

Fractional Calculus: History, Definitions and Applications for the Engineer

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Abstract

This report is aimed at the engineering and/or scientific professional who wishes to learn about Fractional Calculus and its possible applications in his/her field(s) of study. The intent is to first expose the reader to the concepts, applicable definitions, and execution of fractional calculus (including a discussion of notation, operators, and fractional order differential equations), and second to show how these may be used to solve several modern problems. Also included within is a list of applicable references that may provide the reader with a library of information for the further study and use of fractional calculus.

1 Introduction

The traditional integral and derivative are, to say the least, a staple for the technology professional, essential as a means of understanding and working with natural and artificial systems. Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Consider the physical meaning of the exponent. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself is easy to grasp and straight forward. However, this physical definition can clearly become confused when considering exponents of non integer value. While almost anyone can verify

that $x^3 = x \cdot x \cdot x$, how might one describe the physical meaning of $x^{3.4}$, or moreover the transcendental exponent x^π . One cannot conceive what it might be like to multiply a number or quantity by itself 3.4 times, or π times, and yet these expressions have a definite value for any value x , verifiable by infinite series expansion, or more practically, by calculator.

Now, in the same way consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning. Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g. function continuity), completing n integrations can become as methodical as multiplication. But the curious mind can not be restrained from asking the question what if n were not restricted to an integer value? Again, at first glance, the physical meaning can become convoluted (pun intended), but as this report will show, fractional calculus flows quite naturally from our traditional definitions. And just as fractional exponents such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order $1/2$ and beyond can find practical use in many modern problems.

2 History and Mathematical Background

2.1 Brief History

Most authors on this topic will cite a particular date as the birthday of so called 'Fractional Calculus'. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the n th-derivative of the linear function $f(x) = x$, $\frac{D^n x}{Dx^n}$. L'Hopital's posed the question to Leibniz, what would the result be if $n = 1/2$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born.

Following L'Hopital's and Leibniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [2]. Many found, using their own notation and methodology, definitions that fit the concept of

a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus (not yet the world as a whole) are the Riemann-Liouville and Grunwald-Letnikov definition. While the sheer number of actual definitions are no doubt as numerous as the men and women that study this field, they are for the most part variations on the themes of these two and so are addressed in detail in this document.

Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found. The mathematics has in some cases had to change to meet the requirements of physical reality. Caputo reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [1]. As recently as 1996, Kolowankar reformulated again, the Riemann-Liouville fractional derivative in order to differentiate no-where differentiable fractal functions [9].

Leibniz's response, based on studies over the intervening 300 years, has proven at least half right. It is clear that within the 20th century especially numerous applications and physical manifestations of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical. While the physical meaning is difficult (arguably impossible) to grasp, the definitions themselves are no more rigorous than those of their integer order counterparts.

Understanding of definitions and use of fractional calculus will be made more clear by quickly discussing some necessary but relatively simple mathematical definitions that will arise in the study of these concepts. These are The Gamma Function, The Beta Function, The Laplace Transform, and the Mittag-Leffler Function and are addressed in the following four subsections.

2.2 The Gamma Function

As will be clear later, the gamma function is intrinsically tied to fractional calculus by definition (see *The Fractional Integral*). The simplest interpretation of the gamma function

is simply the generalization of the factorial for all real numbers. The definition of the gamma function is given by (1).

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} du, \quad \text{for all } z \in \mathbb{R} \quad (1)$$

The 'beauty' of the gamma function can be found in its properties. First, as seen in (2), this function is unique in that the value for any quantity is, by consequence of the form of the integral, equivalent to that quantity z minus one times the gamma of the quantity minus one.

$$\Gamma(z+1) = z\Gamma(z), \quad \text{also, when } z \in \mathbb{N}_+, \quad \Gamma(z) = (z-1)! \quad (2)$$

This can be shown through a simple integration by parts. The consequence of this relation for integer values of z is the definition for factorial. Figure 1 demonstrates the gamma function at and around zero. Note that at negative integer values, the gamma function goes to infinity, by yet is defined at non-integer values.

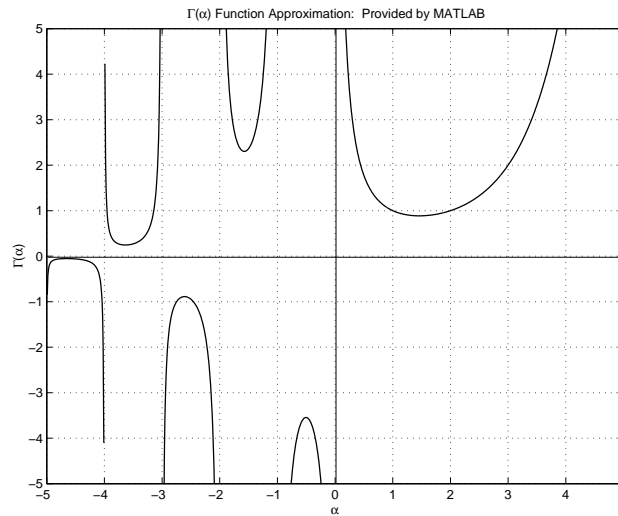


Figure 1: MATLAB Approximation of Gamma Function

Using the gamma function we can also define the function $\Phi(t)$, which later will become useful for showing alternate forms of the fractional integral. $\Phi(t)$ is given by (17).

$$\Phi_\alpha(t) := \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \quad (3)$$

2.3 Beta Function

Also known as the Euler Integral of the First Kind, the Beta Function is in important relationship in fractional calculus. Its solution not is only defined through the use of multiple Gamma Functions, but furthermore shares a form that is characteristically similar to the Fractional Integral/Derivative of many functions, particularly polynomials of the form t^a and the Mittag-Leffler Function (see below). Equation (4) demonstrates the Beta Integral and its solution in terms of the Gamma function.

$$B(p, q) := \int_0^1 (1-u)^{p-1} u^{q-1} du = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(q, p), \quad \text{where } p, q \in \mathbb{R}_+ \quad (4)$$

2.4 Laplace Transformation and Convolution

The Laplace Transform is a function transformation commonly used in the solution of complicated differential equations. With the Laplace transform it is frequently possible to avoid working with equations of different differential order directly by translating the problem into a domain where the solution presents itself algebraically. The formal definition of the laplace transform is given in (5).

$$\mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t) dt = \tilde{f}(s) \quad (5)$$

The Laplace Transform of the function $f(t)$ is said to exist if (5) is a convergent integral. The requirement for this is that $f(t)$ does not grow at a rate higher than the rate at which the exponential term e^{-st} decreases.

