6. (a) State space = 26!

Transition by selecting any 2 positions out of 26 and swapping the letters at those positions

No. of ways = $^{26}C_2 = \frac{26 \times 25}{2} = 325$

$\therefore$ Probability of transitioning from permutation g to specific permutation h by a single unique swap is:

$$P(g,h) = \frac{1}{^{26}C_2} = \frac{1}{325}$$

If h cannot be obtained from g by a single swap, then $P(g,h) = 0$

Total no. of permutations = state space = 26!

$\therefore$ Stationary distribution $\pi(g)$ for any permutation g is

$$\pi(g) = \frac{1}{26!}$$

Chain represents a random walk on symmetric group S_{26}

Random walks on finite groups have uniform stationary distributions.

(b) To show reversibility, we need to prove:

$$\pi(g)\, q(g,h) = \pi(h)\, q(h,g) \quad \forall\, g \neq h$$

$$\Rightarrow \frac{s(g)}{Z} P(g,h) A(g,h) = \frac{s(h)}{Z} P(h,g) A(h,g) \quad \forall\, g \neq h$$ [Z is normalizaⁿ constant]

transition probability $q(g,h)$ consists of:

↳ P of proposing h from g : $P(g,h)$ $\left[= \frac{1}{^{26}C_2} \text{ if one transposition}\right]$

↳ P of accepting proposal : $A(g,h)$

$P(g,h) = P(h,g)$, so eqⁿ reduces to

$$\Rightarrow s(g)\, A(g,h) = s(h)\, A(h,g)$$

$$A(g;h) := \min\left(1, \frac{s(h)}{s(g)}\right)$$

$$\Rightarrow s(g) \min\left(1, \frac{s(h)}{s(g)}\right) = s(h) \min\left(1, \frac{s(g)}{s(h)}\right)$$

$$\Rightarrow \min(s(g), s(h)) = \min(s(h), s(g))$$

LHS = RHS , hence proved

The stationary distribution is proportional to the scores

$$\pi(g) = \frac{s(g)}{\sum_h s(h)}$$