Recurrences

Goal

- Learn how to design and analyze recursive algorithms
- Learn when to use or not to use recursive algorithms
- Derive & solve recurrence equations to analyze recursive algorithms

Recursion

- What is the recursive definition of *n*! ?

$$n! = \begin{cases} 1 & \text{if } n \text{ is 0 or 1} \\ n * ((n-1)!) & \text{otherwise} \end{cases}$$

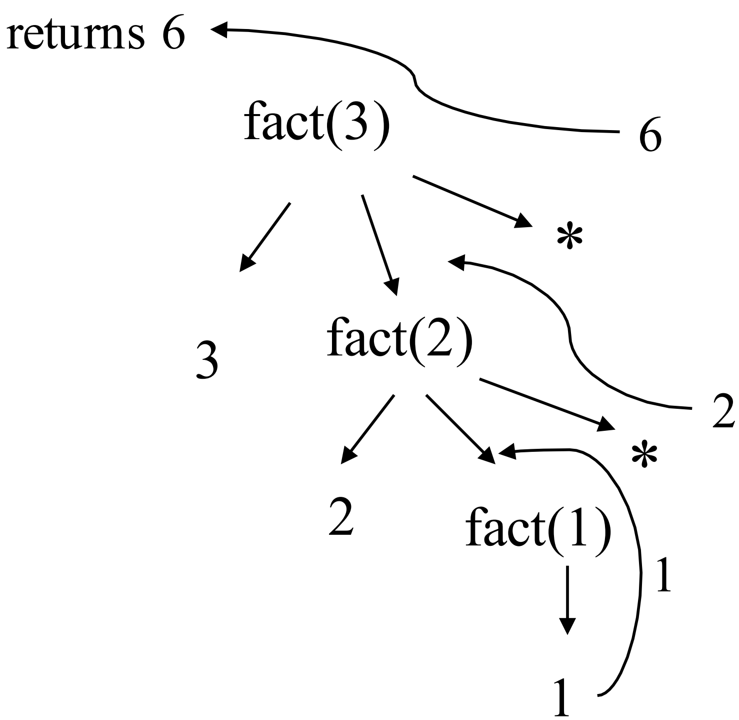
- Program

```
fact(n) {
   if (n<=1) return 1;
   else return n*fact(n-1);
}
// Note '*' is done after returning from fact(n-1)
```

Recursive algorithms

- A recursive algorithm typically contains recursive calls to the same algorithm
- In order for the recursive algorithm to terminate, it must contain code to directly solve some "base case(s)"with no recursive calls
- We use the following notation:
  - *DirectSolutionSize* is the "size" of the base case
  - *DirectSolutionCount* is the number of operations done by the "direct solution"

A Call Tree for fact(3)




```
int fact(int n) {
      if (n<=1) return 1;
else return n*fact(n-1);
}
```

The Run Time Environment

- When a function is called an activation records('ar') is created and pushed on the program stack.
- The activation record stores copies of local variables, pointers to other 'ar' in the stack and the return address.
- When a function returns the stack is popped.

Goal: Analyzing recursive algorithms

- Until now we have only analyzed (derived the count of) non-recursive algorithms.

- In order to analyze recursive algorithms, we must learn to:
    - Derive the recurrence equation from the code
    - Solve recurrence equations

Deriving a Recurrence Equation for a Recursive Algorithm

- Our goal is to compute the count (Time) $T(n)$ as a function of n, where n is the size of the problem
- We will first write a recurrence equation for $T(n)$

  For example, $T(n)=T(n-1)+1$ and $T(1)=0$

- Then we will solve the recurrence equation. What's the solution to $T(n)=T(n-1)+1$ and $T(1)=0$?

Deriving a Recurrence Equation for a Recursive Algorithm

1. Determine the "size of the problem". The count T is a function of this *size*

2. Determine *DirectSolSize*, such that for *size* ≤ *DirectSolSize* the algorithm computes a direct solution, with the *DirectSolCount(s).*

$$T(size) = \begin{cases} DirectSolCount & size \le DirectSolSize \\ GeneralCount & \text{otherwise} \end{cases}$$

Deriving a Recurrence Equation for a Recursive Algorithm

To determine *GeneralCount*:

3. Analyze the total number of recursive calls, *k*, done by a single call of the algorithm and their counts, $T(n_1), \ldots, T(n_k)$ → *RecursiveCallSum* = $\sum_{i=1}^{k} T(n_i)$

4. Determine the "non recursive count" $t(size)$ done by a single call of the algorithm, i.e., the amount of work, excluding the recursive calls done by the algorithm

$$T(size) = \begin{cases} DirectSolCount & size \le DirectSolSize \\ RecursiveCallSum + t(size) & \text{otherwise} \end{cases}$$

Deriving *DirectSolutionCount* for *Factorial*

```
int fact(int n) {
    if (n<=1) return 1;
    else  return n*fact(n-1); }
```

1. *Size* = n
2. *DirectSolSize is* 1 because n<=1
3. *DirectSolCount* is $\theta(1)$

   The algorithm does a small constant number of operations (comparing n to 1, and returning)

Deriving a *GeneralCount* for Factorial

```
int fact(int n) {
  if (n<=1) return 1;
  // Note '*' is done after returning from fact(n-1)
  else
    return n * fact(n-1);
```

Operations counted in t(n)

The only recursive call, requiring $T(n - 1)$ operations

3. *RecursiveCallSum* = $T(n - 1)$

4. $t(n) = \theta(1)$ `(if, *, -, return)`

$$\boxed{T(n) = \begin{cases} \Theta(1) & \text{for } n \leq 1 \\ T(n-1) + \Theta(1) & \text{for } n > 1 \end{cases}}$$

Reminder: Pitfall

- Is the complexity of recursive factorial $\Theta(n)$? Is it a linear time algorithm then?
- No! Recall for algorithms handling arbitrarily big numbers, we need to consider the input size in terms of the number of bits needed to express the input
- In recursive factorial, we have *only one input data, n, but the counts are proportional to the magnitude of $n \geq 2^{s-1}$ where s is the number of bits used to express n* $(s = \lfloor \lg n \rfloor + 1)$
- So, it has *exponential* time complexity! (Chap. 1)

Solving recurrence equations

- Techniques for solving recurrence equations:
  - Recursion tree
  - *Iteration method*
    - $T(n) = T(n-1) + c$
    - $T(n) = T(n/2) + c$ *(See the next slide)*
  - *Master Theorem*
- We discuss these methods with examples.

