CMP502

Mathematics

Notes and Practical Examples

School of Design and Informatics

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Contents

Chapter 1

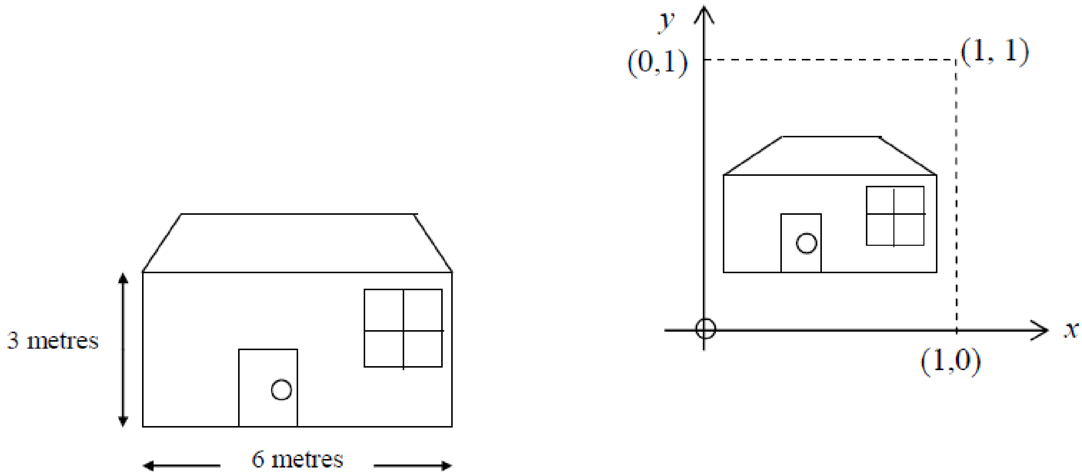
Two-Dimensional Viewing Transformations

Displaying an image of a picture involves mapping the coordinates of the points and lines that form the picture into the appropriate coordinates on the device or workstation where the image is to be displayed. This is done through the use of coordinate transformations known as *viewing transformations*. We use the following names for the various coordinate systems:

The *world coordinate system* (WCS) is the right-handed Cartesian coordinate system in whose coordinates we describe the picture that is to be displayed. The *physical device coordinate system* (PDCS) is the coordinate system that corresponds to the device or workstation where the image of the picture or model is to be displayed. Since there is such a wide variety of display devices, each with its own physical coordinate system, it is convenient to introduce a logical or virtual display surface with a standardised coordinate system.

The *normalised device coordinate system* (NDCS) is a right-handed coordinate system in which the display area of the virtual display device corresponds to the unit (1×1) square whose lower left-hand corner is at the origin of the coordinate system.

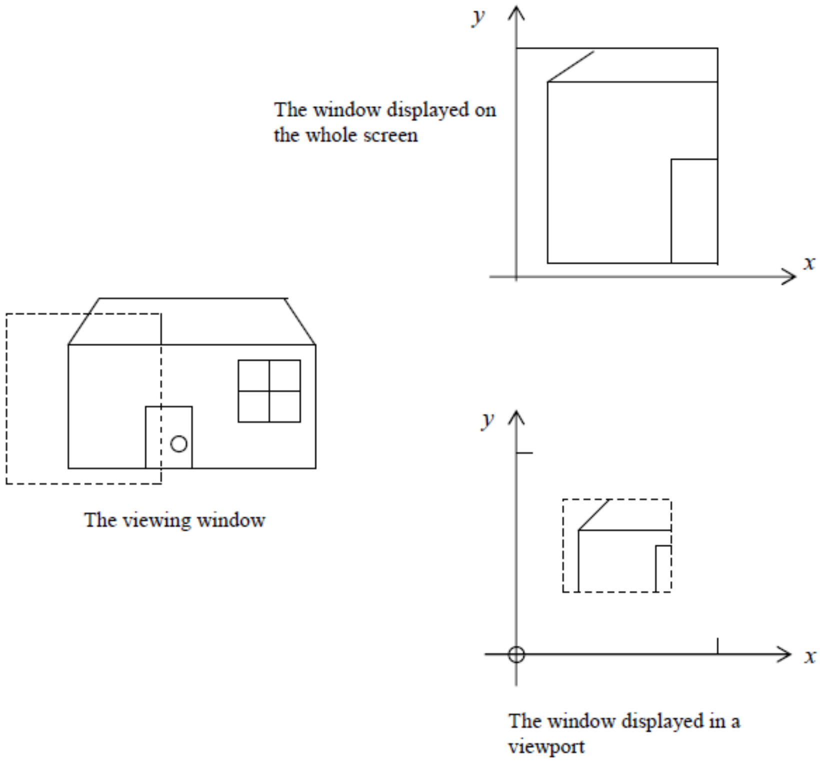
Suppose that we are considering a simplified, two-dimensional view of a real object, e.g. the front of a house. If the whole of this appears in the entire display screen, then we have the situation shown below. The bottom (horizontal) and left (vertical) edges of the screen are interpreted as the x- and y-axes respectively, and the coordinates on the screen are **normalised** so that they all lie between 0 and 1. Thus while positions on the real house might be measured in metres, horizontally and vertically from some point chosen to be an origin, the coordinates of corresponding points on the screen will usually all be positive numbers less than one.



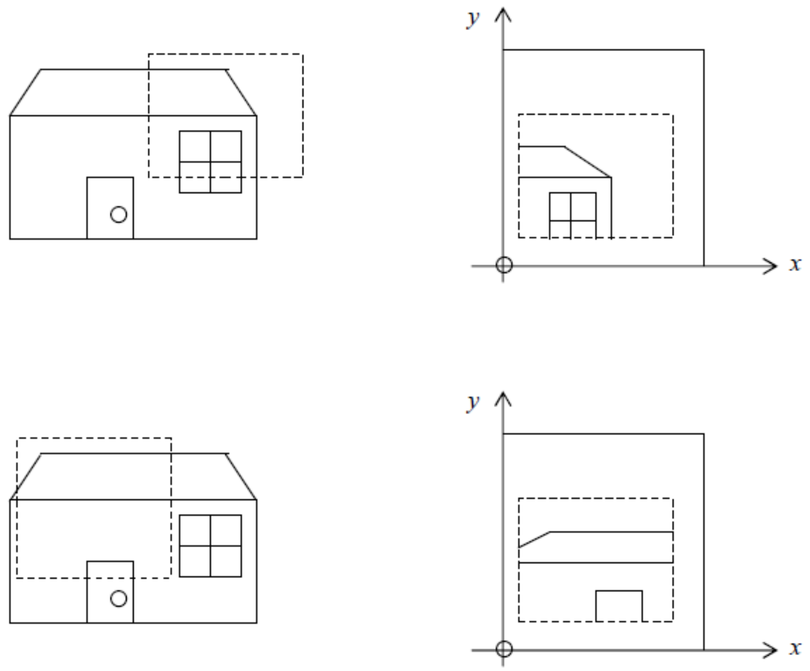
1.1 Viewports and Windows

One of the skills of graphical work is to be selective in the reproduction of images. It is unlikely that the whole of an object, taking up the entire screen display, will be a satisfactory representation for many purposes. It is more likely that attention will be paid to just part of an object, positioned in a particular section of the screen, and to do this we employ what is known as the **viewing transformation**.

Instead of viewing a whole object, it is usual to focus on just part of it, and this is termed the viewing **window**. For real applications of this technique it is obvious that many different shapes for a window will be used; however, for our present purposes we shall concentrate on **rectangular windows**. Sometimes the entire screen will be used to display the contents of a window, but sometimes just part of the screen, called a **viewport**, will be used for the display. These two situations are illustrated in the following figure:

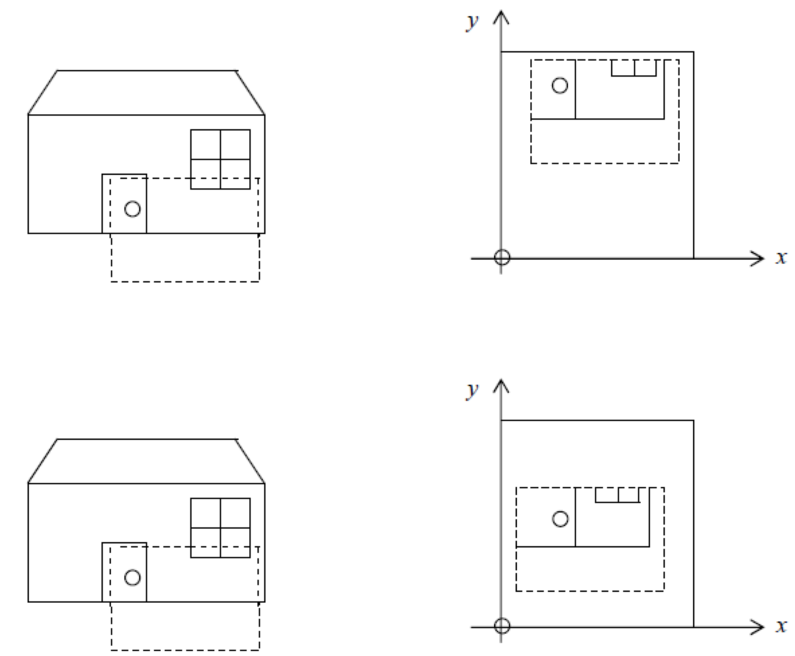


Before considering the calculation of the viewing transformation, consider the illustration below where the window defines the area of interest in the object and the viewport is an area on the viewing screen whose position in the viewing screen is defined.



Different Windows

Same Viewport



Same Windows

Different Viewports

1.2 The Viewing Transformation

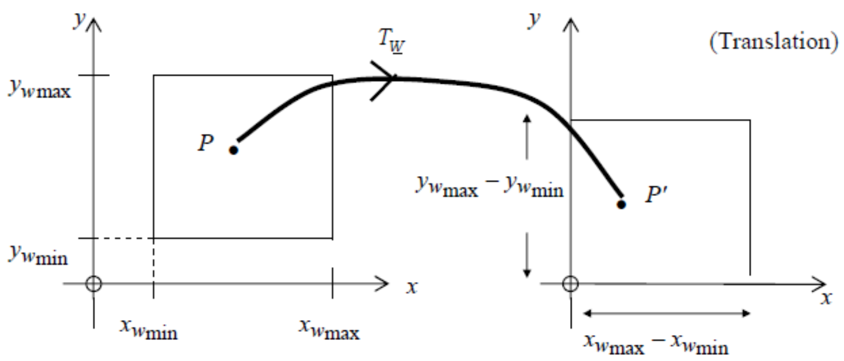
We now consider the viewing transformation itself: by this we take the coordinates of points on that part of the object in a designated window, and transform them to give the coordinates of the corresponding points in the appropriate viewport on the screen. As with all compound transformations, we employ a sequence of simple operations, each of which is performed by one of the matrices we have already met. There are three operations in the sequence.

(a) we shift the lower left corner of the window to the origin on the object, by a transformation

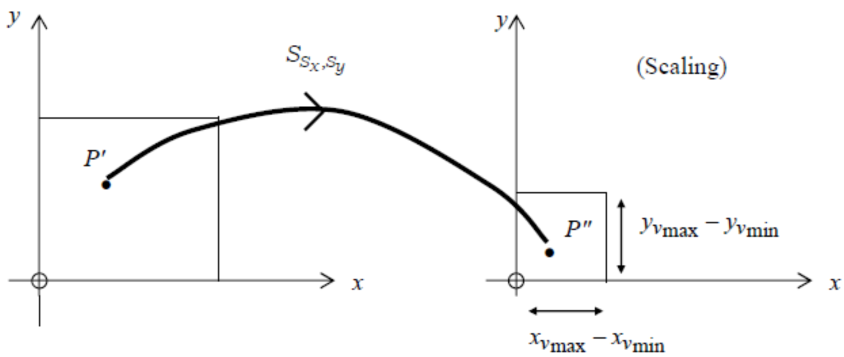
(b) we scale the window dimensions to the dimensions of the viewport, by a transformation , and imagine the two origins (window and viewport) to be coincident, and

(c) we shift the lower left corner of the viewport from the screen origin to its proper position by a transformation

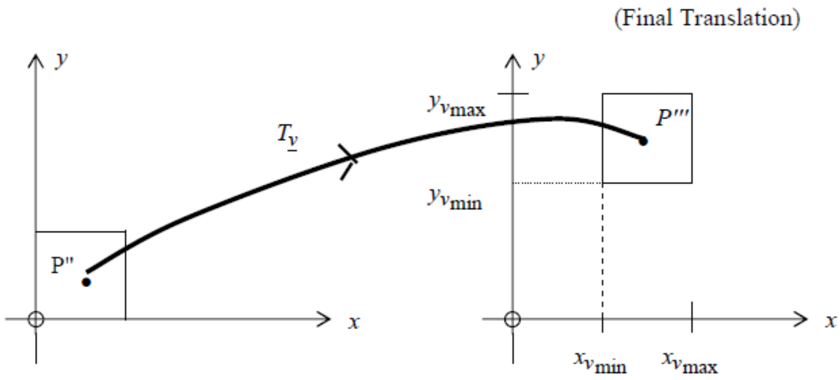
Pictorially, we have



$$[P'] = [P]T_{\underline{w}}$$



$$[P''] = [P']S_{S_x,s_y}$$



$$[P'''] = [P'']T_{\underline{v}}$$

Thus the compound viewing transformation is

$$[P'''] = [P]T_{\underline{w}}S_{S_x,s_y}T_{\underline{v}}$$

where

$$T_{\underline{w}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_{w\min} & -y_{w\min} & 1 \end{pmatrix}, \quad S_{S_x,S_y} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{\underline{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_{v\min} & y_{v\min} & 1 \end{pmatrix}$$

and

$$S_x = \frac{x_{v\max} - x_{v\min}}{x_{w\max} - x_{w\min}}, \quad S_y = \frac{y_{v\max} - y_{v\min}}{y_{w\max} - y_{w\min}}.$$

Notice that $T_{\underline{w}}S_{S_x,s_y}T_{\underline{v}} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ X & Y & 1 \end{pmatrix}$, where

$$X = \frac{\det \begin{pmatrix} x_{w\max} & x_{w\min} \\ x_{v\max} & x_{v\min} \end{pmatrix}}{x_{w\max} - x_{w\min}} \quad \text{and} \quad Y = \frac{\det \begin{pmatrix} y_{w\max} & y_{w\min} \\ y_{v\max} & y_{v\min} \end{pmatrix}}{y_{w\max} - y_{w\min}}$$

Note: it is not recommended to try to remember these formulae – far better to work through the three transformation stages for understanding the method.

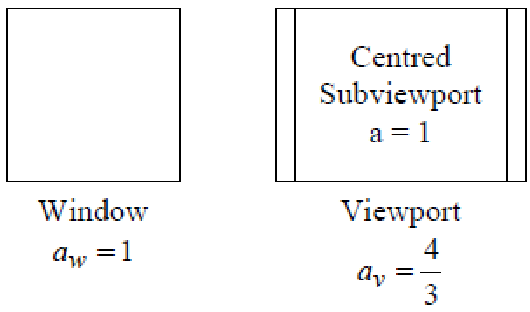
1.3 Aspect Ratio

Since the viewing transformation involves scaling, undesirable distortions may be introduced whenever $S_x \neq S_y$. For example, circles within the window may be displayed as ellipses and squares as rectangles. In considering how these distortions may be avoided, the concept of *aspect ratio* is needed.

The *aspect ratio* of a rectangular window or viewport is defined by

$$a = \frac{x_{\max} - x_{\min}}{y_{\max} - y_{\min}}.$$

If the aspect ratio a_w of the window equals the aspect ratio a_v of the viewport then $S_x = S_y$, and no distortion (other than uniform magnification or compression) occurs. If $a_w \neq a_v$, then distortion occurs. If desired, in this case we can describe a region within the viewport, a *subviewport*, whose aspect ratio is that of the window. We can then redefine the viewing transformation so that the window is transformed onto the subviewport.



1.4 Clipping and Shielding

Quite often we wish to display only a portion of the total picture. In this case a window is used to select that portion of the picture which is to be viewed (much like clipping or cutting out a picture from a magazine). This is known as *clipping*. The process of clipping determines which elements of the picture lie inside the window and so are visible (i.e. displayed). The reverse of clipping is *shielding* – covering up a portion of the picture so that it is not visible.

Examples

1. Find the normalisation transformation which maps a window whose lower left corner is at

$(1, 1)$ and upper right corner is at $(3, 5)$ onto:

(a) a viewport that is the entire normalised device screen;

(b) a viewport that has lower left corner at $(0.2, 0.4)$ and upper right corner at $(0.5, 0.9)$.

Solution

(a)

$$T_{\underline{w}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \quad S_{S_x,S_y} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_{\underline{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(no real need for the last matrix as it is the identity) and so the transformation is

$$T_{\underline{w}} S_{S_x,s_y} T_{\underline{v}} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ -0.5 & -0.25 & 1 \end{pmatrix}$$

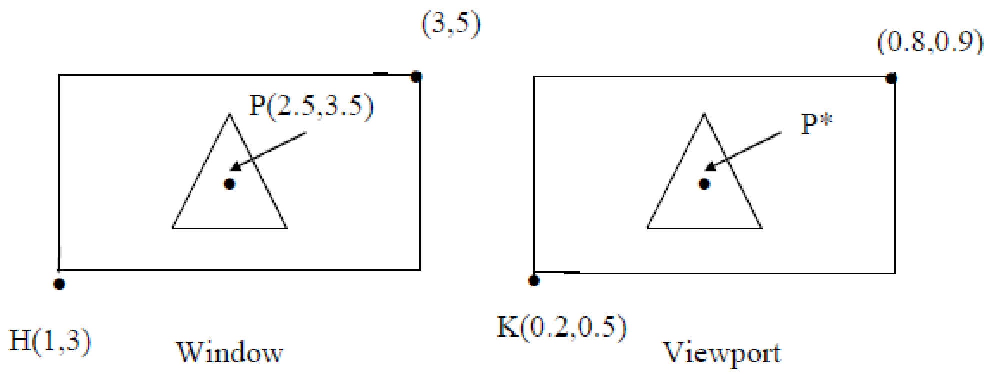
(Check this).

(b) In this case, $T_{\underline{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.2 & 0.4 & 1 \end{pmatrix}$ and $S_{S_x,S_y} = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. These give the normalisation transformation as

$$T_{\underline{w}} S_{S_x,s_y} T_{\underline{v}} = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.125 & 0 \\ 0.05 & 0.275 & 1 \end{pmatrix}$$

(Check this).

2. Part of a triangle (the object) is to be displayed on part of a screen. Describe the transformations to transform the following window to the indicated viewport. In addition, calculate the coordinates of the labelled point where it appears on the screen.



