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Mathematical Formulae

This book is to aid your mathematics studies. However, it is not a substitute for learning material. Notation may differ slightly from lecture notes.

Algebra

Hyperbolic functions: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$; $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

Arithmetic Series: $a + (a+d) + (a+2d) + \ldots + a + (n-1)d$

$$T_n = a + (n-1)d \qquad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Geometric Series: $a + ar + ar^2 + ar^3 + \ldots$

$$T_n = ar^{n-1} \qquad S_n = \frac{a\left(1-r^n\right)}{1-r} \qquad S_\infty = \frac{a}{1-r} \ \left(|r| < 1\right)$$

Trigonometry

Identities: $\tan A = \dfrac{\sin A}{\cos A}$; $\cot A = \dfrac{1}{\tan A}$; $\sec A = \dfrac{1}{\cos A}$

$\operatorname{cosec} A = \dfrac{1}{\sin A}$; $\sin^2 A + \cos^2 A = 1$

Multiple/Double Angles

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Half angle

If $t = \tan\dfrac{\theta}{2}$ then $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, $\tan\theta = \dfrac{2t}{1-t^2}$.

Products-to-Sums

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$; $2\cos A \cos B = \cos(A+B) + \cos(A-B)$;

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Sums-to-Products

$$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

$$\cos Q - \cos P = 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Sine Rule: $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

Cosine Rule: $a^2=b^2+c^2-2bc\cos A$

Area of a Triangle: $\dfrac{1}{2}ab\sin C$

Complex Numbers

$$z=a+ib=r\left(\cos\theta+i\sin\theta\right)$$

where $|z|=r=\sqrt{a^2+b^2}$; $\arg(z)=\tan\theta=\dfrac{b}{a}$

$$r\left(\cos\theta+i\sin\theta\right)=re^{i\theta}\ \text{(Euler's formula)}$$

$$\left[r\left(\cos\theta+i\sin\theta\right)\right]^n=r^n\left(\cos n\theta+i\sin n\theta\right)=r^ne^{in\theta}\quad\text{(De Moivre's theorem)}$$

Calculus

Rules of Differentiation

Let $f'(x)=\dfrac{df}{dx}$.

If $y=h\left(u\left(x\right)\right)$ then $\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$ (Chain rule)

$$\frac{d}{dx}\left\{f\left(x\right)\right\}^n=n\left\{f\left(x\right)\right\}^{n-1}f'\left(x\right)$$

For real numbers λ,μ: $\dfrac{d}{dx}\left(\lambda u+\mu v\right)=\lambda u'+\mu v'$

$\dfrac{d}{dx}\left(u.v\right)=u'v+uv'$ (Product rule)

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{vu'-uv'}{v^2}$ (Quotient rule)

Rules of Integration

$$\int\left[\lambda f\left(x\right)+\mu g\left(x\right)\right]dx=\lambda\int f\left(x\right)dx+\mu\int g\left(x\right)dx$$

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx \qquad \text{(integration by parts)}$$

Mean Value: $$M = \frac{1}{(b-a)} \int_a^b y \, dx$$

Volume of revolution: $$V = \pi \int_a^b y^2 \, dx \qquad (\text{about } x\text{-axis})$$

Pappus's Theorem: $$V = 2\pi \bar{y} A \qquad (\text{about } x\text{-axis})$$

Centroid: $$\bar{x} = \frac{1}{A} \int_a^b xy \, dx, \quad \bar{y} = \frac{1}{2A} \int_a^b y^2 \, dx$$

Standard Derivatives

Many of these may be derived using the chain rule.

$f(x)$	$f'(x)$
1	0
$(ax+b)^n$	$na(ax+b)^{n-1}$
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\tan(ax+b)$	$a\sec^2(ax+b)$
e^{ax}	ae^{ax}
$\ln(ax+b)$	$\frac{a}{ax+b}$
$\ln[f(x)]$	$\frac{f'(x)}{f(x)}$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}, \; x^2 < a^2$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}, \; x^2 < a^2$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$
$\cosh(ax+b)$	$a\sinh(ax+b)$
$\sinh(ax+b)$	$a\cosh(ax+b)$
$\tanh(ax+b)$	$a\,\mathrm{sech}^2(ax+b)$

Standard Integrals

In each case, the constant of integration is omitted.

