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SMFD - ASSIGNMENT 1.1

1) No. of derangements $D_N = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

$P(\text{no letter correct}) = \frac{D_N}{N!}$

So, $P(\text{atleast one correct}) = 1 - \frac{D_N}{N!} = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}$

Taylor's expansion of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

So for large enough k, $\sum_{k=0}^{50} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e}$

$\therefore P(\text{atleast one correct}) = 1 - \frac{1}{e} \approx 0.6321$

(This is a good approximation for $n = 50$)

2) Monty hall problem:

Selected Present 1, host opens & shows present 2

events:

P3 – present 3 is good gift

N2 – present 2 is not the good gift & host opens the present

By Bayes Theorem: $P(P3|N2) = \frac{P(N2|P3) \cdot P(P3)}{P(N2)}$

$P(N2|P3)$ = Probability host opens present 2 given that good gift is ~~second~~ third.

$= 1$ (since we selected P1 and he cannot open)

$P(P3)$ = Probability present 3 is good gift

$= \frac{1}{3}$

$P(N2)$ = Probability host opens present 2

$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 1 + \frac{1}{3} \times 0 = \frac{1}{2}$

$\therefore P(N2|P3) = \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

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$E[x] = \sum x_i \cdot P(x_i)$

Only 2 outcomes possible:

$x_1 \to$ winning 1000 dollars

$x_2 \to$ winning nothing

$E[x] = x_1 \cdot P(x_1) + x_2 \cdot P(x_2)$

$= 1000 \times \frac{2}{3} + 0 \times \frac{1}{3} = \underline{666.6} \to$ expected winnings

4) (a) $E[X]$ is finite but $E[X^2]$ infinite (discretes)

let $X = \{1, 2, 3, \ldots, \infty\}$ with probabilities:

$P(X=n) = \frac{c}{n^3}$ where c is normalizaⁿ constant $= \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^3}}$

$\Rightarrow E[X] = \sum_{n=1}^{\infty} n \cdot \frac{c}{n^{3}} = c \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ $\left(\sum \frac{1}{n^2} \text{ converges}\right)$

$\Rightarrow E[X^2] = \sum_{n=1} n^2 \cdot \frac{c}{n^3} = c \sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$\therefore$ Yes, such a discrete random variable exists

(b) $E[X]$ is finite but $E[X^2]$ infinite (continuous)

let X have prob. distribution function:

$f(x) = \frac{c}{x^3}$ where $c = 2$ to normalize distribution $\left(\int^{\infty} f(x) = 1\right)$

$\Rightarrow E[X] = \int_1^{\infty} x \cdot \frac{c}{x^3} \, dx = 2 \int_1^{\infty} \frac{dx}{x^2} = 2\left[\frac{-1}{x}\right]_1^{\infty} = \underline{2}$ (finite)

$\Rightarrow E[X^2] = \int_1^{\infty} x^2 \cdot \frac{c}{x^3} \, dx = 2 \int_1^{\infty} \frac{dx}{x} = 2\left[\ln x\right]_1^{\infty} = \underline{\infty}$ (infinite)

$\therefore$ Yes, such a continuous random variable exists

(c) $E[X] = 1$ but $E[e^{-X}] < \frac{1}{3}$

By Jensen's inequality, for convex fⁿs:

$E[\phi[X]] \geq \phi[E[X]]$

$\phi(x) = e^{-x}$ is a convex function

$\Rightarrow E[e^{-x}] \geq e^{-E[X]} = e^{-1} = 0.3679 > \frac{1}{3}$

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$\therefore$ such a random variable cannot exist

5) Let went M be maximum prize money obtained

$P(\text{all draws are at most } k) = \dfrac{k^n}{N^n}$

$P(\text{all draws are at most } k-1) = \dfrac{(k-1)^n}{N^n}$

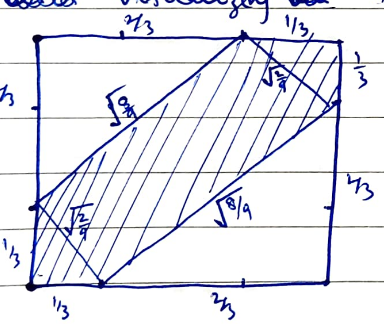
$\therefore P(M=k) = \left(\dfrac{k}{N}\right)^n - \left(\dfrac{k-1}{N}\right)^n$

$$\Rightarrow \boxed{E[M] = \sum_{k=1}^{N} k\left[\left(\frac{k}{N}\right)^n - \left(\frac{k-1}{N}\right)^n\right]}$$

