J. R. Statist. Soc. B (1997)
59, *No.* 3, *pp.* 639–652

Balanced Confidence Regions Based on Tukey's Depth and the Bootstrap

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[Received January 1993. Final revision October 1996]

SUMMARY

We propose and study the bootstrap confidence regions for multivariate parameters based on Tukey's depth. The bootstrap is based on the normalized or Studentized statistic formed from an independent and identically distributed random sample obtained from some unknown distribution in R^q. The bootstrap points are deleted on the basis of Tukey's depth until the desired confidence level is reached. The proposed confidence regions are shown to be second order balanced in the context discussed by Beran. We also study the asymptotic consistency of Tukey's depth-based bootstrap confidence regions. The applicability of the method proposed is demonstrated in a simulation study.

Keywords: BALANCED CONFIDENCE REGION; BOOTSTRAP; SECOND-ORDER BALANCEDNESS; TUKEY'S DEPTH

1. INTRODUCTION

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables in R^q, $q \geqslant 1$, from an unknown distribution $F(\cdot)$. Let θ be a p-dimensional parameter of interest, $p \geqslant 1$. In the case when $p = 1$, the construction of bootstrap confidence intervals for θ has been extensively studied over the past decade. See Efron (1982, 1987) on different bootstrap confidence intervals and also see Hall (1986, 1988a) on the theoretical comparison of these procedures. In the case of higher dimensions ($p > 1$), it is not easy, however, to extend the procedures of constructing bootstrap confidence intervals directly to that of constructing bootstrap confidence regions. To demonstrate the difficulty, let us consider the following example. Suppose that we want to construct a 90% confidence region based on the bootstrap percentile method for $\theta = E(X)$ when $p = q = 2$. We would generate, say, 5000 $\bar{X}_n^*$s, where $\bar{X}_n^*$ is the bootstrap mean. From these 5000 $\bar{X}_n^*$s, we need to delete the 500 most 'exterior' $\bar{X}_n^*$s so that the remaining 4500 $\bar{X}_n^*$s can be used to construct a confidence region for θ. But the question arises: how should we delete the 500 most exterior $\bar{X}_n^*$s in higher dimensions?

We can think of this problem as having a 'cloud' of data points sitting in a higher dimensional space. In what way can we 'trim' the data cloud to the desired size? This thinking leads us to the notion of data depth. Data depth is a concept of ordering data from the centre outwardly especially in higher dimensions. The depth decreases monotonically as we move away from the centre of the distribution in any direction. The idea is to use data depth to order higher dimensional bootstrap points. The points with lower depths can be deleted until the desired size is reached. In our

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0035–9246/97/59639

example, those 5000 $\bar{X}_n^*$s can be ordered by using a certain notion of data depth. The 500 $\bar{X}_n^*$s with the lowest depths, or the most exterior points with respect to the cloud of 5000 $\bar{X}_n^*$s, can be deleted so that the convex hull of the remaining bootstrap points can be used as a confidence region for θ. This paper proposes and investigates the construction of data-depth-based bootstrap confidence regions. In particular, our discussion focuses on Tukey's depth (Tukey, 1975). A formal definition of Tukey's depth is given in Section 2. There are several other depth notions, e.g. Mahalanobis depth (Mahalanobis, 1936), simplicial depth (Liu, 1990) and majority depth (Liu and Singh, 1993). See Liu and Singh (1993) for a review of the definitions.

