Dijkstra's Algorithm

Single source shortest paths for a directed graph with no negative edges

Single-Source Shortest Paths

- We want to find the shortest paths between Binghamton and New York City, Boston, and Washington DC. Given a US road map with all the possible routes how can we determine our shortest paths?

Single Source Shortest Paths Problem

- To solve this problem, we may use Floyd's algorithm that finds all pairs shortest paths via dynamic programming. But, this is an overkill, because we have a single source now.

- Floyd's algorithm is $\Theta(n^3)$. Can we solve the single source shortest paths problem faster than $\Theta(n^3)$?

Dijkstra's algorithm

- Given a weighted digraph and a vertex *s* in the graph, find a shortest path from *s* to an arbitrary node t
- Both for directed and undirected graphs
- No negative edges
- Graph must be connected

Dijkstra's shortest path algorithm

- **Dijkstra's algorithm solves the single source shortest path problem in 2 stages.**
  - Stage 1: A greedy algorithm computes the shortest *distance* from *s* to all other nodes in the graph and saves a *data structure*.
  - Stage 2 : Uses the *data structure* to find a shortest path from *s* to *t*.

Main idea

- Assume that the shortest distances from the starting node *s* to the rest of the nodes are
  $d(s, s) \le d(s, s_1) \le d(\mathrm{s}, \mathrm{s}_2) \le \ldots \le d(\mathrm{s}, \mathrm{s}_{n-1})$

- In this case a shortest path from s to s_i may include any of the vertices $\{s_1, s_2 \ldots s_{i-1}\}$ but cannot include any s_j where $j > i$.

- Dijkstra's main idea is to select the nodes and compute the shortest distances in the order s, s_1, s_2 ,…, s_{n-1}

Example

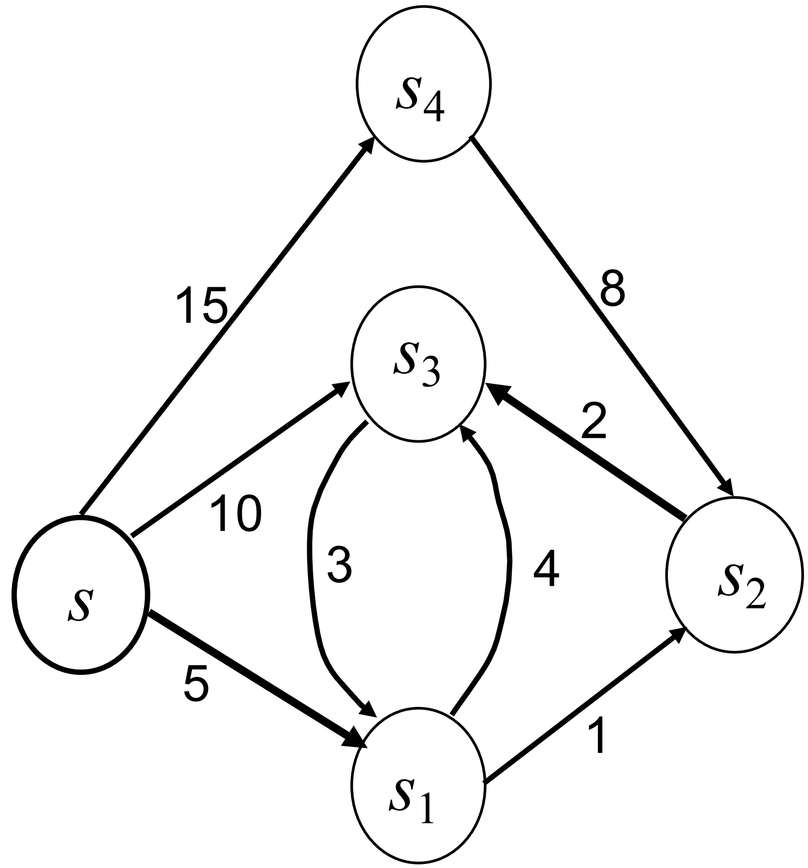
$d(s, s) = 0 \leq$

$d(s, s_1) = 5 \leq$

$d(s, s_2) = 6 \leq$

$d(s, s_3) = 8 \leq$

$d(s, s_4) = 15$



Note: The shortest path from s to s_2 includes s_1 as an intermediate node but cannot include s_3 or s_4.

