Review: Graph Theory and Representation

Graph Algorithms

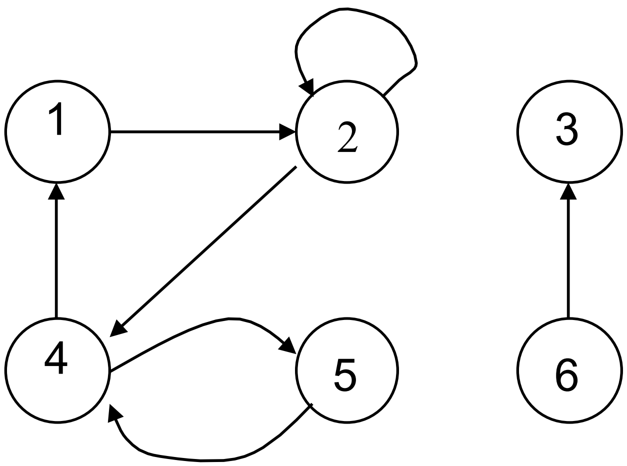
- Graphs and fundamental theorems about Graphs
- Graph implementation
- Graph Algorithms
  - Shortest paths
  - Minimum spanning tree

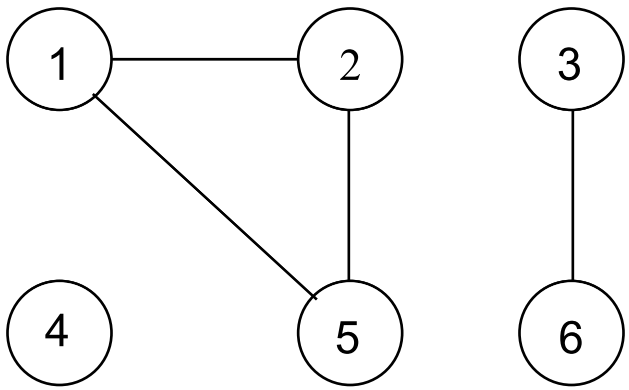
What can graphs model?

- Cost of wiring electronic components together.
- Shortest route between two cities.
- Finding the shortest distance between all pairs of cities in a road atlas.
- Flow of material: liquid flowing through pipes, current through electrical networks, information through communication networks, parts through an assembly line, etc.
- State of a machine.
- Used in Operating systems to model resource handling (deadlock problems).
- Used in compilers for parsing and optimizing the code.

What is a Graph?

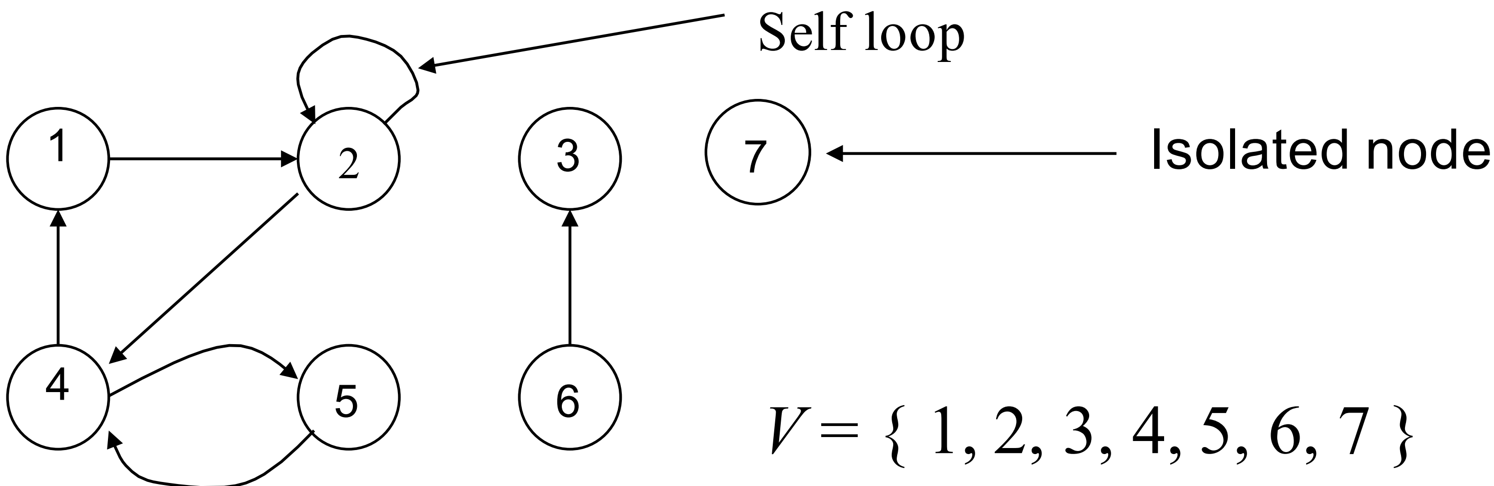
- Informally a *graph* is a set of nodes joined by a set of lines or arrows.





A ***directed graph***, also called a ***digraph* G,** is a pair (V, E) where V is a finite set of vertices and E is a set of *directed edges*.

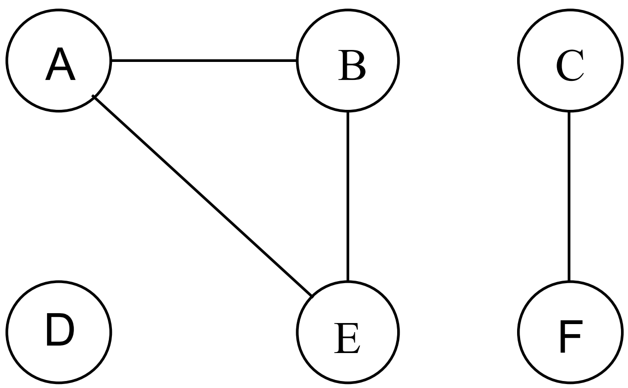
An edge from node a to node b is denoted by the **ordered** pair (a, b).



$$V = \{\ 1, 2, 3, 4, 5, 6, 7\ \}$$
$$|\ V\ | = 7$$

$$E = \{\ (1,2), (2,2), (2,4), (4,5), (4,1), (5,4), (6,3)\ \}$$
$$|\ E\ | = 7$$

Undirected graph $G = (V, E)$: Unlike a digraph, E consists of undirected edges. So, edge (A,B) = edge (B, A).