Solution

In this case we have the translation, $T_{\underline{w}} =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

then the scaling, $S_{S_x,s_y} =$

$$\begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then the translation, $T_{\underline{v}} =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.2 & 0.5 & 1 \end{pmatrix}$$

The composite transformation is $\begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ -0.1 & -0.1 & 1 \end{pmatrix}$ and P^* has co-ordinates $(0.65, 0.6)$. (Check)

$$[P'''] = [P] T_{\underline{w}} S_{S_x,S_y} T_{\underline{v}}$$

$$T_{\underline{w}} S_{S_x S_y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ -0.3 & -0.6 & 1 \end{pmatrix}$$

$$T_{\underline{w}} S_{S_x S_y} T_{\underline{v}} = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ -0.3 & -0.6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.2 & 0.5 & 1 \end{pmatrix} = \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ -0.3+0.2 & -0.6+0.5 & 1 \end{pmatrix}$$

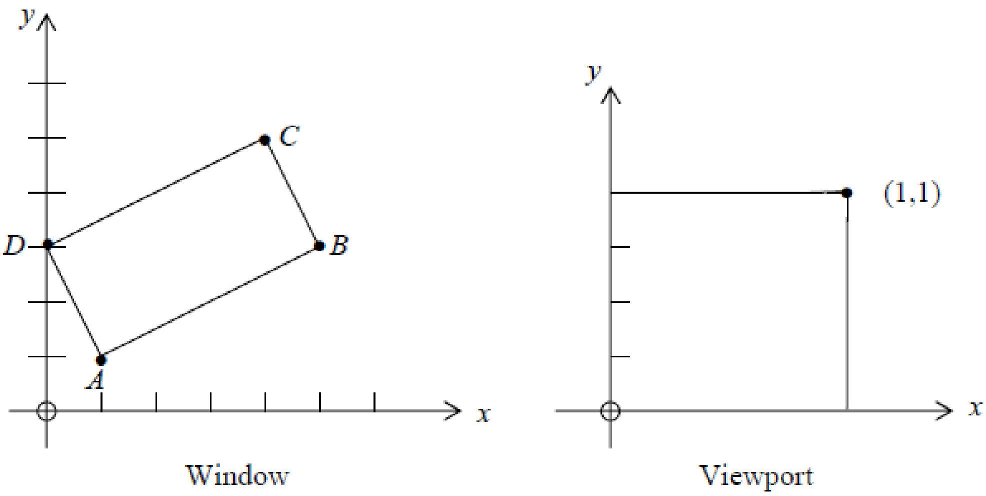
$$= \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ -0.1 & -0.1 & 1 \end{pmatrix}$$

$$[P'''] = (2.5, 3.5, 1) \begin{pmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ -0.1 & -0.1 & 1 \end{pmatrix}$$

$$= (0.75 - 0.1,\ 0.7 - 0.1,\ 1) = \underline{(0.65, 0.6)} = P'''$$

3. Find the normalisation transformation which uses the rectangle $A(1,1)$, $B(5,3)$, $C(4,5)$ and

$D(0,3)$ as a window and the normalised device screen as a viewport, where AB is mapped to the $x-$axis.



Solution

(a) Move A to $(0,0)$ and then rotate by $-\theta$, where $\tan\theta = 1/2$, i.e. use the transformation

$\sin\theta = \frac{2}{2\sqrt{5}}$ $\cos\theta = \frac{4}{2\sqrt{5}}$

$R_\theta = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\sin(-\theta) = -\sin(\theta)$

$\cos(-\theta) = \cos(\theta)$

(b) Calculate the scaling factors S_x and S_y:

scaling: $S_{Sx,Sy}$

window width $= |B-A| = 2\sqrt{5}$

viewport width $= 1$

$Sx = \frac{1}{2\sqrt{5}}$

window height $= |D-A| = \sqrt{5}$

viewport height $= 1$

$Sy = \frac{1}{\sqrt{5}}$

$S_{SxSy} = \begin{pmatrix} \frac{1}{2\sqrt{5}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) Use the scaling transformation on the above to get the final transformation matrix

$$\underline{T_w} R_{-\beta} S_{sx\,sy} = \begin{pmatrix} \frac{2}{\sqrt5} & -\frac{1}{\sqrt5} & 0 \\ \frac{1}{\sqrt5} & \frac{2}{\sqrt5} & 0 \\ -\frac{3}{\sqrt5} & -\frac{1}{\sqrt5} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt5} & 0 & 0 \\ 0 & \frac{1}{\sqrt5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{\begin{pmatrix} \frac{1}{5} & -\frac{1}{5} & 0 \\ \frac{1}{10} & \frac{2}{5} & 0 \\ -\frac{3}{10} & -\frac{1}{5} & 1 \end{pmatrix}}$$

$$\underline{T_v} = \underline{\underline{I}}$$

check answer: map ABCD into normalised coords

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 3 & 1 \\ 4 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} & 0 \\ \frac{1}{10} & \frac{2}{5} & 0 \\ -\frac{3}{10} & -\frac{1}{5} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5}+\frac{1}{10}-\frac{3}{10}, & -\frac{1}{5}+\frac{2}{5}-\frac{1}{5}, & 1 \\ 1+\frac{3}{10}-\frac{3}{10}, & -1+\frac{6}{5}-\frac{1}{5}, & 1 \\ \frac{4}{5}+\frac{5}{10}-\frac{3}{10}, & -\frac{4}{5}+\frac{10}{5}-\frac{1}{5}, & 1 \\ \frac{3}{10}-\frac{3}{10}, & \frac{6}{5}-\frac{1}{5}, & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0, & 0, & 1 \\ 1, & 0, & 1 \\ 1, & 1, & 1 \\ 0 & 1, & 1 \end{pmatrix} \begin{matrix} (A') \\ (B') \\ (C') \\ (D') \end{matrix}$$

✓✓

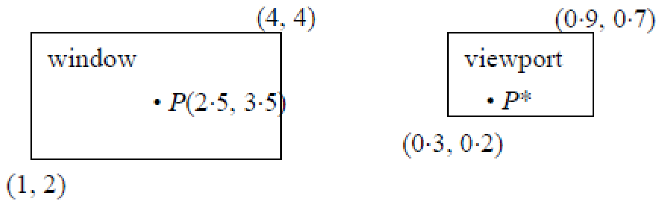
Tutorial Exercises - Viewing Transformations

1. Obtain the normalisation transformation that maps a window whose lower left corner is at $(2, 3)$ and upper right corner is at $(4, 7)$ onto

   (a) a viewport that is the entire normalised device screen;

   (b) a viewport that has a lower left corner at $(0.1, 0.3)$ and upper right corner at $(0.4, 0.5)$.

2. Determine the transformations from the following window to the indicated viewport. What are the coordinates of the labelled point where it appears on the screen?



3. Obtain the complete viewing transformation that

   (a) maps a window with world co-ordinates $1 \leq x \leq 10$, $1 \leq y \leq 10$ onto the viewport with $0.25 \leq x \leq 0.75$, $0 \leq y \leq 0.5$ in normalised device space, and then

   (b) maps a window with $0.25 \leq x \leq 0.5$, $0.25 \leq y \leq 0.5$ in the normalised device space into a viewport with $1 \leq x \leq 10$, $1 \leq y \leq 10$ on the physical display device.

4. Derive a normalisation transformation from the window whose lower left corner is at $(0, 0)$ and whose upper right corner is at $(4, 3)$ onto the normalised device screen so that the aspect ratios are preserved.

5. Determine the workstation transformation which maps the normalised device screen onto a physical device with $0 \leq x \leq 199$, $0 \leq y \leq 639$ where the origin is located at the lower left corner of the device.

6. Calculate the normalisation transformation that uses a circle of radius 5 units and centre $(1, 1)$ as a window, and a circle of radius 0.5 and centre $0.5, 0.5$ as a viewport.

7. Obtain the normalisation that uses the rectangle $A(0, 5)$, $B(2, 7)$, $C(0, 9)$ and $D(-2, 7)$ as a window and the normalised device screen as a viewport.

8. Determine the complete viewing transformation that uses the rectangle with vertices at $A(1, 3)$, $B(3, 5)$, $C(6, 2)$ and $D(4, 0)$ as a window and displays it in a viewing device with A mapped

to the lower left corner at $(5, 5)$ and C mapped to the upper right corner at $(10, 10)$ (AD and BC being mapped parallel to the new horizontal axis).

__Answers__

1. (a) $\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \\ -1 & -0.75 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.05 & 0 \\ -0.2 & 0.15 & 1 \end{pmatrix}$

2. $T = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0.1 & -0.3 & 1 \end{pmatrix}$; $(0.6, 0.575)$

3. First stage, $T_1 = \begin{pmatrix} \frac{1}{18} & 0 & 0 \\ 0 & \frac{1}{18} & 0 \\ \frac{7}{36} & \frac{-1}{18} & 1 \end{pmatrix}$, overall $T = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & -10 & 1 \end{pmatrix}$.

4. $\begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. $\begin{pmatrix} 199 & 0 & 0 \\ 0 & 639 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0.4 & 0.4 & 1 \end{pmatrix}$

7. $\begin{pmatrix} 0.25 & -0.25 & 0 \\ 0.25 & 0.25 & 0 \\ -1.25 & -1.25 & 1 \end{pmatrix}$

8. $\begin{pmatrix} \frac{5}{6} & \frac{5}{4} & 0 \\ \frac{-5}{6} & \frac{5}{4} & 0 \\ \frac{20}{3} & 0 & 1 \end{pmatrix}$

Chapter 2

3-D Graphics Transformations

Manipulation, viewing, and construction of three-dimensional graphic images requires the use of three-dimensional geometric and coordinate transformations. These transformations are formed by composing the basic transformations of translation, scaling, and rotation. As in two-dimensions, each of these transformations can be represented as a matrix transformation. This permits more complex transformations to be built up by use of matrix multiplication or *concatenation*.

Two complementary points of view can be adopted: either the object or picture is manipulated directly through the use of geometric transformations, or the object remains stationary and the viewer's coordinate system is changed by using coordinate transformations. The transformations and concepts introduced here are direct generalisations of those introduced for two-dimensional transformations.

2.1 Geometric Transformations

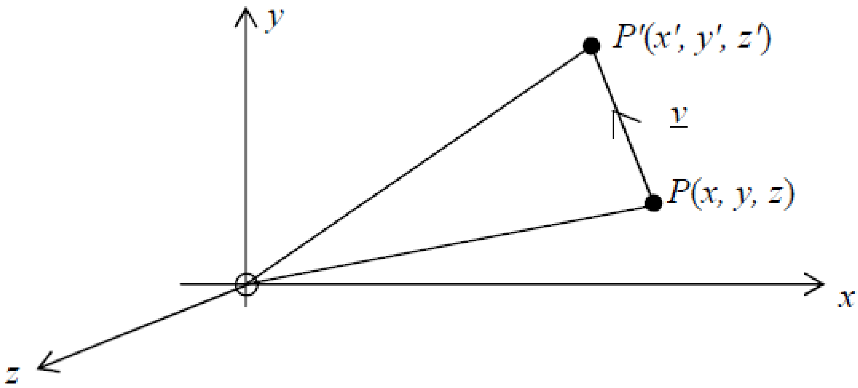
With respect to some three-dimensional coordinate systems, an object Obj is considered as a set of points:

$$\mathrm{Obj} = \{P(x, y, z)\}$$

If the object is moved to a new position, we can regard it as a new object Obj' all of whose coordinate points $P'(x', y', z')$ can be obtained from the original coordinate points $P(x, y, z)$ of Obj through the application of a geometric transformation.

2.2 Translation

An object is displaced a given distance and direction from its original position. The direction and displacement of the translation is prescribed by a vector $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$



The new coordinates of a translated point can be calculated by using the transformation

$$T_{\underline{v}} : \begin{cases} x' = x + a \\ y' = y + b \\ z' = z + c \end{cases}$$

In order to represent this transformation as a matrix transformation, we need to use homogeneous coordinates. The required homogeneous matrix transformation can then be expressed as

$$T_{\underline{v}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

2.3 Scaling

The process of scaling changes the dimensions of an object. The scale factor s determines whether the scaling is a *magnification*, $s > 1$, or a *reduction*, $s < 1$.

Scaling with respect to the origin, where the origin remains fixed, is effected by the transformation

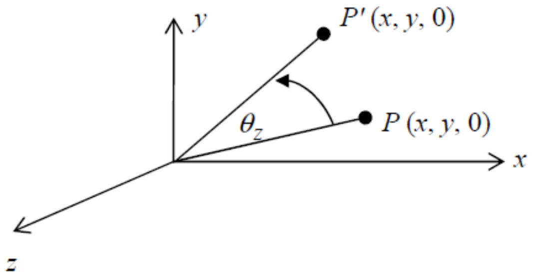
$$S_{S_x, S_y, S_z} : \begin{cases} x' = s_x x \\ y' = s_y y \\ z' = s_z z \end{cases}$$

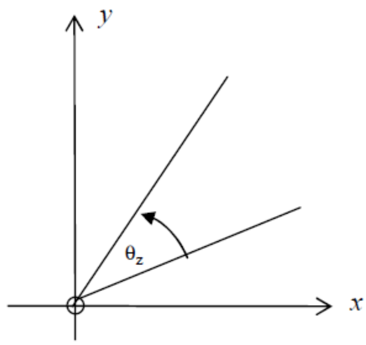
In matrix form this is

$$S_{S_x,S_y,S_z} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_z \end{pmatrix}$$

2.4 Rotation

Rotation in three dimensions is considerably more complex than rotation in two dimensions. In two dimensions, a rotation is prescribed by an angle of rotation θ and a centre of rotation P. Three-dimensional rotations require the prescription of an angle of rotation and an axis of rotation. The *canonical* rotations are defined when one of the positive x, y, or z coordinate axes is chosen as the axis of rotation. Then the construction of the rotation transformation proceeds just like that of a rotation in two dimensions about the origin.





2.4.1 Rotation about the z−axis

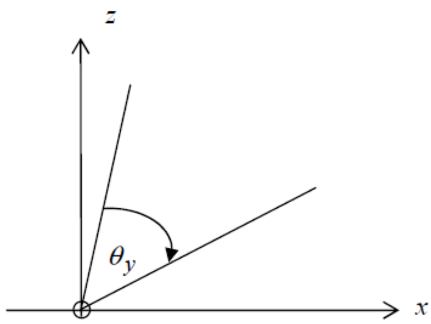
From 2D-transformations we know that

$$R_{\theta_{z,\underline{k}}} : \begin{cases} x' = x\cos(\theta_z) - y\sin(\theta_z) \\ y' = x\sin(\theta_z) + y\cos(\theta_z) \\ z' = z \end{cases}$$

2.4.2 Rotation about the y−axis

An analogous derivation leads to

$$R_{\theta_y,\underline{j}}: \begin{cases} x' = x\cos(\theta_y) + z\sin(\theta_y) \\ y' = y \\ z' = -x\sin(\theta_y) + z\cos(\theta_y) \end{cases}$$



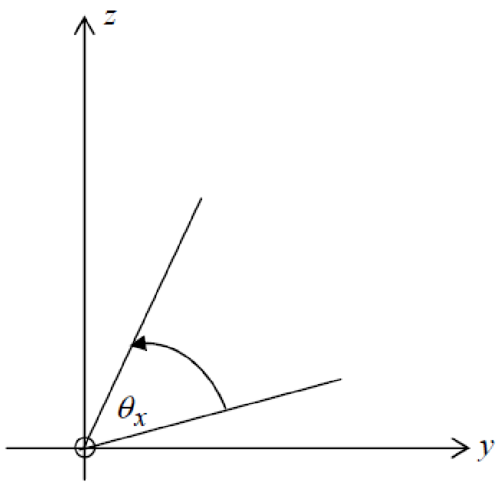
($y-$axis points inward to the $x - z$ plane shown)

2.4.3 Rotation about the x−axis

Similarly:

$$R_{\theta_x,\underline{i}}: \begin{cases} x' = x \\ y' = y\cos(\theta_x) - z\sin(\theta_x) \\ z' = y\sin(\theta_x) + z\cos(\theta_x) \end{cases}$$

($x-$axis points outward from the $y - z$ plane shown)



Note that the direction of a positive angle of rotation is chosen in accordance to the *right-hand rule* with respect to the axis of rotation.

The corresponding matrix transformations are

$$R_{\theta_z,\underline{k}} = \begin{pmatrix} \cos(\theta_z) & \sin(\theta_z) & 0 \\ -\sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_{\theta_y,\underline{j}} = \begin{pmatrix} \cos(\theta_y) & 0 & -\sin(\theta_y) \\ 0 & 1 & 0 \\ \sin(\theta_y) & 0 & \cos(\theta_y) \end{pmatrix}, \qquad R_{\theta_x,\underline{i}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & \sin(\theta_x) \\ 0 & -\sin(\theta_x) & \cos(\theta_x) \end{pmatrix}$$

The general case of rotation about an axis L can be built up from these canonical rotations using matrix multiplication. However, there is a more direct method for doing this as we shall see later.

2.5 Composite Transformations and Matrix Concatenation

More complex geometric and coordinate transformations are formed through the process of *composition of functions.* For matrix functions, however, the process of composition is equivalent to matrix multiplication or **concatenation**.

In order to build these more complex transformations through matrix concatenation, we must be able to multiply translation matrices with rotation and scaling matrices. This necessitates the use of **homogeneous coordinates** where the standard (3×3) matrices of rotation and scaling can be represented as (4×4) homogeneous matrices by adjoining an extra row and column as follows:

$$\begin{pmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These transformations are then applied to points $P(x, y, z)$ having the homogeneous form:

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix}$$

For example, the matrices of translation by (a, b, c) and rotation about the $y-$axis have the respective homogeneous forms:

$$T_{\underline{v}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix} \quad \text{and} \quad R_{\theta_y,\underline{j}} = \begin{pmatrix} \cos(\theta_y) & 0 & -\sin(\theta_y) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta_y) & 0 & \cos(\theta_y) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example

Tilting is defined as a rotation about the $x-$axis followed by a rotation about the $y-$axis. Find the tilting matrix and decide whether the order of performing the rotation matters

Solution

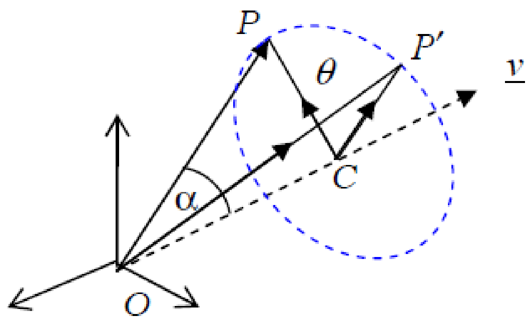
$$R_{\theta_x,\underline{i}}R_{\theta_y,\underline{j}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_x) & \sin(\theta_x) & 0 \\ 0 & -\sin(\theta_x) & \cos(\theta_x) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta_y) & 0 & -\sin(\theta_y) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(\theta_y) & 0 & \cos(\theta_y) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_y) & 0 & -\sin(\theta_y) & 0 \\ \sin(\theta_x)\sin(\theta_y) & \cos(\theta_x) & \sin(\theta_x)\sin(\theta_y) & 0 \\ \cos(\theta_x)\sin(\theta_y) & -\sin(\theta_x) & \cos(\theta_x)\cos(\theta_y) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The order **does** matter as $R_{\theta_x,\underline{i}}R_{\theta_y,\underline{j}} \neq R_{\theta_y,\underline{j}}R_{\theta_x,\underline{i}}$ (try this).

2.6 Rotation by θ about any axis

Consider the point $P(x, y, z)$ to be rotated through θ about the axis direction $\underline{v}$ as shown to give the image $P'(x', y', z')$.



In homogeneous coordinates, the image point is given by

$$P' = (x \quad y \quad z \quad 1)R_{\theta,\hat{\underline{v}}}$$

where

$$\boxed{R_{\theta,\hat{\underline{v}}} = \begin{pmatrix} \alpha^2(1-\cos\theta)+\cos\theta & \alpha\beta(1-\cos\theta)+\gamma\sin\theta & \alpha\gamma(1-\cos\theta)-\beta\sin\theta & 0 \\ \alpha\beta(1-\cos\theta)-\gamma\sin\theta & \beta^2(1-\cos\theta)+\cos\theta & \beta\gamma(1-\cos\theta)+\alpha\sin\theta & 0 \\ \alpha\gamma(1-\cos\theta)+\beta\sin\theta & \beta\gamma(1-\cos\theta)-\alpha\sin\theta & \gamma^2(1-\cos\theta)+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

For the interested student, a derivation of the above rotation matrix $R_{\theta,\hat{\underline{v}}}$ can be found in Appendix A.1.

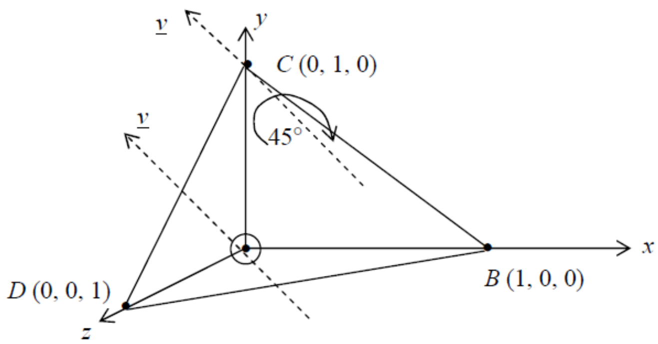
Example

The pyramid defined by co-ordinates $O(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$ and $D(0,0,1)$ is rotated 45° about the line that has direction $\underline{v} = \underline{j} + \underline{k}$ and passes through:

(i) the origin;

(ii) the point $C(0,1,0)$.

