Forced Duffing Oscillator

June 2021

1 Introduction

Duffing Oscillator represent a family of systems with cubic non-linearity. It is used to model many systems in the field of engineering and physics like beams, pendulum,non-linear electronic circuits, superconducting Josephson parametric amplifiers, and ionization waves in plasmas.Study of duffing oscillator is also very useful to prevent catastrophe like plane crash due flutter,and uncontrolled vibrations of structure like bridges alternately fluttering phenomenon can also be used to harvest energy. Duffing oscillator shows various non linear phenomenon like hysteresis and chaos.Effect of hardening and softening spring can also be studied using duffing equation.

2 Solution using harmonic balance method

Let us consider following Duffing equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k_1x + k_3x^3 = f\sin(\omega t) \tag{1}$$

Writing Equation in non dimensional form Let $\tau = \omega_0 t$

$$m\omega_0^2x'' + c\omega_0x' + k_1x + k_3x^3 = f\sin(\frac{\omega}{\omega_0}\tau) \tag{2}$$

Where $x'' = \frac{d^2x}{d\tau^2}$ and $x' = \frac{dx}{d\tau}$
Dividing equation (2) by $m\omega_0^2$

$$x'' + \frac{c}{m\omega_0}x' + \frac{k_1}{m\omega_0^2}x + \frac{k_3}{m\omega_0^2}x^3 = \frac{f}{m\omega_0^2}\sin(\frac{\omega}{\omega_0}\tau)$$

Here

$$\Omega = \frac{\omega}{\omega_0}$$

$$\omega_0 = \sqrt{\frac{k_1}{m}}$$

$$\alpha = 2\zeta = \frac{c}{m\omega_0}$$

$$\beta = \frac{k_1}{m\omega_0^2} = 1$$

$$\gamma = \frac{k_3}{m\omega_0^2} = \frac{k_3}{k_1}$$

$$F = \frac{f}{m\omega_0^2}$$

Therefore

$$x'' + \alpha x' + x + \gamma x^3 = F\sin(\Omega\tau) \tag{3}$$

Let us assume the solution in following form

$$x = A\sin(\Omega\tau) + B\cos(\Omega\tau)$$

$$x' = A\Omega\cos(\Omega\tau) - B\Omega\sin(\Omega\tau)$$
$$x'' = -A\Omega^2\sin(\Omega\tau) - B\Omega^2\cos(\Omega\tau)$$

Substituting above equation in equation (3)

$$-A\Omega^2\sin(\Omega\tau) - B\Omega^2\cos(\Omega\tau) + \alpha A\Omega\cos(\Omega\tau) - \alpha B\Omega\sin(\Omega t)$$
$$+A\sin(\Omega\tau) + B\cos(\Omega\tau) - \frac{1}{4}\gamma A^3\sin(3\Omega\tau) + \frac{3}{4}\gamma A^3\sin(\Omega\tau)$$
$$+\frac{3}{4}\gamma A^2B\cos(\Omega\tau) - \frac{3}{4}\gamma A^2B\cos(3\Omega\tau) + \frac{3}{4}\gamma AB^2\sin(3\Omega\tau) + \frac{3}{4}\gamma AB^2\sin(\Omega\tau)$$
$$+\frac{1}{4}\gamma B^3\cos(3\Omega\tau) + \frac{3}{4}\gamma B^3\cos(\Omega\tau) - F\sin(\Omega\tau) = 0$$

Comparing Coefficient of $\sin(\Omega\tau)$

$$-A\Omega^2 - \alpha B\Omega + A + \frac{3}{4}\gamma A^3 + \frac{3}{4}\gamma AB^2 - F = 0 \tag{4}$$

Comparing Coefficient of $\cos(\Omega\tau)$

$$-B\Omega^2 + \alpha A\Omega + B + \frac{3}{4}\gamma A^2B + \frac{3}{4}\gamma B^3 = 0 \tag{5}$$