Also commonly used is the laplace convolution, given by (6).

$$f(t) * g(t) := \int_0^t f(t-\tau)g(\tau)d\tau = g(t) * f(t) \quad (6)$$

The convolution of two function in the domain of t is sometime complicated to resolve, however, in the laplace domain (s), the convolution results in the simple function multiplication as shown in (7).

$$\mathcal{L}\{f(t) * g(t)\} = \tilde{f}(s)\tilde{g}(s) \quad (7)$$

One final important property of the Laplace transform that should be addressed is the Laplace transform of a derivative of integer order n of the function $f(t)$, given by (8)

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n \tilde{f}(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \quad (8)$$

2.5 The Mittag-Leffler Function

The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. In fact, the exponential function itself is a very specific form, one of an infinite set, of this seemingly ubiquitous function.

The standard definition of the Mittag-Leffler is given in (9)

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0 \quad (9)$$

The exponential function corresponds to $\alpha = 1$. Figure 2.5 demonstrates The Mittag-Leffler function for different values of α .

It is also common to represent the Mittag-Leffler function in two arguments, α and β . Such that

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0 \quad (10)$$

This is the more generalized form of the function, however not always a necessity when used with fractional differential equations.

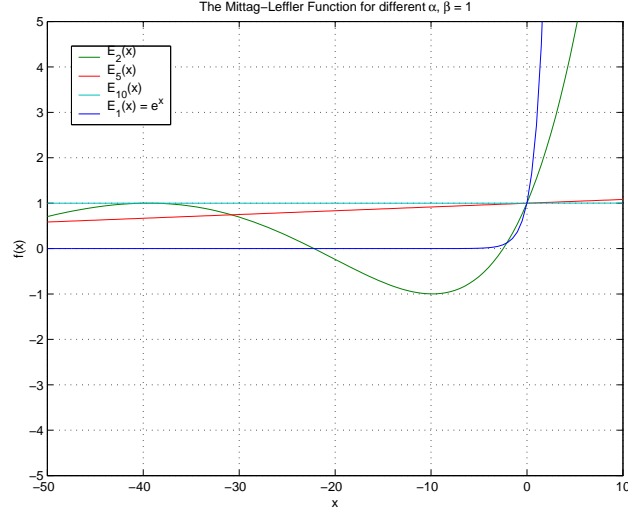


Figure 2: MATLAB Approximation of Mittag-Leffler Function: See Appendix for supporting MATLAB script

3 Definitions and Derivations

3.1 The Fractional Integral

It was touched upon in the introduction that the formulation of the concept for fractional integrals and derivatives was a natural outgrowth of integer order integrals and derivatives in much the same way that the fractional exponent follows from the more traditional integer order exponent. For the latter, it is the notation that makes the jump seem obvious. While one can not imagine the multiplication of a quantity a fractional number of times, there seems no practical restriction to placing a non-integer into the exponential position.

Similarly, the common formulation for the fractional integral can be derived directly from a traditional expression of the repeated integration of a function. This approach is commonly referred to as the Riemann-Liouville approach. (11) demonstrates the formula usually attributed to Cauchy for evaluating the n^{th} integration of the function $f(t)$.

$$\int \dots \int_0^t f(\tau) d\tau = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \quad (11)$$

For the abbreviated representation of this formula, we introduce the operator J^n such as

shown in (12).

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau \quad (12)$$

Often, one will also find another operator, D^{-n} , used in place of J^n . While they represent the same formulation of the repeated integral function, and can be seen as interchangeable, one will find that the use of D^{-n} may become misleading, especially when multiple operators are used in combination (See *The Fractional Derivative: Properties*).

For direct use in (11), n is restricted to be an integer. The primary restriction is the use of the factorial which in essence has no meaning for non-integer values. The gamma function (See *History and Mathematical Background: The Gamma Function*) is however an analytic expansion of the factorial for all reals, and thus can be used in place of the factorial as in (2). Hence, by replacing the factorial expression for its gamma function equivalent, we can generalize (12) for all $\alpha \in \mathbb{R}_+$, as shown in (13).

$$J^\alpha f(t) := f_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (13)$$

3.1.1 Properties

This formulation of the fractional integral carries with it some very important properties, that will later show importance when solving equations involving integrals and derivatives of fractional order. First, we consider integrations of order $\alpha = 0$ to be an identity operator, i.e.

$$J^0 f(t) = f(t). \quad (14)$$

Also, given the nature of the integral's definition, and based on the principle from which it came (Cauchy Repeated Integral Equation), we can see that just as

$$J^n J^m = J^{m+n} = J^m J^n, \quad m, n \in \mathbb{N} \quad (15)$$

so to,

$$J^\alpha J^\beta = J^{\alpha+\beta} = J^\beta J^\alpha, \quad \alpha, \beta \in \mathbb{R}. \quad (16)$$

The one presupposed condition placed upon a function $f(t)$ that needs to be satisfied for these and other similar properties to remain true, is that $f(t)$ be a *causal* function, i.e. that it is vanishing for $t \leq 0$. Although this is a consequence of convention, the convenience of this condition is especially clear in the context of the property demonstrated in (16). The effect is such that $f(0) = f_n(0) = f_\alpha(0) \equiv 0$.