Obtain the vertices of the rotated figure in both cases.



Note that $\underline{v} = \underline{j} + \underline{k} \Rightarrow |\underline{v}| = \sqrt{2}$ and hence $\hat{\underline{v}} = \dfrac{\underline{j} + \underline{k}}{\sqrt{2}} \Rightarrow \alpha = 0,\ \beta = \gamma = \dfrac{1}{\sqrt{2}}$. The rotation matrix $R_{45°,\hat{\underline{v}}}$ is then calculated using $\theta = 45°$, i.e.

$$\begin{aligned}
R_{\theta,\hat{\underline{v}}} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}} & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) & 0 \\ \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) & 0 \\ \frac{1}{2} & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) & \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

(i) To find the coordinates of the rotated figure, we apply the rotation matrix $R_{45°,\hat{\underline{v}}}$ to the matrix of homogeneous co-ordinates of the vertices 0, B, C, D, i.e.

$$(O \quad B \quad C \quad D) \cdot R_{\theta,\underline{\hat{v}}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & 0 \\ \frac{1}{2} & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & 1 \\ \frac{1}{2} & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0.707 & 0.5 & -0.5 & 1 \\ -0.5 & 0.854 & 0.146 & 1 \\ 0.5 & 0.146 & 0.854 & 1 \end{pmatrix}$$

New vertices are at

$O'(0,0,0)$, $B'(0.707, 0.5, -0.5)$,

$C'(-0.5, 0.854, 0.146)$, $D'(0.5, 0.146, 0.854)$

(ii) We now have to translate the axis of rotation passing through $(0,1,0)$ by $-\underline{j}$ before rotation and then translate back by $\underline{j}$ to get the correct image points. Hence the process is defined by[1]

$$(O' \quad B' \quad C' \quad D') = (O \quad B \quad C \quad D) \cdot T_{-\underline{p}} \cdot R_{45^\circ,\underline{\hat{v}}} \cdot T_{\underline{p}},$$

where

$$T_{-\underline{p}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad T_{\underline{p}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

[1] Although most graphics pipelines use this method, a more efficient way is shown in Appendix A.2.

$$\therefore (O' \quad B' \quad C' \quad D') = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \cdot R_{45^\circ,\hat{\underline{v}}} \cdot T_{\underline{p}}$$

$$= \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & 0 \\ \frac{1}{2} & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot T_{\underline{p}}$$

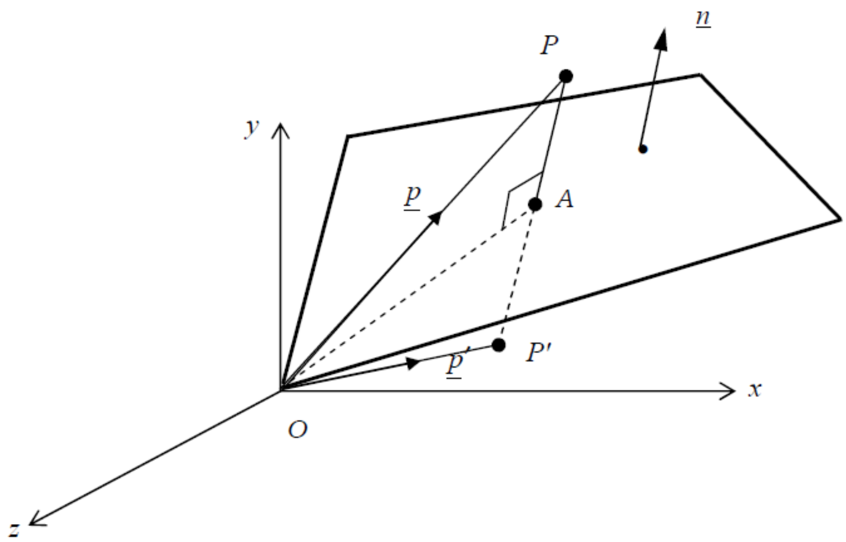
$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}(1+\frac{1}{\sqrt{2}}) & -\frac{1}{2}(1-\frac{1}{\sqrt{2}}) & 1 \\ \frac{1}{\sqrt{2}}+\frac{1}{2} & -\frac{1}{2\sqrt{2}} & -1+\frac{1}{2\sqrt{2}} & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & -\frac{1}{2}(1-\frac{1}{\sqrt{2}}) & 1 \\ \frac{1}{\sqrt{2}}+\frac{1}{2} & 1-\frac{1}{2\sqrt{2}} & -1+\frac{1}{2\sqrt{2}} & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.5 & 0.146 & -0.146 & 1 \\ 1.207 & 0.646 & -0.646 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0.293 & 0.707 & 1 \end{pmatrix}.$$

This gives the image vertices at

$O'(0.5, 0.146, -0.146)$, $B'(1.207, 0.646, -0.646)$,
$C'(0, 1, 0)$, $D'(1, 0.293, 0.707)$



2.7 Reflection in a Given Plane

Consider the plane of reflection, passing through the origin, given by its equation $ax + by + cz = 0$, then the reflection of some point P in the plane has image P' lying directly opposite P on the other side of the plane through some point A on its surface, such that $\overrightarrow{P'P}$ is parallel to the normal $\underline{n} = (a \quad b \quad c)^T$.

In homogeneous coordinates, the image point is given by

$$\boxed{P' = (x \quad y \quad z \quad 1)\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac & 0 \\ -2ab & a^2-b^2+c^2 & -2cb & 0 \\ -2ac & -2bc & a^2+b^2-c^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{pmatrix}}$$

(Notice the *symmetry* of this matrix.)

A derivation for the above reflection matrix can be found in Appendix A.3.

If the plane does not pass through the origin then we have to perform a translation first to satisfy this condition before applying the above transformation matrix and then translate back for the final image.

Examples

1. Obtain the matrix for mirror reflection with respect to the plane passing through the origin and having a normal vector $\underline{n} = \underline{i} + \underline{j} + \underline{k}$.

Solution

Using the above transformation, with $a = b = c = 1$, gives:

$$T = \begin{pmatrix} 1 & -2 & -2 & 0 \\ -2 & 1 & -2 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

2. Calculate the images of the points $(3, 2, 1)$ and $(-1, 0, 1)$ when reflected in the plane $2x + y - 3z = 4$.

Solution

$$R = \begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac & 0 \\ -2ab & a^2-b^2+c^2 & -2bc & 0 \\ -2ac & -2bc & a^2+b^2-c^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{pmatrix}$$ for plane $ax+by+cz=0$

• translate (plane doesn't pass through origin)

$3z = 2x + y - 4$ $x=1$ $y=2$ $\to z=0$

$\therefore$ plane passes $(1,2,0)$

$\therefore$ translate by $(-1,-2,0)$

$$T_w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 6 & -4 & 12 & 0 \\ -4 & 12 & 6 & 0 \\ 12 & 6 & -4 & 0 \\ 0 & 0 & 0 & 14 \end{pmatrix}$$

$$T_w R T_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -4 & 12 & 0 \\ -4 & 12 & 6 & 0 \\ 12 & 6 & 4 & 0 \\ 0 & 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -4 & 12 & 0 \\ -4 & 12 & 6 & 0 \\ 12 & 6 & -4 & 0 \\ 2 & -20 & -24 & 14 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -4 & 12 & 0 \\ -4 & 12 & 6 & 0 \\ 12 & 6 & -4 & 0 \\ 16 & 8 & -24 & 14 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 & -4 & 12 & 0 \\ -4 & 12 & 6 & 0 \\ 12 & 6 & -4 & 0 \\ 16 & 8 & -24 & 14 \end{pmatrix} = \begin{pmatrix} 38 & 26 & 20 & 14 \\ 22 & 18 & -40 & 14 \end{pmatrix} \div 14$$

$$\approx \begin{pmatrix} 2.71, & 1.86, & 1.429, & 1 \\ 1.57, & 1.29, & -2.86, & 1 \end{pmatrix}$$

Tutorial Exercises - 3D Transformations

1. Write down the separate(4×4) matrices that would produce each of the following transformations:

   (a) shift 0.5 in the $x-$direction, 0 in the $y-$direction and -0.2 in the $z-$direction;

   (b) scale the $z-$coordinates to be half as large;

   (c) scale both the $x-$and $y-$coordinates to be twice as large;

   (d) rotate through an angle of $45°$ about the $x-$axis;

   (e) rotate through an angle of $60°$ about the $y-$axis;

   (f) reflect in the xy plane, and then scale overall by a factor of 3;

   (g) rotate through an angle of $180°$ about the line passing through the points $(0, 0, 0)$ and $(1, 0, 1)$.

2. We can define a **tilt** to be the transformation caused by first rotating about the $x-$axis by θ and then rotating about the $y-$axis by ϕ.

   (a) Determine the tilting matrix when $\theta = \phi = 45°$.

   (b) Does the order of performing the rotations matter?

3. Calculate the single matrix which performs translations in the x, y and z directions of -1, -1 and -1 respectively, followed by, successively, a $30°$ rotation about the $x-$axis and a $45°$ rotation about the $y-$axis. Apply this transformation to the unit cube with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and state the coordinates of the vertices of the image formed.

4. Determine the coordinates of the image of the point $(-1, 5, 2)$ by rotating it through $90°$ about the axis $\underline{i} + 2\underline{j} - 2\underline{k}$, passing through the origin.

5. Determine the coordinates of the image of the point $(3, 0, -4)$ by rotating it through $50°$ about the axis $-\underline{i} + 3\underline{j} + 2\underline{k}$, passing through the origin.

6. Obtain the matrix for mirror reflection with respect to the plane $y = 2z$ passing through the origin and having a normal vector $\underline{n} = \underline{j} - 2\underline{k}$. What is the reflection of the point $(6, 2, 5)$ in this plane?

7. Determine the matrix to represent a reflection in the plane through $(5, 6, 0)$ with normal $\underline{n} = 8\underline{i} + 6\underline{j} + 24k$.

8. Obtain the single matrix to represent a rotation through:

(i) 60° about the line through $(1, 0, 2)$ parallel to $\underline{v} = \underline{i} - \underline{j} + \sqrt{2}\underline{k}$;

(ii) 24° about the line $\underline{r} = 5\underline{i} + 6j + t(8\underline{i} + 6j + 24k)$.

9. If the points $(2, -1, 0)$ and $(4, 1, -4)$ are mirror images of each other in a certain plane, determine the equation of the plane.

10. An *alignment* transformation is a rotation of a vector $\underline{v}$ such that it becomes parallel (aligned) to another vector $\underline{u}$. Using this definition with the standard rotation matrix, determine an alignment matrix that aligns $\underline{v} = 2\underline{i} - \underline{j} + 2\underline{k}$ with $\underline{u} = -\underline{i} + \underline{k}$. [Hint: obtain the angle between the vectors and the axis of rotation to substitute in the rotation matrix.]

11. A transformation matrix for rotation about an axis through the origin is given by

$$R = \frac{1}{9}\begin{pmatrix} 4 & 4 & 7 \\ -8 & 1 & 4 \\ 1 & -8 & 4 \end{pmatrix}.$$

(i) Use the *sum of the diagonal elements (trace)* to calculate the rotation angle.

(ii) Use the three *differences of the pairs of reflected off-diagonal* elements to obtain the axis of rotation.

12. The rotation in question 11 is followed by another rotation of 90° about an axis parallel to $3\underline{j} - 4\underline{k}$, passing through the origin. Multiply the transformation matrices out and hence use the technique of question 11 to determine the single equivalent rotation angle and axis of the combined rotations.

13. A rigid body is rotating with angular speed 3 rads/sec about the axis $\underline{i} - 2\underline{j} + 2\underline{k}$, passing through the origin. If the point $(1, 0, -1)$ lies on the body, calculate its image coordinates at times 5 and 10 seconds after the motion starts.

Answers

1.

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0 & -0.2 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.71 & 0.71 & 0 \\ 0 & -0.71 & 0.71 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 0.5 & 0 & -0.87 & 0 \\ 0 & 1 & 0 & 0 \\ 0.87 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2. (a) $\begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/2 & 1/\sqrt{2} & 1/2 & 0 \\ 1/2 & -1/\sqrt{2} & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, (b) Yes.

3. $\begin{pmatrix} 0.71 & 0 & -0.71 & 0 \\ 0.35 & 0.87 & 0.35 & 0 \\ 0.61 & -0.5 & 0.61 & 0 \\ -1.67 & -0.37 & -0.26 & 1 \end{pmatrix}$; points transformed to $\begin{pmatrix} -1.67 & -0.37 & -0.26 & 1 \\ -0.96 & -0.37 & -0.97 & 1 \\ -1.32 & 0.5 & 0.09 & 1 \\ -1.06 & -0.87 & 0.35 & 1 \end{pmatrix}$.

4. $(5.222,\ 1.111,\ 1.222)$

5. $(-0.248,\ -0.433,\ -4.975)$

6. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.6 & 0.8 & 0 \\ 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$; $(6,\ 5.2,\ -1.4)$

7. $\begin{pmatrix} 0.81 & -0.14 & -0.57 & 0 \\ -0.14 & 0.89 & -0.43 & 0 \\ -0.57 & -0.43 & -0.70 & 0 \\ 1.80 & 1.35 & 5.40 & 1 \end{pmatrix}$

8. (i) $\begin{pmatrix} 0.625 & 0.487 & 0.610 & 0 \\ -0.737 & 0.625 & 0.256 & 0 \\ -0.256 & -0.610 & 0.750 & 0 \\ 0.887 & 0.732 & -0.110 & 1 \end{pmatrix}$, (ii) $\begin{pmatrix} 0.92 & 0.38 & -0.07 & 0 \\ -0.37 & 0.92 & 0.14 & 0 \\ 0.12 & -0.11 & 0.99 & 0 \\ 2.61 & -1.42 & -0.51 & 1 \end{pmatrix}$

9. $x + y - 2z = 7$

10. $\begin{pmatrix} 0.056 & -0.013 & 0.998 & 0 \\ 0.458 & 0.889 & -0.013 & 0 \\ -0.887 & 0.458 & 0.056 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

11. (i) 90°, (ii) axis parallel to $2\underline{i} - \underline{j} + 2\underline{k}$.

12. Rotation of 59.86° about an axis parallel to $-3\underline{i} + \underline{j} + 2\underline{k}$.

13. $(-0.52,\ 1.04,\ 0.80)$ and $(-0.60,\ -0.80,\ -1.00)$.

Chapter 3

Mathematics of Projection

For centuries, artists, engineers, designers, drafters, cartographers, and architects have tried to come to terms with the difficulties and constraints imposed by the problem of representing a three-dimensional object or scene in a two-dimensional medium – the problem of *projection.* Volumes have been written describing tools, techniques, and tricks used for this purpose. The ability to plot a computed point enables us to use the screen as a drawing medium. The mathematical description of the projection process allows us to display images of three-dimensional objects and scenes at will.

The two basic methods of projection – *perspective* and *parallel* – were designed to solve the basic but mutually exclusive problems of pictorial representation: showing an object as it appears and preserving its true size and shape.

An important observation (in terms of computer graphics) is that projections preserve lines, i.e. the line joining the projected images of the endpoints of the original line is the same as the projection of that line.

3.1 Parallel Projection

Parallel projection methods are used by drafters and engineers to create working drawings of an object that preserve its scale and shape. The complete representation of these details often requires two or more views (projections) of the object onto different view planes.

Orthographic projections are characterised by the fact that the direction of projection is perpendicular to the view plane. When the direction of projection is parallel to any of the principal axes, this produces the front, top, and side views of mechanical drawings (also referred to as *multiview* *drawings*).

The matrices that produce these transformations are clearly ones that keep unchanged two of the three coordinate values, but equally clearly they make the third coordinate zero and thus lose all the information carried by it. These matrices are as follows:

$$Par_{\underline{k}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{projection onto the } x-y \text{ plane } (z=0)$$

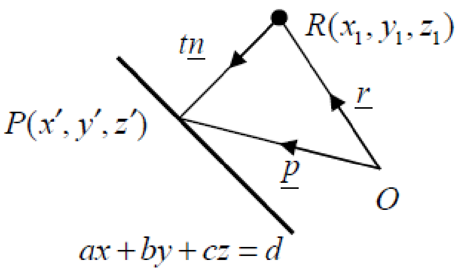
$$Par_{\underline{j}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{projection onto the } x-z \text{ plane } (y=0)$$

$$Par_{\underline{i}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{projection onto the } y-z \text{ plane } (x=0)$$

3.1.1 Orthographic Projection onto any Plane

In many scenarios in computer games, there is need to move the "camera" (viewing position) and so obtain a projected image onto a viewing plane that is at right angles to this viewing point.

To obtain the required transformation matrix, consider the orthogonal projection of the general point at $R(x_1, y_1, z_1)$ onto the plane $ax + by + cz = d$.



The projection is parallel to the normal of the plane and so $\underline{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The coordinates of the projected point at P are then given by $\underline{p} = \underline{r} + t\underline{n}$.

We now want to find where this intersects with the given plane, i.e. substitute $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ into $ax' + by' + cz' = d$, to get $a(x_1 + at) + b(y_1 + bt) + c(z_1 + ct) = d$, giving the value of the parameter t as, $t = \dfrac{d - (ax_1 + by_1 + cz_1)}{a^2 + b^2 + c^2}$. Using this value in the expression for $\underline{p}$ above gives the coordinates of P as:

$$x' = x_1 + \frac{a[d - (ax_1 + by_1 + cz_1)]}{a^2 + b^2 + c^2} = \frac{(b^2 + c^2)x_1 - aby_1 - acz_1 + ad}{a^2 + b^2 + c^2},$$

similarly

$$y' = \frac{-abx_1 + (a^2 + c^2)y_1 - bcz_1 + bd}{a^2 + b^2 + c^2} \quad \text{and} \quad z' = \frac{-acx_1 - bcy_1 + (a^2 + b^2)z_1 + cd}{a^2 + b^2 + c^2}.$$

Putting these together gives the homogeneous coordinate transformation matrix as

$$\boxed{T_{orth} = \begin{pmatrix} (b^2 + c^2) & -ab & -ac & 0 \\ -ab & (a^2 + c^2) & -bc & 0 \\ -ac & -bc & (a^2 + b^2) & 0 \\ ad & bd & cd & (a^2 + b^2 + c^2) \end{pmatrix}}.$$

Example

Calculate the orthogonal projection of the points at $(3, 5, -1)$ and $(-4, 2, 6)$ onto the plane $-2x + 4y - z = 3$ and hence determine the equation of the line segment joining the image points on the plane.

