Exercise 1 (Section 2.1)**.** Recall that the r-th central moment of a discrete random variable Y described by a probability mass function $f(y)$ is given by

$$\mathbb{E}(Y^r) = \sum_{y \in P} y^r f(y).$$

The r-th moment about the mean $\mu = \mathbb{E}(Y)$ of Y is

$$\mathbb{E}((Y-\mu)^r) = \sum_{y \in P} (y-\mu)^r f(y).$$

Let $Y_1, Y_2, \ldots Y_n$ be discrete random variables described by a joint probability mass function $f(y_1, y_2, \ldots, y_n)$. For any function $g(Y_1, Y_2, \ldots, Y_n)$ its expectation value is

$$\mathbb{E}(g(Y_1, Y_2, \ldots, Y_n)) = \sum_{y_1, y_2, \ldots, y_n \in P} g(y_1, y_2, \ldots, y_n) f(y_1, y_2, \ldots, y_n).$$

Show that:

1. $\mathbb{E}(\alpha_0 + \alpha_1 Y_1^r + \alpha_2 Y_2^p) = \alpha_0 + \alpha_1 \mathbb{E}(Y_1^r) + \alpha_2 \mathbb{E}(Y_2^p)$, where $\alpha_0, \alpha_1, \alpha_2$ are constants. In other words, you need to show that $\mathbb{E}$ is a linear operator.
2. $\sigma^2 := \mathbb{E}((Y-\mu)^2) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$.
3. $\text{Cov}(Y_1, Y_2) := \mathbb{E}((Y_1 - \mu_1)(Y_2 - \mu_2)) = \mathbb{E}(Y_1 Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2)$.
4. $\text{Cov}(Y_1, Y_2) = 0$ if Y_1, Y_2 are independent random variables. Hint: use the property that the joint probability mass function factorizes as $f(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$ if Y_1 and Y_2 are independent random variables.

Exercise 2 (Section 2.3)**.** Let Y be a Poisson random variable parameterised by a constant θ. Its probability mass function is $f(y) = \theta^y e^{-\theta}/y!$ where $y = 0, 1, 2, \ldots$

1. Show that $\mathbb{E}(Y) = \theta$. Hint: use Taylor series of the exponential function

$$e^\theta = \sum_{i=0} \frac{\theta^i}{i!} = 1 + \theta/1! + \theta^2/2! + \ldots$$

2. Find the maximum likelihood estimate of θ. Hint: consider a sample y_1, y_2, $\ldots y_n$ drawn from Y_1, Y_2, $\ldots$, Y_n which are i.i.d. Poisson random variables parameterised by θ.
3. Minimize $S = \sum_{i=1}^n (y_i - \theta)^2$ to obtain the least squares estimate of θ.

Trivia: The Poisson distribution, for example, is appropriate for modeling the number of phone calls an office would receive during the noon hour, or the number of goals scored in each game in the World Cup, with θ being the expected value (average). You will study Poisson regression in the 3rd year “Linear Modelling” course.

Exercise 3 (Section 2.4).

1. Let $Y_1 \sim N(1,3)$ and $Y_2 \sim N(2,5)$ be independent random variables.
   Find the distributions of $W_1 = Y_1 + 2Y_2$ and $W_2 = 4Y_1 - Y_2$.

2. Let $Y_1 \sim N(0,1)$ and $Y_2 \sim N(1,9)$ be independent random variables.
   Find the distribution of $W = Y_1^2 + (Y_2 - 1)^2/9$.

3. Let $Y_1 \sim N(0,1)$ and $Y_2 \sim N(3,4)$ be independent random variables.
   Set $\boldsymbol{Y} = \begin{pmatrix} Y_1 \\ (Y_2 - 3)/2 \end{pmatrix}$. Find the distribution of $\boldsymbol{Y}^T\boldsymbol{Y}$.

4. Let $Y_1, \ldots, Y_n$ be independent random variables each with the distribution $N(\mu, \sigma^2)$. Find the distribution of $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$.

Exercise 4 (Sections 2.4, 2.5). Suppose that $\boldsymbol{y} = (y_1, y_2, y_3, y_4)^T \sim N_4(\boldsymbol{\mu}, V)$ is a random 4×1 vector described by a multivariate normal distribution with

$$\boldsymbol{\mu} = \begin{pmatrix} -4 \\ 2 \\ 5 \\ -1 \end{pmatrix}, \qquad V = \begin{pmatrix} 8 & 0 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 5 & 0 \\ 0 & 2 & 0 & 7 \end{pmatrix}.$$

Find

1. the trace of the matrix V;
2. the distribution of y_2 and of y_3;
3. the distribution of $z = y_1 + 2y_2 - y_4$;
4. the joint distribution of y_1 and y_3, i.e. the multivariate normal dist. of $\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}$;
5. the joint distribution of y_1, y_2 and $\frac{1}{2}(y_2 + y_4)$;
6. which pairs of random variables are independent, i.e. are y_1 and y_2 independent? are y_1 and y_3 independent? and so on.