CMP502

Mathematics

Tutorial Question Booklet

School of Design and Informatics

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Contents

1 Two-Dimensional Viewing Transformations **1**

Answers 3

2 3-D Graphics Transformations **4**

Answers 7

3 Mathematics of Projection **9**

Answers 12

4 Complex Numbers **13**

Answers 15

5 Quaternions **16**

Answers 18

6 Rotation by Quaternions **19**

Answers 20

Chapter 1

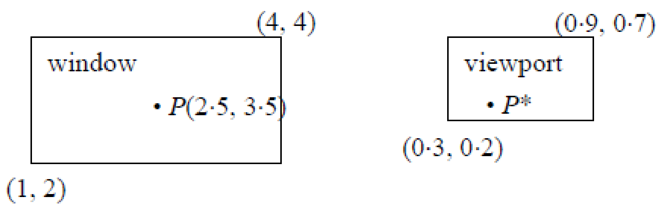
Two-Dimensional Viewing Transformations

1. Obtain the normalisation transformation that maps a window whose lower left corner is at $(2,3)$ and upper right corner is at $(4,7)$ onto

   (a) a viewport that is the entire normalised device screen;

   (b) a viewport that has a lower left corner at $(0.1, 0.3)$ and upper right corner at $(0.4, 0.5)$.

2. Determine the transformations from the following window to the indicated viewport. What are the coordinates of the labelled point where it appears on the screen?



3. Obtain the complete viewing transformation that

   (a) maps a window with world co-ordinates $1 \leq x \leq 10$, $1 \leq y \leq 10$ onto the viewport with $0.25 \leq x \leq 0.75$, $0 \leq y \leq 0.5$ in normalised device space, and then

   (b) maps a window with $0.25 \leq x \leq 0.5$, $0.25 \leq y \leq 0.5$ in the normalised device space into a viewport with $1 \leq x \leq 10$, $1 \leq y \leq 10$ on the physical display device.

4. Derive a normalisation transformation from the window whose lower left corner is at $(0, 0)$ and whose upper right corner is at $(4, 3)$ onto the normalised device screen so that the aspect ratios are preserved.

5. Determine the workstation transformation which maps the normalised device screen onto a physical device with $0 \leq x \leq 199$, $0 \leq y \leq 639$ where the origin is located at the lower left corner of the device.

6. Calculate the normalisation transformation that uses a circle of radius 5 units and centre $(1, 1)$ as a window, and a circle of radius 0.5 and centre $0.5, 0.5$ as a viewport.

7. Obtain the normalisation that uses the rectangle $A(0, 5)$, $B(2, 7)$, $C(0, 9)$ and $D(-2, 7)$ as a window and the normalised device screen as a viewport.

8. Determine the complete viewing transformation that uses the rectangle with vertices at $A(1, 3)$, $B(3, 5)$, $C(6, 2)$ and $D(4, 0)$ as a window and displays it in a viewing device with A mapped to the lower left corner at $(5, 5)$ and C mapped to the upper right corner at $(10, 10)$ (AD and BC being mapped parallel to the new horizontal axis).

Answers

1. (a) $\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ -1 & -0.75 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.05 & 0 \\ -0.2 & 0.15 & 1 \end{pmatrix}$

2. $T = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0.1 & -0.3 & 1 \end{pmatrix}$; $(0.6, 0.575)$

3. First stage, $T_1 = \begin{pmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{18} & 0 \\ \frac{7}{36} & \frac{-1}{18} & 1 \end{pmatrix}$, overall $T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & -10 & 1 \end{pmatrix}$.

4. $\begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 199 & 0 & 0 \\ 0 & 639 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.4 & 0.4 & 1 \end{pmatrix}$

7. $\begin{pmatrix} 0.25 & -0.25 & 0 \\ 0.25 & 0.25 & 0 \\ -1.25 & -1.25 & 1 \end{pmatrix}$

8. $\begin{pmatrix} \frac{5}{6} & \frac{5}{4} & 0 \\ \frac{-5}{6} & \frac{5}{4} & 0 \\ \frac{20}{3} & 0 & 1 \end{pmatrix}$

Chapter 2

3-D Graphics Transformations

1. Write down the separate(4×4) matrices that would produce each of the following transformations:

   (a) shift 0.5 in the $x-$direction, 0 in the $y-$direction and -0.2 in the $z-$direction;

   (b) scale the $z-$coordinates to be half as large;

   (c) scale both the $x-$and $y-$coordinates to be twice as large;

   (d) rotate through an angle of $45°$ about the $x-$axis;

   (e) rotate through an angle of $60°$ about the $y-$axis;

   (f) reflect in the xy plane, and then scale overall by a factor of 3;

   (g) rotate through an angle of $180°$ about the line passing through the points $(0, 0, 0)$ and $(1, 0, 1)$.