Solution

plane $\underline{ax+by+cz=d}$

$$T_{orth} = \begin{pmatrix} (b^2+c^2) & -ab & -ac & 0 \\ -ab & (a^2+c^2) & -bc & 0 \\ -ac & -bc & (a^2+b^2) & 0 \\ ad & bd & cd & (a^2+b^2+c^2) \end{pmatrix}$$

line equation between A & B

$$\underline{r} = \underline{A} + t(\underline{B}-\underline{A})$$

$$T_{orth} = \begin{pmatrix} 16+1 & 8 & -2 & 0 \\ 8 & 4+1 & 4 & 0 \\ -2 & 4 & 16+4 & 0 \\ -6 & 12 & -3 & 16+4+1 \end{pmatrix} = \begin{pmatrix} 17 & 8 & -2 & 0 \\ 8 & 5 & 4 & 0 \\ -2 & 4 & 20 & 0 \\ -6 & 12 & -3 & 21 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 5 & -1 & 1 \\ -4 & 2 & 6 & 1 \end{pmatrix} \begin{pmatrix} 17 & 8 & -2 & 0 \\ 8 & 5 & 4 & 0 \\ -2 & 4 & 20 & 0 \\ -6 & 12 & -3 & 21 \end{pmatrix} = \begin{pmatrix} 87 & 57 & -9 & 21 \\ -70 & 14 & 133 & 21 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{29}{7} & \frac{19}{7} & \frac{-3}{7} & 1 \\ \frac{-10}{3} & \frac{2}{3} & \frac{19}{3} & 1 \end{pmatrix}$$

Image coordinates: $\left(\frac{29}{7}, \frac{19}{7}, \frac{-3}{7}\right)$ & $\left(\frac{-10}{3}, \frac{2}{3}, \frac{19}{3}\right)$

$$\underline{r} = \frac{1}{7}\begin{pmatrix} 29 \\ 19 \\ 3 \end{pmatrix} + t\begin{pmatrix} \frac{-10}{3} - \frac{29}{3} \\ \frac{2}{3} - \frac{19}{3} \\ \frac{19}{3} - \frac{3}{7} \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 29 \\ 19 \\ 3 \end{pmatrix} + t\begin{pmatrix} \frac{-70-87}{21} \\ \frac{14-57}{21} \\ \frac{133-9}{21} \end{pmatrix}$$

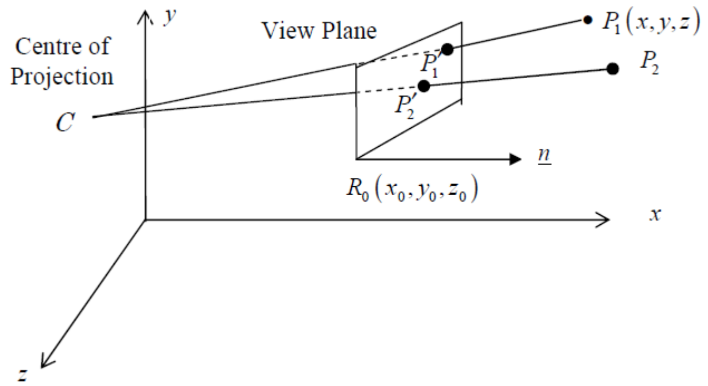
$$= \frac{1}{7}\begin{pmatrix} 29 \\ 19 \\ 3 \end{pmatrix} + \frac{t}{21}\begin{pmatrix} -157 \\ -43 \\ 124 \end{pmatrix}$$

3.2 Perspective Projection

In a great deal of graphical work (for animation and simulation as well as for virtual reality) the prime intention is to present an image that is as realistic as possible, so the viewer's sense of depth perception must be enhanced. The most easily managed types of depth cues for this purpose are those that depend on the techniques of **perspective**. When perspective has been used, lines that are parallel appear to converge to *vanishing points*. As a consequence, the size of an object appears to be reduced according to its distance from the viewing point and foreshortening occurs according to orientation and distance.

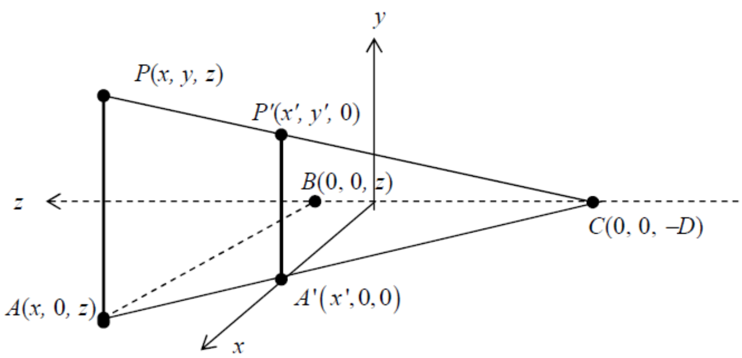
3.2.1 Perspective Transformation

The techniques of perspective projection are generalisations of the principles used by artists in preparing perspective drawings of three-dimensional objects and scenes. The eye of the artist is placed at the *centre of projection* and the canvas, or more precisely the plane containing the canvas, becomes the *view plane*. An image point is located at the intersection of a *projector* (a ray drawn from an object point to the centre of projection) with the view plane.



A perspective transformation is determined by prescribing a centre of projection and a view plane. The view plane is determined by its *view reference point* R_0 and *view plane normal* $\underline{n}$. The *object point* P is located in world coordinates at (x, y, z). The problem is to determine the *image point* coordinates $P'(x', y', z')$.

The standard perspective projection is shown below. Here, the view plane is the $x - y$ plane, and the centre of projection is taken as the point $C(0, 0, -D)$ on the negative $z-$axis.



Using similar triangles ABC and $A'OC$, we find that

$$x' = \frac{xD}{z + D}, \quad y' = \frac{yD}{z + D}, \quad z' = 0.$$

The perspective transformation between object and image point is nonlinear and so cannot be represented as a 3×3 matrix transformation. However, if we use homogeneous coordinates, the perspective transformation can be represented as a 4×4 matrix:

$$\begin{aligned} (x' \quad y' \quad z' \quad 1) &= (xD \quad yD \quad 0 \quad z+D) \\ &= (x \quad y \quad z \quad 1) \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & D \end{pmatrix}. \end{aligned}$$

If the same ideas are applied to projection onto, say, the $x - z$ plane from $y = -D$, we get

$$\begin{aligned} (x' \quad y' \quad z' \quad 1) &= (x \quad y \quad z \quad 1) \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \\ &= (xD \quad 0 \quad zD \quad y+D) \end{aligned}$$

i.e.

$$x' = \frac{xD}{y+D}, \quad y' = 0, \quad z' \frac{zD}{y+D}.$$

[See if you can write down the perspective transformation matrix from $x = -D$ onto the $y - z$ plane.]

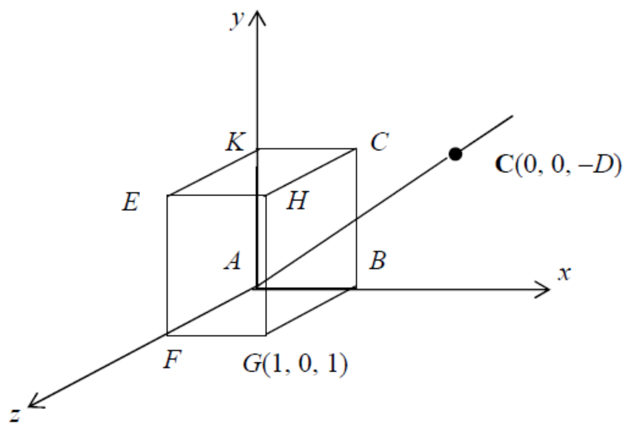
Examples

1. The unit cube is projected into the $x - y$ plane. Note the position of the x, y and z-axes. Draw the projected image using the standard perspective transformation with (a) $D = 1$ and (b) $D = 10$, where D is distance from the view plane.

   Solution

   We represent the unit cube in terms of the homogeneous coordinates of its vertices by the rows in the matrix:

$$V = (ABCKEFGH) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$



The standard perspective matrix is

$$Per_{\underline{k}} = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & D \end{pmatrix}$$

(a) With $D = 1$, the projected coordinates are found by applying the matrix $Per_{\underline{k}}$ to the matrix of coordinates V. Then

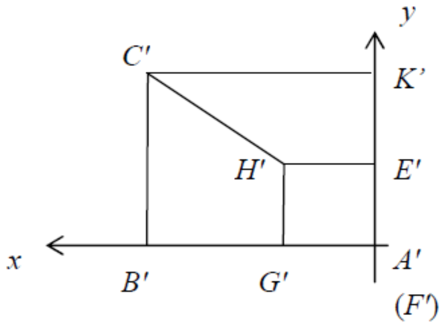
$$V \cdot Per_{\underline{k}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

.

If these homogeneous coordinates are changed to three-dimensional coordinates, the projected image has coordinates:

$$A' = (0,0,0); \quad B' = (1,0,0); \quad C' = (1,1,0); \quad K' = (0,1,0);$$
$$E' = (0,\tfrac{1}{2},0); \quad F' = (0,0,0); \quad G' = (\tfrac{1}{2},0,0); \quad H' = (\tfrac{1}{2},\tfrac{1}{2},0).$$

We draw the projected image by preserving the edge connections of the original object below. Note the vanishing point at $(0,0,0)$.



(b) With $D = 10$, the perspective matrix is

$$Per_{\underline{k}} = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Then we have

$$V \cdot Per_{\underline{k}} = \begin{pmatrix} 0 & 0 & 0 & 10 \\ 10 & 0 & 0 & 10 \\ 10 & 10 & 0 & 10 \\ 0 & 10 & 0 & 10 \\ 0 & 10 & 0 & 11 \\ 0 & 0 & 0 & 11 \\ 10 & 0 & 0 & 11 \\ 10 & 10 & 0 & 11 \end{pmatrix}$$

as the matrix image coordinates in homogenous form. The projected image coordinates are then

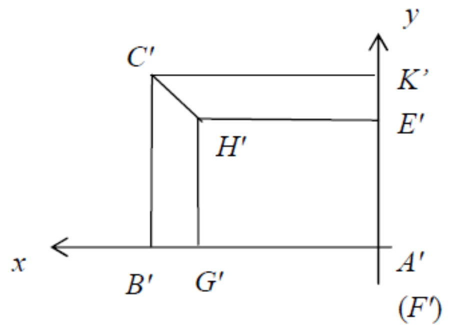
$$A' = (0,0,0); \quad B' = (1,0,0); \quad C' = (1,1,0); \quad K' = (0,1,0);$$
$$E' = (0, \tfrac{10}{11}, 0); \quad F' = (0,0,0); \quad G' = (\tfrac{10}{11}, 0, 0); \quad H' = (\tfrac{10}{11}, \tfrac{10}{11}, 0).$$

Note the different perspectives of the face $E'F'G'H'$. To a viewer standing at the centre of projection $(0, 0, -D)$, this face is the back face of the unit cube.

2. Under the standard perspective transformation $Per_{\underline{k}}$, what is the projected image of (a) a point in the plane $z = -D$ and (b) the line segment joining $P_1(-1, 1, -2D)$ to $P_2(2, -2, 0)$?

Solution

(a) The plane is the plane parallel to the $x - y$ view plane and located at the centre of projection $C(0, 0, -D)$. If $P(x, y, -D)$ is any point in this plane, the line of projection CP does not intersect the $x - y$ view plane. We say that P is **projected out to infinity**



(∞). Note that if P coincides with the viewing point at $(0,0,-D)$ then the projection is **undefined**.

(b) The projected coordinates of P_1 and P_2 on the $x-y$ view plane are given by

$$\begin{pmatrix} -1 & 1 & -2D & 1 \\ 2 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & D \end{pmatrix} = \begin{pmatrix} -D & D & 0 & -D \\ 2D & -2D & 0 & D \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$

i.e. P_1' is at $(1 \quad -1 \quad 0)$ and P_2' is at $(2 \quad -2 \quad 0)$.

The equation of the line segment joining these two points on the plane is then given by

$$r = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad (0 \le t \le 1)$$

or,

$$x = 1 + t, \quad y = -1 - t, \quad z = 0.$$

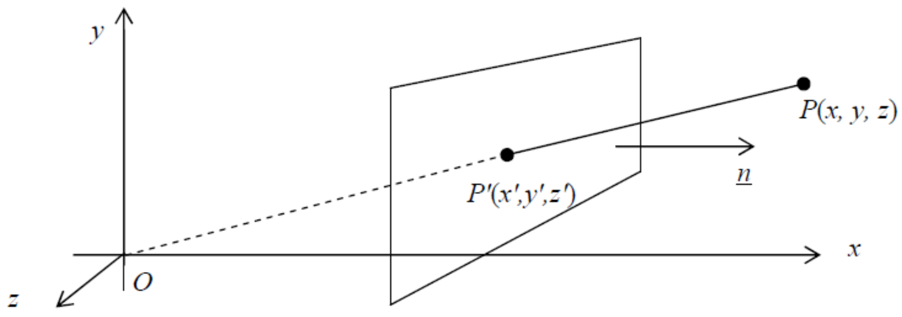
3.2.2 Using the origin as the centre of projection

In this case, the perspective transformation is onto the plane $ax + by + cz = d$.

Let $P(x, y, z)$ be projected onto $P'(x', y', z')$. From O, the vectors $\overrightarrow{PO}$ and $\overrightarrow{P'O}$ have the same direction. Thus there is a number α so that $\overrightarrow{P'O} = \alpha\overrightarrow{PO}$. Comparing components, we have

$$x' = \alpha x, \quad y' = \alpha y, \quad z' = \alpha z.$$

We now find the value of α. Since any point $P'(x', y', z')$ lying on the plane satisfies $ax' + by' + cz' = d$,



substitution of $x' = \alpha x,\ y' = \alpha y$ and $z' = \alpha z$ into this equation gives

$$\alpha = \frac{d}{ax + by + cz}.$$

This projection transformation cannot be represented as a (3×3) matrix transformation. However, by using the homogeneous coordinate representation for three-dimensional points, we can write the projection transformation as a (4×4) matrix:

$$Per_{\underline{n}} \begin{pmatrix} d & 0 & 0 & a \\ 0 & d & 0 & b \\ 0 & 0 & d & c \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Application of this matrix to the homogeneous representation $P(x, y, z, 1)$ of point P gives $P'(xd,\ yd,\ zd,\ ax + by + cz)$, which is the homogeneous representation of $P'(x', y', z')$ found above.

Example

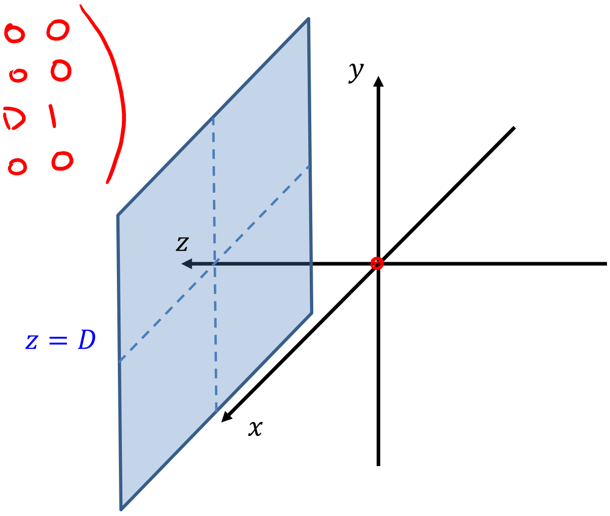
Determine the perspective projection onto the view plane $z = D$ where the centre of projection is the origin $(0, 0, 0)$.

Solution

$$\text{Pern} = \begin{pmatrix} d & 0 & 0 & \\ 0 & d & 0 & b \\ 0 & 0 & d & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad z = D \Rightarrow \quad 0x + 0y + 1z = D$$

$$a = 0 \quad b = 0 \quad c = 1 \quad d = D$$

$$\text{Pern} = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



3.2.3 The General Case

We now consider the general case of any point $C(x_c,\ y_c,\ z_c)$ being the viewing centre by using the results derived for the origin as the viewing centre. The basic idea is to make a transformation of coordinates to move this viewing centre to the origin and then transform back after the projection has taken place.

Using the same notation as before, we want to calculate the coordinates of the image $P'(x', y', z')$ of the point $P(x, y, z)$ under the single-point perspective projection with viewing centre at $C(x_c,\ y_c,\ z_c)$ onto the plane $ax + by + cz = d$.

Making the translation of coordinates: $X = x - x_c$, $Y = y - y_c$, $Z = z - z_c$, gives the viewing centre at $(0, 0, 0)$ and the plane becomes

$$aX + bY + cZ = d - (ax_c + by_c + cz_c)$$

or

$$aX + bY + cZ = d_0 \quad [\text{where } d_0 = d - (ax_c + by_c + cz_c)].$$

The perspective transformation matrix is now of the same form as before. Hence, the complete transformation is given by the concatenation of matrices:

$$Per = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_c & -y_c & -z_c & 1 \end{pmatrix} \begin{pmatrix} d_0 & 0 & 0 & a \\ 0 & d_0 & 0 & b \\ 0 & 0 & d_0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_c & y_c & z_c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d-(by_c+cz_c) & ay_c & az_c & a \\ bx_c & d-(ax_c+cz_c) & bz_c & b \\ cx_c & cy_c & d-(ax_c+by_c) & c \\ -dx_c & -dy_c & -dz_c & -(ax_c+by_c+cz_c) \end{pmatrix}$$

(try verifying this yourself)

Note: with all perspective projections, it is necessary that the ray vector from the viewing centre C to the point P being projected intersects the view plane (otherwise the image would appear *upside-down*). This is always true if C and P are either side of the plane. However, if they are both on the same side of the plane then intersection occurs only if $|d_c| > |d_p|$, where d_c and d_p are the distances of C and P respectively from the plane. In the general case above this should always be checked and can be done in a straightforward way as follows:

- To determine if the points are on the same side of the plane, use the "distance" formula (see 1st year notes or formulae booklet) for a point to a plane, i.e.

  $$D = \frac{ax+by+cz-d}{\sqrt{a^2+b^2+c^2}}.$$

  In fact we only really need the numerator of this expression because the denominator is a constant and will be the same for both points C and P.

- Calculate $D_C = ax_C + by_C + cz_C - d$ for C, and $D_P = ax_P + by_P + cz_P - d$ for P.

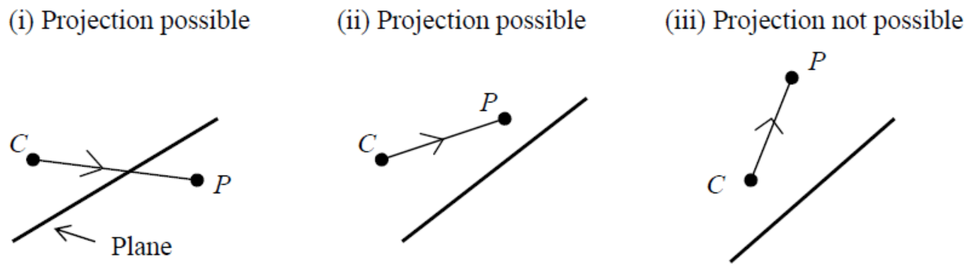
- If D_C and D_P are of different sign then the projection is possible.

- If D_C and D_P are of the same sign then $|D_C| > |D_P|$ for the projection to be possible.

Examples

1. Calculate the image of the point $(2, -3, 4)$ under the single-point perspective projection with viewing centre at $(1, 2, 1)$ onto the plane $x - 2y - 3z = 4$.

   Solution



Checking the possibility of the projection first gives,

$$D_C = 1 - 2(2) - 3(1) - 4 = -10 \quad \text{and} \quad D_P = 2 - 2(-3) - 3(4) - 4 = -8.$$

Hence points lie on the same side of the plane, but $|D_C| > |D_P|$ and so projection is possible.

Using the above perspective matrix, with $a = 1$, $b = -2$, $c = -3$, $d = 4$, the image point has homogeneous coordinates given by

$$\begin{aligned} (x' \quad y' \quad z' \quad 1) &= (2 \quad -3 \quad 4 \quad 1)\begin{pmatrix} 11 & 2 & 1 & 1 \\ -2 & 6 & -2 & -2 \\ -3 & -6 & 7 & -3 \\ -4 & -8 & -4 & 6 \end{pmatrix} \quad \text{(check this)} \\ &= (12 \quad -46 \quad 32 \quad 2). \end{aligned}$$

i.e., the image point has coordinates $(6, -23, 16)$. A quick check shows that this point indeed lies on the plane $x - 2y - 3z = 4$.

2. The points $(2, 4, -1)$ and $(-3, 6, 4)$ are projected onto the plane $3x - y + z = 5$ by a single-point perspective transformation with viewing centre at $(2, 3, 4)$. Find the images of these points and hence determine the parametric equation of the line segment joining these images on the plane.

