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CS 663 Assignment 4

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Q3. Matrix $A \rightarrow m \times n$, $m \le n$.

$P = A^T A$

$Q = A A^T$

(a) To prove: for any vector y, prove $y^t P y \geq 0$.
& for any vector z, prove $z^t Q z \geq 0$

$y^T P y = y^T A^T A y$

$= (Ay)^T (Ay) \quad \because (XY)^T = Y^T X^T$

let $Ay = m$, where m is a vector with appropriate

$= m^T m$

$= \|m\|_2^2 = \|Ay\|_2^2 \geq 0.$

Similarly.

$z^t Q z = z^t A A^T z$

$= (A^T z)^T A^T z$

$= \|A^T z\|_2^2 \geq 0.$

Let u be an eigenvector of P, ie. $Pu = \lambda u$

We know $y^t P y \geq 0$

$\Rightarrow u^t P u \geq 0$

$\Rightarrow \lambda u^t u \geq 0 \quad \because u^t u = 1$

$\Rightarrow \lambda \geq 0$

Similarly, let v be an eigenvector of Q, with eigenvalue μ

i.e. $Qv = \mu v$

We know that, $z^t Q z \geq 0$

$\Rightarrow v^t Q v \geq 0$

$\Rightarrow v^t \mu v \geq 0$

$\Rightarrow \mu \geq 0$

Therefore, eigenvalues of both P and Q are non-negative.

(b) Given u is an eigenvector of P with eigenvalue λ, show Au is eigenvector of Q with eigenvalue λ

$\Rightarrow Pu = \lambda u$

$A^T A u = \lambda u$

Premultiplying both sides with A.

$AA^T (Au) = \lambda (Au)$

$Q(Au) = \lambda (Au)$

$\therefore Au$ is eigenvector of Q with eigenvalue λ.

Similarly, given that v is an eigenvector of Q with eigenvalue μ.

$$\Rightarrow \quad Qv = \mu v$$

$$\Rightarrow \quad AA^T v = \mu v$$

Premultiplying with A^T,

$$\Rightarrow \quad A^T A (A^T v) = \mu (A^T v)$$

$$\Rightarrow \quad P(A^T v) = \mu (A^T v)$$

$\therefore$ $A^T v$ is eigenvector of P with eigenvalue μ.

<u>No. of elements</u>:

u is eigenvector of P.

$$P = \underset{n\times m}{A^T} \underset{m\times n}{A} \qquad \text{size}(P) = n\times n$$

$\therefore$ Number of elements in $u = n$.

v is eigenvector of Q.

$$Q = \underset{m\times n}{A} \underset{n\times m}{A^T} \qquad \text{size}(Q) = m\times m.$$

$\therefore$ Number of elements in $v = m$.

(c) v_i is an eigenvector of Q.

$$\therefore \quad Q v_i = \mu_i v_i$$

where μ_i is the eigenvalue corresponding to the eigenvector v_i of Q.

$$u_i \triangleq \frac{A^T v_i}{\|A^T v_i\|_2}$$

$$A u_i = \frac{A A^T v_i}{\|A^T v_i\|_2}$$

$$= \frac{Q v_i}{\|A^T v_i\|_2} \qquad Q = AA^T$$

$$= \frac{\mu_i v_i}{\|A^T v_i\|_2}$$

$$\boxed{A u_i = \gamma_i v_i}$$

$\gamma_i = \frac{\mu_i}{\|A^T v_i\|_2}$. From (a), we know that $\mu_i \geq 0$.

$\|A^T v_i\|_2$ = magnitude of vector $A^T v_i$ which is non-negative

Thus, γ_i is non-negative.

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A is of size $m \times n$.

(d). Given, $\underset{m\times m}{U} = [v_1 | v_2 | v_3 \cdots v_m]$

$\underset{n\times n}{V} = [u_1 | u_2 | u_3 \cdots u_n]$

Let us consider,

For $i \in \{1, 2, \cdots m\}$, $Au_i = \gamma_i v_i$

For $i \in \{m+1, m+2 \cdots n\}$, $Au_i = 0$

From (c), we have, $Au_i = \gamma_i v_i$

$\Rightarrow AV = U\Gamma$ where Γ is a diagonal matrix with γ_i as diagonal elements.

Pre-multiplying with V^T

$\Rightarrow AVV^T = U\Gamma V^T$ — ①

v_i's are eigenvectors of ϕ, $\therefore$ they are orthogonal to each other and of magnitude 1.

$\therefore VV^T = I$

for u_i's $i \neq j$,

$$u_i^t u_j = \frac{v_i^t A A^T v_j}{\|A^T v_i\|_2 \|A^T v_j\|_2} = \frac{v_i^t q v_j}{\|A^T v_i\|_2 \|A^T v_j\|_2}$$

$$= \frac{\mu v_i^t v_j}{\|A^T v_i\|_2 \|A^T v_j\|_2} = 0$$

$\therefore$ u_i's are orthogonal & magnitude $=1$
by definition.

$$u_i^t v_j = 0$$

from (1) $\quad A V V^T = U \Gamma V^T$

$$\Rightarrow \quad A = U \Gamma V^T \qquad \because V V^T = I$$

This is the Singular Value Decomposition (SVD) for
matrix $A_{m \times n}$.