2. We can define a **tilt** to be the transformation caused by first rotating about the $x-$axis by θ and then rotating about the $y-$axis by ϕ.

   (a) Determine the tilting matrix when $\theta = \phi = 45°$.

   (b) Does the order of performing the rotations matter?

3. Calculate the single matrix which performs translations in the x, y and z directions of -1, -1 and -1 respectively, followed by, successively, a $30°$ rotation about the $x-$axis and a $45°$ rotation about the $y-$axis. Apply this transformation to the unit cube with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and state the coordinates of the vertices of the image formed.

4. Determine the coordinates of the image of the point $(-1, 5, 2)$ by rotating it through $90°$ about the axis $\underline{i} + 2\underline{j} - 2\underline{k}$, passing through the origin.

5. Determine the coordinates of the image of the point $(3, 0, -4)$ by rotating it through 50° about the axis $-\underline{i} + 3\underline{j} + 2\underline{k}$, passing through the origin.

6. Obtain the matrix for mirror reflection with respect to the plane $y = 2z$ passing through the origin and having a normal vector $\underline{n} = \underline{j} - 2\underline{k}$. What is the reflection of the point $(6, 2, 5)$ in this plane?

7. Determine the matrix to represent a reflection in the plane through $(5, 6, 0)$ with normal $\underline{n} = 8\underline{i} + 6\underline{j} + 24k$.

8. Obtain the single matrix to represent a rotation through:

   (i) 60° about the line through $(1, 0, 2)$ parallel to $\underline{v} = \underline{i} - \underline{j} + \sqrt{2}\underline{k}$;

   (ii) 24° about the line $\underline{r} = 5\underline{i} + 6j + t(8\underline{i} + 6j + 24k)$.

9. If the points $(2, -1, 0)$ and $(4, 1, -4)$ are mirror images of each other in a certain plane, determine the equation of the plane.

10. An *alignment* transformation is a rotation of a vector $\underline{v}$ such that it becomes parallel (aligned) to another vector $\underline{u}$. Using this definition with the standard rotation matrix, determine an alignment matrix that aligns $\underline{v} = 2\underline{i} - \underline{j} + 2\underline{k}$ with $\underline{u} = -\underline{i} + \underline{k}$. [Hint: obtain the angle between the vectors and the axis of rotation to substitute in the rotation matrix.]

11. A transformation matrix for rotation about an axis through the origin is given by

$$R = \frac{1}{9}\begin{pmatrix} 4 & 4 & 7 \\ -8 & 1 & 4 \\ 1 & -8 & 4 \end{pmatrix}.$$

   (i) Use the *sum of the diagonal elements (trace)* to calculate the rotation angle.

   (ii) Use the three *differences of the pairs of reflected off-diagonal* elements to obtain the axis of rotation.

12. The rotation in question 11 is followed by another rotation of 90° about an axis parallel to $3\underline{j} - 4\underline{k}$, passing through the origin. Multiply the transformation matrices out and hence use the technique of question 11 to determine the single equivalent rotation angle and axis of the combined rotations.

13. A rigid body is rotating with angular speed 3 rads/sec about the axis $\underline{i} - 2\underline{j} + 2\underline{k}$, passing through the origin. If the point $(1, 0, -1)$ lies on the body, calculate its image coordinates at times 5 and 10 seconds after the motion starts.

Answers

1.

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0 & -0.2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.71 & 0.71 & 0 \\ 0 & -0.71 & 0.71 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 0.5 & 0 & -0.87 & 0 \\ 0 & 1 & 0 & 0 \\ 0.87 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. (a) $\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/2 & 1/\sqrt{2} & 1/2 & 0 \\ 1/2 & -1/\sqrt{2} & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, (b) Yes.

3. $\begin{pmatrix} 0.71 & 0 & -0.71 & 0 \\ 0.35 & 0.87 & 0.35 & 0 \\ 0.61 & -0.5 & 0.61 & 0 \\ -1.67 & -0.37 & -0.26 & 1 \end{pmatrix}$; points transformed to $\begin{pmatrix} -1.67 & -0.37 & -0.26 & 1 \\ -0.96 & -0.37 & -0.97 & 1 \\ -1.32 & 0.5 & 0.09 & 1 \\ -1.06 & -0.87 & 0.35 & 1 \end{pmatrix}$.