$V = \{ A, B, C, D, E, F \}$

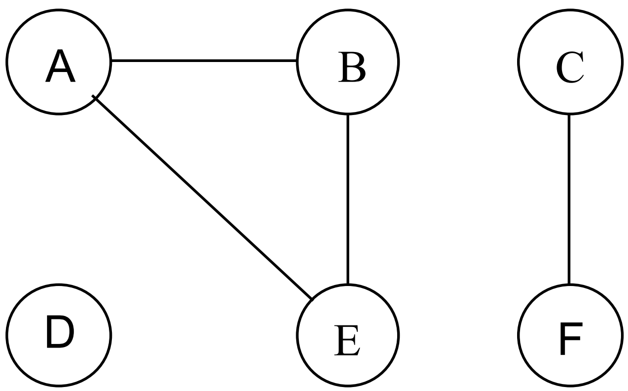
$|V| = 6$

$E = \{ \{A, B\}, \{A,E\}, \{B,E\}, \{C,F\} \}$

$|E| = 4$

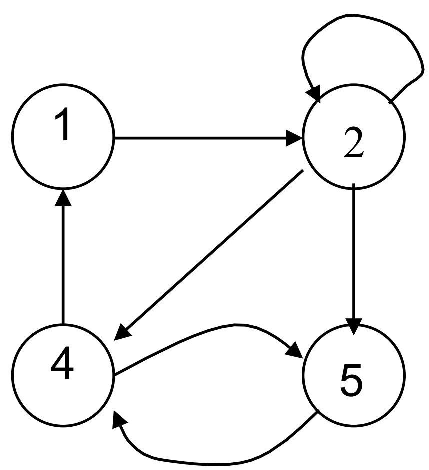
-The *degree* of a vertex in an undirected graph is the number of edges incident on it.

- In a directed graph, the *out degree* of a vertex is the number of edges leaving it and the *in degree* is the number of edges entering it.



The degree of B is 2.

Self-loop



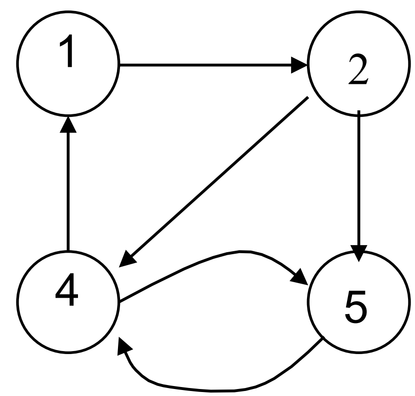
The in degree of 2 is 2 and the out degree of 2 is 3.

Cyclic and Acyclic

A path from a vertex to itself is called a cycle

(e.g., v1 → v2 → v4 → v1)

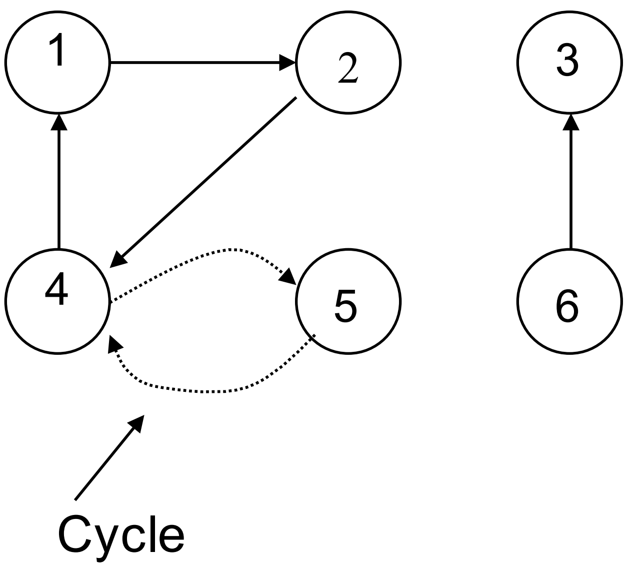
If a graph contains a cycle, it is cyclic

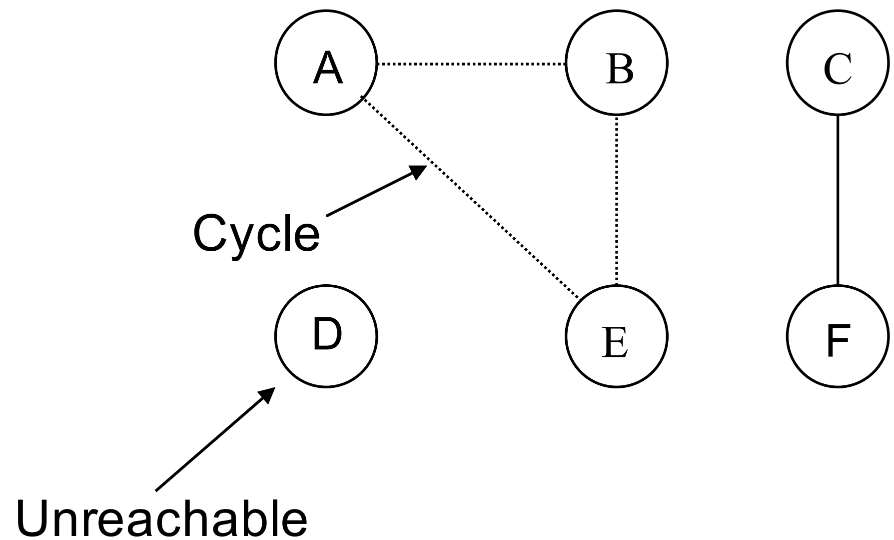


Otherwise, it is acyclic

A path is simple if it never passes through the same vertex twice.

A ***path*** is a sequence of vertices such that there is an edge from each vertex to its successor. A path from a vertex to itself is called a ***cycle***. A graph is called ***cyclic*** if it contains a cycle; otherwise it is called ***acyclic***. A path is ***simple*** if each vertex is distinct.





Simple path from 1 to 5
= (1, 2, 4, 5)
or as in our text
((1, 2), (2, 4), (4,5))

If there is path p from u to v then we say v is **reachable** from u via p.

Simple Graphs

- *Simple graphs* are graphs without multiple edges or self-loops. We will consider only simple graphs.

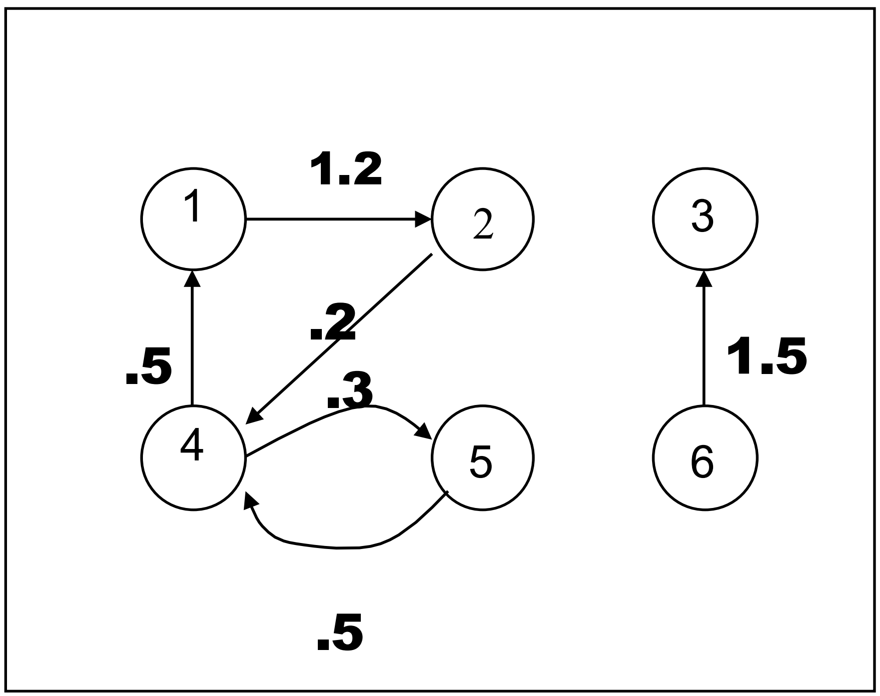
- Proposition: If G is an undirected graph then