Let $Z_n = n^{1/2}\Sigma_{\hat{\theta}_n}^{-1/2}(\hat{\theta}_n - \theta)$ and $T_n = n^{1/2}S_{\hat{\theta}_n}^{-1/2}(\hat{\theta}_n - \theta)$ be the normalized and Studentized statistics respectively, where $\hat{\theta}_n = \hat{\theta}_n(X_1, X_2, \ldots, X_n)$ is a consistent estimator of θ, $\Sigma_{\hat{\theta}_n}$ is the dispersion matrix of $\hat{\theta}_n$ and $S_{\hat{\theta}_n}$ is a consistent estimator of $\Sigma_{\hat{\theta}_n}$. The bootstrap will be based on Z_n or T_n. Let $Z_n^* = n^{1/2}S_{\hat{\theta}_n}^{-1/2}(\hat{\theta}_n^* - \hat{\theta}_n)$ and $T_n^* = n^{1/2}S_{\hat{\theta}_n}^{*-1/2}(\hat{\theta}_n^* - \hat{\theta}_n)$ be the bootstrap counterparts of Z_n and T_n respectively. Here the asterisk is used to denote the statistic computed under the bootstrap sample. Based on Z_n^*, let $W_{n,1-\alpha}^*$ be a region obtained by first deleting $100\alpha\%$ exterior bootstrap points based on Tukey's depth and then forming the convex hull of the remaining points. A $100(1-\alpha)\%$ bootstrap confidence region based on Z_n^* can be obtained as

$$R_{n,1-\alpha}^* = \left\{\hat{\theta}_n - \frac{1}{\sqrt{n}}\Sigma_{\hat{\theta}_n}^{1/2}\omega:\ \omega \in W_{n,1-\alpha}^*\right\}. \tag{1.1}$$

Likewise, let $V_{n,1-\alpha}^*$ be a region formed based on T_n^* in a similar manner to $W_{n,1-\alpha}^*$; the desired confidence region can be readily constructed as

$$A_{n,1-\alpha}^* = \left\{\hat{\theta}_n - \frac{1}{\sqrt{n}}S_{\hat{\theta}_n}^{1/2}\omega:\ \omega \in V_{n,1-\alpha}^*\right\}. \tag{1.2}$$

It is easy to see that in the univariate case Tukey's depth-based bootstrap confidence intervals are equivalent to the so-called two-sided equal-tailed bootstrap confidence intervals. This type of confidence intervals is second order balanced in the sense that the exclusion probabilities on the two sides of the interval are equal up to the order of $O(n^{-1})$. Beran (1988, 1990) has discussed the concept of balanced confidence sets in general settings, although most of the discussions focus on the first-order balancedness. The question is what type of balancing property do the bootstrap confidence regions based on Tukey's depth achieve in higher dimensions? It turns out that our proposed bootstrap confidence region in equation (1.1) is second order balanced along any fixed linear combination of the unknown parameter. Geometrically, consider any line passing through the interior of region (1.1). Move the line in a parallel way to the opposite sides until it is at the verge of exiting the region. The two outer half-closed spaces thus obtained have the same probabilities up to the order of $o(n^{-1/2})$. As for the region defined in equation (1.2), we cannot establish the second-order balancing property because of technical difficulties. However, the region $V_{n,1-\alpha}^*$, whose inversion leads to the region defined in equation (1.2), is second order balanced in the sense described above. Our simulation study suggests that region (1.2) is more balanced than just first order.

The rest of the paper is organized as follows. In Section 2, the methodology of constructing a bootstrap confidence region based on Tukey's depth is presented. In Section 3, the convergence of the proposed confidence regions is studied. We then investigate the second-order balancing property of the proposed confidence regions. All technical details are deferred to Appendix A. Section 4 is devoted to a simulation study in which the method proposed is applied to an asymmetric bivariate data set. The result is compared with the confidence regions obtained under the same data set using several existing methods. The simulation study shows that Tukey's depth-based bootstrap confidence regions produce quite promising results. We close the discussion in Section 5 by stating some open problems for possible future research.

Before closing this section, it should be mentioned that Owen (1990) has presented an example in which a two-dimensional bootstrap confidence region is constructed using an approximated Tukey depth. There the bootstrap is based on $\hat{\theta}_n$ not on Z_n or T_n. The idea of using depth for constructing bootstrap confidence regions in the present study was conceived independently.

2. METHODOLOGY

We begin by defining Tukey's depth of a point x under some distribution $F(\cdot)$.