4. $(5.222,\ 1.111,\ 1.222)$

5. $(-0.248,\ -0.433,\ -4.975)$

6. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.6 & 0.8 & 0 \\ 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$; $(6,\ 5.2,\ -1.4)$

7. $$\begin{pmatrix} 0.81 & -0.14 & -0.57 & 0 \\ -0.14 & 0.89 & -0.43 & 0 \\ -0.57 & -0.43 & -0.70 & 0 \\ 1.80 & 1.35 & 5.40 & 1 \end{pmatrix}$$

8. (i) $$\begin{pmatrix} 0.625 & 0.487 & 0.610 & 0 \\ -0.737 & 0.625 & 0.256 & 0 \\ -0.256 & -0.610 & 0.750 & 0 \\ 0.887 & 0.732 & -0.110 & 1 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} 0.92 & 0.38 & -0.07 & 0 \\ -0.37 & 0.92 & 0.14 & 0 \\ 0.12 & -0.11 & 0.99 & 0 \\ 2.61 & -1.42 & -0.51 & 1 \end{pmatrix}$$

9. $x + y - 2z = 7$

10. $$\begin{pmatrix} 0.056 & -0.013 & 0.998 & 0 \\ 0.458 & 0.889 & -0.013 & 0 \\ -0.887 & 0.458 & 0.056 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

11. (i) $90°$, (ii) axis parallel to $2\underline{i} - \underline{j} + 2\underline{k}$.

12. Rotation of $59.86°$ about an axis parallel to $-3\underline{i} + \underline{j} + 2\underline{k}$.

13. $(-0.52,\ 1.04,\ 0.80)$ and $(-0.60,\ -0.80,\ -1.00)$.

Chapter 3

Mathematics of Projection

1. Determine the image of the point $(-1, 4, 3)$ when it is projected orthogonally onto the plane $5x - 2y + 3z = 2$.

2. A camera is aligned such that it projects the point $(1, 1, 2)$ to the origin by an orthographic projection. Determine the image of the point $(-1, 4, -3)$ under the same projection.

3. A line segment has equation $\underline{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, $(0 \leq t \leq 1)$. It undergoes an orthographic projection onto the plane $4x + y + 2z = 1$. Obtain the equation of its image on the plane.

4. There is a cube whose base is the square $ABCD$, where the coordinates are $A(0, 0, 0)$, $B(0, 0, 3)$, $C(3, 0, 3)$ and $D(3, 0, 0)$. This cube is first transformed by the single point perspective transformation with viewing point at $Z(0, 0, -20)$ and then the result is projected onto the viewing plane $z = 0$.

   (a) Write down the coordinates of E, F, G and H, the other vertices of the cube.

   (b) Write down the matrix of the perspective transformation.

   (c) Calculate the coordinates of the vertices of the image of the cube when transformed by perspective.

   (d) Calculate the vertices of the planar shape, as projected onto the viewing plane.

5. Under the single point perspective transformation viewed from the point Z(0,0,-20), find in homogeneous form the images of:

   (a) the origin, (b) $(0, 0, -19)$, (c) $(1, 1, -20)$.

Are any of these **invariant** (i.e. unchanged) under this transformation?

6. The point $(5, 4, -6)$ is subject to a single-point perspective projection, with viewing centre at the origin, onto the plane $2x + 3y - z = 3$. Calculate its image on this plane.

7. The two points at either end of the line segment

$$\underline{r} = -\underline{i} + 2\underline{j} - 4\underline{k} + t(2\underline{i} + 3\underline{j} - 5\underline{k}), \quad (0 \leq t \leq 1)$$

are subjected to a single-point perspective projection onto the plane $z + 2 = x + y$, with viewing point at the origin. Determine the equation of the image of the line segment on the plane.

8. Determine if the following perspective transformations are possible for the given viewing centre, point of projection and view-plane respectively:

   (i) $C(2, 2, 3),\ P(3, -1, 2),\ x + 2y - z = 2$;

   (ii) $C(-2, 3, 1),\ P(5, 0, -1),\ 3x - y + 2z = 4$;

   (iii) $C(-1, -2, -3),\ P(3, -2, -1),\ -x + 3y + 4z = 1$;

   (iv) $C(2, 3, 2),\ P(4, 3, 3),\ x + 2y + z = 5$.