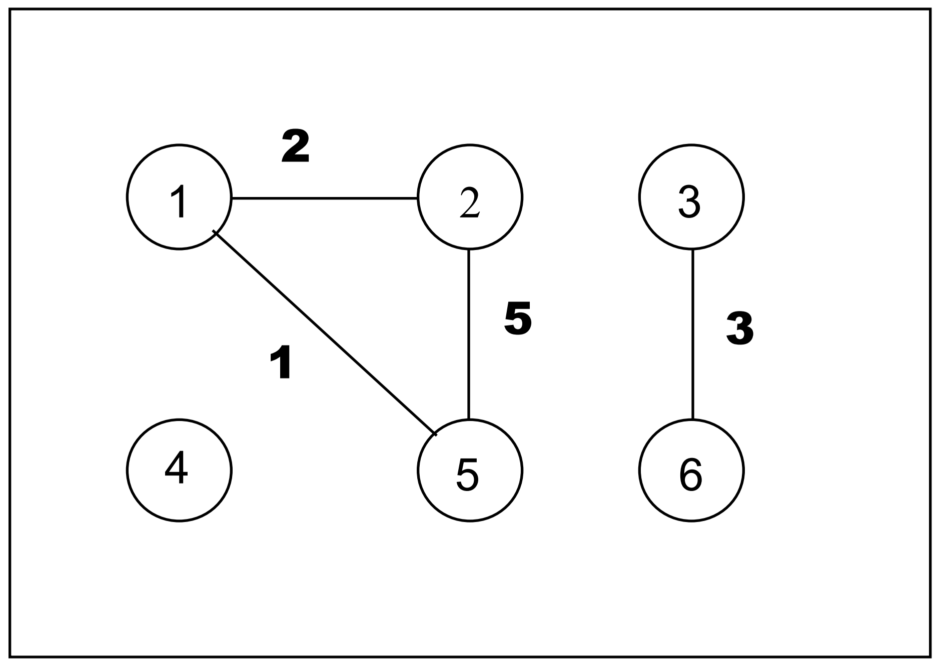
$$\sum_{v \in G} \deg(v) = 2\,|E|$$

- Proposition: If G is a digraph then

$$\sum_{v \in G} \text{indeg}(v) = \sum_{v \in G} \text{outdeg}(v) = |E|$$

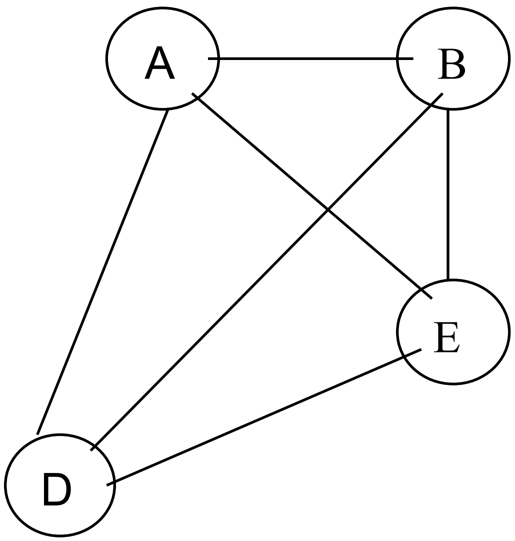
A ***weighted graph*** is a graph for which each edge has an associated ***weight***, usually given by a ***weight function*** $w: E \rightarrow \mathbf{R}$.





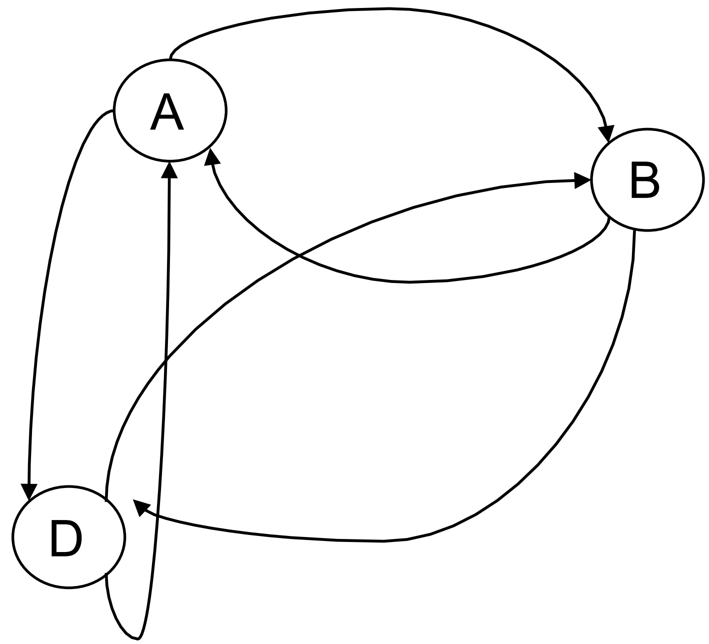
If (u, v) is an edge in a graph G, we say that vertex v is ***adjacent*** to vertex u.

A ***complete graph*** is an undirected/directed graph in which every pair of vertices is adjacent.



4 nodes and (4*3)/2 edges

V nodes and V*(V-1)/2 edges

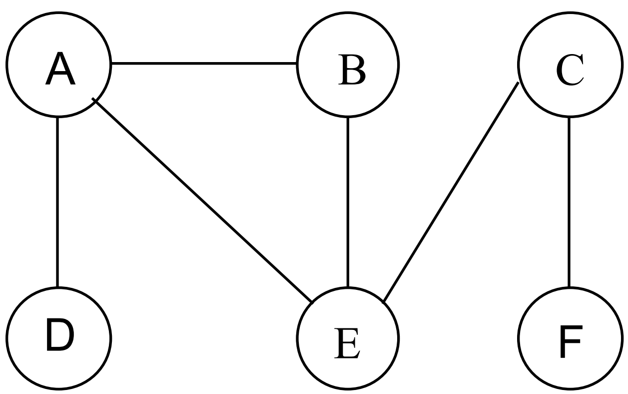


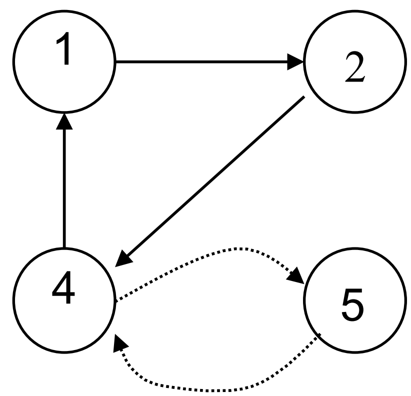
3 nodes and 3*2 edges

V nodes and V*(V-1) edges

An undirected graph is ***connected*** if you can get from any node to any other by following a sequence of edges, i.e., a path.