Definition 1. Tukey's depth of a point x under $F(\cdot)$, denoted by $\mathrm{TD}(F, x)$, is defined as

$$\mathrm{TD}(F, x) = \inf_H \{F(H)\text{: } H \text{ is a half-closed space containing } x\}. \tag{2.1}$$

The sample version of $\mathrm{TD}(F, \cdot)$ can be obtained by replacing F with F_n, the empirical cumulative distribution function (ECDF). In the univariate case, $\mathrm{TD}(F, x) = \min\{F(x), 1 - F(x^-)\}$.

Let K_n and K_n^* be the distribution functions of Z_n and Z_n^* respectively. On the basis of Z_n^*, the following steps can be carried out in practice to construct a $100(1-\alpha)\%$ TD-based bootstrap confidence region.

Step 1: compute $Z_{n,1}^*, Z_{n,2}^*, \ldots, Z_{n,B}^*$ for some large B.

Step 2: compute $\mathrm{TD}(K_B^*, Z_{n,1}^*), \mathrm{TD}(K_B^*, Z_{n,2}^*), \ldots, \mathrm{TD}(K_B^*, Z_{n,B}^*)$ and denote them by $d_{n,1}, d_{n,2}, \ldots, d_{n,B}$, where K_B^* is the ECDF of Z_n^* obtained on the basis of $\{Z_{n,1}^*, Z_{n,2}^*, \ldots, Z_{n,B}^*\}$.

Step 3: order $d_{n,1}, d_{n,2}, \ldots, d_{n,B}$ and denote them by $d_{n,(1)}, d_{n,(2)}, \ldots, d_{n,(B)}$, where $d_{n,(j)} = \mathrm{TD}(K_B^*, Z_{n,(j)}^*)$.

Step 4: let $\tilde{W}_{n,1-\alpha}^*$ be the convex hull constructed on the basis of the set $\{Z_{n,(j)}^*\text{: } \alpha B + 1 \leqslant j \leqslant B\}$. Then a $100(1-\alpha)\%$ bootstrap confidence region for θ can be obtained as

$$\tilde{R}_{n,1-\alpha}^* = \left\{\hat{\theta}_n - \frac{1}{\sqrt{n}} \Sigma_{\hat{\theta}_n}^{1/2} \omega\text{: } \omega \in \tilde{W}_{n,1-\alpha}^*\right\}. \tag{2.2}$$

The steps for constructing a bootstrap confidence region based on T_n^* and Tukey's depth are similar.

Note that $\tilde{W}_{n,1-\alpha}^*$ is a Monte Carlo approximation of $W_{n,1-\alpha}^*$, and likewise $\tilde{R}_{n,1-\alpha}^*$ in equation (2.2). The theoretical results presented in the following section are based on the exact $W_{n,1-\alpha}^*$. Also note that TD is computed on the basis of K_B^*. In the case when $p = 2$, computing $\mathrm{TD}(K_B^*, Z_{n,k}^*)$ is equivalent to first finding the smaller of the proportions of points (with respect to the $Z_{n,j}^*$) that lie on either side of any line passing through $Z^*_{n,k}$, and $\mathrm{TD}(K^*_B, Z^*_{n,k})$ is equal to the infimum of all such proportions of all possible lines passing through $Z^*_{n,k}$. See Donoho (1982) for a detailed discussion.

3. SECOND-ORDER BALANCEDNESS

We first introduce some notation and state the assumptions. Let $\Omega = \{\omega \in R^p: |\omega| = 1\}$ be the p-dimensional unit sphere. Let $\Phi(x)$ and $\phi(x)$ denote the distribution function and density of $\mathrm{MN}_p(0, I)$, a p-dimensional standard normal distribution, respectively. We assume that Z_n and T_n converge to $\mathrm{MN}_p(0, I)$. As for Z^*_n and T^*_n, it is assumed that they converge to $\mathrm{MN}_p(0, I)$ almost surely. We also assume that K_n and K^*_n allow Edgeworth expansions up to the second order, i.e.