Iteration for binary search

$$
\begin{aligned}
W(n) &= 1 + W(\lfloor n/2 \rfloor) \\
&= 1 + (1 + W(\lfloor \lfloor n/2 \rfloor / 2 \rfloor)) = 2 + W(\lfloor n/4 \rfloor) \\
&= 2 + (1 + W(\lfloor n/8 \rfloor)) = 3 + W(\lfloor n/8 \rfloor) \\
&\ \dots \\
&= k + W(\lfloor n/2^k \rfloor) = k + W(1) = k + 1 \\
&= \lfloor \lg n \rfloor + 1 \in \Theta(\lg n)
\end{aligned}
$$

Deriving the count using the recursion tree method

- Recursion trees provide a convenient way to represent the unrolling of a recursive algorithm
- It is not a formal proof but a good technique to compute the count.
- Once the tree is generated, each node contains its "non recursive number of operations" $t(n)$ or *DirectSolutionCount*
- The count is derived by summing the "non recursive number of operations" of all the nodes in the tree
- For convenience, we usually compute the sum for all nodes at each given depth, and then sum these sums over all depths.

Building the recursion tree

- The initial recursion tree has a single node containing two fields:
  - The recursive call, (for example *Factorial*(n)) and
  - the corresponding count $T(n)$.
- The tree is generated by:
  - unrolling the recursion of the node depth 0,
  - then unrolling the recursion for the nodes at depth 1, etc.
- The recursion is unrolled as long as the size of the recursive call is greater than *DirectSolutionSize*

Building the recursion tree

- When the "recursion is unrolled", each current leaf node is substituted by a subtree containing a root and a child for each recursive call done by the algorithm.
  - The root of the subtree contains the recursive call, and the corresponding "non recursive count".
  - Each child node contains a recursive call, and its corresponding count.
- The unrolling continues, until the "size" in the recursive call is *DirectSolutionSize*
- Nodes with a call of *DirectSolutionSize*, are not "unrolled", and their count is replaced by *DirectSolutionCount*

Example: Recursive factorial

$Factorial(n)$	$T(n)$

- Initially, the recursive tree is a node containing the call to *Factorial*(*n*), and count *T*(n).

- When we unroll the computation this node is replaced with a subtree containing a root and one child:

•The root of the subtree contains the call to *Factorial*(*n*) , and the “non recursive count” for this call $t(n)= \Theta(1)$.

•The child node contains the recursive call to *Factorial*(*n*-1), and the count for this call, *T*(*n*-1).

After the first unrolling

		depth	nd	$T(n)$
$Factorial(n)$	$t(n) = \Theta(1)$	0	1	$\Theta(1)$
$Factorial(n\text{-}1)$	$T(n\text{-}1)$	1	1	$T(n\text{-}1)$

nd denotes the number of nodes at that depth

$$T(n) = \begin{cases} \Theta(1) & \text{for } n \le 1 \\ T(n-1) + \Theta(1) & \text{for } n > 1 \end{cases}$$

After the second unrolling

		depth	nd	$T(n)$
$Factorial(n)$	$t(n) = \Theta(1)$	0	1	$\Theta(1)$
$Factorial(n\text{-}1)$	$t(n\text{-}1) = \Theta(1)$	1	1	$\Theta(1)$
$Factorial(n\text{-}2)$	$T(n\text{-}2)$	2	1	$T(n\text{-}2)$

$$T(n) = \begin{cases} \Theta(1) & \text{for } n \le 1 \\ T(n-1) + \Theta(1) & \text{for } n > 1 \end{cases}$$

After the third unrolling

		depth	nd	$T(n)$
$Factorial(n)$	$t(n) = \Theta(1)$	0	1	$\Theta(1)$
$Factorial(n\text{-}1)$	$t(n\text{-}1) = \Theta(1)$	1	1	$\Theta(1)$
$Factorial(n\text{-}2)$	$t(n\text{-}2) = \Theta(1)$	2	1	$\Theta(1)$
$Factorial(n\text{-}3)$	$T(n\text{-}3)$	3	1	$T(n\text{-}3)$

$$T(n) = \begin{cases} \Theta(1) & \text{for } n \le 1 \\ T(n-1) + \Theta(1) & \text{for } n > 1 \end{cases}$$

For *Factorial*

$DirectSolutionSize = 1$ and $DirectSolutionCount = \Theta(1)$

The recursion tree

		depth	nd	$T(n)$
$Factorial(n)$	$t(n) = \Theta(1)$	0	1	$\Theta(1)$
$Factorial(n\text{-}1)$	$t(n\text{-}1) = \Theta(1)$	1	1	$\Theta(1)$
$Factorial(n\text{-}2)$	$t(n\text{-}2) = \Theta(1)$	2	1	$\Theta(1)$
$Factorial(n\text{-}3)$	$t(n\text{-}3) = \Theta(1)$	3	1	$\Theta(1)$
$Factorial(1)$	$T(1) = \Theta(1)$	n-1	1	$T(1) = \Theta(1)$

The sum $T(n) = n^* \, \Theta(1) = \Theta(n)$

Divide and Conquer

- Basic idea: divide a problem into smaller portions, solve the smaller portions and combine the results (if necessary).
- Name some algorithms you already know that employ this technique.
- D&C is a **top down** approach. We often use recursion to implement D&C algorithms.
- The following is an "outline" of a divide and conquer algorithm

Divide and Conquer

- Let *size*(*I*) = *n*
- *DirectSolutionCount* = *DS*(*n*)
- *t*(*n*) = *D(n) + C(n) where:*
  - *D*(*n*) = instruction counts for dividing problem into subproblems
  - *C*(*n*) = instruction counts for combining solutions

$$T(n) = \begin{cases} DS(n) & \text{for } n \le DirectSolutionSize \\ \sum_{i=1}^{k} T(n_i) + D(n) + C(n) & \text{otherwise} \end{cases}$$

Divide and Conquer

- Main advantages
  - Code: simple
  - Algorithm: efficient
  - Implementation
    - Parallel computation is possible, e.g., parallel quick sort, parallel merge sort, parallel convex hull, etc.
    - Parallel algorithm is an advanced topic. If we have time, we can discuss a few representative parallel algorithms in the end of the semester

