Problem 1

a) Since the coin is fair, each RV is from Ber(0.5)

Let X_1, X_2, X_3 be the three flips.

$X = X_1 + X_2$ and $Y = X_2 + X_3$, $E[X_i] = \mu_i$

$Cov(X,Y) = Cov(X_1 + X_2, X_3 + X_2)$

$\Rightarrow Cov(X,Y) = E[(X_1 + X_2 - E[X_1 + X_2])\,(X_3 + X_2 - E[X_3 + X_2])]$

$\Rightarrow Cov(X,Y) = E\big[\big((X_1 - \mu_1) + (X_2 - \mu_2)\big)\big((X_2 - \mu_2) + (X_3 - \mu_3)\big)\big] - By\ LOE\ E[X_1 + X_2] = \mu_1 + \mu_2$

$\Rightarrow Cov(X,Y) = E\big[\big((X_1 - \mu_1) + (X_2 - \mu_2)\big)\big((X_2 - \mu_2) + (X_3 - \mu_3)\big)\big] - By\ LOE\ E[X_1 + X_2] = \mu_1 + \mu_2$

$\Rightarrow\ = E[(X_1 - \mu_1)(X_2 - \mu_2)] + E[(X_1 - \mu_1)(X_3 - \mu_3)] + E[(X_2 - \mu_2)(X_3 - \mu_3)] + E[(X_2 - \mu_2)^2]$

$\Rightarrow\ Cov(X,Y)\ = Cov(X_1, X_2) + Cov(X_1, X_2) + Cov(X_2, X_3) + Var(X_2)$

$\because Flips\ are\ indepentent$

$\Rightarrow Cov(X,Y) = Var(X_2) = \frac{1}{4} \qquad \because X_i \perp X_j \Rightarrow Cov(X_i, X_j) = 0\ and\ X \sim Ber(p) \Rightarrow Var(X) = p(1-p)$

b) $Y = X^2$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[X]E[Y] = 0 \qquad \because \quad E[X] = \frac{1}{5}[0 - 5 - 2 + 2 + 5]$

$E[XY] = E[X^3] = 0$

$\therefore Cov(X,Y) = 0$

c) No. 2b) is a counterexample.
Since covariance only captures linear relationships, dependent random variables with no linear relationship will have zero covariance.

problem 2

Let X be a non-negative RV with mean μ and variance σ^2, and let t > 0 be some real number.

a) Prove the following: $E[X] \geq \int_t^\infty xf(x)dx$

$$E[X] = \int_0^\infty xf(x)dx \qquad \because x \geq 0$$

$$\Rightarrow E[X] = \int_0^t xf(x)dx + \int_t^\infty xf(x)dx$$

$$\int_0^t xf(x)dx \geq 0 \qquad \because t > 0, X \geq 0 \; and \; f(x) \geq 0$$

$$\Rightarrow E[X] \geq \int_t^\infty xf(x)dx \qquad -(1)$$

b) With the help of part(a), prove the following inequality: $\Pr(X > t) \leq \frac{E[X]}{t}$

Sol^n:

$$E[X] \geq \int_t^\infty xf(x)dx$$

$$\Rightarrow E[X] \geq \int_t^\infty tf(x)dx$$

$$\Rightarrow E[X] \geq t\int_t^\infty f(x)dx = tP(X > t) \qquad -(2)$$

$$\Rightarrow P(X > t) \leq \frac{E[X]}{t}$$

c) Using the inequality proved in part (b), prove the following: $\Pr(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$

Sol^n:

$$E[X] \geq tP(X > t) \qquad -From\ (2)$$

$$\Rightarrow P(X > t) \leq \frac{E[X]}{t}$$

$$\Rightarrow P((X-\mu)^2 > t^2) \leq \frac{E[(X-\mu)^2]}{t^2}$$

$$\Rightarrow P((X-\mu)^2 > t^2) \leq \frac{\sigma^2}{t^2}$$

$$\Rightarrow P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

Problem 3

a) $X_1, X_2, \dots, X_k$ are k independent exponential random variables with

$$f_{X_i}(x) = \lambda_i e^{-\lambda_i x},\ x \geq 0 \qquad \forall\ i \in \{1, 2, \dots, k\} \text{ and}$$