$$K_n(x) = \Phi(x) + \frac{1}{\sqrt{n}} q_1(x)\,\phi(x) + O(n^{-1}) \tag{3.1}$$

and

$$K^*_n(x) = \Phi(x) + \frac{1}{\sqrt{n}} \hat{q}_1(x)\,\phi(x) + O_p(n^{-1}), \tag{3.2}$$

where typically $\hat{q}_1(x) - q_1(x) = O_p(n^{-1/2})$. The validity of equations (3.1) and (3.2) has been studied by various researchers, e.g. Singh (1981) and Hall (1986, 1987, 1988a, b, 1992). More generally, if $A \subset R^p$ can be expressed as a finite union of convex sets, then

$$\int_A \mathrm{d}K_n(x) = \int_A \left\{1 + \frac{1}{\sqrt{n}} s_1(x)\right\} \phi(x)\,\mathrm{d}x + O(n^{-1}) \tag{3.3}$$

and

$$\int_A \mathrm{d}K^*_n(x) = \int_A \left\{1 + \frac{1}{\sqrt{n}} \hat{s}_1(x)\right\} \phi(x)\,\mathrm{d}x + O_p(n^{-1}), \tag{3.4}$$

where

$$s_1(x) = \frac{\mathrm{d}}{\mathrm{d}x} \{q_1(x)\,\phi(x)\}\,\phi^{-1}(x),$$

$$\hat{s}_1(x) = \frac{\mathrm{d}}{\mathrm{d}x} \{\hat{q}_1(x)\,\phi(x)\}\,\phi^{-1}(x).$$

The remainders are uniform over all sets that can be expressed as a finite union of convex sets. Note that $s_1(x)$ and $\hat{s}_1(x)$ are odd functions. For a detailed discussion of equations (3.3) and (3.4), see section 4.2 of Hall (1992).

Under $\mathrm{MN}_p(0, I)$, it can easily be shown that TD remains constant on the set $\{x: |x| = \gamma\}$ for a fixed $\gamma > 0$, where $|\cdot|$ denotes the Euclidean norm. This in turn implies that, under $\mathrm{MN}_p(0, I)$, a $100(1-\alpha)\%$ confidence region based on Tukey's depth is equivalent to the set $\{x: |x| \leqslant \gamma_{1-\alpha}\}$ where $\gamma_{1-\alpha} > 0$ depends on α. However, does $W^*_{n,1-\alpha}$, constructed on the basis of Z^*_n, converge to a p-dimensional sphere as we would expect? This is investigated in the following theorem.

Theorem 1. If F is absolutely continuous in R^q and $E_F\|X\|^2 < \infty$, then

$$C^{\text{int}}_{\gamma_{1-\alpha}} \subseteq \varliminf_{n\to\infty} (W^*_{n,1-\alpha}) \subseteq \varlimsup_{n\to\infty} (W^*_{n,1-\alpha}) \subseteq C_{\gamma_{1-\alpha}} \qquad \text{almost surely,} \tag{3.5}$$

where $C_{\gamma_{1-\alpha}}$ is a p-dimensional sphere centred at 0 with radius $\gamma_{1-\alpha}$ and $C^{\text{int}}_{\gamma_{1-\alpha}}$ is the interior of $C_{\gamma_{1-\alpha}}$.

Result (3.5) essentially says that $W^*_{n,1-\alpha}$ converges to the right region. As for T^*_n-based region $V^*_{n,1-\alpha}$, the proof is very similar in establishing the convergence of $V^*_{n,1-\alpha}$ to a p-dimensional sphere.

Theorem 1 only provides the first-order convergence of $W^*_{n,1-\alpha}$. One can certainly question whether there is any advantage of using $W^*_{n,1-\alpha}$ as opposed to the region constructed on the basis of asymptotic normality. We shall argue that the answer is yes from the standpoint that $W^*_{n,1-\alpha}$ is second order balanced. This feature of $W^*_{n,1-\alpha}$ is particularly important in practice since it captures the first-order departure from symmetry. If the underlying distribution is indeed asymmetric, $W^*_{n,1-\alpha}$ would better reflect such asymmetry than the region constructed on the basis of asymptotic normality, and likewise the region defined in equation (1.1).

