DS 203: Assignment 1

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Ex-1 Let's define the following events:

A - Product is manufactured by factory A
B - Product is manufactured by factory B
} exhaustive events

D - Product is defective

Given : $P(A) = 0.8$, $P(B) = 1 - P(A) = 0.2$

& $P(D|A) = 0.3$, $P(D|B) = 0.1$

1) $P(D) = P(A)P(D|A) + P(B)P(D|B)$ Theorem of total probability

$= (0.8)(0.3) + (0.2)(0.1) = \boxed{0.26}$

2) $P(A|D) = \dfrac{P(D|A)P(A)}{P(D)}$ Baye's Theorem

$= \dfrac{(0.3)(0.8)}{0.26} = \boxed{\dfrac{12}{13} \simeq 0.923}$

Ex-2 Events: W - server working,

S - Successful access attempt, S_i (for i^{th} attempt)

Given: $P(W) = 0.8$, $P(S|W) = 0.9$

1) $\underbrace{P(1^{st} \text{ attempt fails})}_{P(S_1^c)} = 1 - P(1^{st} \text{ attempt success})$ → $P(S_1)$

$= 1 - P(W)P(S|W)$

$= 1 - (0.8)(0.9) = \boxed{0.28}$

2) $P(W|S_1^c) = \dfrac{P(S_1^c|W)P(W)}{P(S_1^c)} = \dfrac{(0.1)(0.8)}{0.28}$

$= \boxed{\dfrac{2}{7} \simeq 0.286}$

3) $P(S_2^c \mid S_1^c) = \dfrac{P(S_2^c \cap S_1^c)}{P(S_1^c)} = \dfrac{P(W)\,P(S_2^c \cap S_1^c) + P(W^c)(1)}{P(S_1^c)}$

$= \dfrac{(0.8)(0.1)(0.1) + 0.2}{0.28} = \boxed{0.743}$

4) $P(W \mid S_1^c \cap S_2^c) = \dfrac{P(S_1^c \cap S_2^c \mid W)\,P(W)}{P(S_1^c \cap S_2^c)} = \dfrac{(0.1)^2 \cdot (0.8)}{0.208}$

$= \boxed{0.038}$

Ex-3 Two dice are rolled, X_1 & X_2 are observations

1) $P(\text{at least one of } X_1 \text{ or } X_2 \text{ is } 6) = \dfrac{{}^2C_1 \cdot 6 - 1}{36} = \boxed{\dfrac{11}{36}}$

2) $P(X_1 \text{ or } X_2 = 6 \mid X_1 \neq X_2) = \dfrac{{}^2C_1 \times 5}{6 \times 5} = \dfrac{10}{30} = \boxed{\dfrac{1}{3}}$

Ex-4 Events: C : Color-blind, M : male, F : female

given, $P(C \mid M) = 0.05$, $P(C \mid F) = 0.01$

$P(M) = P(F) = 0.5$

$P(M \mid C) = \dfrac{P(C \mid M)\,P(M)}{P(C \mid M)P(M) + P(C \mid F)P(F)} = \dfrac{0.05}{0.05 + 0.01}$

$= \boxed{\dfrac{5}{6}}$

Ex-5 a) given event E is independent of itself.

Thus $P(E \cap E) = P(E \mid E) \cdot P(E)$

$= (P(E))^2$ as $P(E \mid E) = P(E)$ (given)

But $P(E \cap E) = P(E)$ by default, thus,

$P(E) = (P(E))^2 \Rightarrow P(E)(1 - P(E)) = 0$

$\boxed{\text{Thus, } P(E) = 0 \text{ or } 1}$

b) Given, $P(A) = 0.3$, $P(B) = 0.4$

i) A & B are independent $\Rightarrow P(A \cap B) = P(A)P(B)$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.3 + 0.4 - (0.3)(0.4)$$
$$= \boxed{0.58}$$

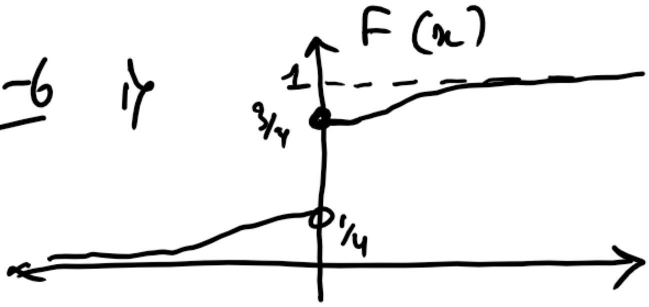
ii) A & B are mutually exclusive $\Rightarrow P(A \cap B) = 0$

Thus, $P(A \cap B) = P(A) + P(B) = \boxed{0.7}$

c) Given $P(A) = 0.6$ & $P(B) = 0.8$.