An additional property of the Reimann-Liouville integral appears after the introduction of the function Φ_α in (17)

$$\Phi_\alpha = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \Rightarrow \Phi_\alpha(t) * f(t) = \int_0^t \frac{(t-\tau)_+^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau \quad (17)$$

t_+ denotes the function vanishes for $t \leq 0$ and hence, (17) is a causal function. From the definition of the Laplace Convolution given in (6), it follows that

$$J^\alpha f(t) = \Phi_\alpha(t) * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (18)$$

Here we will also find the Laplace Transform of the Reimann-Liouville fractional integral. In (17) we showed that the fractional integral could be expressed as the convolution of two terms, Φ_α and $f(t)$. The Laplace transform of $t^{\alpha-1}$ is given by

$$\mathcal{L}\{t^{\alpha-1}\} = \Gamma(\alpha)s^{-\alpha}. \quad (19)$$

Thus, given the convolution relation to the fractional integral through Φ_α demonstrated in (17), and the Laplace transform of the convolution shown in (6), the Laplace transform of the fractional integral is found to be

$$\mathcal{L}\{J^\alpha\} = s^{-\alpha}\tilde{f}(s). \quad (20)$$

3.2 The Fractional Derivative

3.2.1 Left Hand Definition vs Right Hand Definition

Because the Reimann-Louisville approach to the fractional derivative began with an expression for the repeated integration of a function, one's first instinct may be to imitate a similar approach for the derivative. For A.K. Grunwald, and later, Letnikov, this was the preferred method (see *The Fractional Derivative: Grunwald-Letnikov 'differintegral'*). However, it is also possible to formulate a definition for the fractional order derivative using the definition already obtained for the analogous integral.

Consider a differentiation of order $\alpha = 1/2, \alpha \in \mathbb{R}_+$. Now, we select an integer m such that $m - 1 < \alpha < m$. Given these numbers, we now have two possible ways to define the derivative. The first method, which we will call the Left Hand Definition (for reasons that will be clear later), is demonstrated graphically in Figure 3

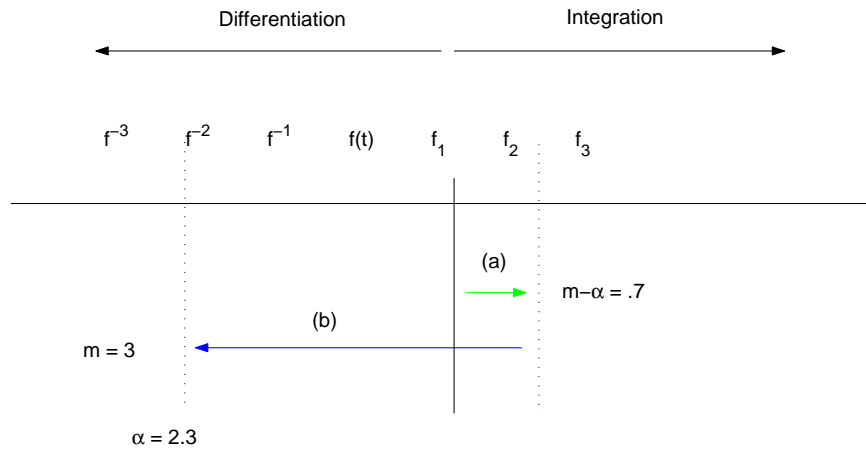


Figure 3: Graphical Representation of Left-Hand Definition Method

The explanation for this procedure is simple. Having found the integer m , the first step of the process is to complete operation (a), i.e. integrate our function $f(t)$ by order $m - \alpha = .7$ where for the figure, $\alpha = 2.3$. Second, we differentiate the resulting function, $f_{.7}(t)$ by order $m = 3$ (operation (b)), thereby achieving a resultant differentiation of order α . This method is shown in its mathematical form in (21).

$$D^\alpha f(t) := \left\{ \begin{array}{ll} \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \right], & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{array} \right\} \quad (21)$$

The second method, referred to here as the Right-Hand Definition, is likewise shown graphically in Figure 4.

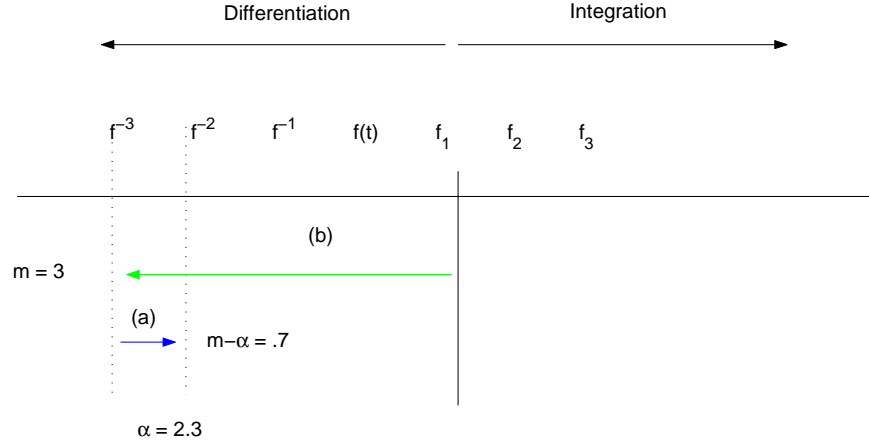


Figure 4: Graphical Representation of Right-Hand Definition Method

The Right-Hand Definition attempts to arrive at the same result using the same operations (a), (b), but in the reverse order. The mathematical result of this is shown in (22).

$$D_*^\alpha f(t) := \left\{ \begin{array}{ll} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{array} \right\} \quad (22)$$

This second definition, although referred to here as the 'Right Hand' Definition, was originally formulated by Caputo, and is therefore, commonly referred to as the Caputo Fractional Derivative.