Now consider two parallel lines that touch the boundary of $W^*_{n,1-\alpha}$ at y^*_1 and y^*_2. Let us denote the two outer half-closed spaces obtained from these two lines by E^*_1 and E^*_2 respectively. We now present the second-order balancing property of $W^*_{n,1-\alpha}$ in the following theorem.

Theorem 2.

$$\int_{E^*_1} \mathrm{d}K_n(x) - \int_{E^*_2} \mathrm{d}K_n(x) = o(n^{-1/2}) \qquad \text{almost surely.} \tag{3.6}$$

Assertion (3.6) basically says that, along any fixed direction, the two outer half-closed spaces touching the boundary of $W^*_{n,1-\alpha}$ have the same probabilities up to the order of $o(n^{-1/2})$. An immediate application of theorem 2 is that equation (1.1), as a $100(1-\alpha)\%$ confidence region for θ, is second order balanced. This can easily be concluded by observing that TD is affine invariant under any non-random linear transformation. The second-order balancing property of $V^*_{n,1-\alpha}$ can readily be established along the lines of theorem 2. However, we cannot conclude that equation (1.2) is second order balanced since $S_{\hat{\theta}_n}$ is random. Since region (1.2) is based on a second-order balanced set $V^*_{n,1-\alpha}$, this bootstrap-based confidence region makes a correction for the population skewness in the right direction. This is also confirmed by our simulation study (see Section 4).

In practice we use $W^*_{n,1-\alpha}$ to approximate $W_{n,1-\alpha}$, a $100(1-\alpha)\%$ Tukey depth-based confidence region under Z_n since K_n is not known. Along the same direction that leads to y^*_1 and y^*_2, there are two parallel lines touching the boundary of $W_{n,1-\alpha}$ at two points denoted by y_1 and y_2 respectively. Let E_1 and E_2 be the two outer half-closed spaces touching $W_{n,1-\alpha}$ at y_1 and y_2. Likewise, along the same direction, we can obtain two outer half-closed spaces H_1 and H_2 touching the boundary of $C_{\gamma_{1-\alpha}}$ at γ_1 and γ_2 respectively, where $C_{\gamma_{1-\alpha}}$ is the $100(1-\alpha)\%$ Tukey depth-based confidence region under $\mathrm{MN}_p(0, I)$. Here we assume that y_i, y^*_i and γ_i are on the same side for $i = 1, 2$. Note that $\mathrm{TD}(K_n, y_1) = \mathrm{TD}(K_n, y_2)$ and $\mathrm{TD}(\Phi, \gamma_1) = \mathrm{TD}(\Phi, \gamma_2)$, and that $|\gamma_1| = |\gamma_2| = \gamma_{1-\alpha}$. To establish assertion (3.6), utilizing equations (3.3) and (3.4), it amounts to determining the order of magnitude of the differences of (for $i = 1, 2$)