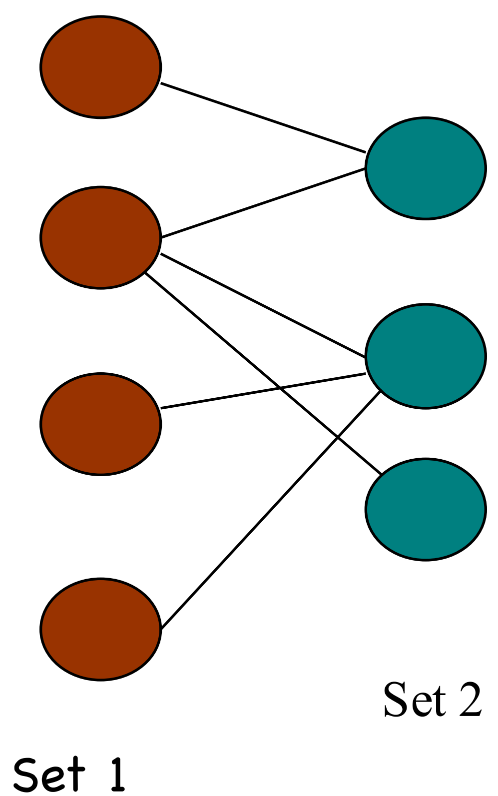
A directed graph is ***strongly connected*** if there is a directed path from any node to any other node.





- A graph is ***sparse*** if $| E | \approx | V |$
- A graph is ***dense*** if $| E | \approx | V |^2$

A ***bipartite graph*** is an undirected graph $G = (V,E)$ in which V can be partitioned into 2 sets V_1 and V_2 such that $(u,v) \in E$ implies either $u \in V_1$ and $v \in V_2$ OR $v \in V_1$ and $u \in V_2$.

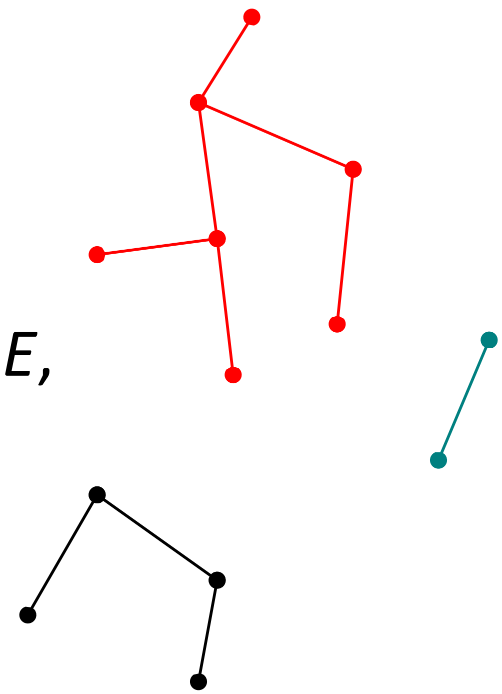


A ***free tree*** is an acyclic, connected, undirected graph. A ***forest*** is an acyclic undirected graph. A ***rooted tree*** is a tree with one distinguished node, ***root***.

Let $G = (V, E)$ be an undirected, acyclic, connected graph.

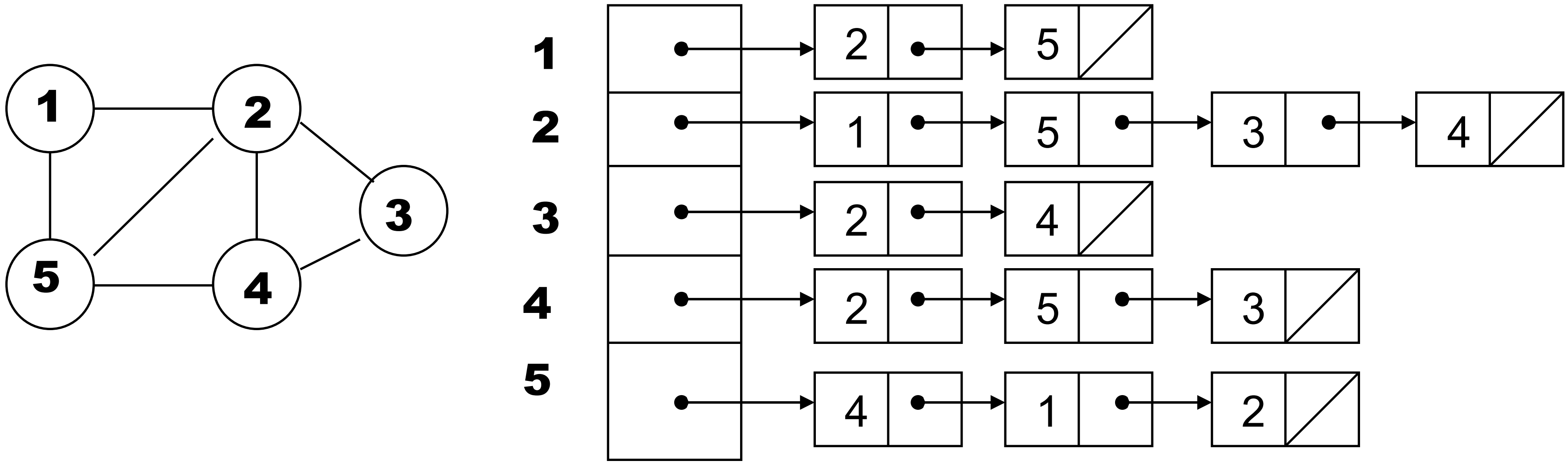
The following statements are equivalent.

- G is a tree
- Any two vertices in G are connected by unique simple path
- G is connected, acyclic, and $|E| = |V| - 1$
- G is connected, but if any edge is removed from E, the resulting graph is disconnected
- G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