$f(x)$	$\int f(x)\,dx$
1	x
$(ax+b)^n$	$\dfrac{(ax+b)^{n+1}}{a(n+1)}$, $n \neq -1$
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b)$
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b)$
$\tan(ax+b)$	$-\dfrac{1}{a}\ln\lvert\cos(ax+b)\rvert$
$\sec^2(ax+b)$	$\dfrac{1}{a}\tan(ax+b)$
e^{ax}	$\dfrac{1}{a}e^{ax}$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\ln\lvert ax+b\rvert$
$\dfrac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert$
$\dfrac{1}{\sqrt{a^2-x^2}}$, $x^2 < a^2$	$\sin^{-1}\left(\dfrac{x}{a}\right)$, $a > 0$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$, $a > 0$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert$
$\dfrac{x}{a+x^2}$	$\dfrac{1}{2}\ln\lvert a+x^2\rvert$
$\sinh(ax+b)$	$\dfrac{1}{a}\cosh(ax+b)$
$\cosh(ax+b)$	$\dfrac{1}{a}\sinh(ax+b)$
$\mathrm{sech}^2(ax+b)$	$\dfrac{1}{a}\tanh(ax+b)$

Numerical Methods

Newton-Raphson method for solving $f(x)=0$: $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$

Trapezium Rule: $\displaystyle\int_a^b f(x)\,dx=\frac{h}{2}\left[f_0+f_n+2\left(f_1+f_2+\ldots+f_{n-1}\right)\right]+O\left(h^2\right)$

Simpson's Rule

$$\int_a^b f(x)\,dx=\frac{h}{3}\left[f_0+f_n+4\left(f_1+f_3+\ldots+f_{n-1}\right)+2\left(f_2+f_4+\ldots+f_{n-2}\right)\right]+O\left(h^4\right)$$

(n must be even)

Numerical solution of the differential equation $\dfrac{dy}{dx}=f(x,y)$

Euler: $y_{n+1}=y_n+hf(x_n,y_n)$

Modified Euler: $y_{n+1}=y_n+\dfrac{h}{2}(k_1+k_2)$ with $k_1=f(x_n,y_n)$ and

$k_2=f(x_n+h,y_n+hk_1)$

Predictor-Corrector:

$y^{\mathrm{p}}_{n+1}=y_n+hf(x_n,y_n)$; $y^{\mathrm{c}}_{n+1}=y_n+\dfrac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},y^{\mathrm{p}}{}_{n+1})\right]$ (*Euler-trapezium*)

$y^{\mathrm{p}}_{n+1}=y_n+hf(x_n,y_n)$; $y^{\mathrm{c}}_{n+1}=y_{n-1}+\dfrac{h}{3}\left[f(x_{n-1},y_{n-1})+4f(x_n,y_n)+f(x_{n+1},y^{\mathrm{p}}{}_{n+1})\right]$

(*Euler-Simpson*)

Runge-Kutta (4th order): $y_{n+1}=y_n+\dfrac{h}{6}(k_1+2k_2+2k_3+k_4)$ with

$k_1=f(x_n,y_n)$; $k_2=f\left(x_n+\dfrac{h}{2},\ y_n+\dfrac{hk_1}{2}\right)$; $k_3=f\left(x_n+\dfrac{h}{2},\ y_n+\dfrac{hk_2}{2}\right)$;

$k_4=f(x_n+h,y_n+hk_3)$

Verlet Integration - for determining position x_{n+1} and velocity v_n at each time-step, $t_{n+1}=t_n+h$, given the acceleration $a_n=\dfrac{F_n}{m}$:

$x_{n+1}=2x_n-x_{n-1}+a_nh^2$; $\qquad v_n=\dfrac{x_{n+1}-x_{n-1}}{2h}$

Series Expansions

Binomial expansion:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots + nab^{n-1} + b^n$$

$$= \sum_{\mathrm{r}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{r}} a^{n-r}b^n, \text{ where } \binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$$

MacLaurin's expansion:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \qquad |x| < 1$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots; \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots; \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, -1 < x \le 1$$

Taylor's expansion:

One variable: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$

or, by replacing x by $(x + a)$ everywhere,

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \ldots$$

Matrices and Vectors

Determinants: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrix Inverse: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. In general, $A^{-1} = \frac{1}{\det A}adj(A)$.