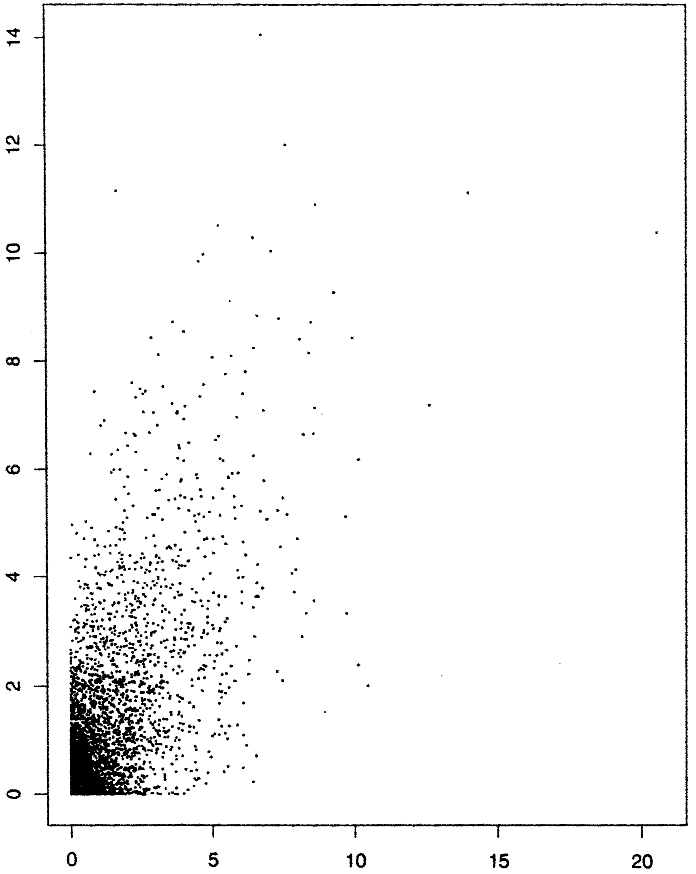

Fig. 1. Sample of size 5000 from the population

(a) $$\int_{\{y_i\}^\perp} \mathrm{d}K_n - \mathrm{TD}(K_n, y_i),$$

(b) $$\int_{E_i} \mathrm{d}K_n - \int_{\{y_i\}^\perp} \mathrm{d}K_n \text{ and}$$

(c) $$\int_{E_i^*} \mathrm{d}K_n - \int_{E_i} \mathrm{d}K_n.$$

Here $\{x\}^\perp$ denotes the outer half-closed space tangent to x. For the univariate confidence interval, we only need to examine difference (c) since both (a) and (b) are equal to 0. The results are summarized in the following three lemmas.

Lemma 1. For $i = 1, 2,$

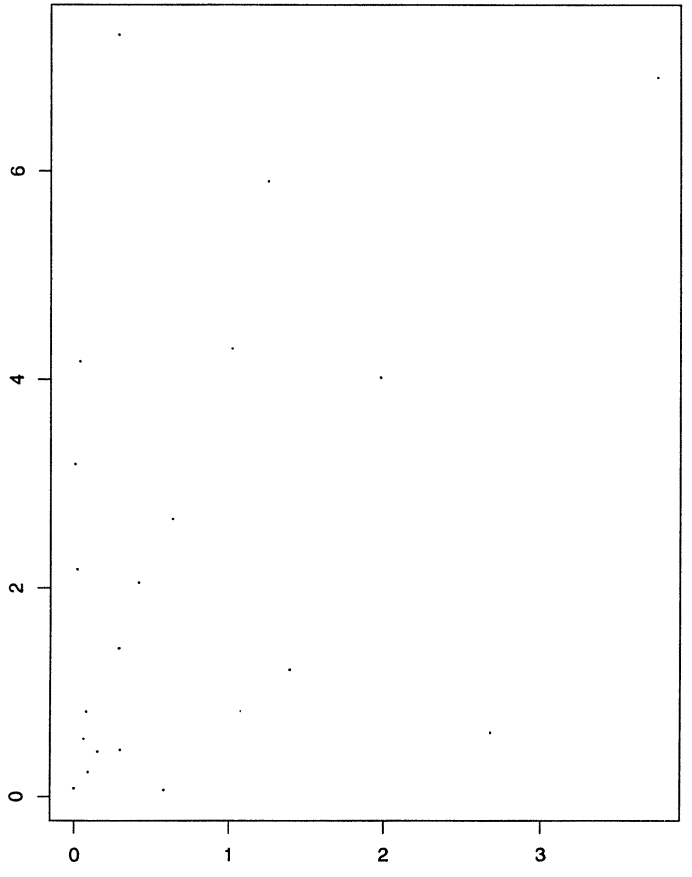

Fig. 2. Sample of size 20 from the population

$$\int_{\{y_i\}^{\perp}} \mathrm{d}K_n(x) - \mathrm{TD}(K_n, y_i) = o(n^{-1/2}) \tag{3.7}$$

and

$$\int_{\{y_i^*\}^{\perp}} \mathrm{d}K_n^* - \mathrm{TD}(K_n^*, y_i^*) = o_p(n^{-1/2}). \tag{3.8}$$