Solution

viewing centre: $D_c = ax_c + by_c + cz_c - d$

point: $D_p = ax_p + by_p + cz_p - d$

$$P_{er} = \begin{pmatrix} d-(by_c+cz_c) & ay_c & az_c & a \\ bx_c & d-(ax_c-cz_c) & bz_c & b \\ cx_c & cy_c & d-(ax_c+by_c) & c \\ -dx_c & -dy_c & -dz_c & -(ax_c+by_c+cz_c) \end{pmatrix}$$

$$\begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \qquad a=3,\ b=-1,\ c=1,\ d=5$$

$D_c =$ $3\times 2 - 1\times 3 + 1\times 4 - 5 = \underline{2}$

$D_{p_1} = 3\times 2 - 1\times 4 - 1\times 1 - 5 = 6-4-1-5 = \underline{-4}$

$D_{p2} = -3\times 3 - 1\times 6 + 1\times 4 - 5 = -9-6+4-5 = \underline{-16}$

Sign change ∴ projection possible

$$P_{er} = \begin{pmatrix} 5-(-3+4) & 3\times 3 & 12 & 3 \\ -1\times 2 & 5-(6+4) & -1\times 4 & -1 \\ 1\times 2 & 1\times 3 & 5-(6-3) & 1 \\ -5\times 2 & -5\times 3 & -5\times 4 & -(6-3+4) \end{pmatrix} = \begin{pmatrix} 4 & 9 & 12 & 3 \\ -2 & -5 & -4 & -1 \\ 2 & 3 & 2 & 1 \\ -10 & -15 & -20 & -7 \end{pmatrix}$$

Images:

$$\begin{pmatrix} 2 & 4 & -1 & 1 \\ -3 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 9 & 12 & 3 \\ -2 & -5 & -4 & -1 \\ 2 & 3 & 2 & 1 \\ -10 & -15 & -20 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -12 & -20 & -14 & -6 \\ -26 & -60 & -72 & -18 \end{pmatrix} = \begin{pmatrix} 2 & \frac{10}{3} & \frac{7}{3} & 1 \\ \frac{13}{9} & \frac{10}{3} & 4 & 1 \end{pmatrix}$$

points: $\left(2, \frac{10}{3}, \frac{7}{3}\right)$ & $\left(\frac{13}{9}, \frac{10}{3}, 4\right)$

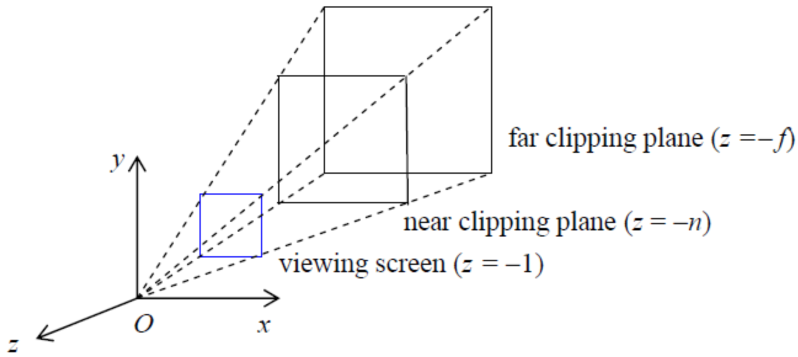
line segment:
$$\underline{r} = \frac{1}{3}\begin{pmatrix} 6 \\ 10 \\ 7 \end{pmatrix} + t\begin{pmatrix} \frac{13}{9} - 2 \\ \frac{10}{3} - \frac{10}{3} \\ 4 - \frac{7}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ 10 \\ 7 \end{pmatrix} + \frac{t}{9}\begin{pmatrix} -5 \\ 0 \\ 15 \end{pmatrix}$$

$$x = 2 - \frac{5}{9}t \qquad y = \frac{10}{3} \qquad z = \frac{7}{3} + \frac{5}{3}t$$

3.2.4 Clipped Volume

In **Open GL**, use is made of perspective transformations for 3D-object projection onto a viewing screen, with account taken of *clipping* out any objects lying outside of a pre-determined volume. This clipped volume takes the form of a standardized **frustum** of a right pyramid as illustrated below.

The viewing screen's centre is on the (negative) $z-$axis at $(0, 0, -1)$, with bottom-left and top-right



corner coordinates at $(-1,-1,-1)$ and $(1,1,-1)$ respectively. The viewing centre is at the origin and the volume in the frustum between the near and far clipping planes is to be projected onto the viewing screen.

Notice that this is the *standardized clipped* volume used in the Open GL graphics pipeline procedure. This means that often some transformations may be required to get the required clipped volume into this standardized form, which we shall not consider here.

The idea now is to transform the clipped volume into a symmetrical cube of side 2. The main reason for this is that a cube aligned to the axes is the easiest shape to test for whether points lie outside its bounding volume and hence can be rejected for projection purposes.

We know from previous work that a simple perspective transformation matrix for a viewing plane at $z = -1$ with centre of projection at the origin is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now let us consider the projection matrix

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & -1 \\ 0 & 0 & \beta & 0 \end{pmatrix}$$

which is a variation on M. Applying N to transform a point gives the new homogeneous coordinates as

$$(x' \quad y' \quad z' \quad 1) = (x \quad y \quad z \quad 1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & -1 \\ 0 & 0 & \beta & 0 \end{pmatrix} = (x \quad y \quad \alpha z + \beta \quad -z)$$

$$\text{i.e.} \quad x' = -\frac{x}{z}, \quad y' = -\frac{y}{z}, \quad z' = -\left(\alpha + \frac{\beta}{z}\right).$$

Now if we consider the original clipped frustum volume given above we notice that it is formed by the intersection of the planes $x = \pm z$ (the *left* and *right* side planes), $y = \pm z$ (the *bottom* and *top* planes), $z = -n$ and $z = -f$ (the *end* planes). So using these in the transformation equations give

$$x' = \mp 1, \quad y' = \mp 1, \quad z' = -\left(\alpha - \frac{\beta}{n}\right) \quad \text{and} \quad z' = -\left(\alpha - \frac{\beta}{f}\right).$$

[Notice that all of these plane sides remain in the same *relative* order as before.]

This means that the frustum volume has been transformed to a cuboid bounded by these planes. Further, if we want the bounding "$z-$planes" to be at $z' = \pm 1$, then we can choose $-\alpha + \frac{\beta}{n} = 1$ and $-\alpha + \frac{\beta}{f} = -1$, which solve to give

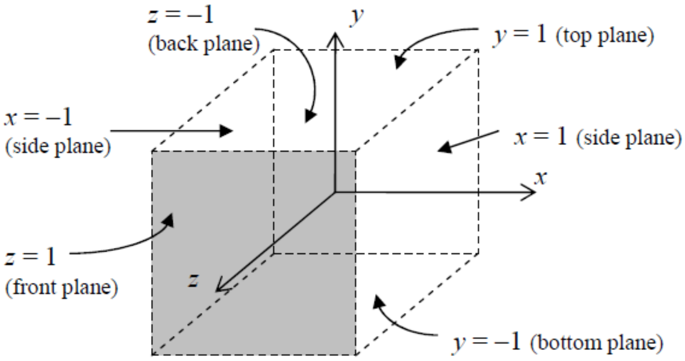
$$\alpha = \frac{f+n}{f-n} \quad \text{and} \quad \beta = \frac{2fn}{f-n}.$$

Hence, the clipped frustum volume between the planes $z = -n$ and $z = -f$ is transformed into the cube (symmetrical about the origin with side 2) by the transformation matrix

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -1 \\ 0 & 0 & \frac{2fn}{f-n} & 0 \end{pmatrix}.$$

Any points satisfying $|x| > 1$ or $|y| > 1$ or $|z| > 1$ can then be rejected for the clipped volume. Orthographic projection parallel to the $z-$axis then completes the process – note that the new screen position is not really important because it is always parallel to the one at $z = -1$ and hence will have the same (x, y) coordinates for the image points. Further, the positions of the points relative to each other in the $z-$direction are retained in the transformation *within the clipped volume*, so that if $z_1 > z_2$ then $z'_1 > z'_2$.

Finally, if the origin and aspect ratio of the final viewport is to be different to those of the above ($(0, 0, 0)$ and 1 respectively) and the $z-$coordinate range is to be scaled to the resolution of the



$z-$buffer for sorting and hidden points/surface removal, then we need to use further transformation matrices (scaling and translation) to give a concatenated transformation equivalent to

$$
\begin{aligned}
T &= \begin{pmatrix} \frac{1}{2}w & 0 & 0 & 0 \\ 0 & \frac{1}{2}h & 0 & 0 \\ 0 & 0 & \frac{1}{2}ZR & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ X_0 & Y_0 & \frac{1}{2}ZR & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}w & 0 & 0 & 0 \\ 0 & \frac{1}{2}h & 0 & 0 \\ 0 & 0 & \frac{1}{2}ZR & 0 \\ X_0 & Y_0 & \frac{1}{2}ZR & 1 \end{pmatrix} \\
&= \qquad\qquad (\text{scale}) \qquad\qquad\qquad (\text{translate})
\end{aligned}
$$

where w and h are the screen's width and height respectively, ZR is the $z-$buffer range and (X_0, Y_0) are the coordinates of the new origin. The product $(x \quad y \quad z \quad 1)NT$ will give the required image points.

Example

The standardized pyramid frustum with cut-off planes at$z = -2$ and $z = -4$ is projected onto a viewport at $z = -1$. Obtain the images of the following points on the viewport:

(i) $(3, -2, -3)$ (ii) $(-2.4, 0.5, -2.6)$ (iii) $(1, 3.5, 0.5)$.

If the viewport is now changed such that the lower left corner of the screen is moved to $(-0.5, 0.5)$ and the aspect ratio is to be 4:3, keeping the $x-$scale the same as before, then calculate the new image coordinates of the points in the clipped volume on this screen.

Solution

clipped frustum, $z \in [-n, -f]$: $N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{f+n}{f-n} & -1 \\ 0 & 0 & \frac{2fn}{f-n} & 0 \end{pmatrix}$

$n = 2 \quad f = 4 \quad N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 8 & 0 \end{pmatrix}$

$$\begin{array}{c} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \end{array} \begin{pmatrix} 3 & -2 & -3 & 1 \\ -2.4 & 0.5 & -2.6 & 1 \\ 1 & 3.5 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 8 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 & 3 \\ -2.4 & 0.5 & 0.2 & 2.6 \\ 1 & 3.5 & 9.5 & -0.5 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & 1 \\ -0.923 & 0.192 & 0.077 & 1 \\ -2 & -7 & -19 & 1 \end{pmatrix}$$

iii) lies outside range $x, y \in [-1, 1]$ $\therefore$ no image onscreen

i) & ii) found at $\underline{\left(1, -\frac{2}{3}\right)}$ and $\underline{(-0.923, 0.192)}$

Tutorial Exercises - Mathematics of Projection

1. Determine the image of the point $(-1, 4, 3)$ when it is projected orthogonally onto the plane $5x - 2y + 3z = 2$.

2. A camera is aligned such that it projects the point $(1, 1, 2)$ to the origin by an orthographic projection. Determine the image of the point $(-1, 4, -3)$ under the same projection.

3. A line segment has equation $\underline{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$, $(0 \leq t \leq 1)$. It undergoes an orthographic projection onto the plane $4x + y + 2z = 1$. Obtain the equation of its image on the plane.

4. There is a cube whose base is the square $ABCD$, where the coordinates are $A(0, 0, 0)$, $B(0, 0, 3)$, $C(3, 0, 3)$ and $D(3, 0, 0)$. This cube is first transformed by the single point perspective transformation with viewing point at $Z(0, 0, -20)$ and then the result is projected onto the viewing plane $z = 0$.

   (a) Write down the coordinates of E, F, G and H, the other vertices of the cube.

   (b) Write down the matrix of the perspective transformation.

   (c) Calculate the coordinates of the vertices of the image of the cube when transformed by perspective.

   (d) Calculate the vertices of the planar shape, as projected onto the viewing plane.

5. Under the single point perspective transformation viewed from the point Z(0,0,-20), find in homogeneous form the images of:

   (a) the origin, (b) $(0, 0, -19)$, (c) $(1, 1, -20)$.

   Are any of these **invariant** (i.e. unchanged) under this transformation?

6. The point $(5, 4, -6)$ is subject to a single-point perspective projection, with viewing centre at the origin, onto the plane $2x + 3y - z = 3$. Calculate its image on this plane.

7. The two points at either end of the line segment

$$\underline{r} = -\underline{i} + 2\underline{j} - 4\underline{k} + t(2\underline{i} + 3\underline{j} - 5\underline{k}), \quad (0 \leq t \leq 1)$$

   are subjected to a single-point perspective projection onto the plane $z+2 = x+y$, with viewing point at the origin. Determine the equation of the image of the line segment on the plane.

8. Determine if the following perspective transformations are possible for the given viewing centre, point of projection and view-plane respectively:

   (i) $C(2,2,3),\ P(3,-1,2),\ x+2y-z=2;$

   (ii) $C(-2,3,1),\ P(5,0,-1),\ 3x-y+2z=4;$

   (iii) $C(-1,-2,-3),\ P(3,-2,-1),\ -x+3y+4z=1;$

   (iv) $C(2,3,2),\ P(4,3,3),\ x+2y+z=5.$

9. If the viewing point is changed to $(1,-1,1)$ in question 6, calculate the new image coordinates of the point.

10. The line segment joining the points $(0,2,4)$ to $(-2,5,1)$ is projected onto the plane $x+y+z=4$ by a single-point perspective transformation with viewing centre at $(-4,3,-6)$. Determine the images of these end-points under the transformation and hence derive the equation of the projected line segment.

11. A standardized pyramid frustum volume, bounded by the planes $x=\pm z$, $y=\pm z$, $z=-1.5$ and $z=-4.5$, is to be projected onto the plane $z=-1$. Determine:

    (i) the 2D image of the point $(0.2,-1.4,-3.2)$ on the screen;

    (ii) the further transformation matrix needed to place the centre of the screen at $(1,2)$, with width 3 and height 1.5, and to give the z values a range of 10;

    (iii) the final 2D image of the point in (i) after the transformation in (ii).

12. A standardized clipped volume, bounded by the planes $x=\pm z$, $y=\pm z$, $z=-3$ and $z=-6$, is to be projected onto a plane parallel to the $x-y$ plane. The centre of the image screen is to be placed at $(2,5)$, of width 4 and height 3, with the $z-$values projected within the clipped volume having a range of 8. Obtain the final screen image coordinates of the points $(0.5,1,-5)$ and $(1,4,-3)$ respectively, determining whether they appear on the screen or not.

13. A point $P(x,y,z)$ is to be projected onto the plane $ax+by+cz=d$ in a direction parallel to the vector $\underline{v}=(v_x \quad v_y \quad v_z)$. Show that the transformation matrix in homogeneous coordinates that does this is given by

$$T_{par}=\begin{pmatrix} bv_y+cv_z & -av_y & -av_z & 0 \\ -bv_x & av_x+cv_z & -bv_z & 0 \\ -cv_x & -cv_y & av_x+bv_y & 0 \\ dv_x & dv_y & dv_z & av_x+bv_y+cv_z \end{pmatrix}.$$

Answers

1. $\left(-\dfrac{4}{19},\dfrac{70}{19},\dfrac{66}{19}\right)$

2. $(-0.5, 4.5, -2)$

3. $\underline{r} = \dfrac{1}{21}\begin{pmatrix}-10\\-13\\37\end{pmatrix} + \dfrac{u}{21}\begin{pmatrix}-46\\20\\82\end{pmatrix}$, $(0 \le u \le 1)$

4. (a) $(0,3,0)$, $(0,3,3)$, $(3,3,3)$, $(3,3,0)$.

   (b) $\begin{pmatrix}20 & 0 & 0 & 0\\0 & 20 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 20\end{pmatrix}$

   (c) Matrix of homogeneous coordinates= $\begin{pmatrix}0 & 0 & 0 & 20\\0 & 0 & 0 & 23\\60 & 0 & 0 & 23\\60 & 0 & 0 & 20\\60 & 60 & 0 & 20\\60 & 60 & 0 & 23\\0 & 60 & 0 & 23\\0 & 60 & 0 & 20\end{pmatrix}$

   (d) $(0,0)$, $(0,0)$, $(60/23,0)$, $(3,0)$, $(3,3)$, $(60/23,60/23)$, $(0,60/23)$, $(0,3)$.

5. (a) $(0,0,0)$ invariant, (b) $(0,0,0)$, (c) projected out to infinity.

6. $(15/28, 3/7, -9/14)$

7. $\underline{r} = \frac{2}{5}(-\underline{i} + 2\underline{j} + 4\underline{k}) + \frac{2}{15}(4\underline{i} - \underline{j} + 3\underline{k})$

8. (i) yes, (ii) yes, (iii) yes, (iv) no.

9. $(5/3, -1/6, -1/6)$

10. $(-8/13, 28/13, 32/13)$, $(-2,5,1)$; $\underline{r} = (-8\underline{i} + 28\underline{j} + 32\underline{k} + t[-18\underline{i} + 37\underline{j} - 19\underline{k}])/13$.

11. (i) $(0.0625, -0.4375)$, (ii) $N = \begin{pmatrix}1.5 & 0 & 0 & 0\\0 & 0.75 & 0 & 0\\0 & 0 & 5 & 0\\1 & 2 & 5 & 1\end{pmatrix}$, (iii) $(1.094, 1.672)$.

12. $(2.2, 5.3)$ and $(2.67, 7)$ - last one outside screen.

Chapter 4

Complex Numbers

Just as integers are a subset of the set of real numbers, real numbers are a subset of the set of complex numbers.

4.1 Notation

$\mathbb{N}$ the set of Natural Numbers (or positive integers) 1, 2, 3, ...

$\mathbb{Z}$ the set of all Integers 0, ± 1, ± 2, ± 3, ...

$\mathbb{Q}$ the set of Rational Numbers $= \left\{ \dfrac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$ (fractions)

$\mathbb{R}$ the set of Real Numbers

$\mathbb{C}$ the set of Complex Numbers

Note

The equation $x^2 - 4 = 0$ has the two solutions $-2, 2 \in \mathbb{Z}$.

The equation $x^2 - \dfrac{9}{4} = 0$ has the two solutions $-\dfrac{3}{2}, \dfrac{3}{2} \in \mathbb{Q}$.

The equation $x^2 - 3 = 0$ has the two solutions $-\sqrt{3}, \sqrt{3} \in \mathbb{R}$.

If we now consider the equation $x^2 + 1 = 0$, then $x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$ and no solution exists in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

The equation may be solved by considering a more general number system: the Complex Number system, where we denote $\sqrt{-1}$ by i {or j in some Engineering texts}. Then the solution of the equation may be written

$$x = \pm i.$$

Example

Solve the equation $x^2 - 4x + 13 = 0$.

Solution

Use the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for solving the quadratic equation $ax^2 + bx + c = 0$, which $a = 1$, $b = -4$, $c = -13$:

$$\Rightarrow x = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

A number such as $2 \pm 3i$ is known as a Complex Number.

In general, we denote a Complex Number by $a + bi$, or the ordered pair (a, b), where a and b are both Real numbers.

4.2 Addition, Subtraction and Multiplication of Complex Numbers

We define

$$\begin{aligned}
(a + bi) + (c + di) &= (a + c) + (b + d)i \quad \text{(add the components)} \\
(a + bi) - (c + di) &= (a - c) + (b - d)i \quad \text{(subtract the components)}
\end{aligned}$$

$$\begin{aligned}
(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\
&= ac + (ad + bc)i + bd(-1) \quad \text{using } i^2 = -1 \\
&= (ac - bd) + (ad + bc)i
\end{aligned}$$

Examples

1. $(2 + 3i) + (4 + 5i) =$

   equate real and imaginary parts:

   $(2+4) + (3+5)i = \underline{6+8i}$

2. $(2 + 3i) - (4 + 5i) =$

   $(2-4) + (3-5)i = \underline{-2-2i}$

3. $(2 + 3i)(4 + 5i) =$

   $(2)(4) + (3i)(4) + (2)(5i) + (3i)(5i)$

   $= 8 + 12i + 10i + 15i^2$

   $= (8-15) + (12+10)i$

   $= \underline{-7 + 22i}$

4.3 Powers of i

Powers of i may be expressed in terms of ± 1 or $\pm i$. For example

$$i^2 = i.i = -1$$

$$i^3 = (i.i).i = (-1)i = -i$$

$$i^4 = (i.i).(i.i) = (-1).(-1) = 1$$

$$i^5 = (i.i).(i.i).i = (-1)(-1)i = i$$

$$-1 \xrightarrow{\times i} -i \xrightarrow{\times i} 1 \xrightarrow{\times i} i \xrightarrow{\times i} -1$$

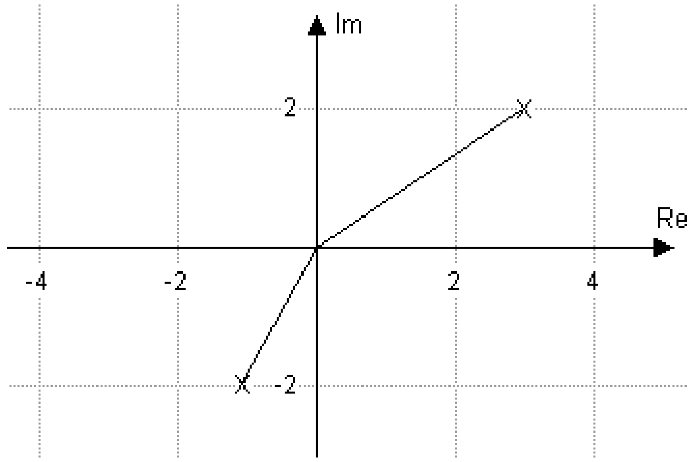
etc, with the sequence repeating itself cyclically.