(i)

Let $Z = \min(X_1, X_2, \dots, X_k)$

$F_Z(z) = P(Z \leq z)$

$F_Z(z) = P(\min(X_1, X_2, \dots, X_k) \leq z)$

$F_Z(z) = 1 - P(\mathrm{X}_1 > z, \mathrm{X}_2 > z, \dots, \mathrm{X}_\mathrm{k} > z)$

$F_Z(z) = 1 - P(\mathrm{X}_1 > z, \mathrm{X}_2 > z, \dots, \mathrm{X}_\mathrm{k} > z)$

$F_Z(z) = 1 - \prod_{i=1}^{k} P(X_i > z)$

$F_Z(z) = 1 - \prod_{i=1}^{k} \exp(-\lambda_i z)$

$F_Z(z) = 1 - \exp(-z(\lambda_1 + \lambda_2 + \dots + \lambda_k))$

$f_Z(z) = \dfrac{\partial F_z(z)}{\partial z}$

$f_Z(z) = (\lambda_1 + \lambda_2 + \dots + \lambda_k) \exp(-z(\lambda_1 + \lambda_2 + \dots + \lambda_k))$

$\therefore Z \sim \exp(\lambda_1 + \lambda_2 + \dots + \lambda_k)$

(ii) $\because Z \sim \exp(\lambda_1 + \lambda_2 + \dots + \lambda_k)$

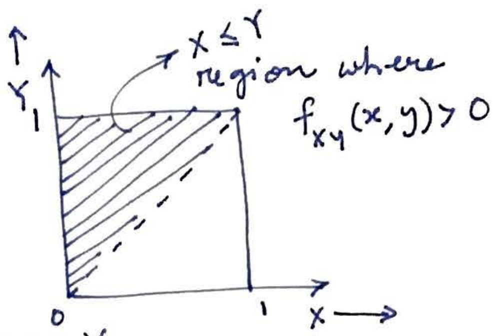
$$\therefore E[Z] = \frac{1}{(\lambda_1 + \lambda_2 + \dots + \lambda_k)}$$

(iii) $\because Z \sim \exp(\lambda_1 + \lambda_2 + \dots + \lambda_k)$

$$\therefore Var(Z) = \frac{1}{(\lambda_1 + \lambda_2 + \dots + \lambda_k)^2}$$

3b

$$f_{XY}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$Z = XY$

$F_Z(z) = P(Z \le z)$

$= P(XY \le z) = 1 - P(XY > z)$

For finding the lower limit of integration over Y

$0 \le X \le Y \le 1$

For any value of $\underline{Z \ge z}$

$Z \ge z$

$\Rightarrow XY \ge z$ ——— ①

$Y \ge X$ [Given]

$\Rightarrow Y^2 \ge XY$ ——— ②

$\Rightarrow Y^2 \ge XY \ge z$ [From ① and ②]

$\Rightarrow Y \ge \sqrt{z}$

$\therefore$ For any any value of z

$$F_Z(z) = 1 - \int_{y=\sqrt{z}}^{1} \int_{x=\frac{z}{y}}^{y} f_{XY}(x,y)\,dx\,dy$$

$$F_Z(z) = 1 - 2\int_{\sqrt{z}}^{1} \int_{z/y}^{y} dx\,dy$$

$$F_Z(z) = 1 - 2\int_{y=\sqrt{z}}^{1} \left(y - \frac{z}{y}\right) dy$$

$$F_Z(z) = 1 - 2\left(\frac{1-(\sqrt{z})^2}{2} - z\left(0 - \frac{\ln(z)}{2}\right)\right)$$

$$\boxed{\therefore F_Z(z) = z - z\ln(z)}$$

$$\therefore f_Z(z) = \frac{d F_Z(z)}{dz}$$

$$\boxed{\therefore f_Z(z) = -\ln(z) \quad \forall z \in (0,1)}$$

Problem 4.