Comparing Coefficient of $\sin(3\Omega\tau)$

$$-\frac{1}{4}\gamma A^3 + \frac{3}{4}\gamma AB^2 = 0 \tag{6}$$

Comparing Coefficient of $\cos(3\Omega\tau)$

$$-\frac{3}{4}\gamma A^2B + \frac{1}{4}\gamma B^3 = 0 \tag{7}$$

Solving equation (4) and (5) using newton raphson method

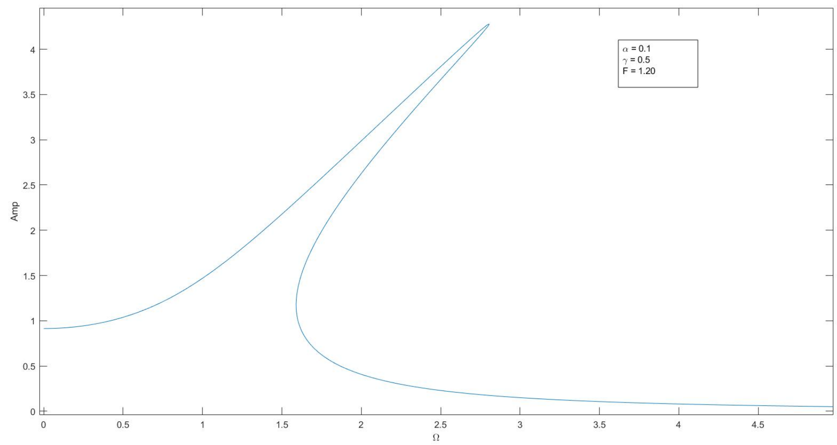

Figure 1: Ω Vs amplitude curve

Here figure 1 shows typical response of Ω and amplitude with hardening spring. As compared to linear case where there is no cubic non linearity curve is straight with maximum amplitude at $\Omega = 1$ condition here the curve is bend towards right and it shows that the amplitude of vibrations is dependent upon the value of Ω