4.4 Argand Diagram

Since a complex number is an ordered pair, it may be represented by a point in a plane – this plane is known as the Argand Diagram. The x-axis is frequently called the 'real' axis and the y-axis the 'imaginary' axis.

Example

Represent the complex numbers $3 + 2i$ and $-1 - 2i$ in the Argand diagram.



4.5 Notation

If the complex number $a + bi$ is denoted by a single letter z i.e. $z = a + bi$, then

$a = \Re(z)$ or $\mathrm{Re}(z)$: the real part of z;

$b = \Im(z)$ or $\mathrm{Im}(z)$: the imaginary part of z

4.6 Conjugate Complex Numbers

The conjugate of a complex number $z = a + bi$, denoted by $\bar{z}$, is defined by

$$\bar{z} = a - bi$$

For example,

the conjugate of $z = 2 - 3i$ is $\bar{z} = 2 + 3i$,

the conjugate of $z = 5i$ is $\bar{z} = -5i$.

Note:

The product of a complex number and its conjugate is always real, since

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2.$$

Exercise: check this.

4.7 Division of Complex Numbers

To divide a pair of complex numbers, write the quotient as a fraction and then multiply the numerator and denominator of the fraction by the conjugate of the denominator.

Example

Divide $3 + 2i$ by $1 + 2i$.

Solution

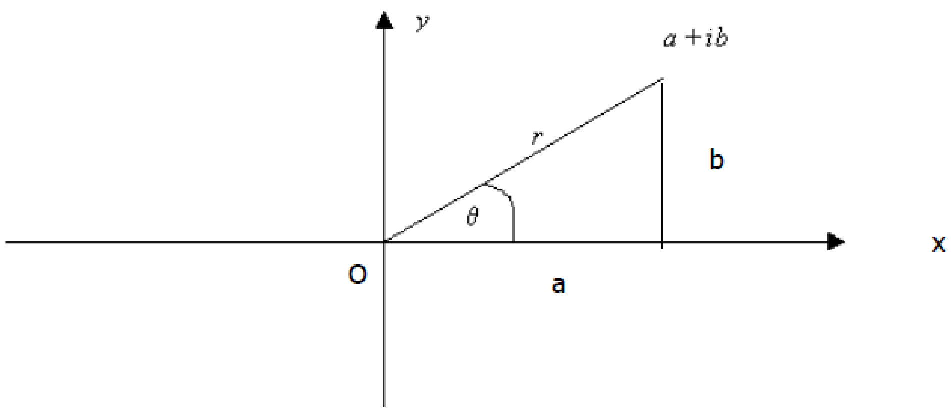
$$\frac{3+2i}{1+2i} = \frac{(3+2i)(1-2i)}{(1+2i)(1-2i)} = \frac{3-6i+2i-4i^2}{1+2i-2i-4i^2} = \frac{3-4i-4(-1)}{1-4(-1)} = \frac{7-4i}{5} = \frac{7}{5} - \frac{4}{5}i$$

4.8 Polar Representation

An alternative representation of a complex number is the polar representation, denoted by

$$r(\cos\theta + i\sin\theta) \quad \text{or} \quad r[\theta]$$

as indicated in the following diagram.



From the diagram it follows that:

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \\ r &= \sqrt{a^2+b^2} \\ \tan\theta &= \frac{b}{a} \end{aligned}$$

Therefore,

$$a + ib = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

4.9 Modulus and Argument

In the polar representation shown above we call

(i) $r = |a + ib| = \sqrt{a^2 + b^2}$, the modulus or absolute value of $a + ib$;

(ii) θ an argument of $a + ib$.

Note:

If θ is an argument of $a + ib$ then so is $\theta + 2n\pi$, where n is any integer.

The particular value of θ for which $-\pi < \theta < \pi$ is called the *principal argument.*

(Some texts use $0 < \theta < 2\pi$)

Example

$z = 2 + 2i = 2\sqrt{2}(\cos 45° + i\sin 45°) = 2\sqrt{2}[45°]$.

Alternatively, the argument can be expressed in radians (rather than degrees) and we can also write z as $2\sqrt{2}\left[\dfrac{\pi}{4}\right]$.

Example

Express the complex number $-3 + i\sqrt{3}$ in polar form.

Check your answer using the Rectangular to Polar conversion on your calculator.

Solution

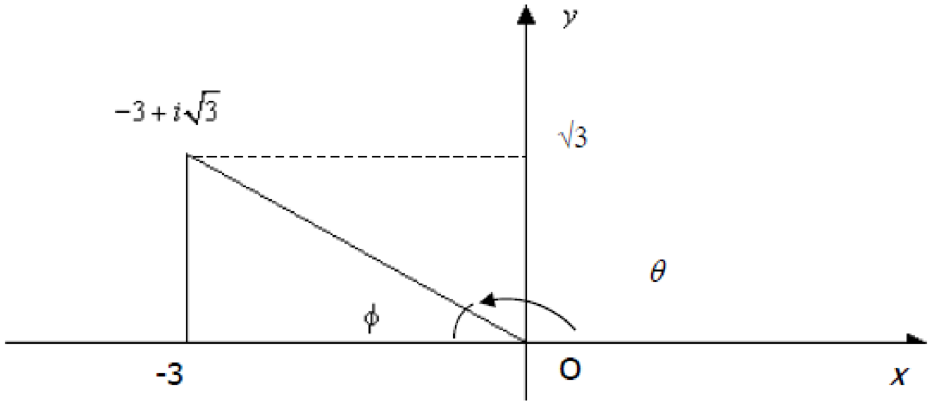
Let $-3 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$ or $r[\theta]$, then

$$r = |-3 + i\sqrt{3}| = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$$

$$\tan\phi = \frac{\sqrt{3}}{3} \quad \Rightarrow \quad \phi = 30°$$

Therefore, the required argument is given by $\theta = 180° - 30° = 150°$

$$\Rightarrow -3 + i\sqrt{3} = 2\sqrt{3}(\cos 150° + i\sin 150°) = 2\sqrt{3}[150°]$$



Checking your answer on your calculator:

You may have to refer to your calculator's manual for instructions specific to your model. On the calculator shown on the right, we can convert $-3 + i\sqrt{3}$ to polar form by inputting the following:

$\boxed{\text{Pol(}}\ \boxed{\text{-}}\ \boxed{3}\ \boxed{,}\ \boxed{\sqrt{\ }}\ \boxed{3}\ \boxed{)}\ \boxed{=}$

The output shown on the screen on the right is the value of $r = 2\sqrt{3} \approx 3.46$.

The values of r and θ are stored in the variables 'E' and 'F' in this calculator (printed above the $\boxed{\text{cos}}$ and $\boxed{\text{tan}}$ buttons).

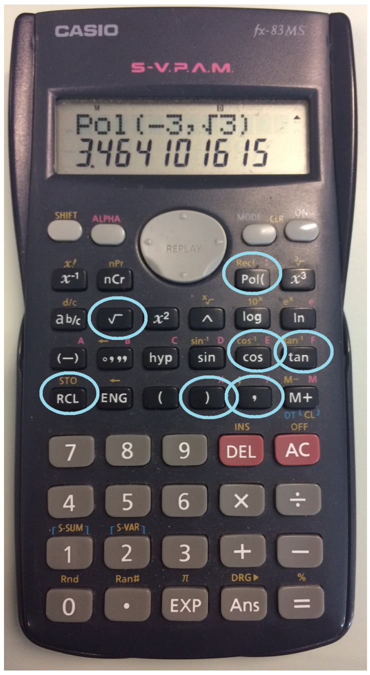
We can see the value of θ by inputting: $\boxed{\text{RCL}}\ \boxed{\text{tan}}$

We can switch back to the value of r by inputting: $\boxed{\text{RCL}}\ \boxed{\text{cos}}$

We can also perform the inverse calculation by using the Polar to Rectangular conversion (accessed with the 'Shift' button here):

$\boxed{\text{Shift}}\ \boxed{\text{Pol(}}\ \boxed{2}\ \boxed{\sqrt{\ }}\ \boxed{3}\ \boxed{,}\ \boxed{150}\ \boxed{)}\ \boxed{=}$

The calculator now stores the values -3 and $\sqrt{3}$ in the variables 'E' and 'F'.



Tutorial Exercises - Complex Numbers

1. Express in the form $a + bi$

   (a) $(4 - 3i) + (2i - 8)$ (b) $3(-1 + 4i) - 2(7 - i)$

   (c) $(3 + 2i)(2 - i)$ (d) $(i - 2)\{2(1 + i) - 3(i - 1)\}$

   (e) $\dfrac{2 - 3i}{4 - i}$ (f) $(4 + i)(3 + 2i)(1 - i)$

   (g) $\dfrac{(2 + i)(3 - 2i)(1 + 2i)}{(1 - i)^2}$ (h) $(2i - 1)^2 \left\{ \dfrac{4}{1 - i} + \dfrac{2 - i}{1 + i} \right\}$

2. If $z_1 = 2 + i$, $z_2 = -3 + 2i$ and $z_3 = 1 - 2i$, evaluate:

   (a) $\dfrac{1}{z_1} + \dfrac{1}{z_2}$ (b) $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}$

3. If $z_1 = 1 - i$, $z_2 = -2 + 4i$ and $z_3 = \sqrt{3} - 2i$, evaluate:

   (a) $z_1^2 + 2z_1 - 3$ (b) $(z_3 - \bar{z_3})^5$

   (c) $\mathrm{Re}\{-2z_1^3 + 3z_2^2 - 5z_3^2\}$ (d) $\mathrm{Im}\left\{ \dfrac{z_1 z_2}{z_3} \right\}$

   (e) $\overline{(z_2 + z_3)(z_1 - z_3)}$ (f) $\dfrac{1}{2}\left(\dfrac{z_3}{\bar{z_3}} + \dfrac{\bar{z_3}}{z_3} \right)$

4. Determine the modulus and principal value of the argument $(-180° < \theta < 180°)$ of the following complex numbers:

   (a) $-\sqrt{3} + 2i$ (b) $-3 - 5i$

   (c) $\dfrac{2 + 3i}{1 + 2i}$ (d) $\dfrac{1 + i}{(1 - i)(2 + i)}$

5. Express each of the following complex numbers in polar form, using the principal value of the argument $(-180° < \theta < 180°)$:

(a) $-\sqrt{3}-3i$ (b) $3+4i$

(c) $2-2i$ (d) $-1+\sqrt{3}i$

(e) $2\sqrt{2}+2\sqrt{2}i$ (f) $-i$

Answers

1. (a) $-4-i$ (b) $-17+14i$ (c) $8+i$ (d) $-9+7i$

   (e) $\dfrac{11}{17}-\dfrac{10}{17}i$ (f) $21+i$ (g) $-\dfrac{15}{2}+5i$ (h) $-\dfrac{11}{2}-\dfrac{23}{2}i$

2. (a) $\dfrac{11}{65}-\dfrac{23}{65}i$ (b) $\dfrac{24}{65}+\dfrac{3}{65}i$

3. (a) $-1-4i$ (b) $-1024i$ (c) -27

   (d) $\dfrac{6\sqrt{3}+4}{7}$ (e) $-7+3\sqrt{3}+\sqrt{3}i$ (f) $-\dfrac{1}{7}$

4. (a) $2.65,\ 130.89°$ (b) $5.83,\ -120.96°$

   (c) $\sqrt{\dfrac{13}{5}}\approx 1.61,\ -7.13°$ (d) $0.45,\ 63.43°$

5. (a) $2\sqrt{3}[-120°]=2\sqrt{3}(\cos(-120°)+i\sin(-120°))$ (b) $5[53.13°]$

   (c) $2\sqrt{2}[-45°]$ (d) $2[120°]$

   (e) $4[45°]$ (f) $1[-90°]$

Chapter 5

Quaternions

A **quaternion** a can be considered as an extension of a complex number $a_0 + a_1 i$ to an entity with four components

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

where a_1, a_2 and a_3 are coefficients of the **hyper-imaginary** quantities i, j and k, denoting three *independent* values for $\sqrt{-1}$. These came about when the famous Irish mathematician, William Hamilton, in 1843 put forward the idea of geometrical representation of complex numbers in 2D to 3D by considering the "extended" complex number

$$z = a + bi + cj$$

where j is another independent value for $\sqrt{-1}$. He argued that this extended form must also obey the rule that when it is multiplied by its *conjugate* the result must be equal to a real number (the square of its modulus). Performing this operation gives

$$\begin{aligned} z\bar{z} &= (a + bi + cj)(a - bi - cj) \\ &= a^2 + b^2 + c^2 - ijbc - jibc. \qquad (\text{using } i^2 = j^2 = -1) \end{aligned}$$

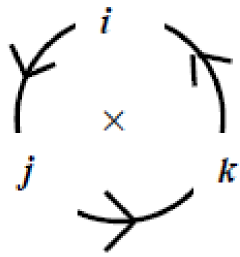
which implies that $ji = -ij$ for this to be a real number. He then considered what this product ij really was by observing that

$$ij.ij = ij(-ji) = -ij^2 i = -i(-1)i = i^2 = -1.$$

This means that ij is another square root of (-1), which we can call k. Hence, in general we have the quaternion (4 terms needed for complete description) stated above to describe the hyper-complex numbers.

Quaternions can be added, subtracted and multiplied by real numbers and by each other using the usual laws of algebra but subject to the following laws of multiplication

$$\begin{array}{ll} i\,j = k = -j\,i, & i^2 = -1 \\ j\,k = i = -k\,j, & j^2 = -1 \\ k\,i = j = -i\,k, & k^2 = -1 \end{array}$$



(Note the cyclic order of multiplication − similar to vector products).

The above laws can be written more compactly as

$$i^2 = j^2 = k^2 = ijk = -1.$$

Examples

1. $3(4i + 2j) = 12i + 6j = (4i + 2j).3$, indicating that multiplication by a real number can be done in any order.

2. $j(2i + 4) = -2k + 4j$, but $(2i + 4)j = 2k + 4j$, indicating that quaternions *do not* commute − similar to matrices.

3.

$$\begin{aligned} (3i + 4j)^2 &= (3i + 4j)(3i + 4j) \\ &= 3i(3i + 4j) + 4j(3i + 4j) \\ &= -9 + 12k - 12k - 16 \\ &= -25. \end{aligned}$$

4. Simplify the following Quaternions:

   (i) $(ai + bj)^2$ (ii) $(2i + 3)(1 + j + k)$ (iii) $(1 + i + j + k)(1 - i)$.

Solution

i) $(ai+bj)(ai+bj) = ai(ai+bj) + bj(ai+bj)$

$= a^2i^2 + abij + baji + b^2j^2 = \underline{-a^2-b^2}$

ii) $(2i+3)(1+j+k)$

$= 2i(1+j+k) + 3(1+j+k) = 2i + 2ij + 2ik + 3 + 3j + 3k$

$= 2i + 2k - 2j + 3 + 3j + 3k = \underline{3 + 2i + j + 5k}$

iii) $(1+i+j+k)(1-i)$

$= (1-i) + i(1-i) + j(1-i) + k(1-i)$

$= 1 - \cancel{i} + \cancel{i} - i^2 + j - ji + k - ki$

$= 2 + \cancel{j} + k + k - \cancel{j} = \underline{2+2k}$

5.1 Notation

For any quaternion given by

$$a = a_0 + a_1i + a_2j + a_3k$$

a_0 is called the **scalar** or **real** part, and is sometimes written $S(a)$. The remainder, $a_1i + a_2j + a_3k$, is called the **vector** or **pure** part, and is written $V(a)$ or a'.

A quaternion is said to be **pure** or a **pure vector** if its scalar part is zero.

The **conjugate** of a quaternion a is written $\bar{a}$ and is defined by

$$\bar{a} = a_0 - a_1i - a_2j - a_3k.$$

Examples

1. If $a = 3 + 4i + 2j + 5k$, then $S(a) = 3$, $a' = 4i + 2j + 5k$ and $\bar{a} = 3 - 4i - 2j - 5k$.

2. Show that $(a')^2 = -(a_1^2 + a_2^2 + a_3^2)$ for a general quaternion.

$a = a_0 + a_1i + a_2j + a_3k \qquad a' = a_1i + a_2j + a_3k$

$(a')^2 = (a_1i + a_2j + a_3k)(a_1i + a_2j + a_3k)$

$= a_1i(a_1i + a_2j + a_3k) + a_2j(a_1i + a_2j + a_3k) + a_3k(a_1i + a_2j + a_3k)$

$= a_1^2i^2 + \cancel{a_1a_2ij} + \cancel{a_1a_3ik} + \cancel{a_2a_1ji} + a_2^2j^2 + \cancel{a_2a_3jk} + \cancel{a_3a_1ki} + \cancel{a_3a_2kj} + a_3^2k^2$

$= -a_1^2 - a_2^2 - a_3^2 = \underline{-(a_1^2 + a_2^2 + a_3^2)}$

3. By writing $a = a_0 + a'$, show that $a\bar{a} = \bar{a}a$.

$a = a_0 + a' \qquad \bar{a} = a_0 - a'$

$a\bar{a} = (a_0 + a')(a_0 - a') = a_0(a_0 - a') + a'(a_0 - a')$

$= a_0^2 - \underline{a_0 a' + a' a_0} - (a')^2$

$\underline{a_0 \text{ is constant}} \quad \therefore \quad a_0 a' = a' a_0$

$\therefore \quad a\bar{a} = a_0^2 - (a')^2$

$= \underline{(a_0 - a')(a_0 + a') = \bar{a}a}$

5.2 Norm of a Quaternion

The **norm** or **modulus** of a quaternion is written as $|a|$ and is defined by

$$|a|^2 = a\bar{a} = \bar{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

(using previous examples' results)

Notice that $|a| \geq 0$ for all quaternions and $|a| = 0$ means that $a = 0$.

A quaternion is said to be a **unit** quaternion if $|a| = 1$.

If $\lambda > 0$ then $|\lambda a| = \lambda |a|$ (easily verified).

Examples

1. The norm of $a = 2 - i + 2j - 3k$ is $|a| = \sqrt{18}$. Hence, $\dfrac{2 - i + 2j - 3k}{\sqrt{18}}$ is a unit quaternion.

