdot \frac{1}{bx} b = -\frac{a}{x}$. $\Rightarrow \boxed{f''\left(\frac{1}{be}\right) = -abe}$ $(a, b, e > 0)$

Since $\boxed{f''\left(\frac{1}{be}\right) < 0}$, the $\underline{\text{turning point is a maximum}}$.



i.) At the $\underline{\text{maximum}}$ $\boxed{x = \frac{1}{be}}$.

$\Rightarrow \boxed{f\left(\frac{1}{be}\right) = -\frac{a}{be}\ln\left(\frac{1}{e}\right) = \frac{a}{be}}$

ii.) Since $\boxed{e \simeq 2.72}$, the position of the maximum is $\underline{\text{asymmetric}}$ (not halfway between $\boxed{x_c = 0}$ and $\boxed{x_c = b^{-1}}$)

iii.) $\underline{\text{Tumour size is scaled by } b^{-1}}$!

The Allee Effect (Warder Clyde Allee)

[Effective growth rate of a species is highest for intermediate values of population size, x.]

$\boxed{\frac{\dot{x}}{x} = r - a(x-b)^2} \Rightarrow \boxed{\dot{x} = f(x) = x\left[r - a(x-b)^2\right]}$

Fixed points: $\boxed{\dot{x} = f(x_c) = 0} \Rightarrow \boxed{x_c = 0}$ and $(x_c - b)^2 = r/a \Rightarrow \boxed{x_c = b \pm \sqrt{r/a}}$. Three fixed points ($r$ and a must have same signs for all the fixed points to be real).

Linear Stability Analysis: $\boxed{\dot{x} = f(x) = \left[r - a(x-b)^2\right]x}$

$\Rightarrow f(x) = x\left[r - a(x^2 - 2bx + b^2)\right] = x\left[r - ax^2 + 2abx - ab^2\right]$

$\Rightarrow \boxed{\dot{x} = f(x) = (r - ab^2)x + 2abx^2 - ax^3} \rightarrow$ cubic polynomial

$\Rightarrow \boxed{f'(x) = (r - ab^2) + 4abx - 3ax^2} \rightarrow$ quadratic

i.) When $\boxed{x_c = 0}$: $\Rightarrow \boxed{f'(0) = r - ab^2}$. For $\boxed{x_c = 0}$ to be a stable fixed point, $\boxed{r - ab^2 < 0} \Rightarrow \boxed{\sqrt{r/a} < b}$.

ii.) When $\boxed{x_c = b - \sqrt{\frac{r}{a}}}$: $\Rightarrow f'\left(b - \sqrt{\frac{r}{a}}\right) = (r - ab^2) + 4ab\left(b - \sqrt{\frac{r}{a}}\right) - 3a\left(b - \sqrt{\frac{r}{a}}\right)^2$

$\Rightarrow f'\left(b - \sqrt{\frac{r}{a}}\right) = r - ab^2 + 4ab^2 - 4b\sqrt{ar} - 3a\left(b^2 - 2b\sqrt{\frac{r}{a}} + \frac{r}{a}\right)$

$\Rightarrow f'\left(b - \sqrt{\frac{r}{a}}\right) = r - \cancel{ab^2} + \cancel{4ab^2} - 4b\sqrt{ar} - 3\cancel{a}b^2 + 6ab\sqrt{\frac{r}{a}} - 3r$

$\Rightarrow \boxed{f'\left(b - \sqrt{\frac{r}{a}}\right) = -2r + 2b\sqrt{ar} = 2\sqrt{ar}\left[b - \sqrt{\frac{r}{a}}\right]}$

If $\boxed{\sqrt{r/a} < b}$, then $\boxed{f'(b - \sqrt{r/a}) > 0} \Rightarrow$ unstable fixed point

(P.T.O.)