a)

$$E[X] = E[X|East]P(East) + E[X|West]P(West)$$

$$E[X] = \frac{1}{2}E[X|East] + \frac{1}{2}\big(E[X|West, West]P(West) + E[X|West, South]P(South)\big) - (1)$$

$$E[X] = \frac{1}{2}E[X+20] + \frac{1}{2}(\frac{1}{2}.5 + \frac{1}{2}E[X+10])$$

$$E[X] = \frac{1}{2}E[X+20] + \frac{1}{2}(\frac{1}{2}.5 + \frac{1}{2}E[X+10])$$

$$E[X] = \frac{55}{4} + \frac{3E[X]}{4}$$

$$E[X] = 55$$

b) Similar to (1)

$$E[X^2] = \frac{1}{2}E[X^2|East] + \frac{1}{2}(E[X^2|West, West]P(West) + E[X^2|West, South]P(South))$$

$$E[X^2] = \frac{1}{2}E[(X+20)^2] + \frac{1}{2}(\frac{1}{2}.5^2 + \frac{1}{2}E[(X+10)^2])$$

$$E[X^2] = \frac{1}{2}E[400 + 40X + X^2] + \frac{1}{2}(\frac{1}{2}.25 + \frac{1}{2}E[X^2 + 100 + 20X])$$

$$E[X^2] = \frac{925}{4} + 25E[X] + \frac{3}{4}E[X^2]$$

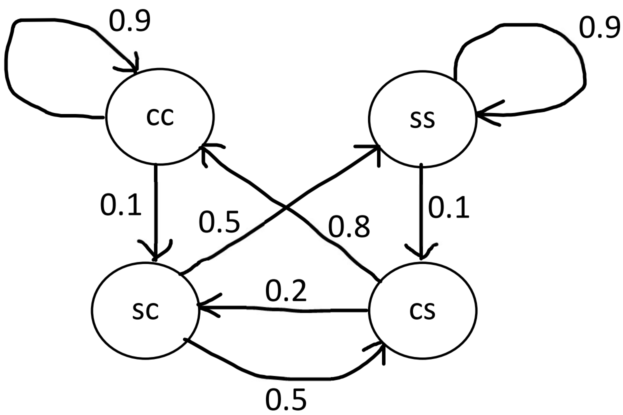
$$E[X^2] = 100E[X] + 925 = 625$$

$$Var(X) = E[X^2] - E^2[X]$$

$$Var(X) = 6425 - 55^2 = 3400$$

Problem 5

$$\begin{array}{c|cccc} & cc & sc & cs & ss \\ \hline cc & 0.9 & 0.1 & 0.0 & 0.0 \\ sc & 0 & 0.0 & 0.5 & 0.5 \\ cs & 0.8 & 0.2 & 0.0 & 0.0 \\ ss & 0 & 0.0 & 0.1 & 0.9 \end{array}$$



Given:

$Pr[c|cc] = 0.9$ $\quad Pr[c|cs] = 0.8$ $\quad Pr[c|sc] = 0.5$ $\quad Pr[c|ss] = 0.1$

$Pr[s|cc] = 0.1$ $\quad Pr[s|cs] = 0.2$ $\quad Pr[s|sc] = 0.5$ $\quad Pr[s|ss] = 0.9$

(a)The Markov chain and transition matrix will be as shown on the figure.

Solving the following stationary equations:

$$\pi_{cc} = 0.9\pi_{cc} + 0.8\pi_{cs}$$

$$\pi_{cs} = 0.5\pi_{sc} + 0.1\pi_{ss}$$

$$\pi_{sc} = 0.1\pi_{cc} + 0.2\pi_{cs}$$

$$\pi_{ss} = 0.5\pi_{sc} + 0.9\pi_{ss}$$

$$\pi_{sc} + \pi_{cc} + \pi_{cs} + \pi_{ss} = 1$$

$$\therefore \pi_{sc} = \frac{1}{15};\ \pi_{cc} = \frac{8}{15};\pi_{cs} =;\frac{1}{15}\pi_{ss} = \frac{5}{15}$$

(b) $\qquad P = \pi_{sc} + \pi_{ss} = \frac{1}{15} + \frac{5}{15} = \frac{6}{15}$

(c) Transition matrix raised to the power 100.

$$\begin{array}{c|cccc} & cc & sc & cs & ss \\ \hline cc & .53 & .06 & .06 & .33 \\ sc & .53 & .06 & .06 & .33 \\ cs & .53 & .06 & .06 & .33 \\ ss & .53 & .06 & .06 & .33 \end{array}$$

Q6.