Dijkstra's greedy selection rule

- Assume s_1, s_2 … s_{i-1} have been selected, and their shortest distances have been stored in Solution

- Select node s_i and save d(s, s_i) if s_i has the shortest distance from s on a path that may include only s_1, s_2 … s_{i-1} as intermediate nodes. We call such paths *special*

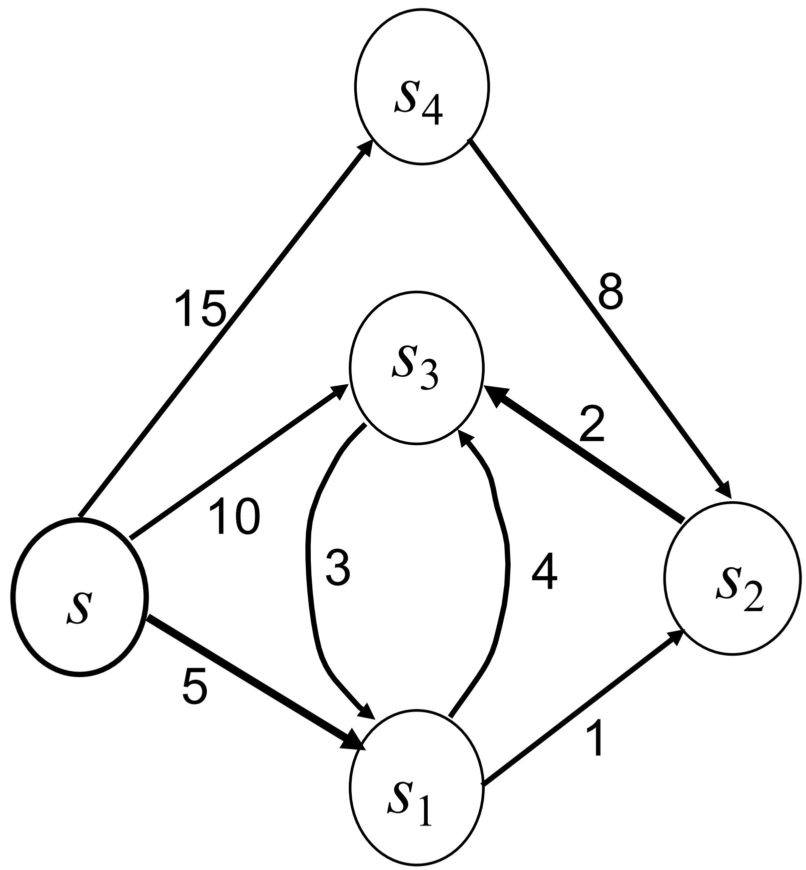
- To apply this selection rule efficiently, we need to maintain for each unselected node v the *distance of the shortest special path* from s to v, $D[v]$.

Application Example

Solution = {(s, 0)}
D[s_1]=5 for path [s, s_1]
D[s_2]= ∞ for path [s, s_2]
D[s_3]=10 for path [s, s_3]
D[s_4]=15 for path [s, s_4].

Solution = {(s, 0), (s_1, 5) }
D[s_2]= 6 for path [s, s_1, s_2]
D[s_3]=9 for path [s, s_1, s_3]
D[s_4]=15 for path [s, s_4]

Solution = {(s, 0), (s_1, 5), (s_2, 6) }
D[s_3]=8 for path [s, s_1, s_2, s_3]
D[s_4]=15 for path [s, s_4]



Solution = {(s, 0), (s_1, 5), (s_2, 6),(s_3, 8), (s_4, 15) }

Implementing the selection rule

- **Node *near* is selected and added to *Solution* if D(*near*) ≤ D(*v*) for any *v* ∉ *Solution*.**