Binary search

- Assumption: The list $S[low \ldots high]$ is sorted, and x is the search key
- If the search key *x* is in the list, *x* == S[*i*], and the index *i* is returned.
- If *x* is not in the list a *NoSuchKey* is returned

Binary search

- The problem is divided into 3 cases
  - $x=S[mid]$, $x\in S[low,..,mid-1]$, $x\in S[mid+1,..,high]$

- The first case $x=S[mid]$ is easily solved

- The other cases
  $x\in S[low,..,mid-1]$, or $x\in S[mid+1,..,high]$ require a recursive call

- When the array is empty the search terminates with a "non-index value"

```
BinarySearch(S, x, low, high)
    if low > high then
        return NoSuchKey
    else
        mid ← floor ((low+high)/2)
        if (x == S[mid])
            return mid
        else if  (x < S[mid]) then
            return BinarySearch(S, x, low, mid-1)
        else
            return BinarySearch(S, x,  mid+1, high)
```

Worst case analysis

- A worst input (what is it?) causes the algorithm to keep searching until low>high
- Assume $2^k \leq n < 2^{k+1}$ $k = \lfloor \lg n \rfloor$
- $T(n)$: worst case number of comparisons for the call to $BS(n)$

$$T(n) = \begin{cases} 0 \text{ for } & n = 0 \\ 1 \text{ for } & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for } & n > 1 \end{cases}$$

Recursion tree for BinarySearch (*BS*)

$BS(n)$	$T(n)$

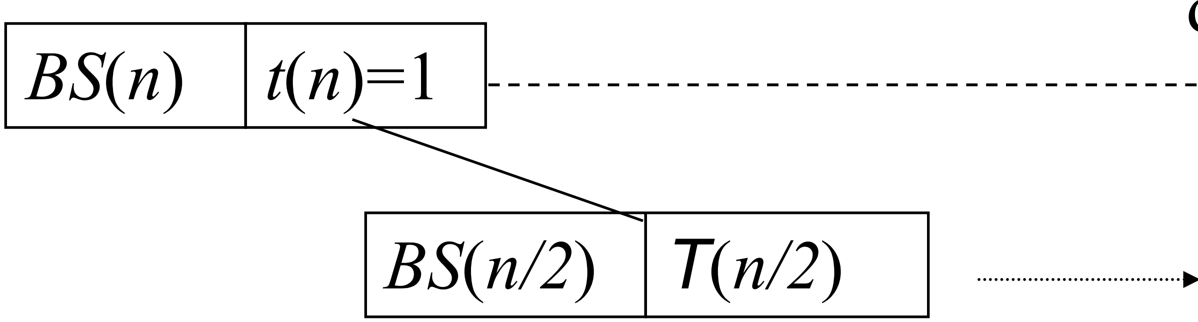
- Initially, the recursive tree is a node containing the call to *BS*(*n*), and total amount of work in the worst case, *T*(n).

- When we unroll the computation this node is replaced with a subtree containing a root and one child:

•The root of the subtree contains the call to *BS*(*n*) , and the “nonrecursive work” for this call *t*(*n*).

•The child node contains the recursive call to *BS*(*n*/2), and the total amount of work in the worst case for this call is *T*(*n*/2).