2.1 Effect of forcing

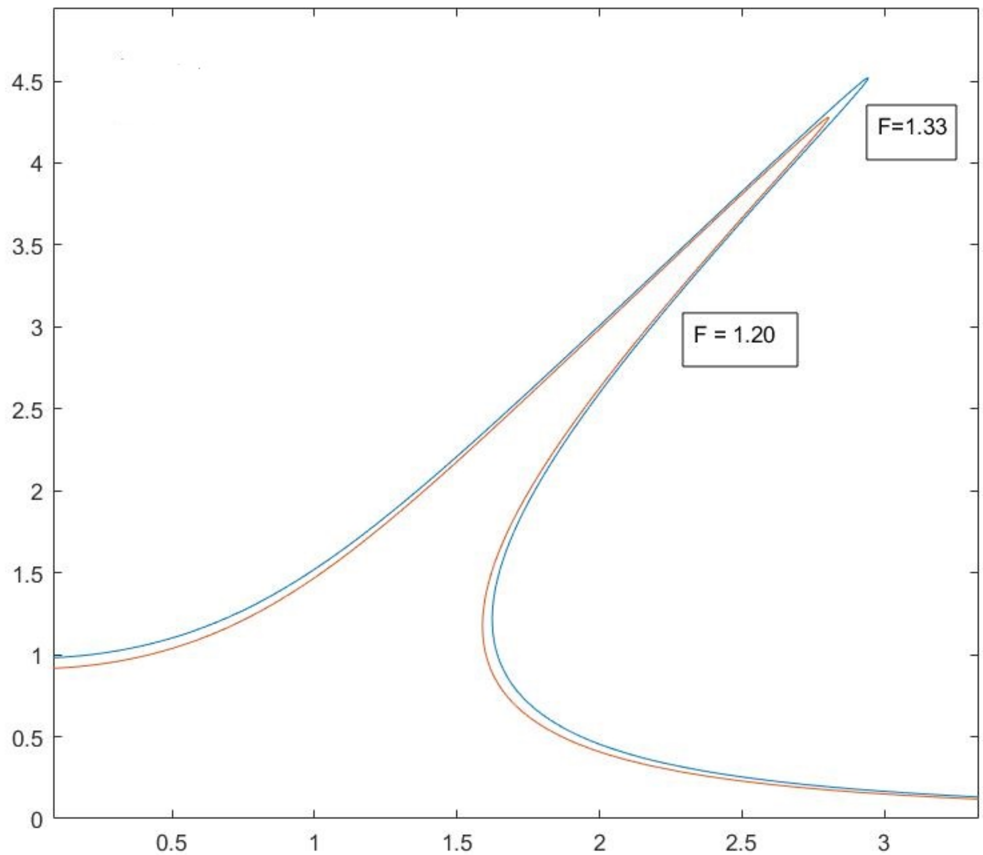

Figure 2: Effect of Forcing term

2.2 Effect of damping

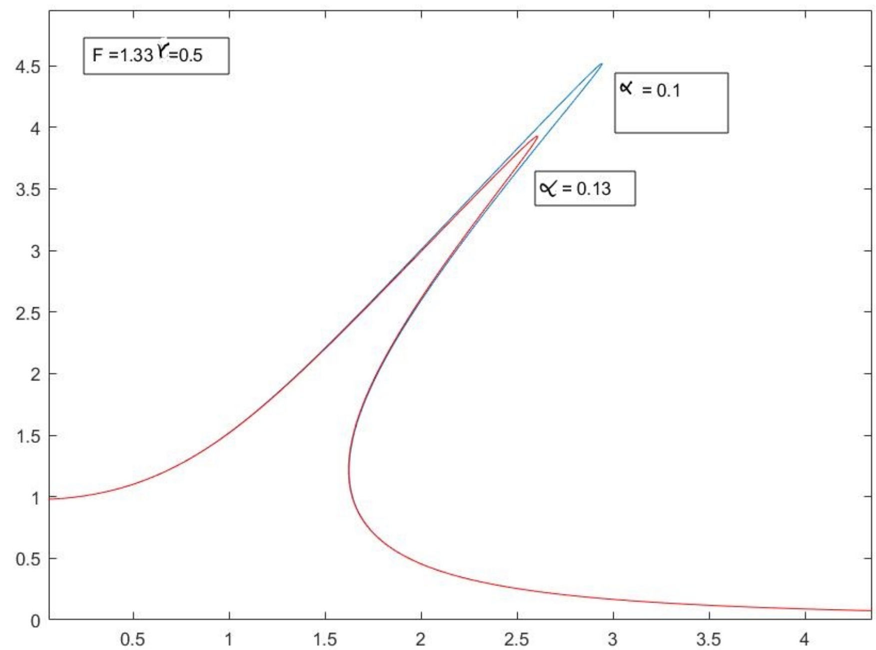

Figure 3: Effect of damping term

As shown in figure increase in damping value decreases the maximum amplitude of oscillations as increase in damping causes more energy to dissipate from the system.

2.3 Jump Up and jump down phenomenon

Here considering a single degree of freedom system with mass m, and dashpot with damping coefficient c, and non linear spring and external forcing term $F\cos(\omega t)$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + k_1x + k_3x^3 = F\cos(\omega t) \tag{8}$$

Writing Equation in non-dimensional form

$$y'' + 2\zeta y' + y + \alpha y^3 = cos(\Omega\tau) \tag{9}$$

where
$\omega_0^2 = \frac{k_1}{m}$, $\tau = \omega_0 t$, $\eta = \frac{c}{2m\omega_0}$, $y = \frac{x}{x_0}$, $\alpha = \frac{k_3}{k_1}x_0^2$, $\Omega = \frac{\omega}{\omega_0}$

Here x_0 is defined as $x_0 = \frac{F}{k_1}$ when $k_3 = 0$ and $\omega = 0$ By Harmonic balance method

$$y = Y\cos(\Omega\tau + \phi) \tag{10}$$

$$y' = -Y\Omega\sin(\Omega\tau + \phi) \tag{11}$$

$$y'' = -Y\Omega^2\cos(\Omega\tau + \phi) \tag{12}$$

substituting equation (10),(11) and (12) in equation (9)

$$-Y\Omega^2\cos(\Omega\tau+\phi) - 2\eta Y\Omega\sin(\Omega\tau+\phi) + Y\cos(\Omega\tau+\phi) + \alpha Y^3\cos^3(\Omega\tau+\phi) = cos(\Omega\tau)$$

$$-Y\Omega^2\cos(\Omega\tau+\phi) - 2\eta Y\Omega\sin(\Omega\tau+\phi) + Y\cos(\Omega\tau+\phi) + \frac{1}{4}\alpha Y^3\cos 3(\Omega\tau+\phi) + \frac{3}{4}\alpha Y^3\cos(\Omega\tau+\phi) = cos(\Omega\tau)$$