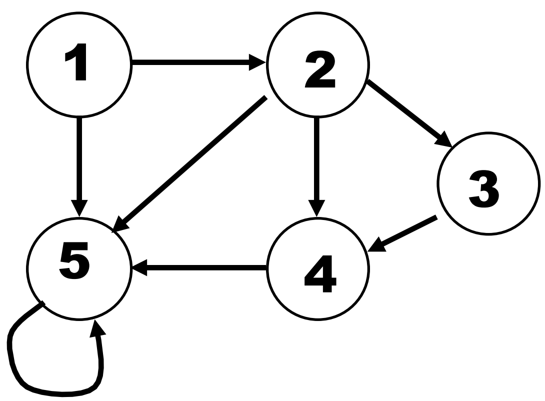


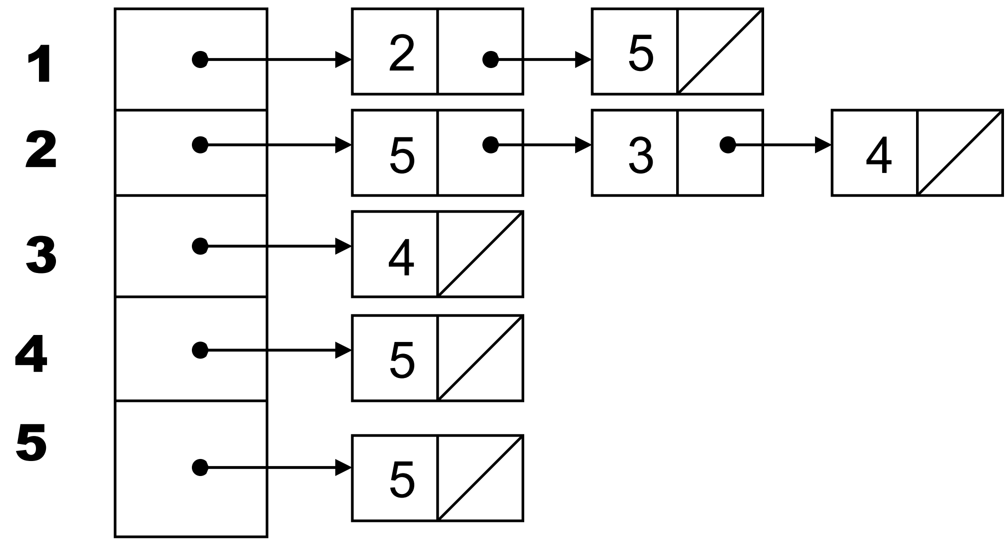
Implementation of a Graph.

- ***Adjacency-list representation*** of a graph $G = (V, E)$ consists of an array ADJ of $|V|$ lists, one for each vertex in V. For each $u \in V$, $ADJ[u]$ points to all its adjacent vertices.



Adjacency-list representation for a directed graph.





Variation: Can keep a second list of edges coming into a vertex.

Adjacency lists

- Property
  - Saves space for sparse graphs. Most graphs are sparse.
  - “Visit” edges that start at v
    - Must traverse linked list of v
    - Size of linked list of v is degree(v)
    - Order: Θ(degree(v))

Adjacency List

- Storage
  - We need V pointers to linked lists
  - For a directed graph the number of nodes (or edges) contained (referenced) in all the linked lists is

$$\sum_{v \in V} (\text{out-degree}(v)) = |E|.$$

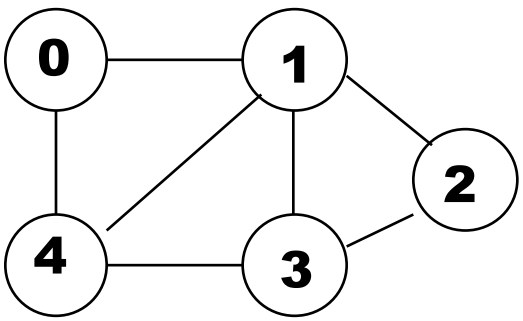
So we need $\Theta(V + E)$

  - For an undirected graph the number of nodes is

$$\sum_{v \in V} (\text{degree}(v)) = 2|E|$$

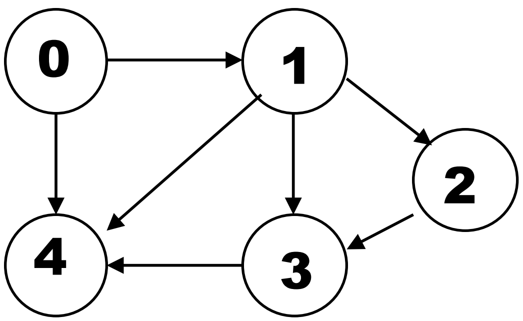
Also $\Theta(V + E)$

Adjacency-matrix ***representation*** of a graph $G = (V, E)$ is a $|V| \times |V|$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $(i, j) \in E$ and 0 otherwise.



$$
\begin{array}{c c}
 & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{ccccc}
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

Adjacency Matrix Representation for a Directed Graph



$$
\begin{array}{c|ccccc|}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

Adjacency Matrix Representation

- Advantage:
  - Saves space on pointers for dense, un-weighted graphs
  - Just one bit per matrix element
  - Faster lookup
    - Is there an edge (v, u)? ⇔ adjacency [i] [j] == true?
    - So $\theta(1)$
- Disadvantage:
  - Waste space for sparse, weighted graphs
    - Size of the adjacency matrix is $|V|^2$
  - "Visit" all the edges that start at v
    - Row v of the matrix must be traversed.
    - So $\theta(|V|)$.

Adjacency Matrix Representation

- Storage
  - $\Theta(V^2)$
  - For undirected graphs you can only use $1/2(V^2)$ storage, since the adjacency matrix of an undirected graph is symmetric

Graph traversals

- Breadth first search
- Depth first search

Breadth first search

- Given a graph G=(V,E) and a *source vertex s*, BFS explores the edges of G to visit each node of G reachable from *s*.

- Idea - Expand a *frontier* one step at a time.

- *Frontier* is a FIFO queue
  - O(1) time to update

Breadth first search

- Computes the *shortest distance* (*dist*) from *s* to any reachable node
- Computes a *breadth first tree* (of *parents*) with root *s* that contains all the reachable vertices from *s*
- To get $O(|V|+|E|)$ if we use an adjacency list representation. If we used an adjacency matrix, it would be $O(|V|^2)$

Coloring the nodes

- We use colors (***white***, ***gray*** and ***red)*** to denote the state of the node during the search

- A node is ***white*** if it has not been reached (visited)

- *Visited* nodes are *gray* or *red*. ***Gray*** nodes are at the frontier of the search. ***Red*** nodes are fully explored nodes

