ASSIGNMENT-3

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Ans 1. Probability that essay is written by AI = P(A)

Probability that essay is written by Human = P(H)

Probability that an test result is positive = $P(T^+)$

Probability that test is positive given that essay is written by AI = $P(T^+|A) = 0.87$ (Given)

Probability that test is positive given that essay is written by human = $P(T^+|H) = 0.36$ (Given)

(a.) Probability of positive test result :-

This can be calculated using law of Total probability.

$$P(T^+) = P(T^+|A) \cdot P(A) + P(T^+|H) \cdot P(H)$$

$P(A) = 1 - P(H)$ [Probability that essay is written by AI]

$P(H) = 0.61$ [Probability that essay is written by Human (Given)]

$P(A) = 1 - 0.61 = 0.39 \Rightarrow P(A) = 0.39$

$$P(T^+) = 0.87 \times 0.39 + 0.36 \times 0.61$$

$$P(T^+) = 0.3393 + 0.2196 = 0.5589$$

$$\boxed{P(T^+) = 0.5589}$$

(b) Probability that essay is written by human given that positive test result :-

We need to calculate $P(H/T^+)$. Using Bayes's Theorem,

$$P(H|T^+) = \frac{P(T^+|H) \cdot P(H)}{P(T^+)}$$

Put the given values,

$$P(H|T^+) = \frac{0.36 \times 0.61}{0.5589} = \frac{0.2196}{0.5589}$$

$$P(H|T^+) = \frac{0.2196}{0.5589} = 0.39291465$$

$$\boxed{P(H|T^+) \approx 0.3929}$$

(c) Proportion of essay written by AI when 44% Test are Positive :-

44% Test are Positive (Given) $\Rightarrow P(T^+) = 0.44$

we need to calculate $P(A)$, using Total probability law.

$$P(T^+) = P(T^+|A) \cdot P(A) + P(T^+|H) \cdot P(H)$$

$$P(T^+) = 0.87 \times P(A) + 0.36 \times (1 - P(A))$$

$$0.44 = 0.87\,P(A) + 0.36 - 0.36\,P(A)$$

$$0.44 = 0.36 + 0.51\,P(A)$$

$$0.44 - 0.36 = 0.51\,P(A)$$

$$P(A) = \frac{0.08}{0.51} = 0.15686274$$

$$\boxed{P(A) \approx 0.1569}$$

Ans2. Possible outcomes for each roll are (X_i):-

→ 4 (on 3 side of dice)
→ 5 (on 2 side of dice)
→ 6 (on 1 side of dice)

X_i represent outcomes at i^{th} roll, where $i = 1, 2, 3$
Each roll is independent.

Probability distribution at each roll :-

$P(X_i = 4) = \frac{3}{6} = \frac{1}{2}$; $P(X_i = 5) = \frac{2}{6} = \frac{1}{3}$; $P(X_i = 6) = \frac{1}{6}$

Roll the die 3 times, To solve the question we need to find the sum of 3 rolls, $S = X_1 + X_2 + X_3$

maximum possible score $= 6 + 6 + 6 = 18$

minimum possible score $= 4 + 4 + 4 = 16$

We need to find out all cases for sum 16 or more

∴ We need to find out all cases for sum 16, 17, 18.

⇒ Total score of 16 :-

$S \to Sum$

$S = 16$

$\to (6, 6, 4)$

$\to (6, 5, 5)$

Permutations possible ⇒ $(6,6,4), (6,4,6), (4,6,6) \Rightarrow$ 3 ways

$\Rightarrow (6,5,5), (5,6,6), (6,5,6) \Rightarrow$ 3 ways

Total cases for $S = 16$ is <u>6</u>

⇒ Total score of 17 :-

$S = 17$

$\to (6, 6, 5)$

Permutations Possible ⇒ $(6,6,5), (5,6,6), (6,5,6) \Rightarrow$ 3 ways

Total cases for $S = 17$ is <u>3</u>

⇒ Total score of 18 :-

$S = 18$

$\to (6, 6, 6)$

No permutation possible

Total cases of $S = 18$ is <u>1</u>

We want to calculate the probability of the total score from 3 rolls being 16 or more. ($P(Sum \geq 16)$)

$$\Rightarrow P(Sum \geq 16) = \frac{\text{Total score from 3 rolls being 16 or more}}{\text{Total possible outcomes from 3 rolls}}$$

$$P(\text{Sum} \geq 16) = \frac{(\text{sum} \geq 16)}{\text{Total possible out comes from 3 rolls}}$$