Comparing coefficient of cos terms and sin terms
cos terms

$$((1-\Omega^2)Y + \frac{3}{4}\alpha Y^3) = 1 \tag{13}$$

sin terms

$$-2\eta Y\Omega = 0 \tag{14}$$

Squaring and adding equation (6) and (7)

$$((1-\Omega^2)Y + \frac{3}{4}\alpha Y^3)^2 + 4\eta^2\Omega^2Y^2 = 1 \tag{15}$$

Simplifying equation (8)

$$\frac{9}{16}\alpha^2Y^6 + \frac{3}{2}\alpha y^4 - \frac{3}{2}\alpha\Omega^2Y^4 + 4\zeta^2\Omega^2Y^2 + Y^2 - 2\Omega^2Y^2 + \Omega^4Y^2 - 1 = 0$$

$$Y^2\Omega^4 - Y^2(\frac{3}{2}\alpha Y^2 - 4\zeta^2 + 2)\Omega^2 + Y^2(\frac{9}{16}\alpha Y^4 + \frac{3}{2}\alpha Y^2 + 1) - 1 = 0$$

$$Y^2\Omega^4 - Y^2(\frac{3}{2}\alpha Y^2 + 2(1-2\zeta^2))\Omega^2 + Y^2(\frac{3}{4}\alpha Y^2 + 1)^2 - 1 = 0 \tag{16}$$

Solving Equation (16) for value of Ω

$$\Omega_1 = -\sqrt{\frac{4y - 4\sqrt{-3\alpha y^4\zeta^2 + 1} + 3\alpha y^3 - 8y\zeta^2}{4y}}$$

$$\Omega_2 = -\sqrt{\frac{4y + 4\sqrt{-3\alpha y^4\zeta^2 + 1} + 3\alpha y^3 - 8y\zeta^2}{4y}}$$

$$\Omega_3 = \sqrt{\frac{4y - 4\sqrt{-3\alpha y^4\zeta^2 + 1} + 3\alpha y^3 - 8y\zeta^2}{4y}}$$

$$\Omega_4 = \sqrt{\frac{4y + 4\sqrt{-3\alpha y^4\zeta^2 + 1} + 3\alpha y^3 - 8y\zeta^2}{4y}}$$