3.2.2 Properties

It is easy to see that in some situations, the Right Hand definition for the fractional derivative is in a sense, more restrictive than the Left Hand definition. We see that this is the case when we consider the restrictions to the Riemann-Liouville fractional integral.

It was mentioned above that it is required that $f(t)$ be a causal function, i.e. vanishing at $t \leq 0$. For the Left Hand definition, as long as the initial function of t satisfies this condition, as a necessity so must all other integers of order $\alpha > 0$, and so no problems of this sort can potentially arrive. However, for the Right Hand definition, because one differentiates the function $f(t)$ by order m , it must be that not only must $f(0) = 0$, but also $f^{(1)} = f^{(2)} \dots f^{(m)} = 0$.

In the mathematical world, this vulnerability of the Right Hand definition may be debilitating, and one may ask why the Right Hand definition is necessary at all. The answer to this question comes when solving non-integer order differential equations. In the mathematical sense, it is possible to use the Left Hand definition in the solution to this problem given the proper initial conditions. As it happens, however, the initial conditions required for the solution of fractional order Diff.EQ's are themselves of a non-integer order. Also, the fractional derivative of a constant using the Left Hand definition is not a 0, and in fact is equal to

$$D^\alpha C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}. \quad (23)$$

In the physical world, these properties of the LHD present a substantial problem. While today we are well familiar with the interpretation of the physical world in integer order equations, we do not (currently) have a practical understanding of the world in a fractional order. Our mathematical tools go beyond the practical limitations of our understanding.

Yet, in the Right Hand (Caputo) definition we find a link between what is possible and what is practical. The consequence of the slight reorganization of the RHD is to allow integer order initial conditions to be used in the solution of fractional Diff.EQ's. In addition, the RHD fractional derivative of a constant is 0.

Demonstrating the practicality of the RHD over the LHD is conveniently simple. In the *Properties* subsection of *The Fractional Integral* we demonstrated the Laplace transform of the fractional integral (20). Using this definition, we may find similarly the Laplace transform of the LHD fractional derivative. The fractional derivative of this form may be written in the form

$$D^\alpha f(t) = g^{(m)}(t), \quad \text{where} \quad g(t) = J^{m-\alpha} f(t) \quad m-1 \leq \alpha < m. \quad (24)$$

Using the property shown in (8) and the definition of the fractional integral Laplace transform, we find that

$$\mathcal{L}\{D^\alpha f(t)\} = s^m \tilde{g}(s) - \sum_{k=0}^{m-1} s^k g^{(m-k-1)}(0) = s^\alpha \tilde{f}(s) - \sum_{k=0}^{m-1} s^k D^{(\alpha-k-1)} f(0). \quad (25)$$

Notice that the required initial conditions are for all k to $n-1$ terms, fractional order derivatives of $f(t)$.

For the ~~LHD~~, we start by writing the derivative in the form

$$\text{RHD} \quad D_*^\alpha f(t) = J^{m-\alpha} g(t), \quad \text{where} \quad g(t) = f^{(m)}(t), \quad m-1 \leq \alpha < m. \quad (26)$$

The Laplace transform is then, by (20)

$$\mathcal{L}\{D_*^\alpha f(t)\} = s^{-(m-\alpha)} \tilde{g}(s) = s^\alpha \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0). \quad (27)$$

In this formulation, the order α does not appear in the derivatives of $f(t)$, but rather in the preceding multiplier $s^{\alpha-k-1}$, opposite the location to that for the LHD. So, quite conveniently, integer order derivatives $f(t)$ (i.e. $f(t), f^{(1)}, f^{(2)}$, and so on) are used as the initial conditions, and therefore easily interpreted from physical data and observations.

3.3 Other Approaches

3.3.1 Grunwald-Letnikov 'differintegral'

Unlike the Reimann-Liouville approach, which derives its definition from the repeated integral, the Grunwald-Letnikov formulation approaches the problem from the derivative side. For this, we start from the fundamental definition a derivative, shown in (28)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (28)$$

Applying this formula again, we can find the second derivative.

$$f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h_1 \rightarrow 0} \frac{\lim_{h_2 \rightarrow 0} \frac{f(x+h_1+h_2) - f(x+h_1)}{h_2} - \lim_{h_2 \rightarrow 0} \frac{f(x+h_2) - f(x)}{h_2}}{h_1} \quad (29)$$

By choosing the same value of h , i.e. $h = h_1 = h_2$, the expression simplifies to

$$f''(x) = \lim_{h \rightarrow 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} \quad (30)$$

For the n^{th} derivative, this procedure can be consolidated into the summation in (31). We introduce the operator d^n to represent the n -repetitions of the derivative.

$$d^n f(x) = \lim_{h \rightarrow 0} \frac{1}{h^n} \sum_{m=0}^n (-1)^m \binom{n}{m} f(x - mh) \quad (31)$$

This expression can be generalized for non-integer values for n with $\alpha \in \mathbb{R}$ provided that the binomial coefficient be understood as using the Gamma Function in place of the standard factorial. Also, the upper limit of the summation (no longer the integer, n) goes to infinity as $\frac{t-a}{h}$ (where t and a are the upper and lower limits of differentiation, respectively). We are left with the generalized form of the Grunwald-Letnikov fractional derivative.

$$d^\alpha f(x) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{m=0}^{\frac{t-a}{h}} (-1)^m \frac{\Gamma(\alpha+1)}{m! \Gamma(\alpha-m+1)} f(x - mh) \quad (32)$$

It is conceivable that just as the Riemann-Liouville definition for the fractional integral could be used to define the fractional derivative, that the above form of the GL derivative could be altered for use in an alternate definition of the fractional integral. The most natural alteration of this form is to consider the GL derivative for negative α . If we revert to the (31) form the most immediate problem is that $\binom{-n}{m}$ is not defined using factorials. Expanded mathematically, $\binom{-n}{m}$ is given by (33).