Vectors: For $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ we have

$$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\underline{a}||\underline{b}|\cos\theta \text{ (Scalar Product)}$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \qquad (\text{Vector Product})$$

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\theta$$

Lines and Planes

A straight line through $\underline{a}$ in the direction $\underline{b}$ has

Vector equation: $\underline{r} = \underline{a} + \lambda \underline{b}$

Cartesian equation: $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$.

Parametric equation: $x = a_1 + \lambda b_1$, $y = a_2 + \lambda b_2$, $z = a_3 + \lambda b_3$

A plane with normal $\underline{n} = a\,\underline{i} + b\underline{j} + c\underline{k}$ has

Cartesian equation: $ax + by + cz = d$ where
$d = ax_0 + by_0 + cz_0$ and (x_0, y_0, z_0) is any point on the plane.

Vector equation: $\underline{r} \cdot \underline{n} \; = \; d$

The perpendicular distance from point $\underline{s}$ to line $\underline{r} = \underline{a} + \lambda \underline{b}$ is $\dfrac{|(\underline{s} - \underline{a}) \times \underline{b}|}{|\underline{b}|}$

The perpendicular distance from point $\underline{s}$ to plane $\underline{r}.\underline{n} \; = \; \underline{a}.\underline{n}$ is $\dfrac{|(\underline{s} - \underline{a}).\underline{n}|}{|\underline{n}|}$

The shortest distance between two skew lines $\underline{r} = \underline{a_1} + \lambda_1 \underline{b_1}$ and $\underline{r} = \underline{a_2} + \lambda_2 \underline{b_2}$ is

$$\frac{|(\underline{a_2} - \underline{a_1}).(\underline{b_1} \times \underline{b_2})|}{|\underline{b_1} \times \underline{b_2}|}$$

Matrix Transformations: Coordinates (row vectors) are *post-multiplied* by transformation matrices.

2D **anticlockwise rotation** through θ about the origin: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

2D **reflection** in the line $y = \tan\phi\, x$: $\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$

2D **translation** by $u\underline{i} + v\underline{j}$ in homogeneous coordinates: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix}$

2D anticlockwise **rotation** through θ about (c,d) in homogeneous coordinates:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ c(1-\cos\theta)+d\sin\theta & d(1-\cos\theta)-c\sin\theta & 1 \end{pmatrix}$$

2D **reflection** in the line $y = x\tan\phi + c$: $\begin{pmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) & 0 \\ -c\sin(2\phi) & c+c\cos(2\phi) & 1 \end{pmatrix}$

3D **scaling** about the origin: $\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$

3D standard **rotations:** $R_{\theta_z,\underline{k}} = \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R_{\theta_y,\underline{j}} = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$,

$$R_{\theta_x,\underline{i}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

3D **translation** by $u\underline{i} + v\underline{j} + w\underline{k}$ in homogeneous coordinates: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u & v & w & 1 \end{pmatrix}$

Aligning vector $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$ with vector $\underline{k}$:

$$A_{\underline{v},\underline{k}} = \begin{pmatrix} \frac{ac}{\lambda|\underline{v}|} & \frac{-b}{\lambda} & \frac{a}{|\underline{v}|} & 0 \\ \frac{bc}{\lambda|\underline{v}|} & \frac{a}{\lambda} & \frac{b}{|\underline{v}|} & 0 \\ \frac{-\lambda}{|\underline{v}|} & 0 & \frac{c}{|\underline{v}|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \lambda = \sqrt{a^2+b^2} \ , \quad |\underline{v}| = \sqrt{a^2+b^2+c^2} \ .$$