2. Determine the values of the scalar d such that $a = \dfrac{1 - i + 2j + k}{d}$ is a unit quaternion.

$|a| = 1 \qquad \dfrac{1^2 + (-1)^2 + 4 + 1^2}{d^2} = 1 \qquad \dfrac{7}{d^2} = 1 \qquad \underline{d = \pm\sqrt{7}}$

3. Determine the values of c such that $\dfrac{-2 + ci - j + 2k}{c + 1}$ has modulus 2.

$$\left|\frac{-2+ci-j+2k}{c+1}\right| = 2 \Rightarrow \sqrt{\frac{(-2)^2+c^2+(-1)^2+2^2}{(c+1)^2}} = 2$$

$$\sqrt{\frac{c^2+9}{(c+1)^2}} = 2 \quad = \quad 4(c+1)^2 = c^2+9$$

$$4c^2+4+8c = c^2+9$$

$$3c^2+8c-5=0$$

$$c = \frac{-8 \pm \sqrt{64+60}}{6}$$

$$= 0.523 \text{ or } -3.189$$

5.3 Inverse

The **inverse** of a quaternion a is defined as

$$a^{-1} = \frac{\bar{a}}{|a|^2},$$

because by definition

$$a.\frac{\bar{a}}{|a|^2} = \frac{\bar{a}}{|a|^2}.a = 1.$$

Example

The inverse of the quaternion $a = 2 + 3j - 2k$ is $\dfrac{2 - 3j + 2k}{17}$ (check this).

$$\bar{a} = 2-3j+2k \qquad |a| = \sqrt{2^2+(-3)^2+2^2} = \sqrt{4+9+4} = \sqrt{17}$$

$$|a|^2 = 17$$

$$\text{so} \quad a^{-1} = \frac{\bar{a}}{|a|^2} = \frac{2-3j+2k}{17}$$

5.4 Matrix method to compute general quaternion products

For general quaternions, it is useful to have a methodical way of calculating products. This can be done by the following procedure.

Let two quaternions be given by $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$. The product of these can be seen to be given by the matrix product:

$$\boxed{\begin{aligned} a\,b &= \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ -b_1 & b_0 & -b_3 & b_2 \\ -b_2 & b_3 & b_0 & -b_1 \\ -b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \\ &= \begin{pmatrix} r_0 & r_1 & r_2 & r_3 \end{pmatrix} = r_0 + r_1 i + r_2 j + r_3 k. \end{aligned}}$$

(Notice that the order of multiplication is important.)

Example

If $a = 2 + 3i - j + 2k$ and $b = -1 + 2i + j - 3k$ then the product is given by:

$$a\,b = \begin{pmatrix} 2 & 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 & 1 & -3 \\ -2 & -1 & 3 & 1 \\ -1 & -3 & -1 & -2 \\ 3 & -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 16 & -3 \end{pmatrix} = -1 + 2i + 16j - 3k.$$

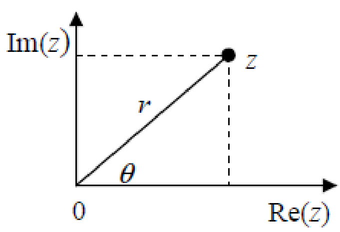
For further quaternion properties, see Appendix **??**.

5.5 Polar Form of a Quaternion

We recall that the polar form of an ordinary complex number can be obtained from the **Argand Diagram** (see below) and written as

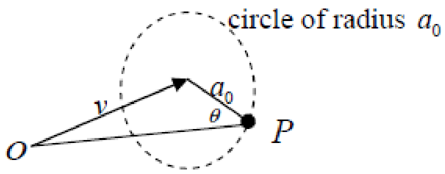
$$z = r(\cos\theta + i\sin\theta)$$

where r is the modulus and θ is the argument of the complex number.



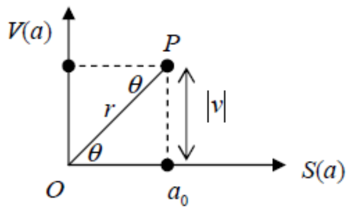
We will now show that we can extend this form to quaternions in general by considering a *generalised* Argand Diagram for quaternions.

Suppose we have a quaternion defined by $a = a_0 + a_1 i + a_2 j + a_3 k$, where $a_0 = S(a)$ and $v = a_1 i + a_2 j + a_3 k = V(a)$. The vector part v can be represented as a point in space and the scalar part a_0 can be represented by any point perpendicular to v, a distance a_0 from its end-point (see below).



Notice that the modulus of a by definition is $\sqrt{a_0^2 + |v|^2}$ and hence the reason why a_0 must be perpendicular to v. This gives the quaternion representation as OP in the above diagram, where P can be *anywhere* on the given circle perpendicular to v as shown (which also shows that we cannot represent a by a single vector in space).

If we now view the above representation in two dimensions, by always using $V(a)$ as the "vertical" axis direction and $S(a)$ as the "horizontal" axis direction, we have the diagram:



Note that if $a_0 < 0$, then this is represented by a point on the above circle diametrically opposite to that given when $a_0 > 0$ (we have to choose a positive and negative direction here, so we use the usual convention).
This can then be thought of as a *generalised* Argand Diagram for the quaternion a, where

$$a = a_0 + v,$$

with modulus

$$r = OP = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

and *argument* θ (angle OP makes with the scalar axis) , where

$$\cos\theta = \frac{a_0}{r}.$$

Further, we notice that

$$a_0 = r\cos\theta$$

and

$$v = |v|I = (r\sin\theta)I,$$

where

$$I = \frac{v}{|v|} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

is a *unit* pure quaternion (vector) in the direction of v. This gives the **polar form** of a as

$$\boxed{a = r(\cos\theta + I\sin\theta) \quad (I^2 = -1)}$$

[Notice in the special case of the complex number $a = x + iy$, we have $I = i$.]

Examples

Express the following quaternions in polar form:

1. $a = 1 + i - 2j + k$
2. $b = -1 + i + 2j - 2k$
3. $c = i - 2j$
4. $d = i + j + k$

Solutions

1. The norm, or modulus is given by $r = |a| = \sqrt{1+1+4+1} = \sqrt{7}$, and the argument is given by $\cos\theta = \dfrac{S(a)}{r} = \dfrac{1}{\sqrt{7}}$ (1st quadrant angle) $\Rightarrow\ \theta = 67.79°$.

   Also, $\sin\theta = \sqrt{1 - 1/7} = \sqrt{\dfrac{6}{7}}$, giving the polar form as

$$a = \sqrt{7}(\cos 67.79° + I\sin 67.79°) \quad \text{where} \quad I = \frac{a'}{|a'|} = \frac{i - 2j + k}{\sqrt{6}}.$$

2. $|b| = \sqrt{1+1+4+4} = \sqrt{10}$, $\cos\theta = \dfrac{-1}{\sqrt{10}}\ \Rightarrow\ \theta = 108.43°$ and $\sin\theta = \dfrac{3}{\sqrt{10}}$.

   This gives the polar of b form as

$$b = \sqrt{10}(\cos 108.43° + I\sin 108.43°) \quad \text{where} \quad I = \frac{i + 2j - 2k}{3}.$$

3. Notice that c is a pure (vector) quaternion and consequently the argument is $90°$, because $\cos\theta = 0$ (no real part). The modulus is $r = \sqrt{5}$ and so the polar form is

$$c = \sqrt{5}(\cos 90° + I\sin 90°) \quad \text{where} \quad I = \frac{i - 2j}{\sqrt{5}}.$$

4. d

$$\cos\theta = \frac{d_0}{r} = 0 \qquad \Rightarrow \qquad \theta = 90°$$

$$v = d = i - 2j, \qquad |v| = r = \sqrt{3}, \qquad I = \frac{i + j + k}{\sqrt{3}}$$

$$a = \sqrt{3}\left(\cos 90° \ + \left(\frac{i + j + k}{\sqrt{3}}\right)\sin 90° \ \right)$$

Tutorial Exercises - Quaternions

1. For each of the following quaternions, write down its conjugate and norm/modulus respectively:

   (i) $2 + 5i + 7j + 4k$ (ii) $-3 + 2i + 2j + k$ (iii) $1 + 2i - 3j - 3k$

   (iv) $-1 + i + k$ (v) $\dfrac{2 - 3i + j}{7}$ (vi) $\dfrac{1}{2} + \dfrac{i}{3} - \dfrac{2j}{3} + \dfrac{k}{4}$.

2. If $a = 2 - i + 2j$ and $b = 2 + 2i + j - 3k$ then determine:

   (i) $2a + 3b$ (ii) $a\,b$ (iii) $b\,a$ (iv) a^{-1} (v) b^{-2}.

3. Express the following quaternions in polar form:

   (i) $2 - i + k$ (ii) $\dfrac{1 + i - j - k}{2}$ (iii) $2i + j - 3k$

   (iv) $-1 + j + 2k$.

4. Prove the result: $(\cos\theta + I\sin\theta)^2 \equiv \cos(2\theta) + I\sin(2\theta)$ for polar forms of quaternions. Hence determine the polar forms of the following:

   (i) $(1 + i - k)^2$ (ii) $\left(\dfrac{i - j + k}{\sqrt{3}}\right)^2$.

5. The **square root** s of a quaternion a is defined by: $s^2 = a$. Using the polar form property in question 4, determine the square roots of the quaternion $a = 2 + i - 2j + 3k$ in cartesian form.

Answers

1. (i) $2 - 5i - 7j - 4k; \quad \sqrt{94}$ (ii) $-3 - 2i - 2j - k; \quad 3\sqrt{2}$

(iii) $1 - 2i + 3j + 3k; \quad \sqrt{23}$ (iv) $-1 - i - k; \quad \sqrt{3}$

(v) $\dfrac{2 + 3i - j}{7}; \quad \sqrt{\dfrac{2}{7}}$ (vi) $\dfrac{1}{2} - \dfrac{i}{3} + \dfrac{2j}{3} - \dfrac{k}{4}; \quad \dfrac{5\sqrt{5}}{12}.$

2. (i) $10 + 4i + 7j - 9k$ (ii) $4 - 4i + 3j - 11k$

(iii) $4 + 8i + 9j - k$ (iv) $\dfrac{2 + i - 2j}{9}$

(v) $\dfrac{-5 - 4i - 2j + 6k}{162}.$

3. (i) $\sqrt{6}\left[\cos 35.26^\circ + \left(\dfrac{-i + k}{\sqrt{2}}\right)\sin 35.26^\circ\right]$ (ii) $\cos 60^\circ + \left(\dfrac{i - j - k}{\sqrt{3}}\right)\sin 60^\circ$

(iii) $\sqrt{14}\left[\cos 90^\circ + \left(\dfrac{2i + j - 3k}{\sqrt{14}}\right)\sin 90^\circ\right]$ (iv) $\sqrt{6}\left[\cos 114.09^\circ + \left(\dfrac{j + 2k}{\sqrt{5}}\right)\sin 114.09^\circ\right].$

4. (i) $3\left[\cos 109.47^\circ + \left(\dfrac{i - k}{\sqrt{2}}\right)\sin 109.47^\circ\right]$ (ii) $\cos 180^\circ + \left(\dfrac{i - j + k}{\sqrt{3}}\right)\sin 180^\circ.$

5. $\pm\left\{\sqrt{\dfrac{3\sqrt{2} + 2}{2}} + I\sqrt{\dfrac{3\sqrt{2} - 2}{2}}\right\},$ where $I = \dfrac{i - 2j + 3k}{\sqrt{14}}$

$\approx \pm(1.7667 + 0.2830\, i - 0.5660\, j + 0.8490\, k).$

Chapter 6

Rotation by Quaternions

A very useful property of quaternions is that we can use them as an alternative way to rotate shapes in 3D space about a specified axis passing through the origin. This can sometimes cut down on the amount of calculations when compared to using matrix transformations.

The result is stated here without proof and is verified by a few simple examples. However, a full proof is given in Appendix A.4 for the interested reader.

6.1 Main Result

Given a **unit** quaternion $q = \cos\theta + I\sin\theta$, the transformation $q\,p\,q^{-1}$ rotates the point with position vector p (pure quaternion) through an angle $\mathbf{2}\theta$ about the axis I.

[Notice that this remarkable result depends on the fact that $q\,p\,q^{-1}$ is *always* a pure quaternion.]

Examples

1. Find the co-ordinates of the point $(2, 3, 1)$ after a rotation through $90°$ about the $x-$axis.
2. Find the co-ordinates of the point $(1, 0, -2)$ after a rotation through $60°$ about a line through the origin parallel to the vector $\underline{i} + 2\underline{j} - \underline{k}$.
3. Find the co-ordinates of the point given in example 1 after a rotation through $90°$ about an axis parallel to the $x-$axis passing through the point $(-1, 2, 1)$.
4. Repeat question 2 but with the axis of rotation passing through the point $(2, 1, 3)$.

Solutions

1. Notice that rotation about the $x-$axis by $90°$ means that $\theta = 45°$ and $I = i$, so that $q = \cos 45° + i \sin 45° = \dfrac{1+i}{\sqrt{2}}$. Also, we have $p = 2i + 3j + k$, the position vector of the point, and so the transformed position vector is given by

$$\begin{aligned} q\,p\,q^{-1} &= \frac{(1+i)}{\sqrt{2}}(2i+3j+k)\frac{(1-i)}{\sqrt{2}} \\ &= \frac{1}{2}(-2+2i+2j+4k)(1-i) \\ &= 2i - j + 3k. \end{aligned}$$

This gives the new co-ordinates as (2, -1, 3) as expected by simple observation.

2. In this example, $\theta = 30°$ and $I = \dfrac{i+2j-k}{\sqrt{6}}$, so that

$$\begin{aligned} q &= \cos 30° + \left(\frac{i+2j-k}{\sqrt{6}}\right)\sin 30° \\ &= \frac{\sqrt{3}}{2} + \frac{i+2j-k}{2\sqrt{6}} \\ &= \frac{3\sqrt{2}+i+2j-k}{2\sqrt{6}} \end{aligned}$$

The transformed point is then given by

$$\begin{aligned} q\,p\,q^{-1} &= \left(\frac{3\sqrt{2}+i+2j-k}{2\sqrt{6}}\right)(i-2k)\left(\frac{3\sqrt{2}-i-2j+k}{2\sqrt{6}}\right) \\ &= \frac{1}{24}\begin{pmatrix} 3\sqrt{2} & 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & -1 & -2 & 1 \\ 1 & 3\sqrt{2} & -1 & -2 \\ 2 & 1 & 3\sqrt{2} & 1 \\ -1 & 2 & -1 & 3\sqrt{2} \end{pmatrix} \\ &= \frac{1}{24}\begin{pmatrix} -3 & (3\sqrt{2}-4) & 1 & (-6\sqrt{2}-2) \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & -1 & -2 & 1 \\ 1 & 3\sqrt{2} & -1 & -2 \\ 2 & 1 & 3\sqrt{2} & 1 \\ -1 & 2 & -1 & 3\sqrt{2} \end{pmatrix} \\ &= \frac{1}{24}\Big[(18-24\sqrt{2})i + (12+6\sqrt{2})j - (30+12\sqrt{2})k\Big] \\ &\approx -0.644\,i + 0.854\,j - 1.957\,k \end{aligned}$$

i.e., the new co-ordinates are at $(-0.664, 0.854, -1.957)$.

3. For any rotation about a line not passing through the origin, as with matrix transformations, we perform a linear translation first to make the axis pass through the origin, then perform the same operations as above. Finally, we translate our co-ordinates back to their correct positions.

   The axis passes through $(-1, 2, 1)$ and so we shift the position vector of p by adding $-(-i + 2j + k)$ to it, i.e.

$$p_T = (2i + 3j + k) - (-i + 2j + k) = 3i + j.$$

   As before, $\theta = 45°$ and $I = i$, giving $q = \cos 45° + i \sin 45° = \dfrac{1+i}{\sqrt{2}}$. The transformation is then effected by

$$\begin{aligned} q\, p_T\, q^{-1} &= \frac{(1+i)}{\sqrt{2}}(3i + j)\frac{(1-i)}{\sqrt{2}} \\ &= \frac{1}{2}(-3 + 3i + j + k)(1 - i) \\ &= 3i + k. \end{aligned}$$

   Performing the inverse translation $q\, p_T\, q^{-1} + (-i + 2j + k)$ gives the quaternion vector $p = 2i + 2j + 2k$, i.e., the point $(2, 3, 1)$ maps to $(2, 2, 2)$.

Q2: $\underline{qpq^{-1}} = \left(\dfrac{3\sqrt{2} + i + 2j - k}{2\sqrt{6}}\right)(i - 2k)\left(\dfrac{3\sqrt{2} - i - 2j + k}{2\sqrt{6}}\right)$

4.

$p_T = \underline{p - 2i - j - 3k} = -i - j - 5k$ ← remember to translate back!

$$q_1 p q_1^{-1} = \left(\frac{3\sqrt{2} + i + 2j - k}{2\sqrt{6}}\right)(-i - j - 5k)\left(\frac{3\sqrt{2} - i - 2j + k}{2\sqrt{6}}\right)$$

$$= \frac{1}{24}\begin{pmatrix}3\sqrt{2} & 1 & 2 & -1\end{pmatrix}\begin{pmatrix}0 & -1 & -1 & -5\\ 1 & 0 & 5 & -1\\ 1 & -5 & 0 & 1\\ 5 & 1 & -1 & 0\end{pmatrix}\begin{pmatrix}3\sqrt{2} & -1 & -2 & 1\\ 1 & 3\sqrt{2} & -1 & -2\\ 2 & 1 & 3\sqrt{2} & 1\\ -1 & 2 & -1 & 3\sqrt{2}\end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix}1+2-5 & -3\sqrt{2}-10-1 & -3\sqrt{2}+5+1 & -15\sqrt{2}-1+2\end{pmatrix}\begin{pmatrix}\cdots\end{pmatrix}$$

$$= \frac{1}{24}\begin{pmatrix}-2 & -3\sqrt{2}-11 & -3\sqrt{2}+6 & -15\sqrt{2}+1\end{pmatrix}\begin{pmatrix}3\sqrt{2} & -1 & -2 & 1\\ 1 & 3\sqrt{2} & -1 & -2\\ 2 & 1 & 3\sqrt{2} & 1\\ -1 & 2 & -1 & 3\sqrt{2}\end{pmatrix}$$

$$= \frac{1}{24}\left(-6\sqrt{2} - 3\sqrt{2} - 11 - 6\sqrt{2} + 12 + 15\sqrt{2} - 1,\ 2 - 18 - 33\sqrt{2} - 3\sqrt{2} + 6 - 30\sqrt{2} + 2,\ 4 + 3\sqrt{2} + 11 - 18 + 18\sqrt{2} + 15\sqrt{2} - 1,\ -2 + 6\sqrt{2} + 22 - 3\sqrt{2} + 6 - 90 + 3\sqrt{2}\right)$$

$$= \frac{1}{20}\begin{pmatrix}0 & -8-66\sqrt{2} & -4+36\sqrt{2} & -64+6\sqrt{2}\end{pmatrix}$$

$$\simeq -4.222i + 1.955j - 2.313k$$

→ translate back $p_I = -4.222i + 1.955j - 2.313k + (2i + 3j + 3k)$

$$= -2.222i + 2.955j + 0.687k$$

$\boxed{\text{Image point: } (-2.222,\ 2.955,\ 0.687)}$

6.2 Composition of Rotations (Concatenation)

For the case of multiple rotations about possible different axes the above process can be adapted as follows.

Suppose that we want to follow the first rotation, given by the transformation $q_1\, p\, q_1^{-1}$, by a second rotation of $2\theta_2$ about some axis I_2 where $q_2 = \cos\theta_2 + I_2 \sin\theta_2$. Notice that, from the first rotation, $p \to q_1\, p\, q_1^{-1}$ and so operating on this position vector for the second rotation gives

$$q_2(q_1\, p\, q_1^{-1})q_2^{-1} = (q_2 q_1)\, p\, (q_1^{-1} q_2^{-1}) = (q_2 q_1)\, p\, (q_2 q_1)^{-1}$$

This result shows that for multiple rotations, we simply multiply the quaternions together in the order of rotation from right to left, i.e. the latest rotation will be the *left multiple* in the sequence.

Example

Find the position of the point $(-1, 0, 1)$ after a rotation about the $z-$axis of $120°$ followed by a rotation of $-90°$ about the line parallel to $\underline{i} - \underline{k}$, passing through the origin.