9. If the viewing point is changed to $(1, -1, 1)$ in question 6, calculate the new image coordinates of the point.

10. The line segment joining the points $(0, 2, 4)$ to $(-2, 5, 1)$ is projected onto the plane $x+y+z = 4$ by a single-point perspective transformation with viewing centre at $(-4, 3, -6)$. Determine the images of these end-points under the transformation and hence derive the equation of the projected line segment.

11. A standardized pyramid frustum volume, bounded by the planes $x = \pm z$, $y = \pm z$, $z = -1.5$ and $z = -4.5$, is to be projected onto the plane $z = -1$. Determine:

    (i) the 2D image of the point $(0.2, -1.4, -3.2)$ on the screen;

    (ii) the further transformation matrix needed to place the centre of the screen at $(1, 2)$, with width 3 and height 1.5, and to give the z values a range of 10;

    (iii) the final 2D image of the point in (i) after the transformation in (ii).

12. A standardized clipped volume, bounded by the planes $x = \pm z$, $y = \pm z$, $z = -3$ and $z = -6$, is to be projected onto a plane parallel to the $x - y$ plane. The centre of the image screen is to be placed at $(2, 5)$, of width 4 and height 3, with the $z-$values projected within the clipped volume having a range of 8. Obtain the final screen image coordinates of the points $(0.5, 1, -5)$ and $(1, 4, -3)$ respectively, determining whether they appear on the screen or not.

13. A point $P(x, y, z)$ is to be projected onto the plane $ax+by+cz = d$ in a direction parallel to the vector $\underline{v} = (v_x \quad v_y \quad v_z)$. Show that the transformation matrix in homogeneous coordinates that does this is given by

$$T_{par} = \begin{pmatrix} bv_y + cv_z & -av_y & -av_z & 0 \\ -bv_x & av_x + cv_z & -bv_z & 0 \\ -cv_x & -cv_y & av_x + bv_y & 0 \\ dv_x & dv_y & dv_z & av_x + bv_y + cv_z \end{pmatrix}.$$

Answers

1. $\left(-\dfrac{4}{19}, \dfrac{70}{19}, \dfrac{66}{19}\right)$

2. $(-0.5, 4.5, -2)$

3. $\underline{r} = \dfrac{1}{21}\begin{pmatrix} -10 \\ -13 \\ 37 \end{pmatrix} + \dfrac{u}{21}\begin{pmatrix} -46 \\ 20 \\ 82 \end{pmatrix}$, $(0 \leq u \leq 1)$

4. (a) $(0,3,0)$, $(0,3,3)$, $(3,3,3)$, $(3,3,0)$.

   (b) $\begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 20 \end{pmatrix}$

   (c) Matrix of homogeneous coordinates= $\begin{pmatrix} 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 23 \\ 60 & 0 & 0 & 23 \\ 60 & 0 & 0 & 20 \\ 60 & 60 & 0 & 20 \\ 60 & 60 & 0 & 23 \\ 0 & 60 & 0 & 23 \\ 0 & 60 & 0 & 20 \end{pmatrix}$

   (d) $(0,0)$, $(0,0)$, $(60/23,0)$, $(3,0)$, $(3,3)$, $(60/23,60/23)$, $(0,60/23)$, $(0,3)$.

5. (a) $(0,0,0)$ invariant, (b) $(0,0,0)$, (c) projected out to infinity.

6. $(15/28, 3/7, -9/14)$

7. $\underline{r} = \frac{2}{5}(-\underline{i} + 2\underline{j} + 4\underline{k}) + \frac{2}{15}(4\underline{i} - \underline{j} + 3\underline{k})$

8. (i) yes, (ii) yes, (iii) yes, (iv) no.

9. $(5/3, -1/6, -1/6)$

10. $(-8/13, 28/13, 32/13)$, $(-2,5,1)$; $\underline{r} = (-8\underline{i} + 28\underline{j} + 32\underline{k} + t[-18\underline{i} + 37\underline{j} - 19\underline{k}])/13$.

11. (i) $(0.0625, -0.4375)$, (ii) $N = \begin{pmatrix} 1.5 & 0 & 0 & 0 \\ 0 & 0.75 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & 2 & 5 & 1 \end{pmatrix}$, (iii) $(1.094, 1.672)$.

12. $(2.2, 5.3)$ and $(2.67, 7)$ - last one outside screen.