$$\binom{-n}{m} = \frac{-n(-n-1)(-n-2)(-n-3)\dots(-n-m+1)}{m!} \quad (33)$$

This form can be rewritten as

$$\binom{-n}{m} = (-1)^m \frac{n(n+1)(n+2)(n+3)\dots(n+m-1)}{m!} = (-1)^m \frac{(n+m-1)!}{(n-1)!m!}. \quad (34)$$

The factorial expression in (34) can be generalized for negative reals using the gamma function, thus

$$\binom{-\alpha}{m} = (-1)^m \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)m!} \quad (35)$$

Using (35) we can now rewrite (32) for $-\alpha$ and thus are left with the Grunwald-Letnikov fractional integral.

$$d^{-\alpha} f(x) = \lim_{h \rightarrow 0} h^\alpha \sum_{m=0}^{\frac{t-a}{h}} \frac{\Gamma(\alpha+m)}{m! \Gamma(\alpha)} f(x-mh) \quad (36)$$

At this point, two formulations of fractional calculus have been presented, that proposed by Riemann and Liouville, and by Grunwald and Letnikov, respectively. The existence of different formulations of the same concept begs the question, are these also equivalent. The short answer to this question is, yes. The definitive mathematical proof to achieve this affirmation is lengthy however, and so is demonstrated in [1]. A fairly good understanding of the underlying mathematics is required for the understanding of the proof involved, but is not required to demonstrate, in a limited way, the agreement of these definitions. By virtue of its form, the Riemann-Liouville definition for the the fractional integral and derivative lends itself to finding the analytical solution for relatively simple functions ($x^a, e^x, \sin(x)$ etc.). Conversely, the Grunwald-Letnikov definition is very easily utilized for numerical evaluation. Thus, the easiest and quickest way to demonstrate the equivalence of these two definitions is comparison of their respective analytical and numerical results. This is demonstrated later in the *Numerical Methods* section of *Applications*.

3.4 Fractional Integral Equations

3.4.1 First Kind

The first form of the Fractional Integral Equation is given by the form in (37)

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau = f(t), \quad 0 < \alpha < 1 \quad (37)$$

This may also be written as

$$J^\alpha u(t) = f(t). \quad (38)$$

The solution of this kind is straightforward, and written

$$u(t) = D^\alpha f(t). \quad (39)$$

One may be tempted to use the Right Hand or Caputo definition for the fractional derivative in this situation interchangeably with the LHD, however it must be underscored that not in every situation does $D_*^\alpha J^\alpha f(t) = f(t)$. In fact, it will be shown below that from a solution arrived at through use of the Laplace transform, a remainder term arises when the RHD is used to solve (37).

In the Laplacian domain, integral equations of the first kind assume the form below.

$$J^\alpha u(t) = \Phi_\alpha(t) * u(t) \implies \mathcal{L}\{\Phi_\alpha(t) * u(t)\} = \frac{\tilde{u}(s)}{s^\alpha} \quad (40)$$

Algebraically, we can reorder the result in (40) into one of two forms.

$$\tilde{u}(s) = s^\alpha \tilde{f}(s) \implies s \left[\frac{\tilde{f}(s)}{s^{1-\alpha}} \right] \quad (41)$$

or

$$\tilde{u}(s) = s^\alpha \tilde{f}(s) \implies \frac{1}{s^{1-\alpha}} [s \tilde{f}(s) - f(0)] + \frac{f(0)}{s^{1-\alpha}} \quad (42)$$

Inverting the first form back into the time domain, we get

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau = f(t), \quad (43)$$

which is equivalent to solution of the equation with the LHD. The second form can be similarly inverted to yield

$$u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau = f(t) + f(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \quad (44)$$

The first element of this result is the RHD, but as mentioned above, one must include a remainder term that is dependent on the value of the function at 0.

3.4.2 Second Kind

Integral equations of the second kind follow the form in (45).

$$u(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d\tau = f(t) \implies (1 + \lambda J^\alpha)u(t) = f(t) \quad (45)$$

The solution to (45) is found to be

$$u(t) = (1 - \lambda J^\alpha)^{-1} f(t) = (1 - \sum_{n=1}^{\infty} (-\lambda)^n J^{\alpha n}) f(t) = f(t) + \left(\sum_{n=1}^{\infty} (-\lambda)^n \Phi_{\alpha n} \right) * f(t). \quad (46)$$

From (9), we can show that

$$E_\alpha(-\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}. \quad (47)$$

By taking the first derivative of (47) we eliminate the first term in the $E_\alpha(-\lambda t^\alpha)$ expansion and recover the form of found in (46). Thus, the solution to the integral equation of the second kind can be formally written as

$$u(t) = f(t) + \frac{d}{dt} \left[E_\alpha(-\lambda t^\alpha) \right] * f(t). \quad (48)$$

The same solution can be reached by using Laplace Transforms. Start by taking the laplace transform of (45).

$$\mathcal{L}\{(1 + \lambda)^\alpha u(t)\} = \mathcal{L}\{f(t)\} \rightarrow \left[1 + \frac{\lambda}{s^\alpha} \right] \tilde{u}(s) = \tilde{f}(s) \quad (49)$$

Equation (49) can be rearranged in many ways, but one in particular leads us back to the result presented in (48).

$$\tilde{u}(s) = \left[s \frac{s^{\alpha-1}}{s^\alpha + \lambda} - 1 \right] \tilde{f}(s) + \tilde{f}(s) \quad (50)$$

Equation (50) is next inverted back into the normal function domain. In order to do this one must address comprehend the Laplace transform of a special form of the Mittag-Leffler function, given in (51).

$$\mathcal{L}\{E_\alpha(-\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha + \lambda} \quad (51)$$

By the relationship given in (8), it is clear that what is between the brackets in the LHS of (50) is simply the Laplace transform of the first derivative of the LHS of (51), i.e.

$$\mathcal{L}\{E_\alpha^{(1)}(-\lambda t^\alpha)\} = s \frac{s^{\alpha-1}}{s^\alpha + \lambda} - 1. \quad (52)$$

From the definition of the Laplace convolution given in (6), is easily seen how the inverse of (50) would yield the same result shown in (48).