(continued)

iii) When $\boxed{x_c = b+\sqrt{\frac{r}{a}}}$: $\Rightarrow f'\left(b+\sqrt{\frac{r}{a}}\right) = (r-ab^2) + 4ab\left(b+\sqrt{\frac{r}{a}}\right) - 3a\left(b+\sqrt{\frac{r}{a}}\right)^2$

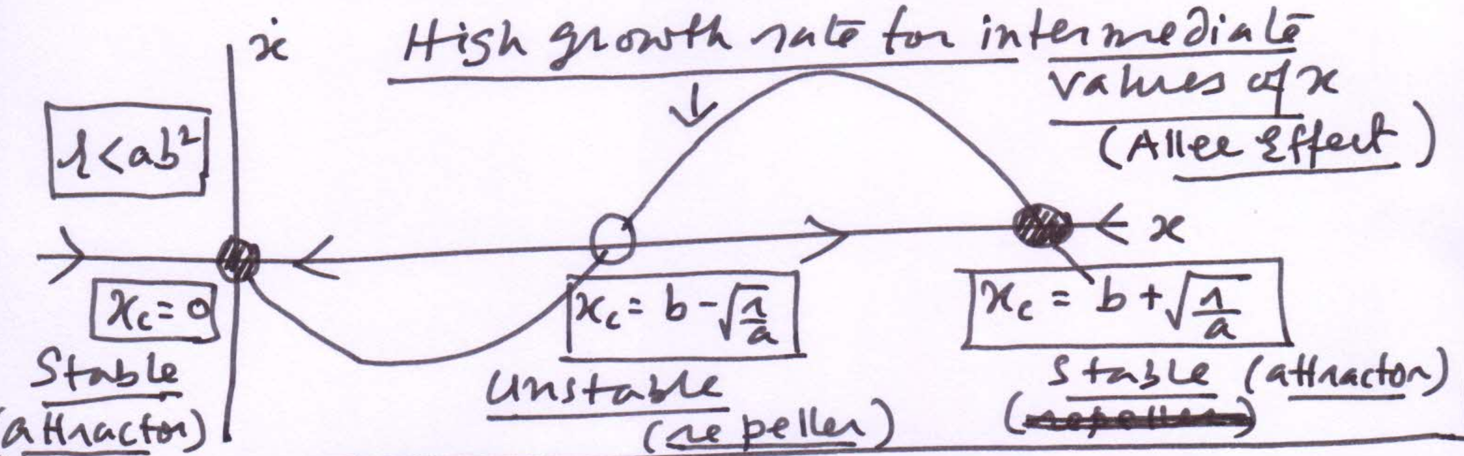
$$\Rightarrow f'\left(b+\sqrt{\tfrac{r}{a}}\right) = r - ab^2 + 4ab^2 + 4b\sqrt{ar} - 3a\left(b^2 + 2b\sqrt{\tfrac{r}{a}} + r/a\right)$$

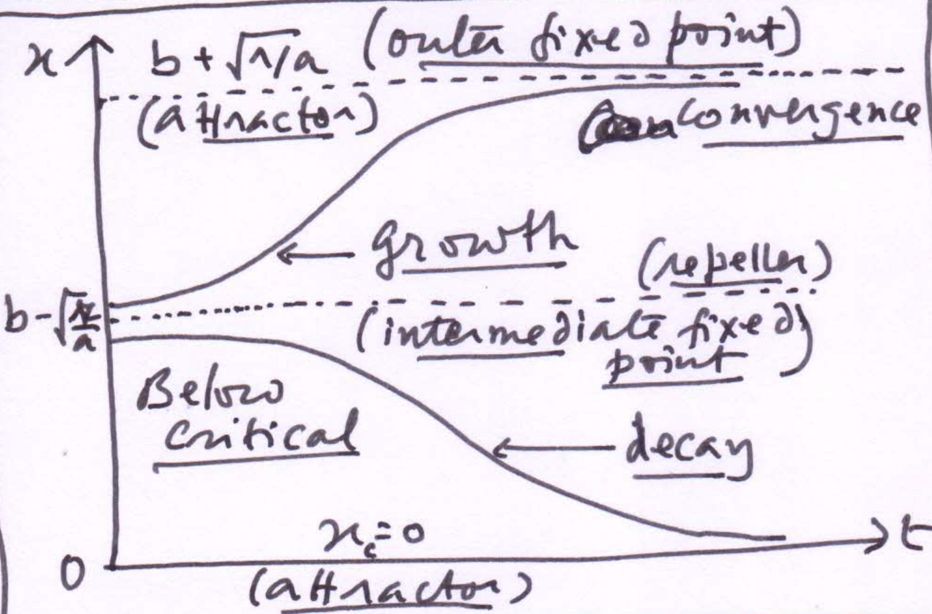
$$\Rightarrow f'\left(b+\sqrt{\tfrac{r}{a}}\right) = r + \cancel{3ab^2} + 4b\sqrt{ar} - \cancel{3ab^2} - 6b\sqrt{ar} - 3r$$