Chapter 4

Complex Numbers

1. Express in the form $a + bi$

(a) $(4-3i)+(2i-8)$ (b) $3(-1+4i)-2(7-i)$

(c) $(3+2i)(2-i)$ (d) $(i-2)\{2(1+i)-3(i-1)\}$

(e) $\dfrac{2-3i}{4-i}$ (f) $(4+i)(3+2i)(1-i)$

(g) $\dfrac{(2+i)(3-2i)(1+2i)}{(1-i)^2}$ (h) $(2i-1)^2\left\{\dfrac{4}{1-i}+\dfrac{2-i}{1+i}\right\}$

2. If $z_1 = 2+i$, $z_2 = -3+2i$ and $z_3 = 1-2i$, evaluate:

(a) $\dfrac{1}{z_1}+\dfrac{1}{z_2}$ (b) $\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}$

3. If $z_1 = 1-i$, $z_2 = -2+4i$ and $z_3 = \sqrt{3}-2i$, evaluate:

(a) $z_1^2+2z_1-3$ (b) $(z_3-\bar{z_3})^5$

(c) $\mathrm{Re}\{-2z_1^3+3z_2^2-5z_3^2\}$ (d) $\mathrm{Im}\left\{\dfrac{z_1z_2}{z_3}\right\}$

(e) $\overline{(z_2+z_3)(z_1-z_3)}$ (f) $\dfrac{1}{2}\left(\dfrac{z_3}{\bar{z_3}}+\dfrac{\bar{z_3}}{z_3}\right)$

4. Determine the modulus and principal value of the argument $(-180° < \theta < 180°)$ of the fol-

lowing complex numbers:

(a) $-\sqrt{3}+2i$ (b) $-3-5i$

(c) $\dfrac{2+3i}{1+2i}$ (d) $\dfrac{1+i}{(1-i)(2+i)}$

5. Express each of the following complex numbers in polar form, using the principal value of the argument $(-180^\circ < \theta < 180^\circ)$:

(a) $-\sqrt{3}-3i$ (b) $3+4i$

(c) $2-2i$ (d) $-1+\sqrt{3}i$

(e) $2\sqrt{2}+2\sqrt{2}i$ (f) $-i$

Answers

1. (a) $-4-i$ (b) $-17+14i$ (c) $8+i$ (d) $-9+7i$

   (e) $\dfrac{11}{17}-\dfrac{10}{17}i$ (f) $21+i$ (g) $-\dfrac{15}{2}+5i$ (h) $-\dfrac{11}{2}-\dfrac{23}{2}i$

2. (a) $\dfrac{11}{65}-\dfrac{23}{65}i$ (b) $\dfrac{24}{65}+\dfrac{3}{65}i$

3. (a) $-1-4i$ (b) $-1024i$ (c) -27

   (d) $\dfrac{6\sqrt{3}+4}{7}$ (e) $-7+3\sqrt{3}+\sqrt{3}i$ (f) $-\dfrac{1}{7}$

4. (a) $2.65,\ 130.89°$ (b) $5.83,\ -120.96°$

   (c) $\sqrt{\dfrac{13}{5}} \approx 1.61,\ -7.13°$ (d) $0.45,\ 63.43°$

5. (a) $2\sqrt{3}[-120°] = 2\sqrt{3}(\cos(-120°)+i\sin(-120°))$ (b) $5[53.13°]$

   (c) $2\sqrt{2}[-45°]$ (d) $2[120°]$

   (e) $4[45°]$ (f) $1[-90°]$

Chapter 5

Quaternions

1. For each of the following quaternions, write down its conjugate and norm/modulus respectively:

   (i) $2 + 5i + 7j + 4k$ (ii) $-3 + 2i + 2j + k$ (iii) $1 + 2i - 3j - 3k$

   (iv) $-1 + i + k$ (v) $\dfrac{2 - 3i + j}{7}$ (vi) $\dfrac{1}{2} + \dfrac{i}{3} - \dfrac{2j}{3} + \dfrac{k}{4}$.

2. If $a = 2 - i + 2j$ and $b = 2 + 2i + j - 3k$ then determine:

   (i) $2a + 3b$ (ii) $a\,b$ (iii) $b\,a$ (iv) a^{-1} (v) b^{-2}.

3. Express the following quaternions in polar form:

   (i) $2 - i + k$ (ii) $\dfrac{1 + i - j - k}{2}$ (iii) $2i + j - 3k$

   (iv) $-1 + j + 2k$.