Yes, A & B can be independent as in that case, $P(A \cap B) = 0.6 + 0.8 - (0.6)(0.8) = 0.92$ which is possible. No, A & B cannot be mutually exclusive, as $P(A \cap B) = 0.6 + 0.8 = 1.4$ is impossible

<u>Ex-6</u> i)



✓ $F(x)$ is non-decreasing

✓ $F(x) \in [0,1] \ \forall x$

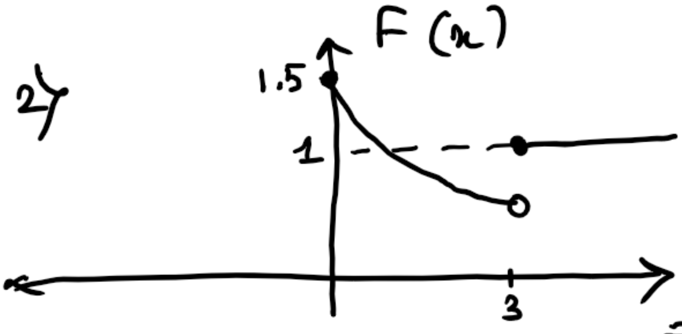
✓ $F(x)$ is right continuous

✓ $\lim\limits_{x \to \infty} F(x) = 1$ & $\lim\limits_{x \to \infty} F(x) = 0$

$\Rightarrow$ CDF is Valid

$$P(X^2 > 5) = P(|X| > \sqrt{5}) = P(X > \sqrt{5}) + P(X < -\sqrt{5})$$
$$= e^{-5}/4 + e^{-5}/4 = \boxed{e^{-5}/2}$$

2)

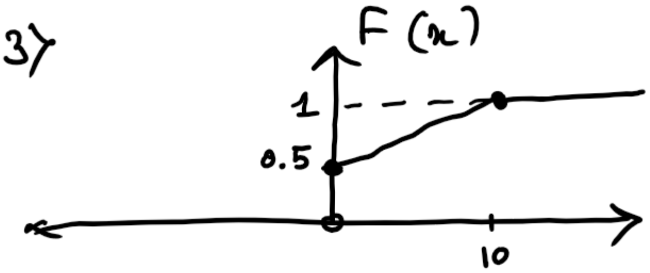


✗ $F(x) \notin [0,1] \ \forall X$
$(F(x) > 1 \ \forall x \in [0, \ln(2)])$

✗ $F(x)$ is not non-decreasing

$\Rightarrow$ CDF is Invalid

3)



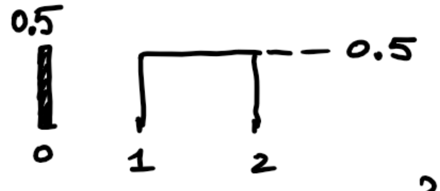
$\Rightarrow$ CDF is Valid (Same reasons as 1))

$$P(X^2 > 5) = P(|X| > \sqrt{5}) = P(X > \sqrt{5}) + P(X < -\sqrt{5})$$

$$= 1 - \left(0.5 + \frac{\sqrt{5}}{20}\right) + 0 = \boxed{0.388}$$

Ex-7 1) $P(x \leq 0.8) = F(0.8) = \boxed{0.5}$

2) CDF to mixed PMF - PDF :



$$\Rightarrow E(X) = (0.5)(0) + \int_1^2 x \; 0.5 \, dx$$

$$= 0 + 0.75 = \boxed{0.75}$$

3) $$Var(X) = E[(X - E(X))^2]$$

$$= (0.5)(0 - 0.75)^2 + \int_1^2 (x - 0.75)^2 (0.5) \, dx$$

$$= \boxed{0.604}$$

Ex-8 $$f(x) = \begin{cases} ce^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_0^{\infty} ce^{-2x} \, dx = c\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} \Rightarrow \boxed{c = 2}$$

$$P(x > 2) = \int_2^{\infty} 2e^{-2x} \, dx = \left[e^{-2x}\right]_{\infty}^{2} = \boxed{e^{-4} = 0.018}$$