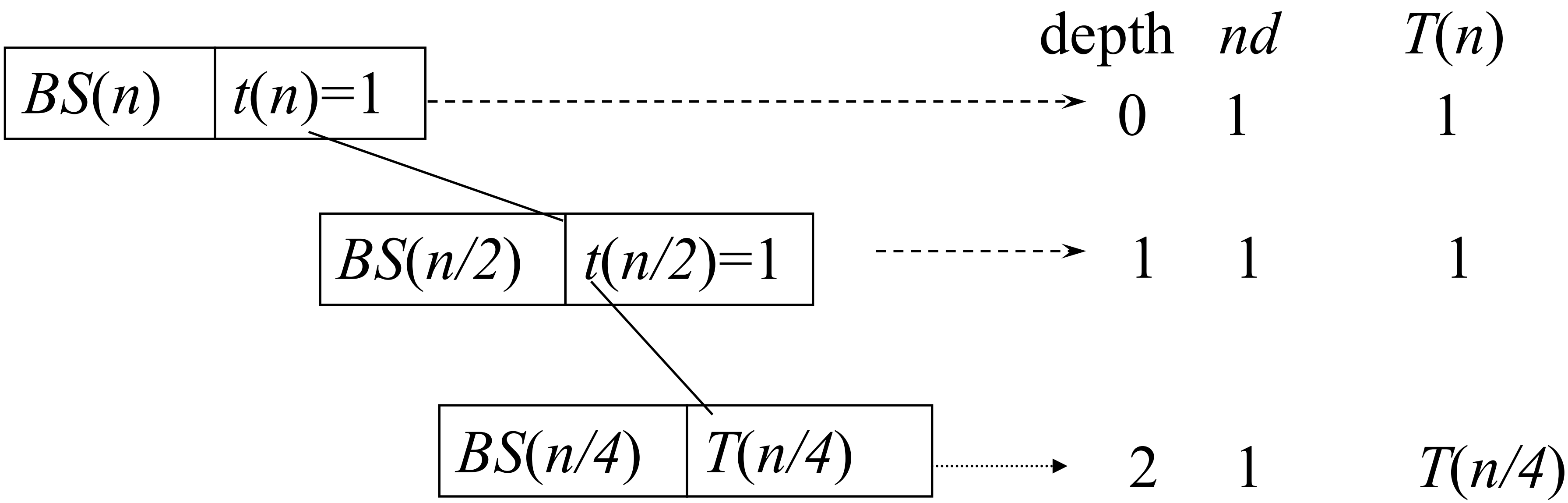
After first unrolling



depth	nd	$T(n)$
0	1	1
1	1	$T(n/2)$

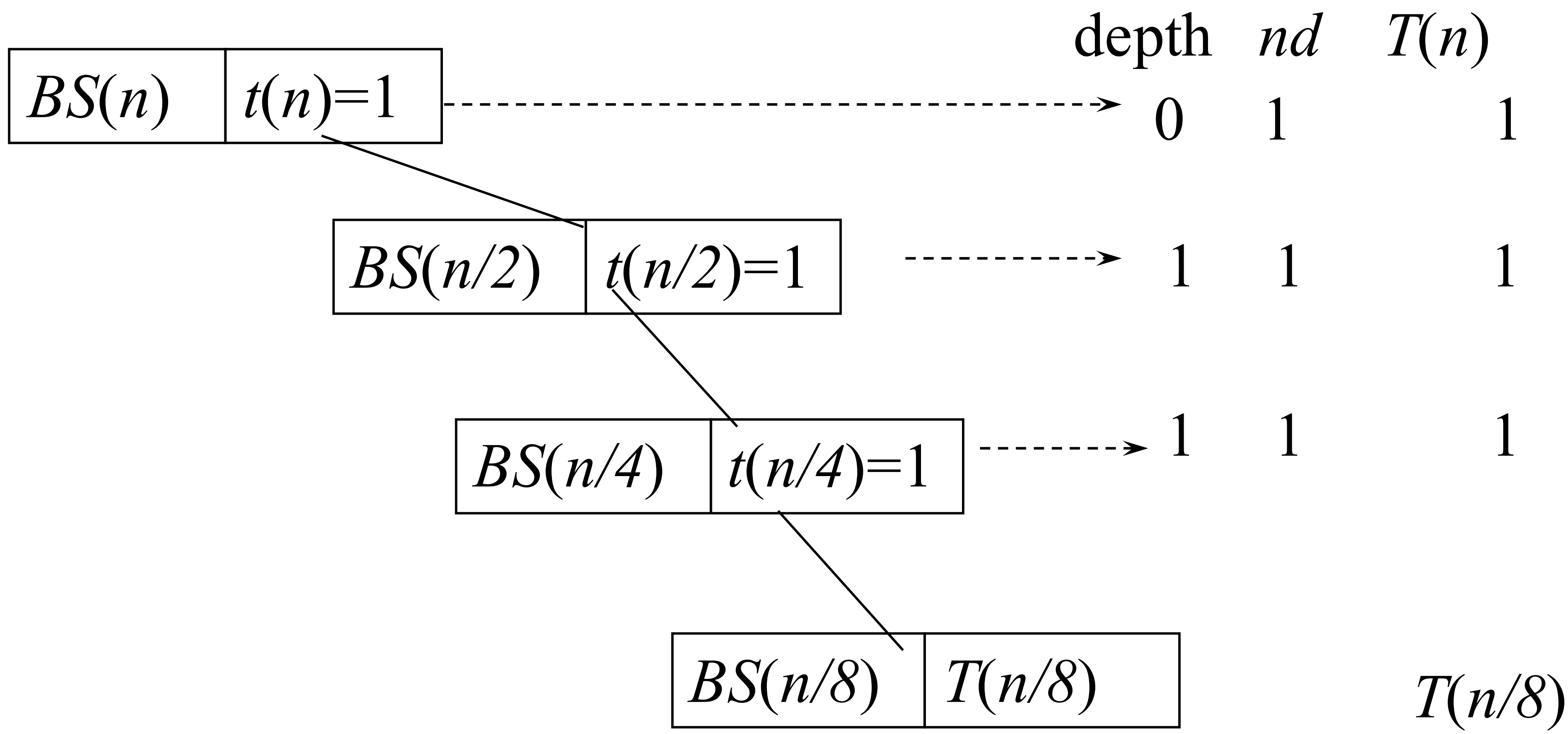
$$T(n) = \begin{cases} 0 \text{ for } & n = 0 \\ 1 \text{ for } & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for } & n > 1 \end{cases}$$

After second unrolling



$$T(n) = \begin{cases} 0 \text{ for } & n = 0 \\ 1 \text{ for } & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for } & n > 1 \end{cases}$$