Figure (4) shows the response

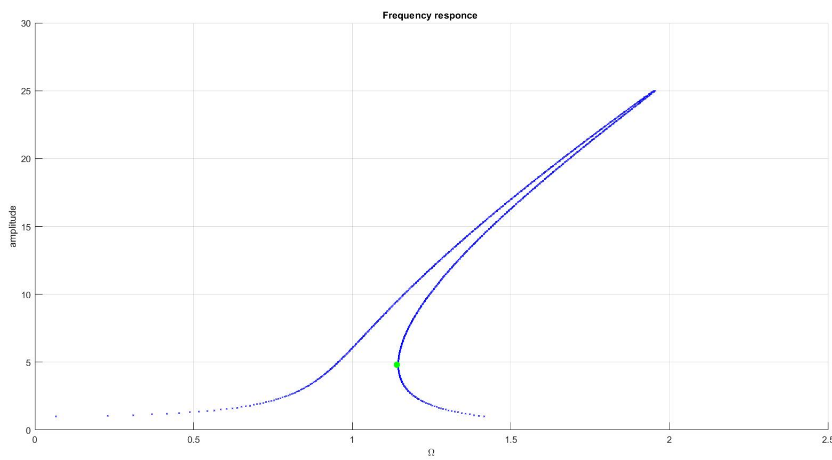

Figure 4: Frequency Response Curves $\alpha = 0.006\zeta = 0.01$

2.4 Jump Up and jump down simulation using RK4 method

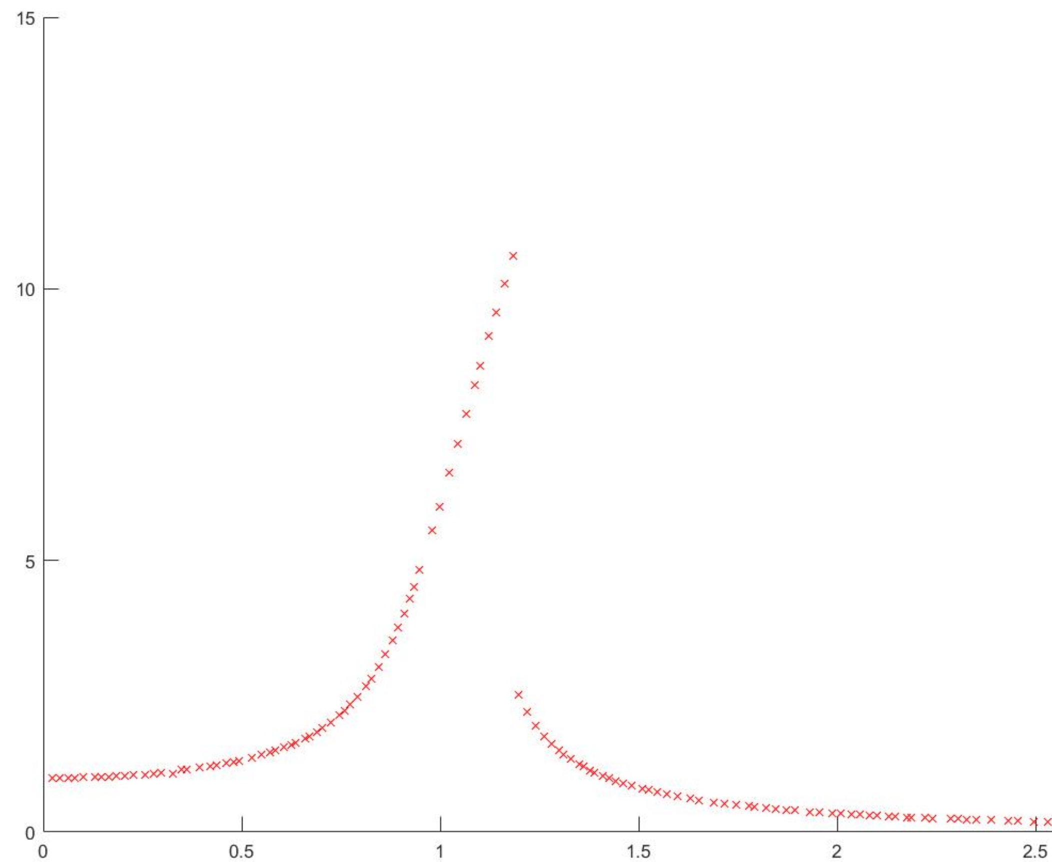

Figure 5: Jump Down $\alpha = 0.006\zeta = 0.01$

As we increase value of Ω amplitude increases at $\Omega = 1.3$ amplitude of oscillations suddenly drops to value around 2 and the value or Ω at which amplitude suddenly decreases is known as jump down frequency. As shown in analytical results in figure(4) jump down takes place at tip of curve but figure(5) is showing jump near $\Omega = 1.3$ this is due to the fact that jump down phenomenon is dependent upon the initial conditions taken.