3D **rotation** by θ about an axis $\hat{\underline{v}} = \alpha\underline{i} + \beta\underline{j} + \gamma\underline{k}$ (unit vector) passing through (0, 0, 0)

$$R_{\theta,\hat{\underline{v}}} = \begin{pmatrix} \alpha^2(1-\cos\theta)+\cos\theta & \alpha\beta(1-\cos\theta)+\gamma\sin\theta & \alpha\gamma(1-\cos\theta)-\beta\sin\theta & 0 \\ \alpha\beta(1-\cos\theta)-\gamma\sin\theta & \beta^2(1-\cos\theta)+\cos\theta & \beta\gamma(1-\cos\theta)+\alpha\sin\theta & 0 \\ \alpha\gamma(1-\cos\theta)+\beta\sin\theta & \beta\gamma(1-\cos\theta)-\alpha\sin\theta & \gamma^2(1-\cos\theta)+\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reflection in the plane $ax+by+cz=0$:

$$R=\begin{pmatrix} -a^2+b^2+c^2 & -2ab & -2ac & 0 \\ -2ab & a^2-b^2+c^2 & -2bc & 0 \\ -2ac & -2bc & a^2+b^2-c^2 & 0 \\ 0 & 0 & 0 & a^2+b^2+c^2 \end{pmatrix}$$

Orthogonal Projection – homogeneous coordinates

Onto the plane $ax+by+cz=d$, $(x \quad y \quad z \quad 1) \to (x \quad y \quad z \quad 1)T_{orth}$, where

$$T_{orth}=\begin{pmatrix} (b^2+c^2) & -ab & -ac & 0 \\ -ab & (a^2+c^2) & -bc & 0 \\ -ac & -bc & (a^2+b^2) & 0 \\ ad & bd & cd & (a^2+b^2+c^2) \end{pmatrix}$$

Perspective Projection

Viewpoint at (x_c, y_c, z_c) onto the viewing plane $ax+by+cz=d$

$$Per=\begin{pmatrix} d-(by_c+cz_c) & ay_c & az_c & a \\ bx_c & d-(ax_c+cz_c) & bz_c & b \\ cx_c & cy_c & d-(ax_c+by_c) & c \\ -dx_c & -dy_c & -dz_c & -(ax_c+by_c+cz_c) \end{pmatrix}$$

Quaternions

Multiplication in matrix form: if $\mathbf{p}=s_0+x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$, $\mathbf{q}=s+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ then

$$\mathbf{pq}=(s_0 \quad x_0 \quad y_0 \quad z_0)\begin{pmatrix} s & x & y & z \\ -x & s & -z & y \\ -y & z & s & -x \\ -z & -y & x & s \end{pmatrix}$$

Multiplication in vector form: if $\mathbf{p}=(s_1,\mathbf{v}_1)$, $\mathbf{q}=(s_2,\mathbf{v}_2)$ then

$$\mathbf{pq}=(s_1s_2-\mathbf{v}_1\cdot\mathbf{v}_2,\ s_1\mathbf{v}_2+s_2\mathbf{v}_1+\mathbf{v}_1\times\mathbf{v}_2)$$

Polar form of $\mathbf{q}=s+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

$\mathbf{q}=|\mathbf{q}|(\cos\theta+\sin\theta\,\mathbf{I})$, where

$|\mathbf{q}|=(s^2+x^2+y^2+z^2)^{1/2}$, $\cos\theta=\dfrac{s}{|\mathbf{q}|}$, $\mathbf{I}=\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{1/2}}$

Rotation Theorem: When a point with position vector $\vec{p}$ is rotated by angle θ about axis $\mathbf{I}$ (a unit vector through the origin), the result is the vector part of the quaternion $\mathbf{qpq}^{-1}$, where $\mathbf{q}=\cos\dfrac{\theta}{2}+\sin\dfrac{\theta}{2}\mathbf{I}$ and $\mathbf{p}=(0,\vec{p})$.

Bilinear Surface Patches: $\underline{q}(u,v)=(1-v)(1-u)\underline{a}+(1-v)u\ \underline{b}+v(1-u)\underline{c}+vu\ \underline{d}$

(AB and CD are opposite edges).

Reflection and Refraction

Reflected direction vector: $\underline{v}=\underline{u}-\dfrac{2(\underline{u}.\underline{n})}{|\underline{n}|^2}\underline{n}$ ($\underline{u}$ incident ray, $\underline{n}$ normal)

Refracted direction vector: $\underline{v}=\underline{u}-\dfrac{\sqrt{(\mu^2-1)|\underline{u}|^2|\underline{n}|^2+(\underline{u}.\underline{n})^2}-|\underline{u}.\underline{n}|}{|\underline{n}|^2}\underline{n}$

(μ refractive index) for $\underline{u}.\underline{n}<0$, otherwise replace $\underline{n}$ by $-\underline{n}$ in formula.