After third unrolling

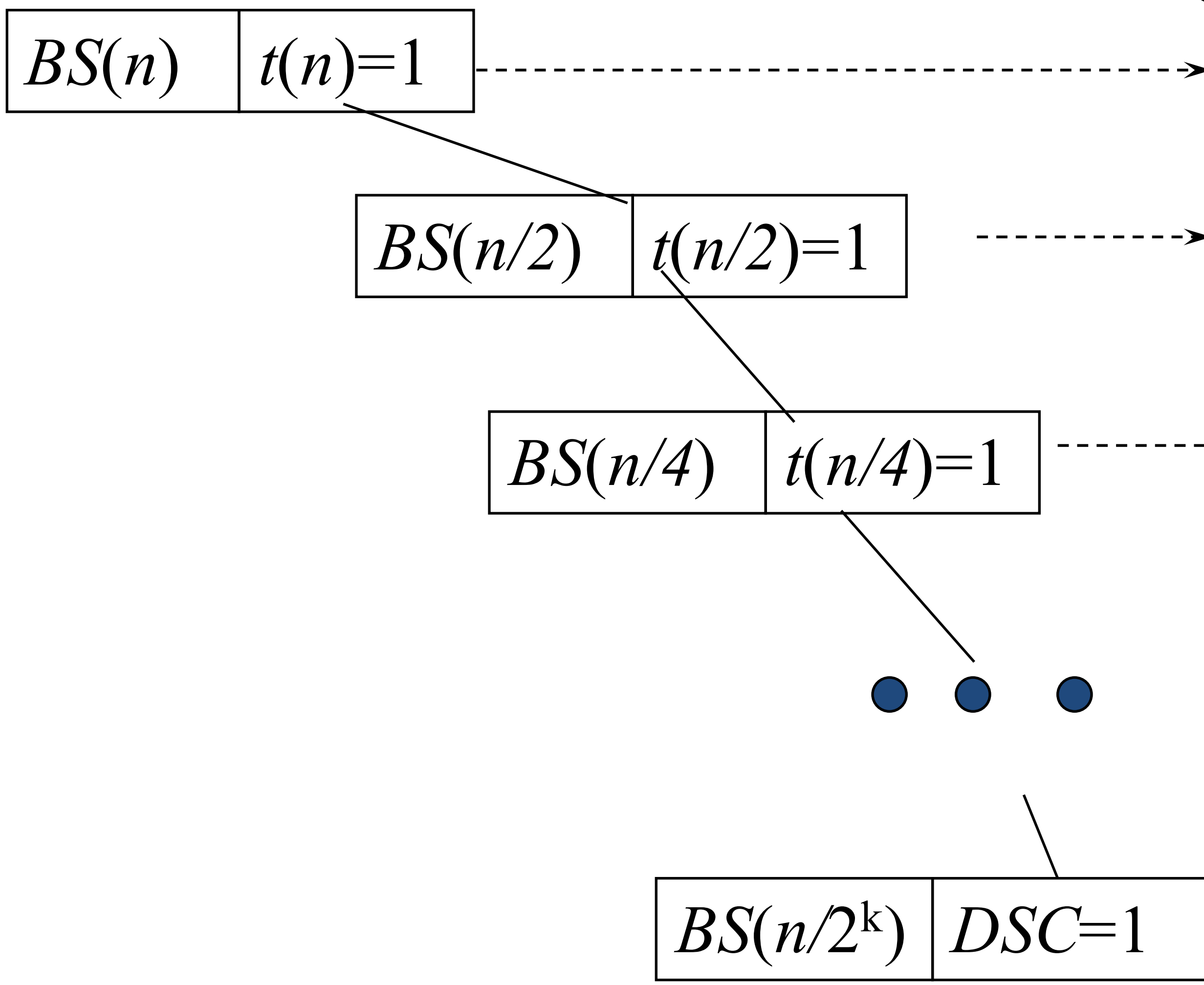


For *BinarySearch*, *DirectSolutionSize* = 0 or 1
and *DirectSolutionCount* = 0 for 0 and 1 for 1

Terminating the unrolling

- Let $2^k \le n < 2^{k+1}$
- $k = \lfloor \lg n \rfloor$
- When a node has a call to $BS(n/2^k)$, (or to $BS(n/2^{k+1})$):
  - The size of the list is *DirectSolutionSize* since $\lfloor n/2^k \rfloor = 1$, (or $\lfloor n/2^{k+1} \rfloor = 0$)
  - In this case the unrolling terminates, and the node is a leaf containing *DirectSolutionCount* (*DSC*) = 1, (or 0)

The recursion tree



depth	nd	$T(n)$
0	1	1
1	1	1
2	1	1
k	1	1

$$T(n)=k+1=\lfloor \lg n \rfloor+1$$

Merge Sort

Input: S of size n.

Output: a permutation of S, such that if $i > j$ then $S[\, i\,] \geq S[\, j\,]$

Divide: If S has at least 2 elements, divide it into S_1 *and* S_2. S_1 contains the the first $\lceil n/2 \rceil$ elements of S. S_2 has the last $\lfloor n/2 \rfloor$ elements of S.

Recursion: Recursively sort S_1 and S_2.

Conquer: Merge sorted S_1 and S_2 into S.

Merge Sort Example

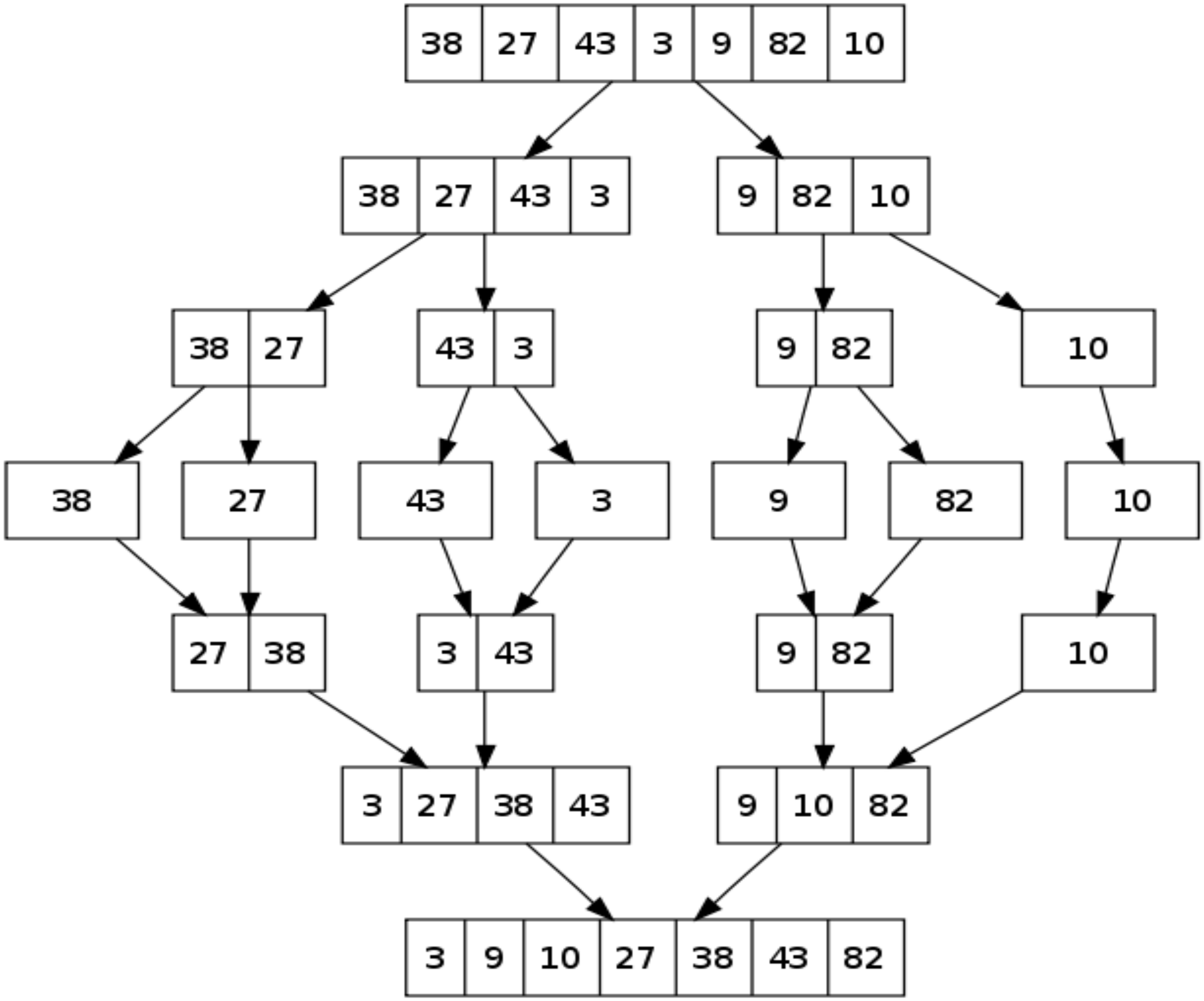


Image source: http://en.wikipedia.org/wiki/File:Merge_sort_algorithm_diagram.svg

Deriving a recurrence equation for Merge Sort

```
Sort(S)
    if (n ≥ 2)
        // Divide S into S1 and S2
        Sort(S1)                // recursion
        Sort(S2)                // recursion
        Merge( S1, S2, S )      // conquer
```

DirectSolutionSize is $n < 2$

DirectSolutionCount is $\theta(1)$

RecursiveCallSum is $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$

The non-recursive count $t(n) = \theta(n)$ ➔ Merge

Merge Sort Algorithm

Algorithm 2.2

Mergesort

Problem: Sort n keys in nondecreasing sequence.