Solution

For the first rotation we have, $q_1 = \cos 60^\circ + I_1 \sin 60^\circ$, with $I_1 = k$, i.e.

$$q_1 = \cos 60^\circ + k \sin 60^\circ = \frac{1 + k\sqrt{3}}{2}.$$

For the second rotation we have, $q_2 = \cos(-45^\circ) + I_2 \sin(-45^\circ)$, with $I_2 = \dfrac{i-k}{\sqrt{2}}$, i.e.

$$q_2 = \cos 45^\circ - \left(\frac{i-k}{\sqrt{2}}\right) \sin 45^\circ = \frac{\sqrt{2} - i + k}{2}.$$

Hence for the combined rotation we have

$$q_2 q_1 = \left(\frac{\sqrt{2} - i + k}{2}\right)\left(\frac{1 + k\sqrt{3}}{2}\right) = \frac{\sqrt{2} - \sqrt{3} - i + j\sqrt{3} + (1 + \sqrt{6})k}{4}.$$

With $p = -i + k$, we have the position vector of the transformed point given by

$$\begin{aligned}(q_2 q_1)\, p\, (q_1 q_2)^{-1} &= \frac{(\sqrt{2} - \sqrt{3} - i + j\sqrt{3} + (1+\sqrt{6})k)}{4}(-i+k)\frac{(\sqrt{2} - \sqrt{3} + i - j\sqrt{3} - (1+\sqrt{6})k)}{4} \\ &\approx 0.36\, i + 1.06\, j + 0.86\, k.\end{aligned}$$

Tutorial Exercises - Rotations by Quaternions

1. Determine the quaternion $q = \cos\theta + I\sin\theta$ that performs the rotation in each of the following:

   (i) $60°$ about the $z-$axis, (ii) $-30°$ about the $x-$axis, (iii) $45°$ about the $y - axis$,

   (iv) $90°$ about the $y-$axis followed by $180°$ about the $x-$axis.

2. A point is rotated by $60°$ about the $y-$axis and then rotated by $90°$ about the $z-$axis. Determine the equivalent *single* rotation angle about an axis that gives the same image point.

3. Determine the co-ordinates of the point $(2, 1, 0)$ after the rotation:

   (i) $50°$ about the line parallel to $\underline{j} + \underline{k}$ passing through the origin,

   (ii) as in part (i) but followed by a rotation of $60°$ about the $z-$axis,

   (iii) $-120°$ about the line with vector equation $\underline{r} = \underline{k} + t(\underline{j} - \underline{k})$,

   (iv) $90°$ about the line with vector equation $\underline{r} = 2\underline{i} - \underline{k} + t(\underline{i} + \underline{j} + \underline{k})$.

Answers

1. (i) $\cos 30° + k\sin 30°$, (ii) $\cos 15° - i\sin 15°$,

   (iii) $\cos 22.5° + j\sin 22.5°$, (iv) $\cos 90° + \left(\dfrac{i+k}{\sqrt{2}}\right)\sin 90°$.

2. $104.48°$ about the axis $(-\underline{i} + \underline{j} + \sqrt{3}\underline{k})$ through the origin.

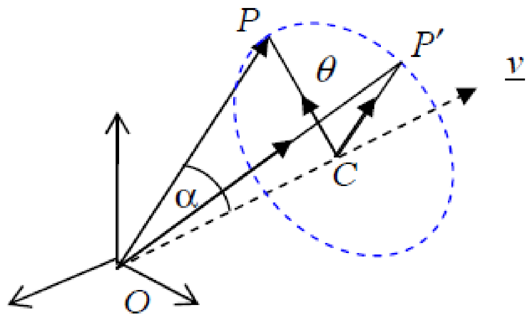
3. (i) $(0.744, 1.905, -0.905)$, (ii) $(-1.278, 1.597, -0.905)$,

   (iii) $(-1, 2.225, 1.225)$, (iv) $(2.667, 0.089, 0.224)$.

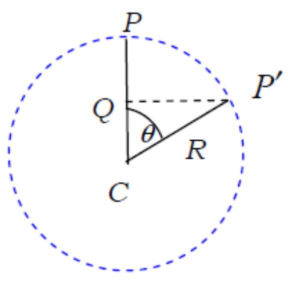
Appendix A

A.1 Derivation of Matrix for Rotation by θ about any Axis

Consider the point $P(x, y, z)$ to be rotated through θ about the axis direction $\underline{v}$ as shown to give the image $P'(x', y', z')$.



If we look **along** the direction of $\underline{v}$ we will see the plane containing P, C and P' as follows:



If $\underline{p}$, $\underline{c}$ and $\underline{p}'$ are the position vectors of P, C and P' respectively, then we see that the point we want is given by

$$\underline{p}' = \underline{c} + \overrightarrow{CP'} = \underline{c} + \overrightarrow{CQ} + \overrightarrow{QP'}. \tag{A.1}$$

Now, $\underline{c}$ is the orthogonal projection of $\underline{p}$ onto the direction of $\underline{v}$, i.e., $\underline{c} = (\underline{p} \cdot \hat{\underline{v}})\hat{\underline{v}}$, where $\hat{\underline{v}}$ is the unit vector in the direction of $\underline{v}$. This gives $\overrightarrow{CP} = \underline{p} - \underline{c} = \underline{p} - (\underline{p} \cdot \hat{\underline{v}})\hat{\underline{v}}$, whose modulus is R in the above diagram.

Further notice that

$$\overrightarrow{CQ} = R\cos\theta \frac{\overrightarrow{CP}}{R} = (\underline{p} - (\underline{p} \cdot \hat{\underline{v}})\hat{\underline{v}})\cos\theta$$

and that $\overrightarrow{QP'}$ is in the direction of $\hat{\underline{v}} \times \underline{p}$, which has modulus R, hence

$$\overrightarrow{QP'} = R\sin\theta\left(\frac{\hat{\underline{v}} \times \underline{p}}{R}\right) = (\hat{\underline{v}} \times \underline{p})\sin\theta.$$

Using these is Equation (A.1) gives the image point as

$$\boxed{\underline{p}' = (\underline{p} \cdot \hat{\underline{v}})\hat{\underline{v}} + [\underline{p} - (\underline{p} \cdot \hat{\underline{v}})\hat{\underline{v}}]\cos\theta + (\hat{\underline{v}} \times \underline{p})\sin\theta} \tag{A.2}$$

Now using $\underline{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\hat{\underline{v}} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$, where $\alpha^2 + \beta^2 + \gamma^2 = 1$, we have, $\underline{p} \cdot \hat{\underline{v}} = \alpha x + \beta y + \gamma z$ and $\hat{\underline{v}} \times \underline{p} = \begin{pmatrix} \beta z - \gamma y \\ \gamma x - \alpha z \\ \alpha y - \beta x \end{pmatrix}$, giving Equation (A.2) as

$$\begin{aligned}
\underline{p}' &= (\alpha x + \beta y + \gamma z)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}\cos\theta - (\alpha x + \beta y + \gamma z)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}\cos\theta + \begin{pmatrix} \beta z - \gamma y \\ \gamma x - \alpha z \\ \alpha y - \beta x \end{pmatrix}\sin\theta \\
&= (1-\cos\theta)(\alpha x + \beta y + \gamma z)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\sin\theta \\
&= (1-\cos\theta)\begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\sin\theta
\end{aligned}$$

i.e.,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \alpha^2(1-\cos\theta) + \cos\theta & \alpha\beta(1-\cos\theta) - \gamma\sin\theta & \alpha\gamma(1-\cos\theta) + \beta\sin\theta \\ \alpha\beta(1-\cos\theta) + \gamma\sin\theta & \beta^2(1-\cos\theta) + \cos\theta & \beta\gamma(1-\cos\theta) - \alpha\sin\theta \\ \alpha\gamma(1-\cos\theta) - \beta\sin\theta & \beta\gamma(1-\cos\theta) + \alpha\sin\theta & \gamma^2(1-\cos\theta) + \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We see immediately that this description gives the transformation matrix in homogeneous coordinate form as

$$(x' \quad y' \quad z' \quad 1) = (x \quad y \quad z \quad 1)R_{\theta,\hat{\underline{v}}}$$

where

$$\boxed{R_{\theta,\underline{\hat{v}}} = \begin{pmatrix} \alpha^2(1-\cos\theta)+\cos\theta & \alpha\beta(1-\cos\theta)+\gamma\sin\theta & \alpha\gamma(1-\cos\theta)-\beta\sin\theta & 0 \\ \alpha\beta(1-\cos\theta)-\gamma\sin\theta & \beta^2(1-\cos\theta)+\cos\theta & \beta\gamma(1-\cos\theta)+\alpha\sin\theta & 0 \\ \alpha\gamma(1-\cos\theta)+\beta\sin\theta & \beta\gamma(1-\cos\theta)-\alpha\sin\theta & \gamma^2(1-\cos\theta)+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

A.2 More Efficient Rotation about a given Axis

Instead of using homogeneous coordinates to cope with rotations about a given axis, which typically involves multiplying the coordinate vector by three (4×4) matrices, another method is given for interest that requires less computation.

Suppose we want to rotate a point $P(x, y, z)$ through an angle θ about an axis parallel to the *unit* vector $\hat{\underline{v}} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$, which passes through the point (p_1, p_2, p_3). To use the rotation matrix derived in Section 2, we have to first translate the point on the axis of rotation to the origin, i.e., $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{pmatrix}$. Now we use the rotation matrix on these new points before finally translating back by adding on the vector $\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$.

The whole process can be summarised by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \alpha^2(1-\cos\theta)+\cos\theta & \alpha\beta(1-\cos\theta)-\gamma\sin\theta & \alpha\gamma(1-\cos\theta)+\beta\sin\theta \\ \alpha\beta(1-\cos\theta)+\gamma\sin\theta & \beta^2(1-\cos\theta)+\cos\theta & \beta\gamma(1-\cos\theta)-\alpha\sin\theta \\ \alpha\gamma(1-\cos\theta)-\beta\sin\theta & \beta\gamma(1-\cos\theta)+\alpha\sin\theta & \gamma^2(1-\cos\theta)+\cos\theta \end{pmatrix} \begin{pmatrix} x-p_1 \\ y-p_2 \\ z-p_3 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

Note that there is no need for homogeneous coordinates in this method and so no (4×4) matrices are required. Further, there are 21 calculations (additions and multiplications) needed to obtain the image point (x', y', z') in the method (once the rotation matrix has been found), whereas the homogeneous coordinate/matrix concatenation method requires 84 calculations to obtain the same point - this can be reduced to around 45 calculations if use is made of the special structures of the homogeneous matrices. However, we see in general that the given method can give a big saving in computational time when hundreds or thousands of points are required to be transformed in a typical game scenario.

Example From Section 2, part (ii) using only point $O(0, 0, 0)$ for illustration:

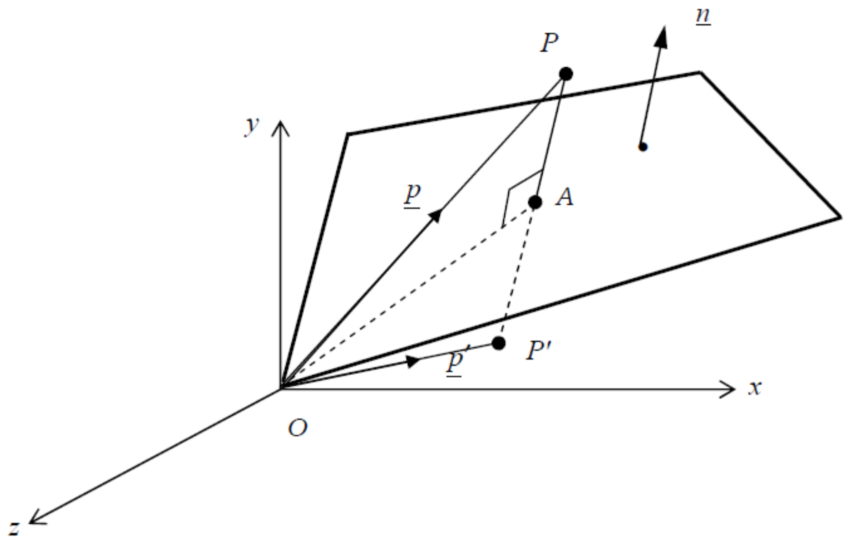
> The pyramid defined by co-ordinates $O(0, 0, 0)$, $B(1, 0, 0)$, $C(0, 1, 0)$ and $D(0, 0, 1)$ is rotated $45°$ about the line that has direction $\underline{v} = \underline{j} + \underline{k}$ and passes through:
>
> (i) the origin;
>
> (ii) the point $C(0, 1, 0)$.

Solution

Notice that $p_1 = 0$, $p_2 = 1$, $p_3 = 0$. The above matrix equation then becomes

$$\begin{aligned}
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \\ -\frac{1}{2} & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) & \frac{1}{2}(1+\frac{1}{\sqrt{2}}) \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2}(1+\frac{1}{\sqrt{2}}) \\ -\frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \\ -\frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{pmatrix} \approx \begin{pmatrix} 0.5 \\ 0.146 \\ -0.146 \end{pmatrix} \quad \text{(as before).}
\end{aligned}$$

A.3 Derivation of Matrix for Reflection in a Given Plane



Consider the plane of reflection, passing through the origin, given by its equation $ax + by + cz = 0$, then the reflection of some point P in the plane has image P' lying vertically opposite P on the other side of the plane through some point A on its surface, such that $\overrightarrow{P'P}$ is parallel to the normal $\underline{n} = (a \quad b \quad c)^T$.

Hence, the position vector of the image point is given by

$$\underline{p}' = \underline{p} - \overrightarrow{P'P}.$$

However, we see that $\overrightarrow{AP}$ is the projection of $\underline{p}$ in the direction of the normal $\underline{n}$ and so $\overrightarrow{AP} = (\underline{p} \cdot \underline{\hat{n}})\underline{\hat{n}}$, giving $\overrightarrow{P'P} = 2\overrightarrow{AP} = 2(\underline{p} \cdot \underline{\hat{n}})\underline{\hat{n}}$. This gives the above as

$$
\begin{aligned}
\underline{p}' &= \underline{p} - \overrightarrow{P'P} = \underline{p} - 2(\underline{p}\cdot\underline{\hat{n}})\underline{\hat{n}} \\
&= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{2(ax+by+cz)}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= \frac{1}{a^2+b^2+c^2}\begin{pmatrix} (-a^2+b^2+c^2)x - 2aby - 2acz \\ -2abx + (a^2-b^2+c^2)y - 2cbz \\ -2acx - 2bcy + (a^2+b^2-c^2)z \end{pmatrix} \\
&= \frac{1}{a^2+b^2+c^2}\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac \\ -2ab & a^2-b^2+c^2 & -2cb \\ -2ac & -2bc & a^2+b^2-c^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}
$$

In homogeneous coordinates, the image point is given by

$$
\boxed{P' = (x \quad y \quad z \quad 1)\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac & 0 \\ -2ab & a^2-b^2+c^2 & -2cb & 0 \\ -2ac & -2bc & a^2+b^2-c^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{pmatrix}}
$$

A.4 Proof of Quaternion result for Rotations

The result used in our work on quaternions that qpq^{-1} gives the coordinates of a point p (pure vector) when rotated through an angle of 2θ about the axis I (unit vector), where $q = \cos\theta + I\sin\theta$, will now be proved.

We shall assume that the axis I passes through the origin (for an axis not passing through the origin, we translate it to the origin before using the stated result and then translate back afterwards). The method of proof followed is to use the matrix form of quaternion multiplication to show an equivalence to the rotation matrix derived earlier.

Suppose that we have the quaternions $q = \cos\theta + I\sin\theta$, where $I = \alpha i + \beta j + \gamma k$, and $p = p_1 i + p_2 j + p_3 k$ – note that $\alpha^2+\beta^2+\gamma^2 = 1$. This also gives $q^{-1} = \cos\theta - I\sin\theta$. Now consider the quaternion product in matrix form given previously,

$$
pq^{-1} = \begin{pmatrix} 0 & -p_1 & -p_2 & -p_3 \\ p_1 & 0 & -p_3 & p_2 \\ p_2 & p_3 & 0 & -p_1 \\ p_3 & -p_2 & p_1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta \\ -\alpha\sin\theta \\ -\beta\sin\theta \\ -\gamma\sin\theta \end{pmatrix} = \begin{pmatrix} p_1\alpha\sin\theta + p_2\beta\sin\theta + p_3\gamma\sin\theta \\ p_1\cos\theta + p_3\beta\sin\theta - p_2\gamma\sin\theta \\ p_2\cos\theta - p_3\alpha\sin\theta + p_1\gamma\sin\theta \\ p_3\cos\theta + p_2\alpha\sin\theta - p_1\beta\sin\theta \end{pmatrix},
$$

which can also be written as

$$pq^{-1} = \begin{pmatrix} \alpha\sin\theta & \beta\sin\theta & \gamma\sin\theta \\ \cos\theta & -\gamma\sin\theta & \beta\sin\theta \\ \gamma\sin\theta & \cos\theta & -\alpha\sin\theta \\ -\beta\sin\theta & \alpha\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The rotation operation is now completed by pre-multiplying this product by q, i.e.

$$qpq^{-1} = \begin{pmatrix} cos\theta & -\alpha\sin\theta & -\beta\sin\theta & -\gamma\sin\theta \\ \alpha\sin\theta & \cos\theta & -\gamma\sin\theta & \beta\sin\theta \\ \beta\sin\theta & \gamma\sin\theta & \cos\theta & -\alpha\sin\theta \\ \gamma\sin\theta & -\beta\sin\theta & \alpha\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha\sin\theta & \beta\sin\theta & \gamma\sin\theta \\ \cos\theta & -\gamma\sin\theta & \beta\sin\theta \\ \gamma\sin\theta & \cos\theta & -\alpha\sin\theta \\ -\beta\sin\theta & \alpha\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ (\alpha^2-\beta^2-\gamma^2)\sin^2\theta+\cos^2\theta & 2\alpha\beta\sin^2\theta-2\gamma\sin\theta\cos\theta & 2\alpha\gamma\sin^2\theta+2\beta\sin\theta\cos\theta \\ 2\alpha\beta\sin^2\theta+2\gamma\sin\theta\cos\theta & (\beta^2-\alpha^2-\gamma^2)\sin^2\theta+\cos^2\theta & 2\beta\gamma\sin^2\theta-2\alpha\sin\theta\cos\theta \\ 2\alpha\gamma\sin^2\theta-2\beta\sin\theta\cos\theta & 2\beta\gamma\sin^2\theta+2\alpha\sin\theta\cos\theta & (\gamma^2-\alpha^2-\beta^2)\sin^2\theta+\cos^2\theta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

which, using the trigonometric identities $\cos^2\theta - \sin^2\theta = \cos 2\theta$, $2\sin^2\theta = 1 - \cos 2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$, simplifies to

$$qpq^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha^2(1-\cos 2\theta)+\cos 2\theta & \alpha\beta(1-\cos 2\theta)-\gamma\sin 2\theta & \alpha\gamma(1-\cos 2\theta)+\beta\sin 2\theta \\ \alpha\beta(1-\cos 2\theta)+\gamma\sin 2\theta & \beta^2(1-\cos 2\theta)+\cos 2\theta & \beta\gamma(1-\cos 2\theta)-\alpha\sin 2\theta \\ \alpha\gamma(1-\cos 2\theta)-\beta\sin 2\theta & \beta\gamma(1-\cos 2\theta)+\alpha\sin 2\theta & \gamma^2(1-\cos 2\theta)+\cos 2\theta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

The first row in the resulting quaternion shows that there is no scalar part and the matrix/vector product representing the image point has a transformation matrix given by the last 3 rows of the above matrix.

Hence, putting this result in its *homogeneous form*, using the transpose of the above 3×3 sub-matrix, gives the transformation matrix as

$$R_{2\theta,I} = \begin{pmatrix} \alpha^2(1-\cos 2\theta)+\cos 2\theta & \alpha\beta(1-\cos 2\theta)+\gamma\sin 2\theta & \alpha\gamma(1-\cos 2\theta)-\beta\sin 2\theta & 0 \\ \alpha\beta(1-\cos 2\theta)-\gamma\sin 2\theta & \beta^2(1-\cos 2\theta)+\cos 2\theta & \beta\gamma(1-\cos 2\theta)+\alpha\sin 2\theta & 0 \\ \alpha\gamma(1-\cos 2\theta)+\beta\sin 2\theta & \beta\gamma(1-\cos 2\theta)-\alpha\sin 2\theta & \gamma^2(1-\cos 2\theta)+\cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which we see to be the standard transformation matrix for a rotation of 2θ about the axis $I = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$.

Hence the result is proved.