Apparent Depth: $l=\dfrac{d}{|\underline{u}.\underline{n}|}\sqrt{(\mu^2-1)|\underline{u}|^2|\underline{n}|^2+(\underline{u}.\underline{n})^2}$ (d actual depth)

Advanced Calculus

Polar Coordinates: Let (x,y) be the Cartesian coordinates of a point on a 2D plane. The polar coordinates (r,θ) are related to the Cartesian by

$$x=r\cos\theta,\ y=r\sin\theta,\ r^2=x^2+y^2,\ \tan\theta=x/y.$$

Spherical Coordinates: Let (x,y,z) be the Cartesian coordinates of a point in 3D space. The spherical coordinates (r,θ,ϕ) are related to the Cartesian by

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta.$$

Here, ϕ is the angle in the (x,y)-plane and θ is the angle made with the z - axis.

Partial Derivatives

Chain Rule: For $f=f(u,v),\ u=u(x,y),\ v=v(x,y)$:

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

Critical or Stationary Points: $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$

$f_{xx}f_{yy}-(f_{xy})^2>0$ and $f_{xx}<0 \Rightarrow$ MAXIMUM POINT

$f_{xx}f_{yy}-(f_{xy})^2>0$ and $f_{xx}>0 \Rightarrow$ MINIMUM POINT

$f_{xx}f_{yy}-(f_{xy})^2<0 \Rightarrow$ SADDLE POINT

Series Expansions: About $x = a$ and $y = b$,

$$f(x,y) = f(a,b) + \left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right) f(a,b) + \frac{1}{2!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right)^2 f(a,b)$$

$$+\frac{1}{3!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} \right)^3 f(a,b) + \ldots$$

where $\dfrac{\partial f(a,b)}{\partial x}$ means $\dfrac{\partial f}{\partial x}$ evaluated at $x = a$ and $y = b$, and so on.

Vector Calculus

$$\text{grad } \phi = \underline{\nabla}\phi = \frac{\partial \phi}{\partial x}\underline{i} + \frac{\partial \phi}{\partial y}\underline{j} + \frac{\partial \phi}{\partial z}\underline{k}, \qquad \text{div } \underline{F} = \underline{\nabla}.\underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{curl } \underline{F} = \underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Green's theorem: $\displaystyle\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) dxdy$

Divergence theorem (Gauss's theorem): $\displaystyle\iint_S \underline{F}.\underline{n}\; dS = \iiint_V \underline{\nabla}\cdot\underline{F}\; dV$

Stokes's theorem: $\displaystyle\iint_S (\underline{\nabla}\times\underline{F}).\,\underline{n}\,dS = \int_C \underline{F}.d\underline{r}$

Ordinary Differential Equations: Solution of 2nd order, constant coefficient, linear, non-homogeneous ODEs of the form $y'' + c_1 y' + c_2 y = f(x)$:

Roots of Aux Eqn	Complementary Fn
Real $m = \alpha, \beta$ with $\alpha \neq \beta$	$y = Ae^{\alpha x} + Be^{\beta x}$
Real $m = \alpha$ repeated root	$y = (Ax + B)e^{\alpha x}$
Complex $m = \alpha \pm \beta i$	$y = e^{\alpha x}(A\sin\beta x + B\cos\beta x)$

$f(x)$	For PI try
$ax^2 + bx + c$	$Ax^2 + Bx + C$
ae^{kx}	Ae^{kx}
$a\sin\omega x + b\cos\omega x$	$A\sin\omega x + B\cos\omega x$

Bézier Curves

Bernstein polynomials of degree *n*: $B_{r,n}(t) = \begin{pmatrix} n \\ r \end{pmatrix} t^r (1-t)^{n-r}$.