Inputs: positive integer n, array of keys S indexed from 1 to n.

Outputs: the array S containing the keys in nondecreasing order.

```
void mergesort (int n, keytype S[])
{
  if (n>1) {
     const int h = ⌊ n/2⌋, m = n − h;
     keytype U[1..h], V[1..m];
     copy S[1] through S[h] to U[1] through U[h];
     copy S[h+1] through S[n] to V[1] through V[m];
     mergesort(h, U);
     mergesort(m, V);
     merge(h, m, U, V, S);
  }
}
```

Merge Algorithm

Algorithm 2.3

Merge

Problem: Merge two sorted arrays into one sorted array.

Inputs: positive integers h and m, array of sorted keys U indexed from 1 to h, array of sorted keys V indexed from 1 to m.

Outputs: an array S indexed from 1 to $h + m$ containing the keys in U and V in a single sorted array.

```
void merge (int h, int m, const keytype U[],
                          const keytype V[],
                                keytype S[])
{
  index i, j, k;

  i = 1; j = 1;  k = 1;
  while (i <= h && j <= m){
     if (U[i] < V[j]) {
        S[k] = U[i];
        i++;
     }
     else {
        S[k] = V[j];
        j++;
     }
     k++;
  }
  if (i>h)
     copy V[j] through V[m] to S[k] through S[h+m];
  else
     copy U[i] through U[h] to S[k] through S[h+m];
}
```

Merge Example

k	U	V	S(Result)
1	**10** 12 20 27	**13** 15 22 25	10
2	10 **12** 20 27	**13** 15 22 25	10 12
3	10 12 **20** 27	**13** 15 22 25	10 12 13
4	10 12 **20** 27	13 **15** 22 25	10 12 13 15
5	10 12 **20** 27	13 15 **22** 25	10 12 13 15 20
6	10 12 20 **27**	13 15 **22** 25	10 12 13 15 20 22
7	10 12 20 **27**	13 15 22 **25**	10 12 13 15 20 22 25
—	10 12 20 27	13 15 22 25	10 12 13 15 20 22 25 27 $\leftarrow$ Final values

Merge Sort

- In-place sort: does not use any extra space beyond that needed to store the input.
- The previous merge sort algorithm is NOT in-place sort.
  - Why?
- Extra space: $n\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)=2(n-1)$.

Merge Sort (with less extra space)

Algorithm 2.4

Mergesort 2

Problem: Sort n keys in nondecreasing sequence.

Inputs: positive integer n, array of keys S indexed from 1 to n.

Outputs: the array S containing the keys in nondecreasing order.

```
void mergesort2 (index low, index high)
{
  index mid;

  if (low < high) {
      mid = ⌊(low + high)/2⌋;
      mergesort2(low, mid);
      mergesort2(mid + 1, high);
      merge2(low, mid, high);
  }
}
```

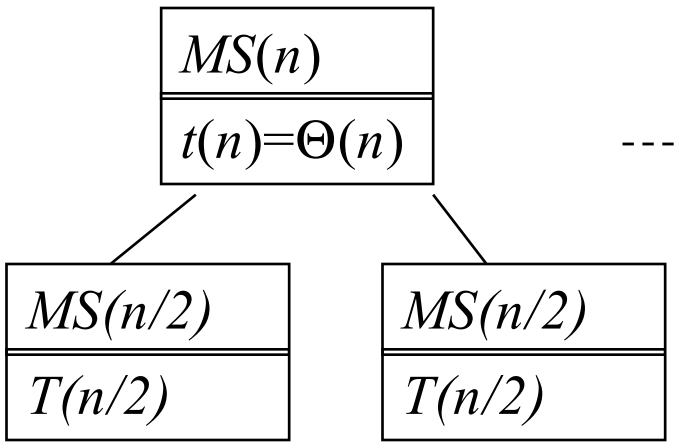
Merge (with less extra space)

```
void merge2 (index low, index mid, index high)
{
  index i, j, k;
  keytype U[low..high];    // A local array needed for the
                           // merging
  i = low; j = mid + 1; k = low;
  while (i ≤ mid && j ≤ high){
      if (S[i]<S[j]){
          U[k] = S[i];
          i++;
      }
      else{
          U[k] = S[j];
          j++;
      }
          k++;
  }
  if (i > mid)
      move S[j] through S[high] to U[k] through U[high];
  else
      move S[i] through S[mid] to U[k] through U[high];
  move U[low] through U[high] to S[low] through S[high];
}
```

Recurrence Equation (cont'd)

- The cost of division is O(1) and *merge* is $\Theta(n)$. So, the total cost for dividing and merging is $\Theta(n)$.
- The recurrence relation for the run time of MergeSort is:
  $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n)$.
  $T(0) = T(1) = \Theta(1)$
- The solution is $T(n) = \Theta(n \lg n)$