$Solution = \{(s, 0)\}$

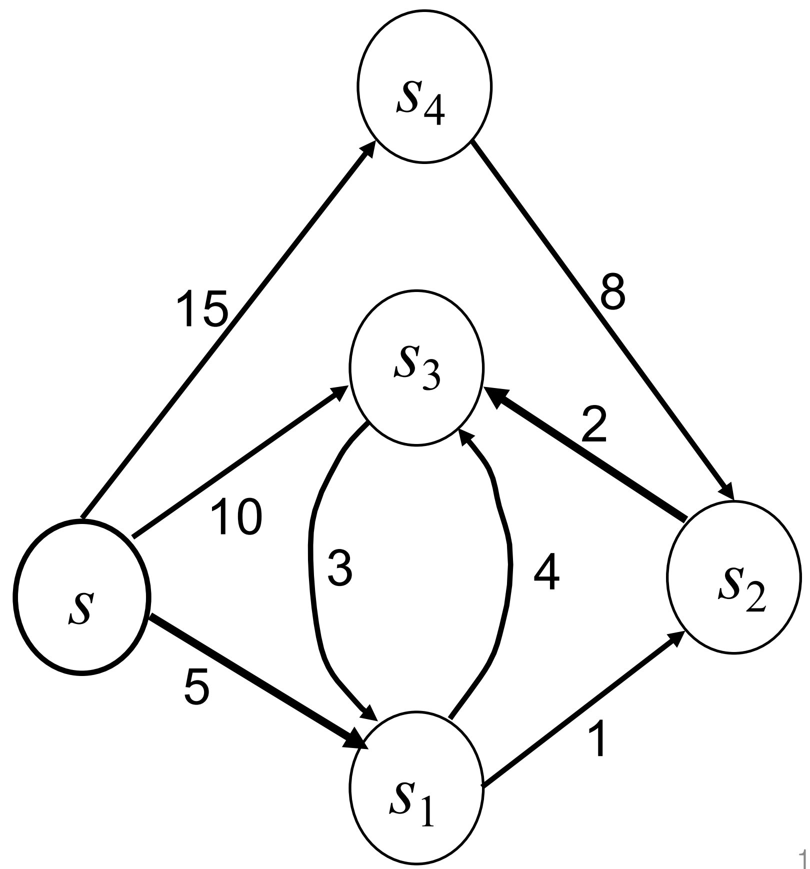
$D[s_1]=5 \leq D[s_2]=\infty$

$D[s_1]=5 \leq D[s_3]=10$

$D[s_1]=5 \leq D[s_4]=15$

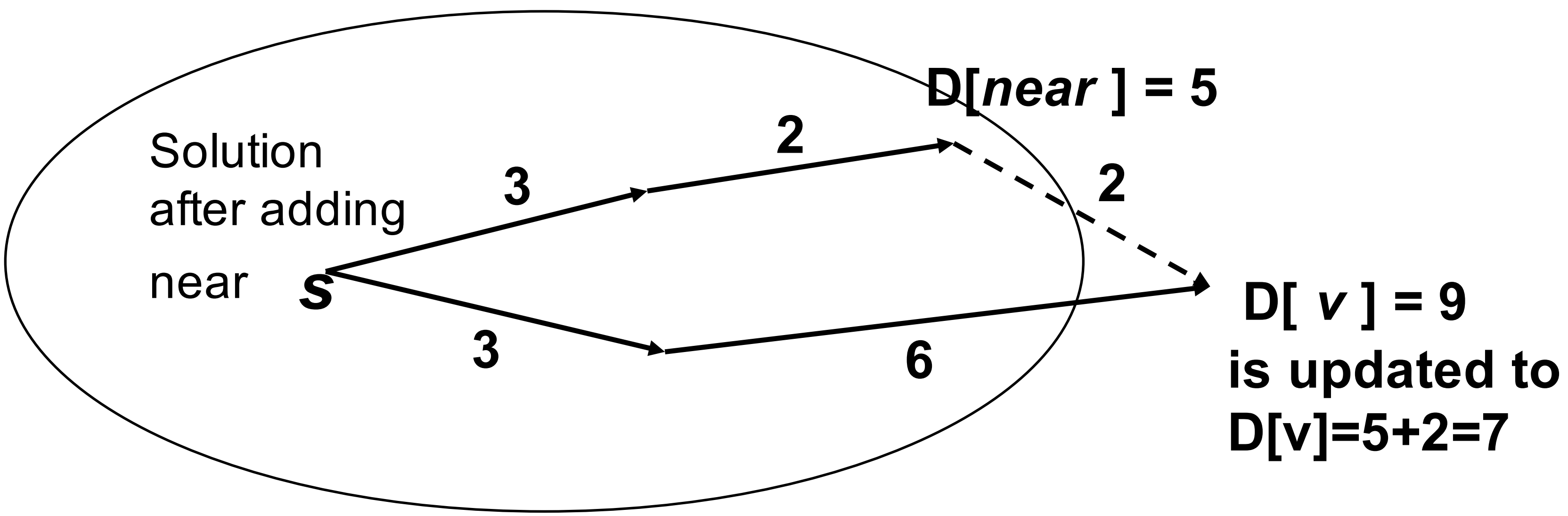
Node s_1 is selected

$Solution = \{(s, 0), (s_1, 5)\}$



Updating D[ ]

- **After adding *near* to *Solution*, D[*v*] of all nodes *v* $\notin$ *Solution are* updated if there is a shorter special path from *s* to *v* that contains node *near*, i.e., if (D[*near*] + w(*near*, *v*) < D[*v*]) then**
  D[v] = D[*near*] + w(*near*, *v*)