BFS - initialize

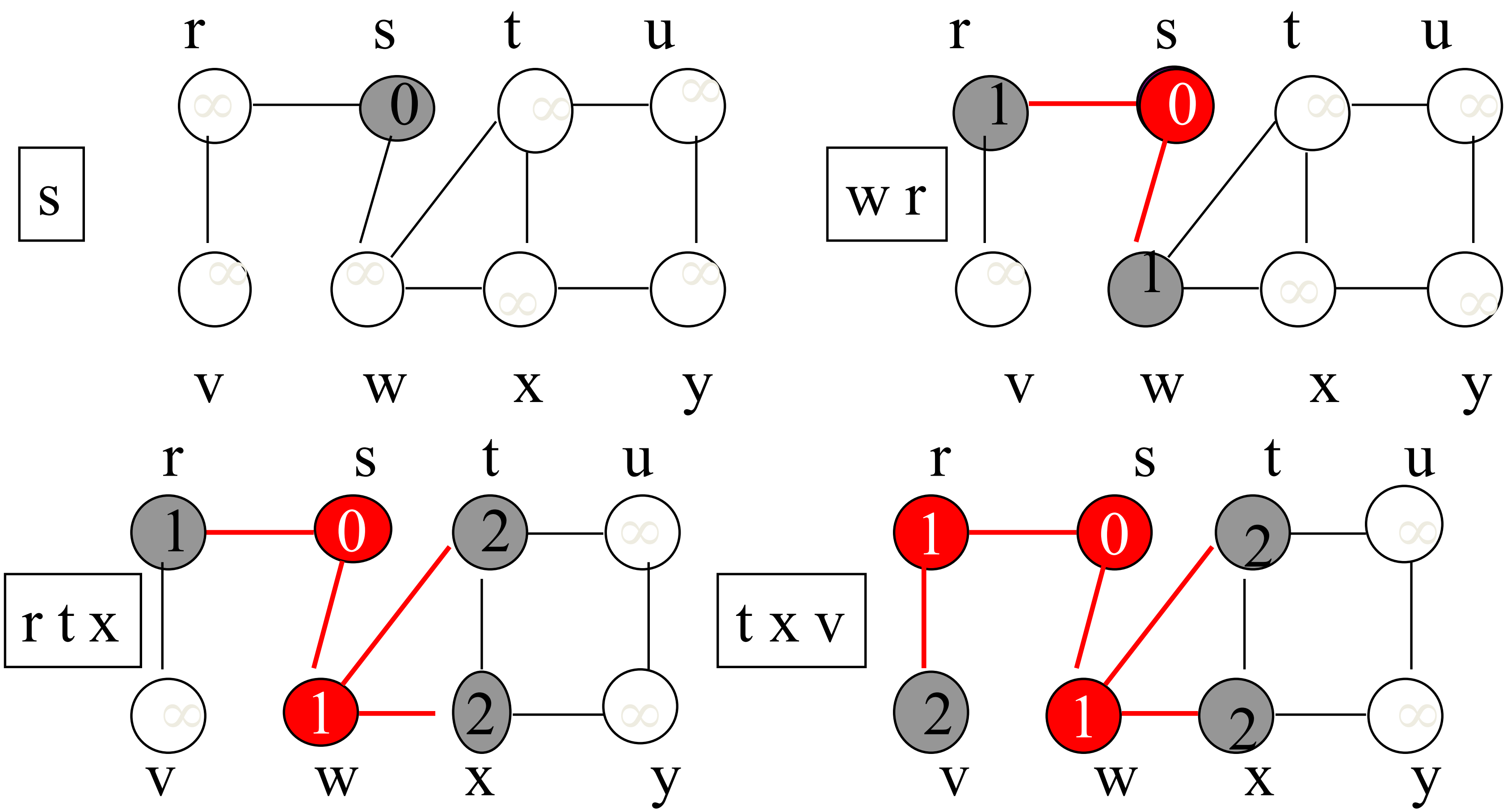
```
procedure BFS(G:graph; s:node; var color:carray;
    dist:iarray; parent:parray);
  for each vertex u do
        color[u]=white; dist[u]=∞;          Θ(V)
        parent[u]=nil;
   end for
  color[s]=gray; dist[s]=0;
  init(Q); enqueue(Q, s);
```

BFS - main

```
while not (empty(Q)) do
    u=head(Q);
    for each v in adj(u) do                O(E)
        if color[v]=white then
            color[v]=gray; dist[v]=dist[u]+1;
            parent[v]=u; enqueue(Q, v);
    dequeue(Q); color[u]=Red; print "u";
end BFS
```

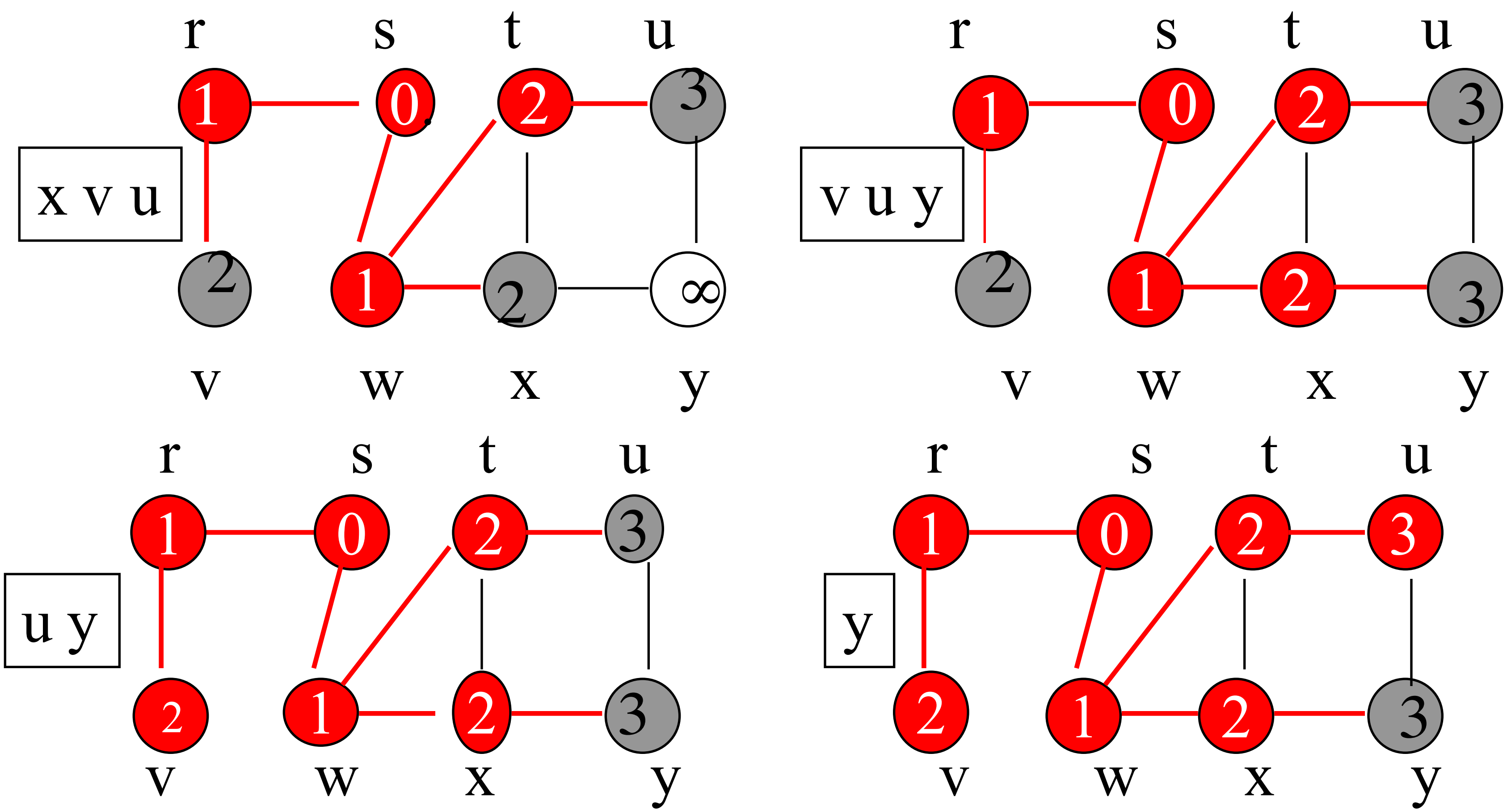
$$\sum_{u\in V}\sum_{v\in ADJ[u]} 1 = \sum_{u\in V} |ADJ[u]| = \sum_{u\in V} \deg ree[u] = O(E)$$

BFS example



r s t u
s
0
v w x y
r s t u
w r
1 0
1
v w x y
r s t u
r t x
1 0 2
1 2
v w x y
r s t u
t x v
1 0 2
2 1 2
v w x y

BFS example



now y is removed from the Q and colored red

Analysis of BFS

- Initialization is $\Theta(|V|)$.
- Each node can be added to the queue at most once (it needs to be white), and its adjacency list is searched only once. At most all adjacency lists are searched.
- If graph is undirected each edge is reached twice, so loop repeated at most $2|E|$ times.
- If a graph is directed each edge is reached exactly once. So the loop is repeated at most $|E|$ times.
- Worst case time $O(|V|+|E|)$

Depth First Search

- Goal: Explore every vertex and edge of G
- We go "deeper" whenever possible.
- *Directed* or *undirected* graph $G = (V, E)$.