(a) Since $X = (X_1, \dots, X_k)$ is multinomial, every linear combination $t_1 X_1 + t_2 X_2 + \dots + t_k X_k$ is Normal. In particular, for $t_j = 1$, and $t_i = 0 \ \forall \ i \neq j$ we have $1 \cdot X_j = X_j$ a Normal distribution

(b) Note that $P(X+Y) = P(X + SX) = P(S=-1)$
$= 1/2$

Since a linear combination of X, Y has non-zero probability mass at -1, the linear combination is not Normal. Therefore (X, Y) can't be Normal.

(c) Since Z, W are i.i.d $N(0,1)$, $t_1 Z \sim N(0, t_1^2)$ and $t_2 W \sim N(0, t_2^2)$ and are independent. Therefore $t_1 Z + t_2 W \sim N(0, t_1^2 + t_2^2)$. Since this holds for any t_1, t_2, (Z, W) is a Multivariate Normal.

Consider $t_1, t_2 \in \mathbb{R}$

We have $t_1(Z + 2W) + t_2(3Z + 5W)$

$= (t_1 + 3t_2) Z + (2t_1 + 5t_2) W$

This is Normal $(0, (t_1 + 3t_2)^2 + (2t_1 + 5t_2)^2)$.

$\therefore (Z + 2W, 3Z + 5W)$ is a multivariate Normal.

(d) Since $X = (X_1, \cdots, X_n)$ and $Y = (Y_1, \cdots, Y_m)$ are Multivariate Normals,

$a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$ and

$b_1 Y_1 + b_2 Y_2 + \cdots + b_m Y_m$ are Normals.

Since X & Y are independent,

$\sum_{i=1}^{m} a_i X_i$ and $\sum_{j=1}^{n} b_j Y_j$ are independent.

Therefore $\sum_{i=1}^{m} a_i X_i + \sum_{j=1}^{n} b_j Y_j$ is Normal

for any a_i's & b_j's

This shows that $(X_1 \cdots, X_n, Y_1, \cdots Y_m)$ is a Multivariate Normal.

(e). From Fact 2, we have

$Cor(X+Y, X-Y)$

$= Var(X) - Cov(X, Y) + Cov(Y, X) - Var(Y)$

$= 0.$

From Fact 1, we have $X+Y$ and $X-Y$ to be independent.

We also have $X+Y \sim N(0,2)$ and

$X-Y \sim N(0,2)$

Therefore $t_1(X+Y) + t_2(X-Y)$ is also Normal.

Problem 7

We model this problem as a sum of geometric random variables which are independent.

$$X = t_1 + t_2 + t_3 + ... + t_d$$

Each t_i represents the number of days to get the i^{th} new pokemon.

(a)

So, $E[t_1]$ will be 1, obviously, because any pokemon we get on the first day will be a new pokemon. And if p_i is the probability of success (that is we get a new pokemon on any day), if we already have found $i-1$ distinct pokemons,

$$p_{i+1} = \frac{d-i}{d},$$

because for i^{th} day, any of the $d-i$ pokemons can be the new pokemon.
Each of these t_i are geometric random variables with probability of success p_i, hence

$$E[t_i] = \frac{1}{p_i}$$

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{d} E[t_i] \\
E[X] &= 1 + \frac{d}{d-1} + \frac{d}{d-2} + ... + \frac{d}{1} \\
E[X] &= d(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{d-2} + \frac{1}{d-1} + \frac{1}{d}) \\
E[X] &= dH_d
\end{aligned}
$$

(b)

$$
Var(t_i) = \frac{1-p_i}{p_i^2} = \frac{d(i-1)}{(d+1-i)^2}
$$

All of the t_i are independent of each other. They depend on i which represents the number of pokemons that have already been found, but the number of days to get a new pokemon is independent. So we have:

$$
\begin{aligned}
Var(X) &= \sum_{i=1}^{d} Var[X_i] \\
&= d\sum_{i=1}^{d} \frac{(i-1)}{(d+1-i)^2} \\
&= d \cdot [\frac{1}{(d-1)^2} + \frac{2}{(d-2)^2} + \cdots + \frac{d-1}{1}]
\end{aligned}
$$