Bézier curve of degree *n* with control points $\underline{b}_0, \underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n$:

$$\underline{B}_n(t) = \sum_{r=0}^{n} B_{r,n}(t)\underline{b}_r$$

4×4 Bez matrix :

$$\text{Bez} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \qquad \text{Bez}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

De Casteljau Algorithm: $\underline{b}_i^{(j)} = (1-t)\underline{b}_i^{(j-1)} + t\ \underline{b}_{i+1}^{(j-1)}$; table: $\begin{matrix} \underline{b}_i^{(j-1)} & & \underline{b}_{i+1}^{(j-1)} \\ & \underline{b}_i^{(j)} & \end{matrix}$

Rational Bézier curve of degree *n*:

$$\underline{B}^*_n(t) = \frac{\sum_{i=0}^{n} w_i \underline{b}_i B_{i,n}(t)}{\sum_{i=0}^{n} w_i B_{i,n}(t)}$$

Control points at $\underline{b}_i$ with weights w_i $(i = 0,1,2,...,n)$ and if $w_i = 0$ then $w_i\underline{b}_i$ is replaced by $\underline{b}_i$.

The ***circle*** $x^2 + y^2 = a^2$ has a parametric form, $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2at}{1+t^2}$.

The ***ellipse*** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has a parametric form, $x = \dfrac{a(1-t^2)}{1+t^2}$, $y = \dfrac{2bt}{1+t^2}$.

The ***hyperbola*** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has a parametric form, $x = \dfrac{a(1+t^2)}{1-t^2}$, $y = \dfrac{2bt}{1-t^2}$.

Bézier surface degree $m+n$: $\underline{B}(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i,n}(u) B_{j,m}(v)\underline{b}_{i,j}$

Splines

Catmull-Rom Splines

$$\underline{P}(t) = \underline{c}_0 + \underline{c}_1 t + \underline{c}_2 t^2 + \underline{c}_3 t^3, \qquad \begin{pmatrix} \underline{c}_0 \\ \underline{c}_1 \\ \underline{c}_2 \\ \underline{c}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\tau & 0 & \tau & 0 \\ 2\tau & \tau-3 & 3-2\tau & -\tau \\ -\tau & 2-\tau & \tau-2 & \tau \end{pmatrix} \begin{pmatrix} \underline{p}_{i-2} \\ \underline{p}_{i-1} \\ \underline{p}_i \\ \underline{p}_{i+1} \end{pmatrix}$$

$\underline{p}_i$ is the i-th control point, τ is the tension parameter and $0 \le t \le 1$ between successive control points.

B-Splines

Given control points $\mathbf{b}_0, \mathbf{b}_1 \ldots \mathbf{b}_n$ and knot vector $(t_0, t_1 \ldots t_m)$, a B-Spline of degree k is given by

$$\mathbf{B}(t) = \sum_{i=0}^{n} N_{i,k}(t)\mathbf{b}_i$$

where $N_{i,k}(t)$ is defined recursively by the de Boor relations

$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t).$$

Tabular form:

$$\begin{array}{llll} N_{0,0}\ \{t_0, t_1\} & N_{0,1}\ \{t_0, t_2\} & N_{0,2}\ \{t_0, t_3\} & \ldots \\ N_{1,0}\ \{t_1, t_2\} & N_{1,1}\ \{t_1, t_3\} & N_{1,2}\ \{t_1, t_4\} & \ldots \\ N_{2,0}\ \{t_2, t_3\} & N_{2,1}\ \{t_2, t_4\} & N_{2,2}\ \{t_2, t_5\} & \ldots \\ \vdots & & & \end{array}$$

where $\{a, b\} \equiv \dfrac{t-a}{b-a}$.

Dynamics

Newton's Law of Restitution: velocity of separation = $-e \times$ velocity of approach

Energy: Rotational $KE = \frac{1}{2} I\omega^2$

Power: Power dissipated in moving a particle at a constant velocity v = v × Force acting on the particle in the direction of v.