3.5 Fractional Differential Equations

In classic linear ODEs, there are typically two forms one may consider, what [5] refers to as the relaxation and oscillation forms, shown in (53) and (54).

$$u'(t) = -u(t) + q(t) \quad (53)$$

$$u''(t) = -u(t) + q(t) \quad (54)$$

When we speak of these forms, we refer to them as being of order one and two, respectively. This distinction may also be made for fractional calculus, refereed to as the *fractional relaxation* and *fractional oscillation* forms of the linear ODE. In the first case, the order of the ODE is $0 < \alpha \leq 1$ and as one might expect the order of the second is $1 < \alpha \leq 2$.

But unlike integer order ODEs, the behavior of first and second order solutions are not so black a white, they rather blend together. Thus there is no specific need to make a

distinction between these two cases. Linear Fractional ODEs can be represented concisely in one form as follows.

$$D_*^\alpha u(t) = D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0) \right) = -u(t) + q(t), \quad m-1 < \alpha \leq m \quad (55)$$

Note the use of the RHD in this definition. As was discussed above in the properties of the RHD and LHD, the choice to use this definition is based upon the ability to use integer order initial conditions in the solution of problems of this kind.

The most straight forward means of solving (55) is by means of Laplace transform. Using equation (27) we can show that (55) can be rearranged to give

$$s^\alpha \tilde{u}(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} u^{(k)}(0) = -\tilde{u}(s) + \tilde{q}(s) \implies \tilde{u}(s) = \sum_{k=1}^{m-1} \frac{s^{\alpha-k-1}}{s^\alpha + 1} u^{(k)}(0) + \frac{1}{s^\alpha + 1}. \quad (56)$$

The terms inside the sum can be rewritten

$$\frac{s^{\alpha-k-1}}{s^\alpha + 1} = \frac{1}{s^k} \frac{s^{\alpha-1}}{s^\alpha + 1} = \mathcal{L}\{J^k E_\alpha(-t^\alpha)\} \quad (57)$$

as can the terms preceding $\tilde{q}(s)$

$$\frac{1}{s^\alpha + 1} = -\left(s \frac{s^{\alpha-1}}{s^\alpha + 1} - 1 \right) = \mathcal{L}\left\{ \frac{d}{dt} \left[E_\alpha(-t^\alpha) \right] \right\}. \quad (58)$$

Finally, using both (57) and (58) to define the inverse laplace transform, it is possible to transform (56) into an expression for $u(t)$, and thus define the solution to the fractional order ODE.

$$u(t) = \sum_{k=0}^{m-1} J^k E_\alpha(-t^\alpha) u^{(k)}(0) - q(t) * E'_\alpha(-t^\alpha) \quad (59)$$

4 Applications

4.1 Numerical Methods

Due to the summation form of the Grunwald-Letnikov definition of the fractional derivative and integral, this formula lends itself to adaptation for use in a computer numerical solver. In

the Appendix of this document there is attached a MATLAB script called *GLdifferintegral.m* designed specifically for this purpose. Given a function, defined in the required *DFunc.m* this script will compute the integer or non-integer order integral or derivative of this function over a user defined domain.

One question that arises in the programming of such a function is, because the Grunwald-Letnikov definition specifies what is essentially an infinite sum, what number of these terms must be computed and summed for an accurate result to be achieved. Because of the speed of modern computing equipment and the relative simplicity of the step by step calculation (except perhaps the calculation of the factorial), one might suppose that anywhere from 10000 to 1000000 steps would provide excellent accuracy without a significant penalty in computation time. In theory, this would be a correct assumption, as even the calculation of Gamma is done approximately in most mathematical software packages including MATLAB.

However the limiting factor to the accuracy of this numerical solver is not the speed of the computer. Rather it is the capacity of the computer to accurately store and move the numbers that are being supplied it. Consider the value of $\Gamma(z)$ for large values of z . At $z = 171$, the approximate value provided by the MATLAB *gamma.m* function is $7.26e306$. At $z = 172$, MATLAB begins to refer to the value $\Gamma(z)$ as 'Inf'. Hence, for *GLdifferintegral.m*, there is no practical use of calculating terms for the GL definition beyond the 171st.

One potential way around this software imposed upper limit is to use gamma function approximations to simplify the multiplication and division of gamma terms necessary for the GL definition. One will find, in fact, that doing so will significantly enhance the ability of MATLAB and similar software to compute higher order terms. But what benefit does this approximation afford the user. When all terms up to the 171st are computed, the error between the analytical derivative of $f(x) = x$, $D^\alpha f(x) = 2\sqrt{\frac{x}{\pi}}$ and the numerical result of *GLdifferintegral.m* is on average less than $1e - 3\%$. Of course the calculation of more terms will lessen this error the limited summing capabilities of the MATLAB Script presented here already maintains a high level of accuracy.

The special benefits of having software that can numerically calculate the fractional derivative and integral is it provides the ability to actually examine the form of fractional order

calculus. Figure 5 demonstrates the non-integer derivatives for $\alpha = 0 \rightarrow 1$ of $f(x) = x$. Figure 6 shows a similar distribution for $f(x) = e^x$.