Total possible outcomes from 3 rolls :-

=> For each roll there are 6 possible outcomes (even through some numbers are repeated, each face of a die is distinct possibility)

=> For indebendent event each roll doesn't affect the other, we multiply the no. of possibilities for each event.

∴ Total possible outcomes from 3 rolls = 6 × 6 × 6

Total possible out comes from 3 rolls = 216

$$P(\text{sum} \geq 16) = \frac{(\text{sum} = 16) + (\text{sum} = 17) + (\text{sum} = 18)}{216}$$

Putting the value of above ~~sum~~ all the 3 sum.

$$P(\text{sum} \geq 16) = \frac{6 + 3 + 1}{216} = \frac{10}{216} = 0.0462963$$

$$\boxed{P(\text{sum} \geq 16) \simeq 0.0463}$$

Ans 3. Poission distribution for the probability of k events when event frequency is λ.

$$P(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$$

where

→ $P(k,\lambda)$ is the probability of exactly k events occuring

→ λ is the expected no. of events in the given time frame.

→ k is the actual number of events.

We are told that large meteors (diameter 1 Km or more) impact mars once every 400000 years, meaning the average rate λ is 1 event per 400000 years.

(a) Probability of atleast 1 large meteor impact on mars in the next 400000 years:-

$\lambda = 1$ for next 400000 years, since the avg. rate is 1 impact per 400000 years.

The probability of at least one impact is compli-ment of the probability of no impact (i.e k=1)

$$P(\text{at least 1 impact}) = 1 - P(0;\lambda)$$

here $\lambda = 1$

$$P(0;1) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-1}\cdot 1^0}{0!} = e^{-1}\cdot 1 = e^{-1}$$

$$P(0;1) = e^{-1}$$

$P(\text{atleast one impact}) = 1 - e^{-1} = 1 - 0.3679$

$$\boxed{P(\text{atleast 1 impact}) = 0.6321}$$

(b) Probability of exactly 1 large meteroid impact on mars in the next 100000 years :-

For this, the expected number of impacts λ over 100000 year can be $\frac{1}{4}$ of 400000 years.

$$\lambda = \frac{1}{4}$$

Probability of exactly 1 impact in 100000 years i.e $k = 1$

$$P(1;\lambda) = \frac{e^{-\lambda}\lambda^{k}}{1!} = \frac{e^{-\frac{1}{4}} \cdot \frac{1}{4}}{1!} = e^{-\frac{1}{4}} \cdot \frac{1}{4}$$

$$\boxed{P(1;\tfrac{1}{4}) = e^{-\frac{1}{4}} \cdot \frac{1}{4}}$$

Above is the explanation and calculation why $P(1;\frac{1}{4}) = e^{-\frac{1}{4}} \cdot \frac{1}{4}$ which is the same as given expression in part b on que 3.

(c) Number of years required for the probability of at least 1 large meteor impact to exceed 3/4 :-

Here, we need to find no. of years (t), such that probability of atleast 1 impact exceeds $\frac{3}{4}$

P(atleast 1 impact) = $1-e^{-\lambda}$

λ is the expected number of impact over time t.
Since λ is one impact per 400000 years,

$$\Rightarrow \lambda = \frac{t}{400000}$$

For this que $\Rightarrow$ P(atleast 1 impact) $> \frac{3}{4}$

$$\Rightarrow 1 - e^{-\frac{t}{400000}} > \frac{3}{4}$$

$$\Rightarrow -e^{-\frac{t}{400000}} > \frac{1}{4}$$

$$\Rightarrow e^{-\frac{t}{400000}} < \frac{1}{4}$$

$\Rightarrow$ apply $\log_e$ on both side

$$\Rightarrow \frac{-t}{400000} < \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow \frac{-t}{400000} < -1.3863$$

$\Rightarrow$ Multiply by $-$ both side

$$\frac{t}{400000} > 1.3863$$

$$t > 1.3863 \times 400000$$

$$\boxed{t > 554520}$$

$\therefore$ The number of years required for the probability of atleast 1 impact to exceed 3/4 is approx 554520.