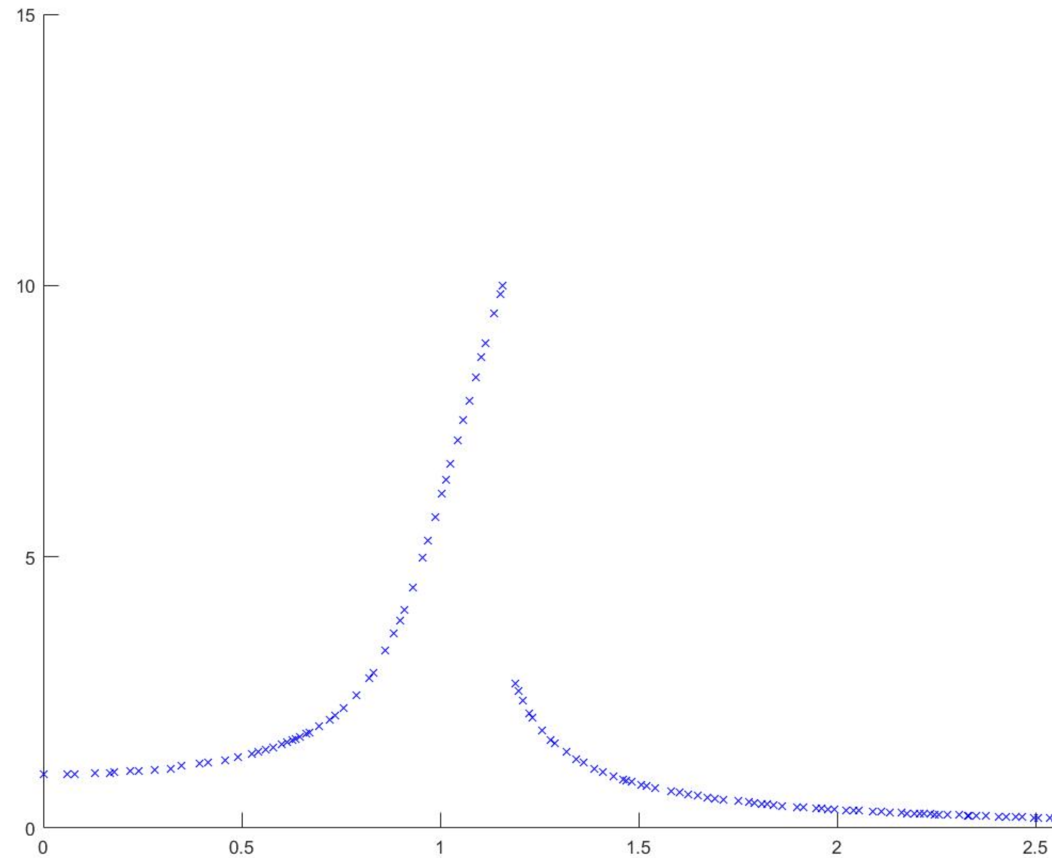

Figure 6: Jump Down $\alpha = 0.006\zeta = 0.01$

As we decrease the value of Ω from value 3 as shown in figure(6) at a point $\Omega = 1.2$ the amplitude os oscillations suddenly increases and compared with the analytical case in figure(4) results are consistentwith the jump up frequency calculated.