4. Prove the result: $(\cos\theta + I\sin\theta)^2 \equiv \cos(2\theta) + I\sin(2\theta)$ for polar forms of quaternions. Hence determine the polar forms of the following:

   (i) $(1 + i - k)^2$ (ii) $\left(\dfrac{i - j + k}{\sqrt{3}}\right)^2$.

5. The **square root** s of a quaternion a is defined by: $s^2 = a$. Using the polar form property in

question 4, determine the square roots of the quaternion $a = 2 + i - 2j + 3k$ in cartesian form.

Answers

1. (i) $2-5i-7j-4k;\quad \sqrt{94}$ (ii) $-3-2i-2j-k;\quad 3\sqrt{2}$

(iii) $1-2i+3j+3k;\quad \sqrt{23}$ (iv) $-1-i-k;\quad \sqrt{3}$

(v) $\dfrac{2+3i-j}{7};\quad \sqrt{\dfrac{2}{7}}$ (vi) $\dfrac{1}{2}-\dfrac{i}{3}+\dfrac{2j}{3}-\dfrac{k}{4};\quad \dfrac{5\sqrt{5}}{12}.$

2. (i) $10+4i+7j-9k$ (ii) $4-4i+3j-11k$

(iii) $4+8i+9j-k$ (iv) $\dfrac{2+i-2j}{9}$

(v) $\dfrac{-5-4i-2j+6k}{162}.$

3. (i) $\sqrt{6}\left[\cos 35.26^\circ+\left(\dfrac{-i+k}{\sqrt{2}}\right)\sin 35.26^\circ\right]$ (ii) $\cos 60^\circ+\left(\dfrac{i-j-k}{\sqrt{3}}\right)\sin 60^\circ$

(iii) $\sqrt{14}\left[\cos 90^\circ+\left(\dfrac{2i+j-3k}{\sqrt{14}}\right)\sin 90^\circ\right]$ (iv) $\sqrt{6}\left[\cos 114.09^\circ+\left(\dfrac{j+2k}{\sqrt{5}}\right)\sin 114.09^\circ\right].$

4. (i) $3\left[\cos 109.47^\circ+\left(\dfrac{i-k}{\sqrt{2}}\right)\sin 109.47^\circ\right]$ (ii) $\cos 180^\circ+\left(\dfrac{i-j+k}{\sqrt{3}}\right)\sin 180^\circ.$

5. $\pm\left\{\sqrt{\dfrac{3\sqrt{2}+2}{2}}+I\sqrt{\dfrac{3\sqrt{2}-2}{2}}\right\},$ where $I=\dfrac{i-2j+3k}{\sqrt{14}}$

$\approx \pm(1.7667+0.2830\,i-0.5660\,j+0.8490\,k).$

Chapter 6

Rotation by Quaternions

1. Determine the quaternion $q = \cos\theta + I\sin\theta$ that performs the rotation in each of the following:

   (i) $60°$ about the $z-$axis, (ii) $-30°$ about the $x-$axis, (iii) $45°$ about the $y - axis$,

   (iv) $90°$ about the $y-$axis followed by $180°$ about the $x-$axis.

2. A point is rotated by $60°$ about the $y-$axis and then rotated by $90°$ about the $z-$axis. Determine the equivalent *single* rotation angle about an axis that gives the same image point.

3. Determine the co-ordinates of the point $(2, 1, 0)$ after the rotation:

   (i) $50°$ about the line parallel to $\underline{j} + \underline{k}$ passing through the origin,

   (ii) as in part (i) but followed by a rotation of $60°$ about the $z-$axis,

   (iii) $-120°$ about the line with vector equation $\underline{r} = \underline{k} + t(\underline{j} - \underline{k})$,

   (iv) $90°$ about the line with vector equation $\underline{r} = 2\underline{i} - \underline{k} + t(\underline{i} + \underline{j} + \underline{k})$.

Answers

1. (i) $\cos 30^\circ + k \sin 30^\circ$, (ii) $\cos 15^\circ - i \sin 15^\circ$,

   (iii) $\cos 22.5^\circ + j \sin 22.5^\circ$, (iv) $\cos 90^\circ + \left(\dfrac{i+k}{\sqrt{2}}\right) \sin 90^\circ$.

2. 104.48° about the axis $(-\underline{i} + \underline{j} + \sqrt{3}\underline{k})$ through the origin.

3. (i) $(0.744, 1.905, -0.905)$, (ii) $(-1.278, 1.597, -0.905)$,

   (iii) $(-1, 2.225, 1.225)$, (iv) $(2.667, 0.089, 0.224)$.