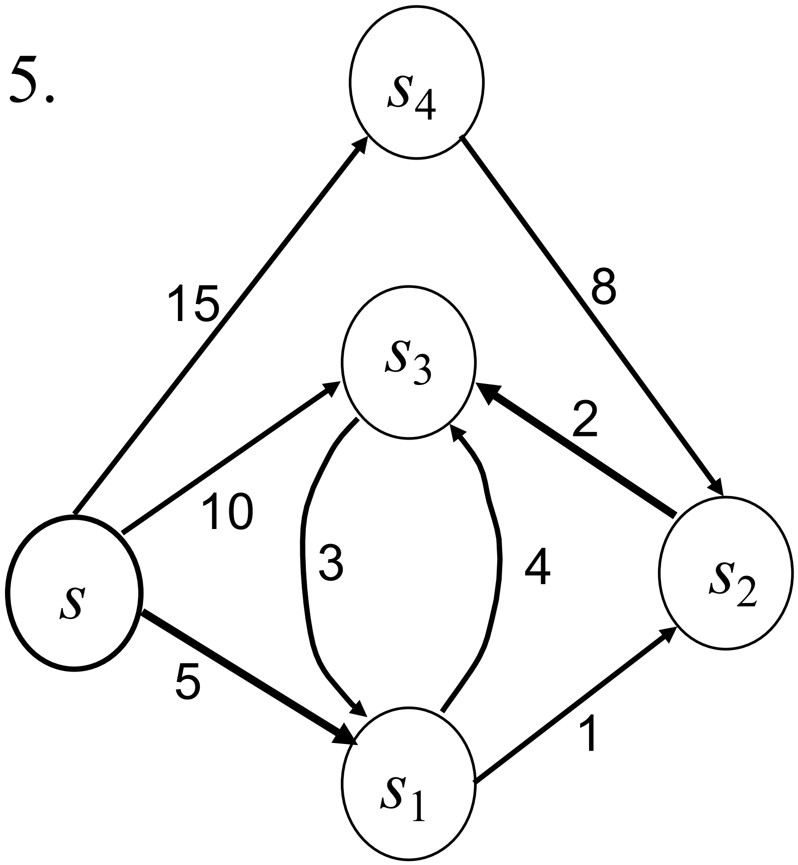


Example: Updating D

$Solution = \{(s, 0)\}$
$D[s_1]=5, D[s_2]=\infty , D[s_3]=10, D[s_4]=15.$

$Solution = \{(s, 0), (s_1, 5) \}$
$D[s_2]= D[s_1]+w(s_1,s_2)=5+1=6,$
$D[s_3]= D[s_1]+w(s_1,s_3)=5+4=9,$
$D[s_4]=15$

$Solution = \{(s, 0), (s_1, 5), (s_2, 6) \}$
$D[s_3]=D[s_2]+w(s_2, s_3)=6+2=8,$
$D[s_4]=15$



$Solution = \{(s, 0), (s_1, 5), (s_2, 6), (s_3, 8), (s_4, 15) \}$

Dijkstra's Algorithm for Finding the Shortest Distance from a Single Source

Dijkstra(G,s)

for each $v \in V$
    do $D[v] \leftarrow \infty$
$D[s] \leftarrow 0$
MH $\leftarrow$ make-MH(D,V) // MH: MinHeap
while MH $\neq \emptyset$
        $near \leftarrow$ MH.extractMin()
        for each $v \in$ Adj($near$)
            if $D[v] > \mathbf{D[near]} + w(near, v)$
            then $D[v] \leftarrow \mathbf{D[near]} + w(near, v)$
            MH.decreaseDistance(D[v], v)
return the label $D[u]$ of each vertex u

Time Complexity Analysis

1. **for each** $v \in V$
2. **do** $D[v] \leftarrow \infty$
3. $D[s] \leftarrow 0$
4. MH $\leftarrow$make-MH(D,V)

5. **while** MH $\neq \emptyset$
6. **do** $near \leftarrow$ MH.extractMin ()
7. **for each** $v \in$ Adj($near$)
8 **if** $D[v] > D[near] +$ $w(near, v)$
9. **then** $D[v] \leftarrow$ $D[near] + w(near, v)$
10. MH.decreaseDistance (D[v], v)
11. **return** the label $D[u]$ of each vertex u

Assume a node in *MH* can be accessed in $O(1)$

Using Heap implementation

Lines 1 - 4 run in $O(V)$

Max Size of *MH* is | V |

(5) Loop = $O(V)$

(6) O(lg V)

(5+6) $O(V \lg V)$

(7, 8, 9) are O(1) and executed O(E) times in total

(10) Decrease- Key operation on the heap takes $O(\lg V)$ time, and is executed O(E) times in total

→ O(E lg V)

So total time is $O(V \lg V + E \lg V)$ = O(E lg V)

Alternative way to implement Dijkstra's algorithm

- Use an array instead of a MinHeap
- Time Complexity
  - O(V) to extract min
  - O(1) for decreaseDistance
  - Thus, $O(V^2)$ in total