2.5 Super-harmonic and sub harmonic response in duffing oscillator

Equation of duffing oscillator

$$\ddot{x} + c\dot{x} + x + ax^3 = F\sin(\omega t) \tag{17}$$

By harmonic balance method
let $x = A\sin(\omega t) + B\cos(\omega t) + P\sin(3\omega t) + Q\cos(3\omega t)$
substituting above equation in equation 10
$(3/4)aP\sin(\omega t)Q^2 + (3/4)aPQ^2\sin(3\omega t) + (3/4)aA^2B\cos(\omega t) - (3/4)aA^2B\cos(3\omega t) + (3/4)aA^2Q\cos(\omega t) + (3/2)aAB\cos(\omega t)P - (3/2)aABP\cos(3\omega t) + (3/2)aA\sin(\omega t)BQ + (3/2)aABQ\sin(3\omega t) + (3/2)aAPQ\cos(\omega t) - (3/2)aAPQ\cos(3\omega t) + (3/2)aBP\sin(\omega t)Q + (3/2)aBPQ\sin(3\omega t) + A\sin(\omega t) + B\cos(\omega t) + P\sin(\omega t) + Q*\cos(\omega t) - F\sin(\omega t) - (3/4)aP^2Q\cos(3\omega t) + cA\omega\cos(\omega t) - cB\omega\sin(\omega t) + cP\omega\cos(\omega t) - cQ\omega\sin(\omega t) + (9/4)aB^2Q\cos(\omega t) + (3/4)aBQ^2 cos(3\omega t) + (9/4)aBQ^2\cos(\omega t) - (3/4)aA^2P\sin(3\omega t) + (9/4)aA^2P\sin(\omega t) - (3/4)aAP^2\sin(3\omega t) + (9/4)aAP^2\sin(\omega t) + (3/4)aB^2Q\cos(3\omega t) - (3/4)aA^2Q cos(3\omega t) + (3/4)aA\sin(\omega t)B^2 + (3/4)aAB^2\sin(3\omega t) + (3/4)aA\sin(\omega t)Q^2 + (3/4)aAQ^2\sin(3\omega t) + (3/4)aB^2P\sin(\omega t) + (3/4)aB^2P\sin(3\omega t) + (3/4)aB\cos(\omega t)P^2 - (3/4)aBP^2 cos(3\omega t) + (3/4)aP^2Q\cos(\omega t) - Aomega^2\sin(\omega t) + (3/4)aP^3\sin(\omega t) - (1/4)aA^3 sin(3\omega t) - B\omega^2\cos(\omega t) - Q\omega^2\cos(\omega t) + (3/4)aB^3\cos(\omega t) + (1/4)aB^3\cos(3\omega t) + (1/4)aQ^3\cos(3\omega t) - P\omega^2\sin(\omega t) - (1/4)aP^3\sin(3\omega t) + (3/4)aQ^3\cos(\omega t) + (3/4)aA^3\sin(\omega t) = 0$

coefficients of $\sin(\omega t)$

$$-P\omega^2 + \frac{3}{4}aA^3 - A\omega^2 + \frac{3}{4}P^3 - cB\omega - cQ\omega + \frac{9}{4}aAP^2 + \frac{9}{4}$$

$$A^2P + \frac{3}{4}aAB^2 + \frac{3}{4}aAQ^2 + \frac{3}{4}B^2P + \frac{3}{4}PQ^2 + \frac{3}{2}ABQ + \frac{3}{4}BPQ - F + A + P = 0$$

coefficients of $\cos(\omega t)$

$$\frac{3}{4}aA^3 - A\omega^2 - \frac{3}{4}aA^2R + \frac{3}{2}aAP^2 + \frac{3}{2}aAR^2 + \frac{3}{2}AS^2 + \frac{3}{4}aAB^2 - cB\omega$$

$$-\frac{3}{4}Q^2R + \frac{3}{2}AQ^2 + \frac{3}{4}P^2R + \frac{3}{4}B^2R - \frac{3}{2}ABS + \frac{3}{2}PQS - F + A = 0$$

coefficients of $\sin(3\omega t)$

$$\frac{3}{4}R^3 - 9R\omega^2 - \frac{1}{4}aA^3 - 3cS\omega + \frac{3}{2}aB^2R + \frac{3}{2}A^2R + \frac{3}{4}AP^2 + \frac{3}{4}AB^2$$

$$-\frac{3}{4}AQ^2 + \frac{3}{4}RS^2 + \frac{3}{2}P^2R + \frac{3}{2}Q^2R + \frac{3}{2}BPQ + R = 0$$

coefficients of $\cos(3\omega t)$

$$\frac{1}{4}B^3 + \frac{3}{4}S^3 - 9S\omega^2 + \frac{3}{2}APQ + S + \frac{3}{2}B^2S + \frac{3}{4}BQ^2 + \frac{3}{2}Q^2S + 3cR\omega$$

$$-\frac{3}{4}BP^2+\frac{3}{2}A^2S-\frac{3}{4}A^2B+\frac{3}{4}R^2S+\frac{3}{2}P^2S=0$$

Using Newton Raphson method to get coefficients A B P Q and ploting $\sqrt{A^2+B^2}$ and $\sqrt{P^2+Q^2}$ Vs ω