Depth First Search

- Until there are no more undiscovered (unvisted) nodes
  - Pick an undiscovered node and start a depth first search from it
  - The search proceeds from the *most recently discovered* node to discover new nodes
  - When the last discovered node *v* is fully explored, backtrack to the node used to discover *v*
  - Eventually, the start node is fully explored

Depth First Search

- In this version *all* nodes are discovered even if the graph is directed, undirected, or not connected
- The algorithm saves:
  - A depth first *forest* of the edges used to discover new nodes.
  - Timestamps for the first time a node u is discovered $d[u]$ and the time when the node is fully explored $f[u]$

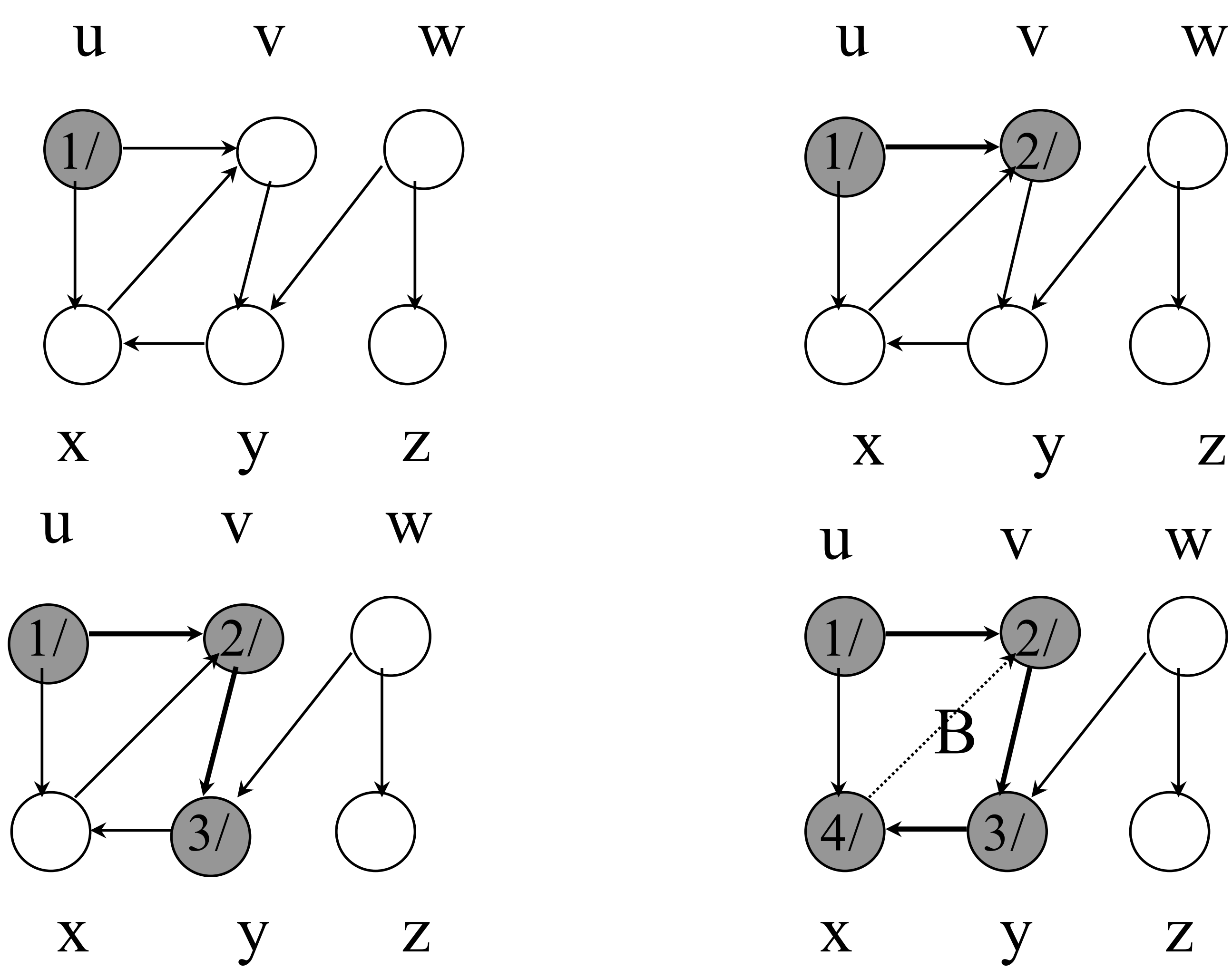
DFS

```
DFS (G:graph; var color:carray;  parent:parray);

  for each vertex u do
    color[u]=white;  parent[u]=nil;                Θ(V)
   end for

   for each vertex u do
    if color[u] == white then
        DFS-Visit(u);
    end if
   end for

end DFS
```

DFS-Visit(u)

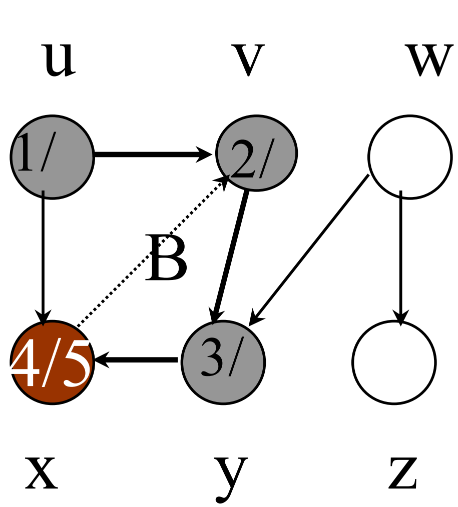
```
DFS-Visit(u)
{
    color[u]=gray;

    for each v in adj[u] do
        if color[v] = white {
            parent[v] = u;
            DFS-Visit(v);
        }

    color[u] = red;
}
```

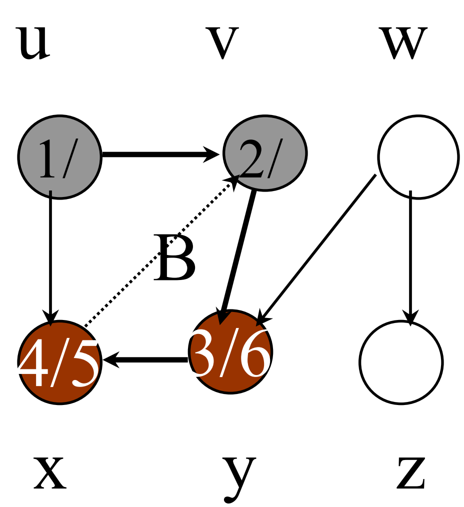
DFS example (1)