$$\Rightarrow \boxed{f'\left(b+\sqrt{\tfrac{r}{a}}\right) = -2r - 2b\sqrt{ar} = -2\sqrt{ar}\left[b+\sqrt{\tfrac{r}{a}}\right]}$$

If $\boxed{\left(b+\sqrt{\frac{r}{a}}\right) > 0}$, then $\boxed{f'\left(b+\sqrt{\frac{r}{a}}\right) < 0}$ $\Rightarrow$ Stable fixed point

Consider $\boxed{r - ab^2 < 0}$ $\boxed{\dot{x} = f(x) = (r-ab^2)x + 2abx^2 - ax^3}$





The difference between the logistic function and the Allee function is the existence of an intermediate state in the latter. Below this state there is only decay to $\boxed{x_c = 0}$.

1/ The Allee Effect is satisfied when $\boxed{r < ab^2}$.

2/ When x is large, $\boxed{\dot{x} \simeq -ax^3}$. (P.T.O.)

(continued)

3/. Hence, for saturation at large values of x, $\boxed{a>0}$ (converge to the outer fixed point).

4/. For high growth rates at intermediate values of x, $\boxed{ab>0}$ $\because$ $\boxed{a>0}$, $\Rightarrow$ $\boxed{b>0}$.

5/. Since, $\boxed{b \pm \sqrt{r/a}}$ are real fixed points, r and a have same signs. $\because$ $\boxed{a>0}$, $\boxed{r>0}$

6/. Hence, all $\boxed{a, b, r > 0}$ (the fixed parameters).

Turning points of $f(x)$: $\Rightarrow$ $\boxed{f'(x) = 0}$

$$\Rightarrow \boxed{f'(x) = (1 - ab^2) + 4abx - 3ax^2 = 0}$$ A quadratic equation

$$\Rightarrow \boxed{3ax^2 - 4abx - (1 - ab^2) = 0}$$ Hence, two turning points exist.

$$x = \frac{4ab \pm \sqrt{16a^2b^2 + 4.3a(1 - ab^2)}}{6a}$$

The discriminant is always positive.

$$\Rightarrow x = \frac{2}{3}b \pm \frac{1}{6a}\sqrt{4a^2b^2 + 12ar\left(\frac{ab^2}{ab^2}\right)}$$

$$\Rightarrow \boxed{x = \frac{2}{3}b \pm \frac{b}{3}\sqrt{1 + \frac{3r}{ab^2}} = \frac{b}{3}\left[2 \pm \left(1 + \frac{3r}{ab^2}\right)^{1/2}\right]}$$

If $\boxed{r \ll ab^2}$ we can expand the discriminant binomially, $\therefore$ $\boxed{\left(1 + \frac{3r}{ab^2}\right)^{1/2} \simeq 1 + \frac{3r}{2ab^2}}$

$$\Rightarrow \boxed{x \simeq \frac{b}{3}\left[2 \pm 1 \pm \frac{3r}{2ab^2}\right]} \Rightarrow \boxed{x \simeq b + b\left(\frac{r}{2ab^2}\right)}$$ (for upper sign)

and $\boxed{x \simeq \frac{b}{3} - b\left(\frac{r}{2ab^2}\right)}$ (for lower sign). Hence, one turning point is at $\boxed{x > b}$ and the other is at $\boxed{0 < x < b}$.