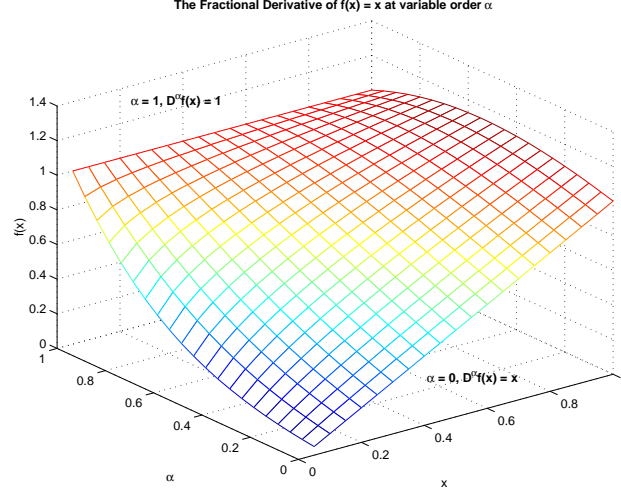


Figure 5: Non-Integer Derivatives of $f(x) = x$

4.2 Signal Processing: Application in a GA Planner [11]

The Genetic Algorithm is another 20th and 21st century tool that uses a very classic concept (that of darwinian/generational evolution) to solve problems that would otherwise be impossible or extremely difficult. GAs work best at solving problems in which a finite number of independent variables have $> 10^{10}$ possible combinations. At such large numbers, it starts to become impractical to direct a computer to try every combination and then evaluate which combination yielded the best result. The time requirements start to become extremely demanding.

A GA, instead of one by one trying every possible combination, works by randomly choosing a small 'population' of possible combinations. The order and composition of each of those combination is now referred to as a 'gene', analogous to a biological gene because they both store coded information (in a computer with 0s and 1s rather than A,T,G, and C) that carries out a function. The GA evaluates the fitness of each of these genes in relation to each other, determining which provide the best answers and which provide the least best.

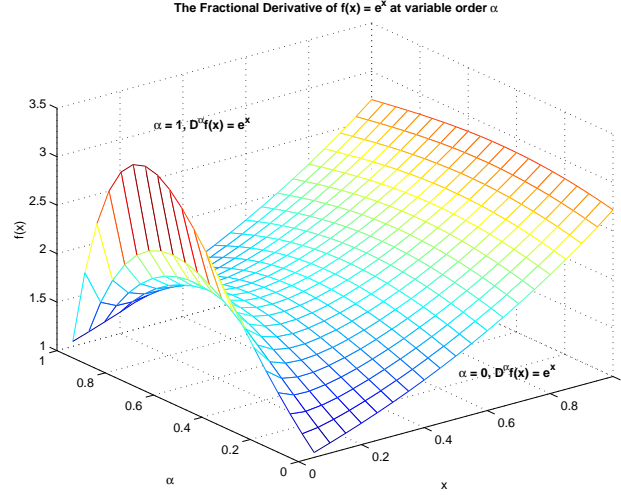


Figure 6: Non-Integer Derivatives of $f(x) = x$

This fitness testing then determines which genes are chosen for 'reproduction'.

In the reproduction phase two things typically occur: crossover reproduction and mutation. In the crossover phase genes exchange parts of their information with another gene. In the mutation phase, a certain probability is assigned to each 1 and 0 within a gene that it will be flipped, just as in biological genes, there is typically a finite probability of a A-T, or G-C pair from being changed.

After the reproduction phase, and new generation is created, built mostly on the genetic pieces that made the previous generation most strong. The process repeats itself until the population becomes homogeneous and, hopefully, has settled on a universal best solution. The idea behind this process (crudely explained here) is that the concept of natural selection and survival of the fitness is a good process not just in the natural world. The random character of the first generation allows the GA to test a variety of different paths, and the mutation keeps these combinations slightly jumbled. But eventually with the best genes continually exchanging code and bettering eachother, they will all become the most fit for their environment, just as in nature.

The subject of [11] is addressing the effect mutation rates in a GA planner have on the fitness of the resulting populations. Just as in examples of fractal surfaces and diffusion,

the apparent link between random occurrences and physical expressions of these occurrences by fractional order expressions are found by the authors of the above document in the GA environment.

Typically in a GA the probability of mutation is a fixed probability. In [11] white noise signal of mutation probability was added into the system over a generational time scale. That is, from generation to generation the probability of mutation was randomly altered. This may be considered the input signal. The output was the effect this had on the fitness of the middle 50th percentile of the population. Their finding was that between input and output, the fractional order transfer function given in (60) provided a good prediction for the reaction of the GA.

$$G_n(s) = k \frac{(s/a)^\alpha + 1}{(s/b)^\beta + 1} \quad (60)$$

In their conclusion they state that "it was shown that fractional-order models capture phenomena and properties that classical integer-order simply neglect. Moreover, for the case under study the signal evolution of similarities to those revealed by chaotic systems which confirms the requirement for mathematical tools well adapted to the phenomena under investigation"

4.3 Tensile and Flexural Strength of Disorder Materials [13]

One particular focus of study in the mechanics of solids is the apparent 'size effect' that occurs in some building materials, particularly those that are aggregates, such as concrete. Although classical solid mechanics dictates that the strength of a material is largely (if not completely) determined by the material properties of that material, and so that scaling should not present a change in relative strength. The size effect however has been a demonstration of how aggregate materials, namely concrete do indeed have strength dependent on the scale of the structure and thus do not follow the rules of classical (and dare i say integer-order) solid mechanics.

In [13], the authors discuss how the microstructure of such aggregate materials have been in the past successfully modeled by fractal sets rather than traditional geometric sets. The

dimension of the fractal sets used to model these structures is also of primary importance as it determines the scaling, and ultimately the sizing effects for that particular material. Fractional calculus is deemed appropriate and necessary for the study of these models as ordinary integer order calculus is ill-equipped to differentiate the fractal functions at work.