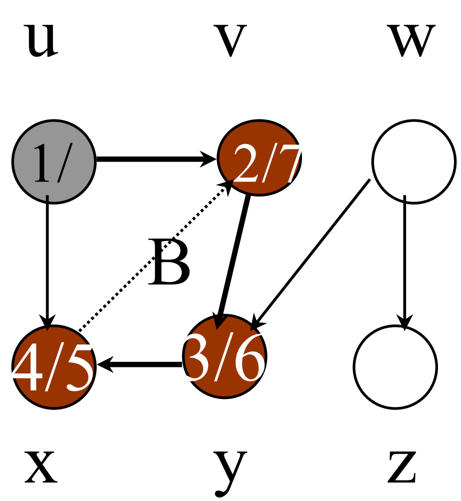
B: Back edge (edge from a node to one of its ancestors)
If back edge → cycle

DFS example (2)

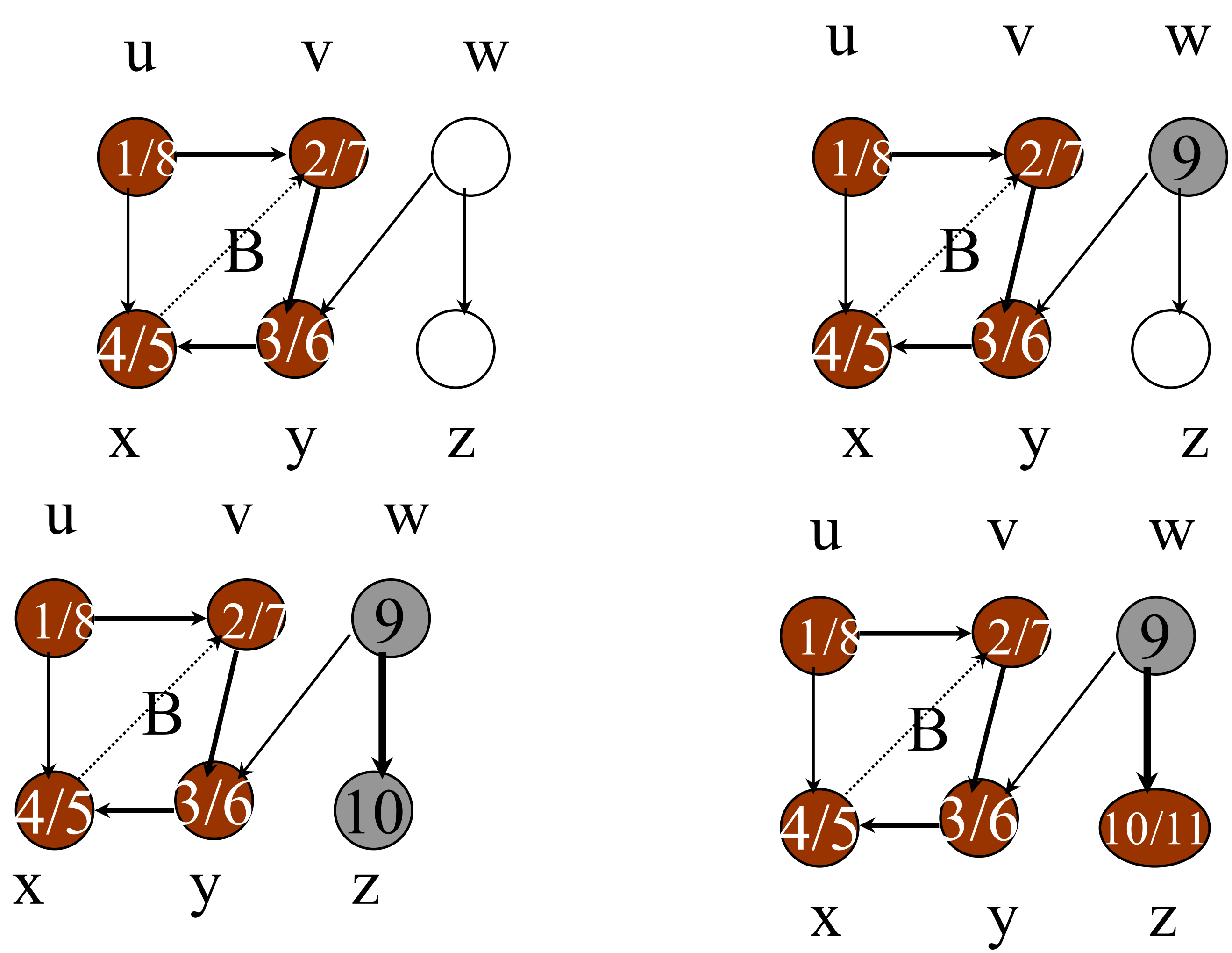




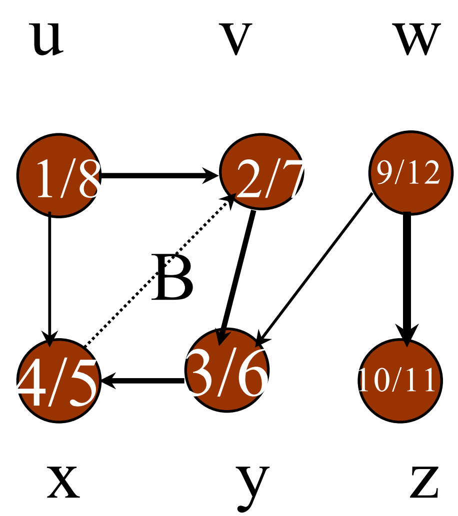
B: back edge (edge from a node to one of its ancestors)



DFS example (3)



DFS example (4)



Analysis

- DFS is $\Theta(|V|)$ (excluding the time taken by the DFS-Visits).
- DFS-Visit is called once for each node v. Its *for* loop is executed $|adj(v)|$ times. The DFS-Visit calls for all the nodes take $\Theta(|E|)$.
- **Worst case time $\Theta(|V|+|E|)$**

Some applications

- **Is undirected G connected?**
  - Do DFS-Visit(v). (Or, do BFS.) If all vertices are reached, return yes. Otherwise, return no. → O(V + E)

- **Find connected components.**
  - Do DFS. Assign a unique component number to the nodes in a single component. $\Theta(V+E)$

More applications

- **Does directed G contain a directed cycle?**
  - Do DFS. If there is one or more back edges, then the answer is yes. Time O(V+E).
  - **Back edges** - edges from a node to an **ancestor** in the tree.
  - Edge (u, v) is:
    - Tree edge – if v is white
    - Back edge – if v is gray
- **Does undirected G contain a cycle?**
  - Same as directed but be careful not to consider (u,v) and (v, u) a cycle.
- **Is undirected G a tree?**
  - Do DFS-Visit(v). If all vertices are reached and there is no back edge and G has |V|-1 edges in total, then G is a tree. O(V)