Elastic strings: For an elastic string of natural length l, mass per unit length λ, extended by length x:

Tension in the string: $\dfrac{\lambda x}{l}$ Energy stored: $\dfrac{\lambda x^2}{2l}$

Centre of mass:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}, \quad \bar{x} = \frac{1}{A}\int_a^b xy\ dx, \quad \bar{y} = \frac{1}{2A}\int_a^b y^2\ dx$$

$$\bar{x} = \int_a^b xy^2 dx \Big/ \int_a^b y^2 dx \quad (\text{solid of revolution})$$

Circular motion: For a particle of mass m moving in a circle radius r at constant angular velocity ω:

Linear velocity $v = \omega r$; Centripetal force = $mr\omega^2$

Relative velocity: For two particles at positions $\underline{p}_1, \underline{p}_2$ moving at velocities $\underline{v}_1, \underline{v}_2$

Distance of closest approach: $d = \dfrac{\left|(\underline{p}_1 - \underline{p}_2) \times (\underline{v}_2 - \underline{v}_1)\right|}{\left|\underline{v}_2 - \underline{v}_1\right|}$

Time taken to get to that point: $t = \dfrac{\left|\left(\underline{p}_1 - \underline{p}_2\right).\left(\underline{v}_2 - \underline{v}_1\right)\right|}{\left|\underline{v}_2 - \underline{v}_1\right|^2}$

Projectiles: For a projectile motion with initial speed V at angle α from the horizontal:

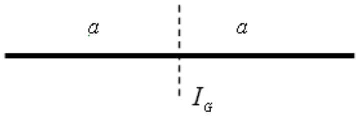
Greatest height attained: $\dfrac{V^2 \sin^2 \alpha}{2g}$ Horizontal range: $\dfrac{V^2 \sin 2\alpha}{g}$

SHM: For a particle performing simple harmonic motion with frequency ω, amplitude A:

Its position, $x(t)$, satisfies $\ddot{x}(t) = -\omega^2 x(t)$
Its velocity, $v(t)$, satisfies $v^2 = \omega^2 (A^2 - x^2)$.

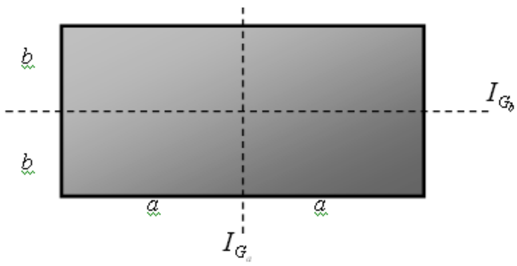
Moments of Inertia - Summary

Uniform **rod** about centre:



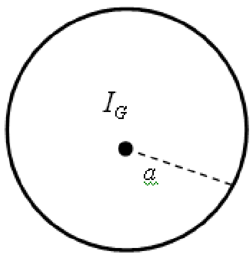
$$I_G = \tfrac{1}{3} ma^2$$

Uniform **rectangle** about axes through centre in its own plane, parallel to sides:



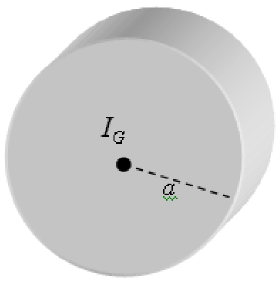
$$I_{G_a} = \tfrac{1}{3} ma^2$$
$$I_{G_b} = \tfrac{1}{3} mb^2$$

Uniform **hoop/thin circular cylinder** about axis through centre perpendicular to its plane:



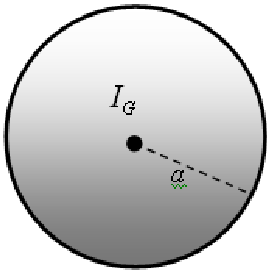
$$I_G = ma^2$$

Uniform **disc/solid circular cylinder** about axis through centre perpendicular to its plane:



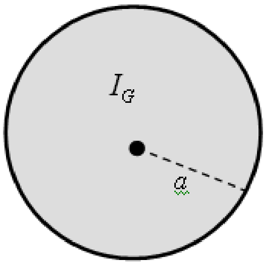
$$I_G = \tfrac{1}{2} ma^2$$

Uniform **solid sphere** about central axis:



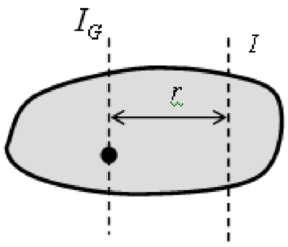
$$I_G = \tfrac{2}{5} ma^2$$

Uniform **hollow thin sphere (shell)** about central axis:



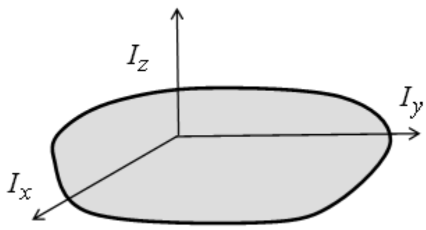
$$I_G = \tfrac{2}{3} ma^2$$

Parallel-axis theorem: moment of inertia about axis parallel to axis through G, centre of mass, distant r from it



$$I = I_G + mr^2$$

Perpendicular-axis theorem: moment of inertia about an axis perpendicular to two other perpendicular axes in plane of lamina



$$I_z = I_x + I_y$$

Rotational equation of motion: $I\ddot{\theta} = \sum$ moments of forces

Motion about a moving axis: Total KE = rotational KE + linear KE = $\tfrac{1}{2} I\omega^2 + \tfrac{1}{2} mv^2$

Impulsive Motion: Change in *angular momentum* = $\sum$ moments of impulsive forces

i.e. $$I\left(\dot{\theta} - \Omega\right) = \sum_i J_i r_i \, .$$

3-D Rigid Body Dynamics

Inertia tensor : $I = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}$ where

$$I_{xx} = \sum_i (y^2 + z^2) m_i, \quad I_{yy} = \sum_i (x^2 + z^2) m_i, \quad I_{zz} = \sum_i (x^2 + y^2) m_i$$

$$I_{xy} = \sum_i xym_i, \quad I_{xz} = \sum_i xzm_i, \quad I_{yz} = \sum_i yzm_i$$

and for continuous bodies,

$$I_{XX} = \iiint (y^2 + z^2)\rho \, dv \qquad I_{YY} = \iiint (x^2 + z^2)\rho \, dv \qquad I_{ZZ} = \iiint (x^2 + y^2)\rho \, dv$$

$$I_{XY} = \iiint xy\rho \, dv \qquad I_{XZ} = \iiint xz\rho \, dv \qquad I_{YZ} = \iiint yz\rho \, dv$$

Body rotating with angular velocity $\underline{\omega}$:

Kinetic Energy $T = ½\, \underline{\omega}^T I \underline{\omega}$, Angular Momentum $\underline{h} = I\underline{\omega}$.

The **Moment of inertia** of a body about a line with unit vector direction $\underline{D}$ is $\underline{D}^T I \underline{D}$, where I is the inertia tensor about a point on the line.

Parallel axis theorem:

$$I_A = I_G + M \begin{pmatrix} Y^2 + Z^2 & -XY & -XZ \\ -XY & X^2 + Z^2 & -YZ \\ -XZ & -YZ & X^2 + Y^2 \end{pmatrix}$$

I_G = inertia tensor of a body of mass M about the centre of gravity *G*,

I_A = inertia tensor of the body about a point *A* (with axes parallel to those at *G*) where the position vector of *A* with respect to *G* is $\underline{i}X + \underline{j}Y + \underline{k}Z$.

Euler's equations:

$$I_1 \frac{d\omega_x}{dt} + \omega_y \omega_z (I_3 - I_2) = N_x; \quad I_2 \frac{d\omega_y}{dt} + \omega_x \omega_z (I_1 - I_3) = N_y; \quad I_3 \frac{d\omega_z}{dt} + \omega_x \omega_y (I_2 - I_1) = N_z$$

where I_1, I_2, I_3 are the **principal moments of inertia**, $\underline{N}$ is the torque and $\underline{\omega}$ is the angular velocity with respect to the principal *body* axes.

General Motion: Acceleration of a point in the body, rotating with *constant* angular velocity $\underline{\omega}$ about an axis through the centre of mass, which is also accelerating at $\underline{A}$ is

$$\underline{a} = \underline{A} + \underline{\omega} \times (\underline{\omega} \times R\tilde{\underline{p}}_0)$$

where $\tilde{\underline{p}}_0$ is the initial position vector of the point *relative to the centre of mass* and *R* is the rotation matrix with $\theta = \omega t$.