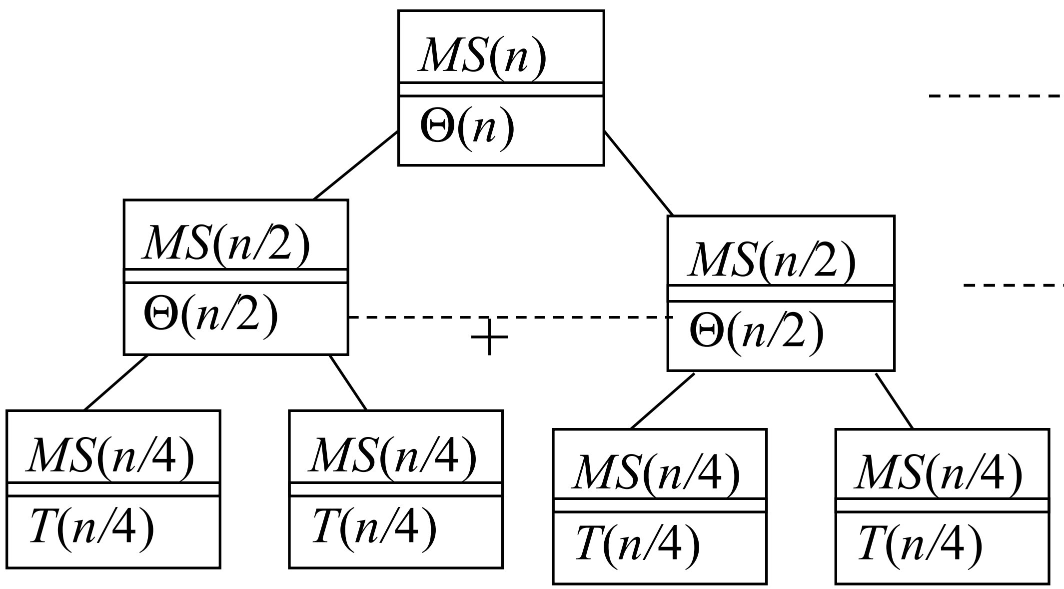
After first unrolling of mergeSort



depth	nd	$T(n)$
0	1	$\Theta(n)$
		$2T(n/2)$

$$T(n) = \begin{cases} 1 \text{ for } n \le 1 \\ 2T(n/2) + \theta(n) \text{ for } n > 1 \end{cases}$$

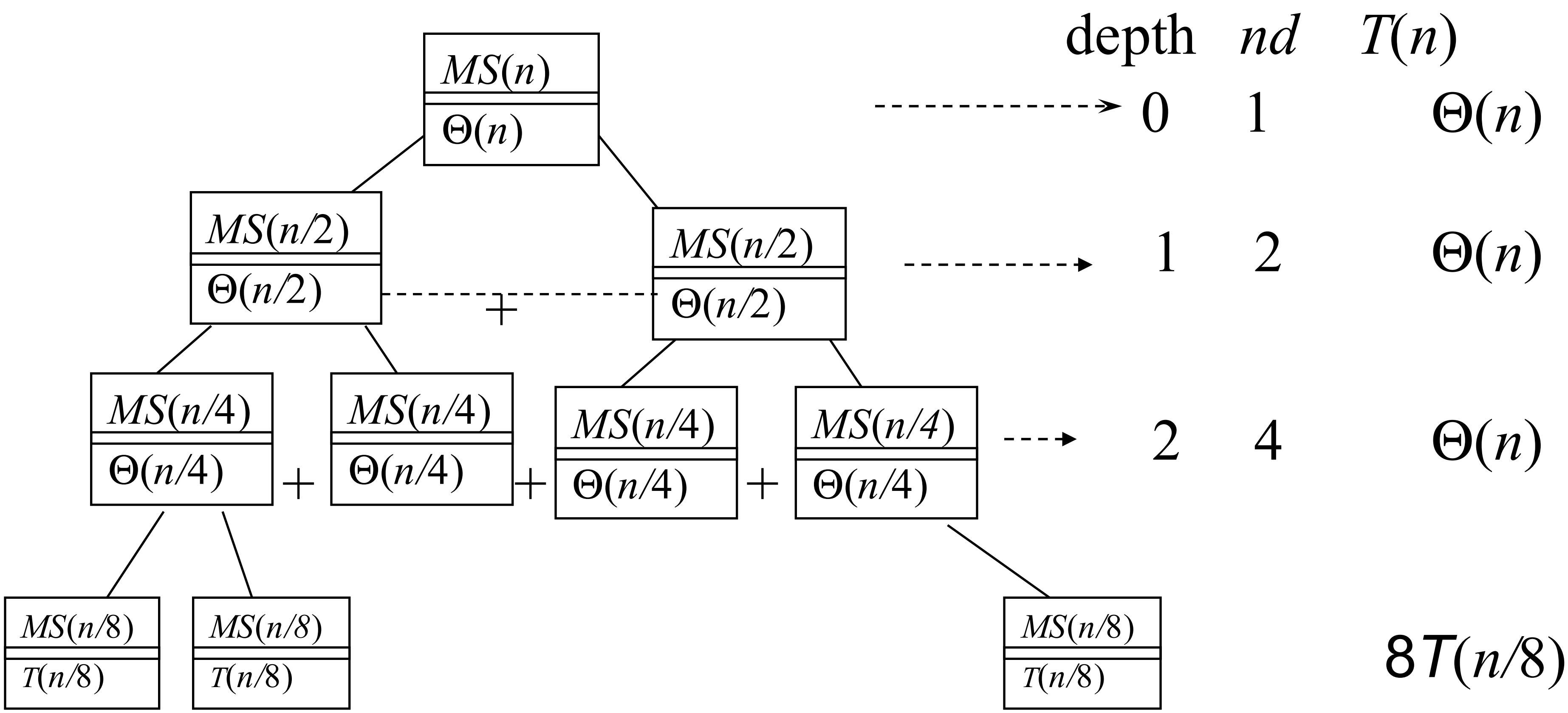
After second unrolling



depth	nd	$T(n)$
0	1	$\Theta(n)$
1	2	$\Theta(n)$
		$4T(n/4)$

$$T(n) = \begin{cases} 1 & \text{for } n \le 1 \\ 2T(n/2) + \theta(n) & \text{for } n > 1 \end{cases}$$