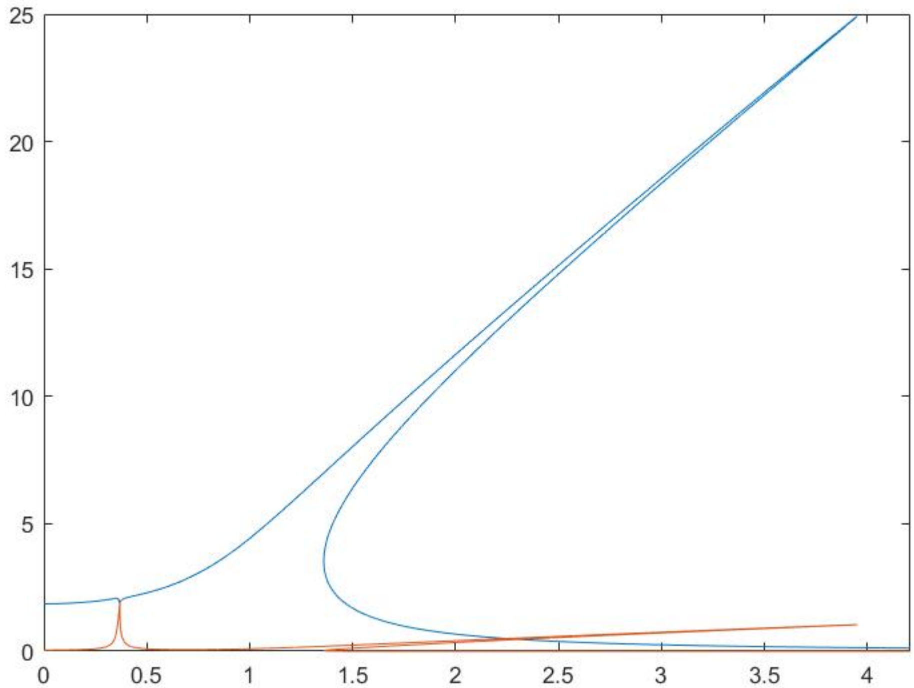

Figure 7: First harmonics and second harmonic responce

Subharmonic Response at $\omega = \frac{1}{3}$

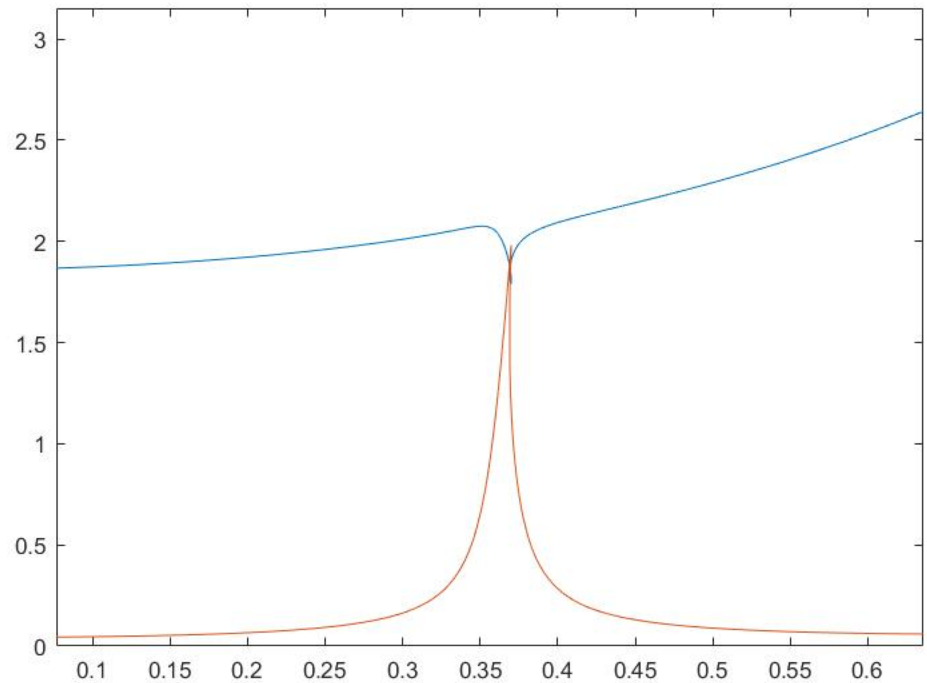

Figure 8: Effect of second harmonics near $\omega = \frac{1}{3}$

In case of when system is excited with $\omega = \frac{1}{3}$ the the contribution of second harmonics increases as shown in figure(8).