The authors take this contention to use with a tensile and flexural analysis of a 2D section of concrete. From experimentally tested results of the composition and stress reactions of concrete, they determine that the microstructure may be approximated by a fractal Cantor set of dimension $\alpha = 0.5$. They compute through use of fractional calculus the tensile and flexural strength of their specimen, and indeed find fractal dimension related dependencies on size that simply do not agree with a classical analysis. These formulas are shown below.

$$(\sigma_u)_{tensile} = \frac{\sigma_u^*}{\Gamma} b^{-(1-\alpha)} \quad (61)$$

$$(\sigma_u)_{flexural} = 2 \left(\frac{2^{1/\alpha} - 1}{2^{1/\alpha} + 1} \right) (\sigma_u)_{tensile} \quad (62)$$

It is the hope of the authors of [13] that their study into the strength of materials with microstructures easily approximated by fractal sets will open the door to a more mathematically rigorous formulation of stress and load relationships. In their own words, "Here, on the one hand, we demonstrate one origin of a stress concentration on fractal sets, viz, the heterogeneity of the aggregates in the concrete. On the other hand, given the existence of the concentration of the stress on a fractal set, we develop a way to make first principle calculations of various strengths. For this purpose we use the concept of fractal integrals. We hope that these will pave the way for more general treatment of these questions."

4.4 Discovering Heat Flux at the Boundary of a Semi-Infinite Rod

4.4.1 Given

Assume you have a semi-infinite bar of unknown radius which extends in length from $x=0 \rightarrow x=\infty$. The Temperature in the bar, given by the function $u(x,t)$, can be expressed by the following partial differential equation.

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= 0 \\ u_t - u_{xx} &= 0\end{aligned}$$

For this application, we suppose that at $t = 0, u(x, 0) = 0$ for $0 < x < \infty$. $x = 0$ corresponds to the boundary across which heat is flowing into/out of the rod.

Furthermore, let us assume that change in temperature with respect to x at the boundary is given by the heat influx function $P(t)$. We may consider this an extension of fourier's law.

$$q = -k \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = u_x(x, t) = P(t)$$

Finally, we suppose that the temperature and the change in temperature with respect to x go to 0 at $x = \infty$. i.e.

$$\lim_{x \rightarrow \infty} u(x, t) = \lim_{x \rightarrow \infty} u_x(x, t) = 0$$

4.4.2 Cosine Transform of Partial Differential Equation

Fractional Calculus arises as useful in this problem only after an appropriate transform is completed on the differential equation. The effect for this problem is to change the style of the problem from its initial partial differential equation form to a form resembling that of Abel's Integral Equation of the First Kind. [4] provides a detailed execution of the this transformation which will be briefly related here.

Fourier Cosine Transform on $u(x, t)$

$$u_c(s, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, t) \cos sxdx$$

of $u_{xx}(x, t)$

$$u_{xxc}(s, t) = -s^2 u_c(s, t) - \sqrt{\frac{2}{\pi}} u_x(0, t).$$

Given these transformations, we can rewrite the partial differential equation for the temperature in the rod as

$$\frac{\partial u_c(s, t)}{\partial t} = -\sqrt{\frac{2}{\pi}} * P(t) - s^2 u_c(s, t).$$

The solution to this first-order non-linear partial differential equation is given by

$$u_c(s, t) = -\sqrt{\frac{2}{\pi}} \int_0^t P(\tau) e^{-s^2(t-\tau)} d\tau.$$

$u_c(s, t)$ is inverted back to the x -domain.

$$\begin{aligned} u(x, t) &= \sqrt{\frac{2}{\pi}} \int_0^\infty u_c(s, t) \cos sx ds \\ u(x, t) &= -\frac{2}{\pi} \int_0^t P(\tau) d\tau \int_0^\infty e^{-s^2(t-\tau)} \cos sx ds \end{aligned}$$

By a relation to Green's Function (see [1]), the transformed solution becomes

$$u(x, t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{P(\tau)}{\sqrt{t-\tau}} e^{-x^2/[4(t-\tau)]} d\tau$$

4.4.3 Implementation of the Fractional Integral Equation

The above equation gives the temperature distribution as a function of time and location for the semi-infinite bar given a known heat flux at the boundary $x = 0$ given by $P(t)$. However, through the introduction of Fractional Calculus, it is possible to obtain, given temperature measurements at the boundary, to arrive at a solution for an unknown heat influx (or efflux, as the case may be).

At the boundary $x = 0^+$, the temperature can be written as $u(0^+, t) = \phi(t)$. Thus,

$$\phi(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{P(\tau)}{\sqrt{t-\tau}} d\tau$$

This equation corresponds to the form of the Abel Integral Equation of the First Kind, and can be re-written with J operator as a fractional integral of order $1/2$. It can then be solved by differentiating by order $1/2$. $P(t)$ can then be written directly in terms of $\phi(t)$.

$$\begin{aligned} J^{1/2}P(t) &= \phi(t) \\ P(t) &= D^{1/2}\phi(t) \\ P(t) &= \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\phi(\tau)}{\sqrt{t-\tau}} d\tau \end{aligned}$$

Although the one dimensionality in this particular example makes its usefulness limited, the fractional-order relationship that appears demonstrates a potential for use of fractional calculus in similar problems. This example shows that for a conductive bar of a length significantly greater than its width, it is possible to determine from temperature measurements along its length to calculate the heat flux at its boundary. By using fractional calculus to express this relationship, it is possible to actually reduce the complexity of turning these temperature measurements into usable results. As *GLdifferentiate.m* in the Appendix demonstrates, it is fairly easy to construct and use a numerical fractional derivative/integral solver, which for this problem could turn out results very quickly.

5 Conclusion

Interest in Fractional Calculus for many years was purely mathematic, and it is not hard to see why. Only the very basic concepts regarding fractional order calculus were addressed here, and yet it is evident that the study fractional calculus opens the mind to entirely new branches of thought. It fills in the gaps of traditional calculus in ways that as of yet, no one completely understands. But the goal of this paper is not only to expose the reader to the basic concepts of fractional calculus, but also to whet his/her appetite with scientific and engineering applications. Despite the infancy of this field, the small sampling of applicable problems given here are merely a tiny fraction of what is currently being studied. It is advised that the reader start with the works cited in the reference to further and enhance

understanding of fractional calculus and then from there branch out from those document's references.

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