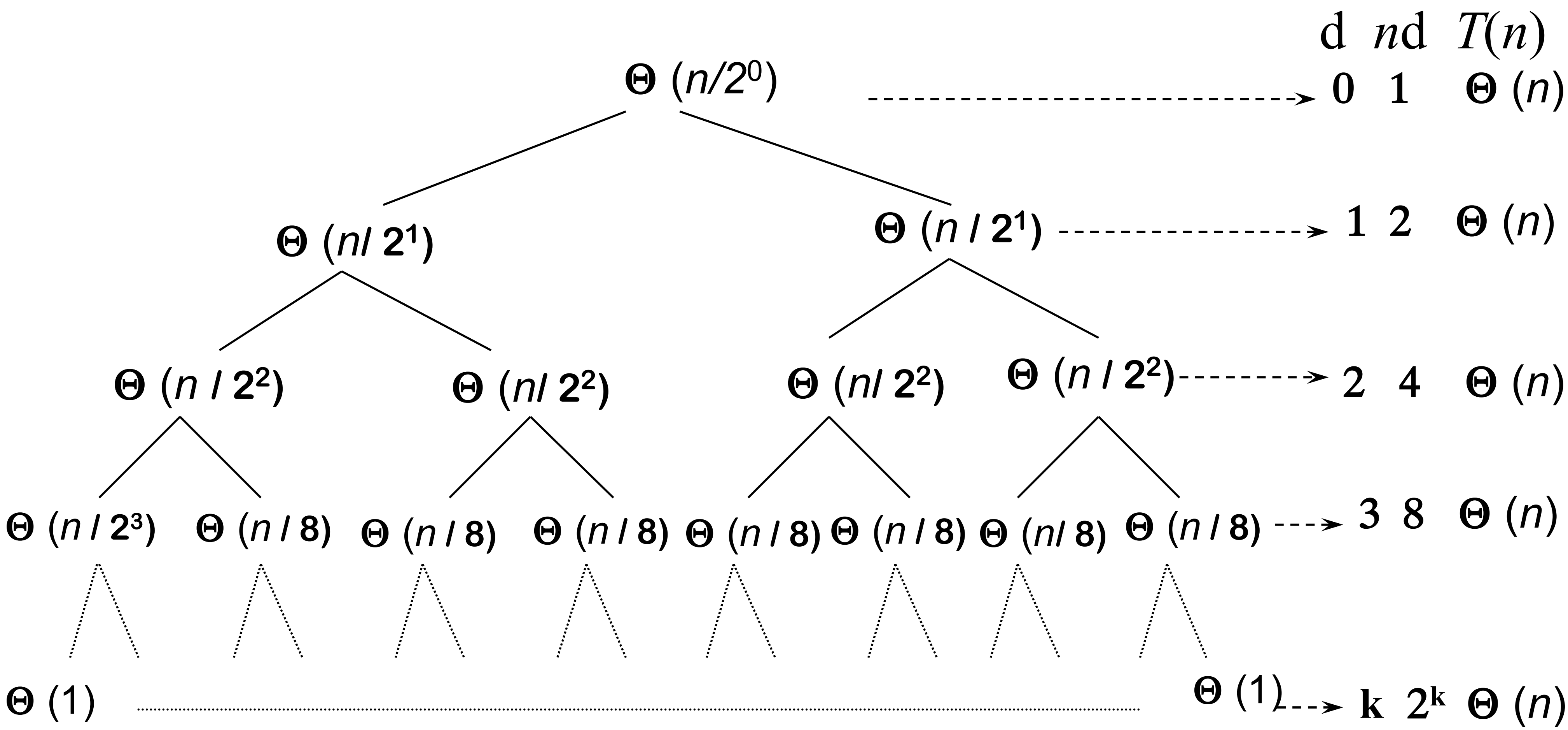
After third unrolling



Terminating the unrolling

- For simplicity let $n = 2^k$
- $\lg n = k$
- When a node has a call to *MergeSort*($n/2^k$):
  - The size of the list to merge sort is *DirectSolutionSize* since $n/2^k = 1$
  - In this case the unrolling terminates, and the node is a leaf containing *DirectSolutionCount* = $\Theta(1)$

The recursion tree



$$T(n) = (k+1)\,\Theta(n) = \Theta(n \lg n)$$

Master Theorem to Solve General Recurrence Equations

- Suppose that (T(n) is eventually nondecreasing):
  - $T(n) = aT(n/b) + cn^k$
  - $T(1) = d$

  where $b \geq 2$ and $k \geq 0$ are constant integers and a, c, d are constants such that $a > 0$, $b > 0$, and $d \geq 0$. Then,

  - $T(n) = \Theta(n^k)$ if $a < b^k$
  - $T(n) = \Theta(n^k \lg n)$ if $a = b^k$
  - $T(n) = \Theta(n^{\log_b a})$ if $a > b^k$

Master method examples

- Case 1:
  - $T(n) = 8T(n/4) + 5n^2$ for $n>1$, n is a power of 4
  - $T(1) = 3$

→ $a=8$, $b=4$, $k=2$

→ As $a < b^k$ (i.e., $8 < 4^2$), $T(n) = \Theta(n^2)$

- Case 2:
  - $T(n) = 8T(n/2) + 5n^3$ for $n>64$, n is a power of 2
  - $T(64) = 200$

→As $a = b^k$ (i.e., $8 = 2^3$), $T(n) = \Theta(n^3 \lg n)$

- Case 3:
  - – T(n) = 9T(n/3) + 5n for n > 1, n is a power of 3
  - – T(1) = 7

→a = 9, b = 3, k = 1

→Since a > b^k , $T(n) = \Theta(n^{\log_3 9}) = \Theta(n^2)$

Generalizing T(n) for n as a power of b

Theorem B.4

- Let b$\geq$ 2 be an integer, let f(n) be a smooth complexity function, and let T(n) be an eventually non-decreasing complexity function. If

$$T(n) \in \theta(f(n)) \quad \text{for n a power of b,}$$

then

$$T(n) \in \theta(f(n)).$$

$$T(n) = \begin{cases} 0 \text{ for } & n = 0 \\ 1 \text{ for } & n = 1 \\ 1 + T(\lfloor n/2 \rfloor) \text{ for } & n > 1 \end{cases}$$

- We can prove $T(n) = lgn + 1 \in \theta(lgn)$ for n a power of 2.
- Need to show T(n) is eventually non-decreasing in order to apply Theorem B.4 to conclude $T(n) = lgn + 1 \in \theta(lgn)$.
- $T(n) = lgn + 1$ is only true for n a power of 2.
- Example B.25 showed that T(n) is eventually non-decreasing using mathematical induction.
  - Non-decreasing: if $1 \leq k < n$, then $T(k) \leq T(n)$.
  - A variation of induction:
  - Assume for all $m \leq n$: if $1 \leq k < m$, then $T(k) \leq T(m)$.
  - Need to prove: for $m = n+1$: if $1 \leq k < m$, then $T(k) \leq T(m)$.

- Only need to show T(n) $\leq$ T(n+1).
- $T(n) = T(\lfloor \frac{n}{2} \rfloor)$+1
- $T(n+1) = T(\lfloor \frac{n+1}{2} \rfloor)$+1
- Can you show $T(\lfloor \frac{n}{2} \rfloor) \leq T(\lfloor \frac{n+1}{2} \rfloor)$?
- $\lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n+1}{2} \rfloor \leq$ n and induction hypothesis.