6) Let the two points be $x, y \in [0, d]$

we need points such that $|x-y| < d/3$

visualizing this using unit square:



- the points can lie anywhere on this square

required area = shaded portion

$P(|x-y| < d/3) = 2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} + \sqrt{\frac{8 \times 2}{9 \times 9}}$

$= \frac{1}{9} + \frac{4}{9}$

$P(|x-y| < d/3) = \frac{5}{9}$

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Q8 $A_1, A_2 \cdots A_m \rightarrow$ They are independent

$\therefore A_1^c, A_2^c \cdots A_m^c$ they are also independent

$$P(\cap A_i^c) = \prod_{i=1}^{m} P(A_i^c) \qquad (\because \text{they are independent})$$

$$= \prod_{i=1}^{m} (1 - P(A_i)) \qquad (P(A_i) + P(A_i^c) = 1)$$

using the inequality →

$$1 - x \le e^{-x}$$

$$\left(\because e^{-x} = 1 - x + \frac{x^2}{2!} \cdots\right)$$

Using this on all $P(A_i)$ →

$$\prod_{i=1}^{m} (1 - P(A_i)) \le e^{-P(A_1) - P(A_2) \cdots}$$

$$\le e^{-\sum P(A_i)}$$

$$\therefore P(\cap A_i^c) \le e^{-\sum P(A_i)}$$

Hence Proved!

Q9 Let $F(x)$ & $G(x)$ be the distribution function & $H(x)$ be convolution.

$$H(x) = (F * G)(x) = \int_{-\infty}^{\infty} F(x-y)\, d(G(y))$$

$$= \int_{-\infty}^{\infty} F(x-y)\, g(y)\, dy$$

1. H is non-decreasing →

if $x_1 \le x_2$ then $F(x_1 - y) \le F(x_2 - y)$ as $F(x)$ is distribution function

$$\therefore (F * G)(x_1) \le (F * G)(x_2)$$

$$H(x_1) \le H(x_2)$$

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2. H is right continuous → as both F & G are right continuous

3. $\lim\limits_{x \to -\infty} = \int_{-\infty}^{\infty} F(x-y)\, g(y)\, dy = 0$

$\lim\limits_{x \to \infty} = \int_{-\infty}^{\infty} F(x-y)\, g(y)\, dy = 1$ as F & G are distribution function

∴ H is a distribution function

Q 10

$(1 - F(X)) = P(X > x)$

we need. $E(X) = \int_0^{\infty} P(X > x)\, dx$

We have → $\int_{\Omega} \left(\int_0^{\infty} I_{[0, X(w)]}(x)\, dx \right) dP(w)$

$I_{[0, X(w)]} = 1$ if $0 \le x \le X(w)$

0 o.w

∴ $\int_{\Omega} \left(\int_0^{X(w)} 1 \cdot dx \right) dP(w)$

$\int_{\Omega} X(w)\, dP(w) = E(X)$

If we change order of integration →

$\int_0^{\infty} \int_{\Omega} I_{[0, X(w)]}(x)$ ~~dx~~ $dP(w)\, dx$

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$I_{[0, X(\omega)]}(x) = 1$ if & only if $x < X(\omega)$

$$\therefore \int_{\Omega} I_{[0, X(\omega)]} \, dP(\omega) = P(X > x)$$
$$= 1 - F(X)$$

$$\therefore \int_0^{\infty} (1 - F(x)) \, dx = E[X]$$

by both the expressions
hence proved.

Q11

$$E[e^{ux}] = \int_{-\infty}^{\infty} e^{ux} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(ux - \frac{(x-\mu)^2}{2\sigma^2}\right)} \, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{1}{2\sigma^2}\left(ux \times 2\sigma^2 - (x-\mu)^2\right)} \, dx$$

$$= \frac{e^{1/2\sigma^2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu-u\sigma^2)^2 + 2\sigma^2\mu u + u^2\sigma^4} \, dx$$

$$= e^{\mu u + \frac{1}{2}u^2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-(\mu+u\sigma^2))^2}{2\sigma^2}} \, dx}_{=1}$$

$= 1$ as normal distribution function.

$$\therefore E[e^{uX}] = e^{\mu u + \frac{1}{2}u^2\sigma^2}$$

b) $E(\phi(X)) = E(e^{\mu X}) = e^{\mu U + \frac{1}{2}U^2\sigma^2}$

$\phi(E(X)) = \phi(\mu) = e^{\mu U}$

$\therefore \quad e^{\mu U + \frac{1}{2}\mu^2\sigma^2} \geq e^{\mu U}$

Hence proved.