Ans 4. We need to find $CBC^T - A^TA$

Step 1:- Define the given matrices

given matrices :-

$$A = \begin{pmatrix} w & -1 \\ 0 & 2 \end{pmatrix} ; \quad B = \begin{pmatrix} -3 & 0 & 0 \\ 0 & x & 0 \\ 1 & 1 & 2 \end{pmatrix} ; \quad C = \begin{pmatrix} y & 0 & z \\ 2 & -1 & 1 \end{pmatrix}$$

Step 2:- Compute CB

For this multiply the matrix C and B

$$CB = \begin{pmatrix} y & 0 & z \\ 2 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} -3 & 0 & 0 \\ 0 & x & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$CB = \begin{pmatrix} -3y + 0 + z & 0 + 0 + z & 0 + 0 + 2z \\ -6 + 0 + 1 & 0 - x + 1 & 0 + 0 + 2 \end{pmatrix}$$

here matrix muliplication is done using this formula

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

where,

→ a_{ik} is the element in the i^{th} row and k^{th} column of matrix C

→ b_{kj} is the element in the k^{th} row and j^{th} column of matrix B

→ n is the number of column in C, which must match the number of rows in matrix B.

$$C \times B = \begin{pmatrix} -3y + z & z & 2z \\ -5 & -x+1 & 2 \end{pmatrix}$$

Step 3 :- Compute C^T

The transpose of matrix $C \Rightarrow C^T$:-

$$C^T = \begin{pmatrix} y & 2 \\ 0 & -1 \\ z & 1 \end{pmatrix}$$

Step 4 :- Compute CBC^T

now multiply CB and C^T

using the previous formula we will do it,

$$CBC^T = \begin{pmatrix} -3y+z & z & 2z \\ -5 & -x+1 & 2 \end{pmatrix} \times \begin{pmatrix} y & 2 \\ 0 & -1 \\ z & 1 \end{pmatrix}$$

$$CBC^T = \begin{pmatrix} -3y^2+yz+2z^2 & -6y+2z-z+2z \\ -5y+2z & -10+x-1+2 \end{pmatrix}$$

$$CBC^T = \begin{pmatrix} -3y^2+yz+2z^2 & -6y+3z \\ -5y+2z & x-9 \end{pmatrix}$$

Step 5 :- Calculate A^TA

here A^T is transpose of A

$$A^T = \begin{pmatrix} w & 0 \\ -1 & 2 \end{pmatrix}$$

Now multiply A^T with A using the previous formula

$$A^T \cdot A = \begin{pmatrix} \omega & 0 \\ -1 & 2 \end{pmatrix} \times \begin{pmatrix} \omega & -1 \\ 0 & 2 \end{pmatrix}$$

$$A^T \cdot A = \begin{pmatrix} \omega^2 + 0 & -\omega + 0 \\ -\omega + 0 & 1+4 \end{pmatrix}$$

$$A^T \cdot A = \begin{pmatrix} \omega^2 & -\omega \\ -\omega & 5 \end{pmatrix}$$

Step 5:- Now compute the final result which is $\Rightarrow CBC^T - A^TA$

$$CBC^T - A^TA = \begin{pmatrix} -3y^2 + yz + 2z^2 & -6y + 3z \\ -5y + 2z & x - 9 \end{pmatrix} - \begin{pmatrix} \omega^2 & -\omega \\ -\omega & 5 \end{pmatrix}$$

For matrix ~~addtion~~ ~~add~~ subtraction subtract the corresponding elements. eg:- $A - B \Rightarrow C_{ij} = A_{ij} - B_{ij}$

$$\boxed{CBC^T - A^TA = \begin{pmatrix} -3y^2 + yz + 2z^2 - \omega^2 & -6y + 3z + \omega \\ -5y + 2z + \omega & x - 14 \end{pmatrix}}$$