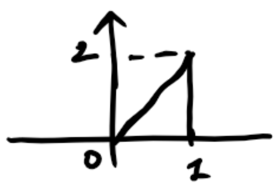
Ex-9 Given situation leads to binomial dist with $n=3$ & $p=0.7$

Thus,

$$\left.\begin{aligned} p(X=0) &= \binom{3}{0} 0.7^0\, 0.3^3 = 0.027 \\ p(X=1) &= \binom{3}{1} 0.7^1\, 0.3^2 = 0.189 \\ p(X=2) &= \binom{3}{2} 0.7^2\, 0.3 = 0.441 \\ p(X=3) &= \binom{3}{3} 0.7^3\, 0.3^0 = 0.343 \end{aligned}\right\} \text{sum up to } 1$$

This is the desired PMF.

Ex-10 pdf



$$P(X \geq 0.4 \mid X \leq 0.8) = \frac{P(0.4 \leq X \leq 0.8)}{P(X \leq 0.8)}$$

$$= \frac{0.5 \times 2(0.4+0.8) \times (0.8-0.4)}{0.5 \times 0.8 \times 2 \times 0.8}$$

$$= \boxed{0.75}$$

Ex-11 Given, $X \sim Exp(\lambda)$

$$P(X > a+b \mid X > a) = \frac{\int_{a+b}^{\infty} \lambda e^{-\lambda x}\, dx}{\int_{a}^{\infty} \lambda e^{-\lambda x}\, dx}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = \boxed{e^{-\lambda b}}$$

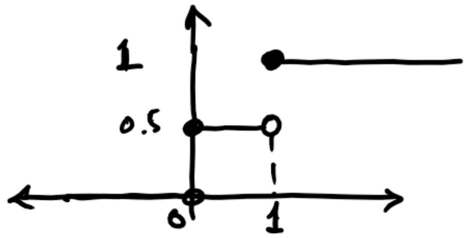
This example demonstrates the memory-lessness property of the exponential distribution.

Ex-12 $I_E = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E^c \text{ occurs} \end{cases}$, where E denotes all heads on toss of 5 fair coins.

$\Rightarrow$ $I_E = 1$ when $(1,1,1,1,1)$ is observed from the sample space $\{x_1, x_2, x_3, x_4, x_5\}$ such that $x_i \in \{0, 1\}$ for all i, $1 \Rightarrow$ head, $0 \Rightarrow$ tail

Thus $P(\{I_E = 1\}) = \left(\frac{1}{2}\right)^5 = \boxed{\frac{1}{32}}$ All 5 coin tosses are independent

Ex-13 Given CDF:



$$\Rightarrow \quad PMF(X): \begin{cases} 0.5 & x = 0 \\ 0.5 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Ex-14 Given, $P(\text{white}) = p = 0.5$, $P(\text{black}) = 1 - p = 0.5$

$$P(\text{two white of first } 4) = \binom{4}{2} 0.5^2\, 0.5^2 = \boxed{0.375}$$

Ex-15 If we get r^{th} head with prob. p on the n^{th} toss, we have exactly $r-1$ heads in 1^{st} $n-1$ tosses $n \geq r$

the Probability that n tosses are required:

$$P(X = n) = P(r-1 \text{ heads in } n-1 \text{ tosses})\, \underbrace{P(\text{head})}_{n^{th}\text{ toss}}$$

$$= \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \cdot p$$

$$= \boxed{\binom{n-1}{r-1} p^r (1-p)^{n-r}}$$ which is what we had to show

Ex-16 A poisson dist. with param λ has PMF:

$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$ s.t. k is a whole number, given $\lambda = 1$

$$\Rightarrow P(\text{at least } 1) = 1 - P(X = 0)$$

$$= 1 - \frac{e^{-1}\, 1^0}{0!} \approx \boxed{0.632}$$