Since the $|y_i|$s and $|y_i^*|$s all converge to $\gamma_{1-\alpha}$, let us express y_1 and y_i^* $(i = 1, 2)$ as

$$y_1 = \left\{\gamma_{1-\alpha} + \frac{1}{\sqrt{n}} v_i + o(n^{-1/2})\right\} \omega_i \tag{3.9}$$

and

$$y_i^* = \left\{\gamma_{1-\alpha} + \frac{1}{\sqrt{n}} \hat{v}_i + o_p(n^{-1/2})\right\} \omega_i^*, \tag{3.10}$$

for some ω_1, ω_2, ω_1^* and $\omega_2^* \in \Omega$.

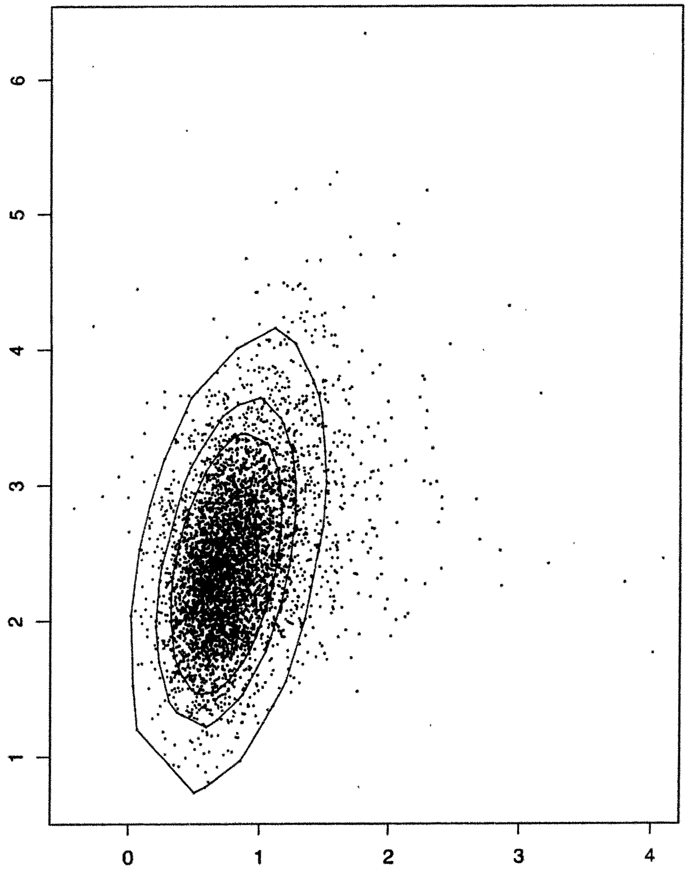

Fig. 3. 75%, 85% and 95% symmetric confidence regions

Lemma 2. For $i = 1, 2,$

$$\int_{E_i} \mathrm{d}K_n - \int_{\{y_i\}^\perp} \mathrm{d}K_n = o(n^{-1/2}) \tag{3.11}$$

and

$$\int_{E_i^*} \mathrm{d}K_n^* - \int_{\{y_i^*\}^\perp} \mathrm{d}K_n^* = o_p(n^{-1/2}). \tag{3.12}$$

Lemma 3. For $i = 1, 2,$

$$\int_{E_i^*} \mathrm{d}K_n - \int_{E_i} \mathrm{d}K_n = o(n^{-1/2}) \qquad \text{almost surely.} \tag{3.13}$$

For lemmas 1–3, second-order balancedness of $W_{n,1-\alpha}^*$ follows immediately (see Appendix A).

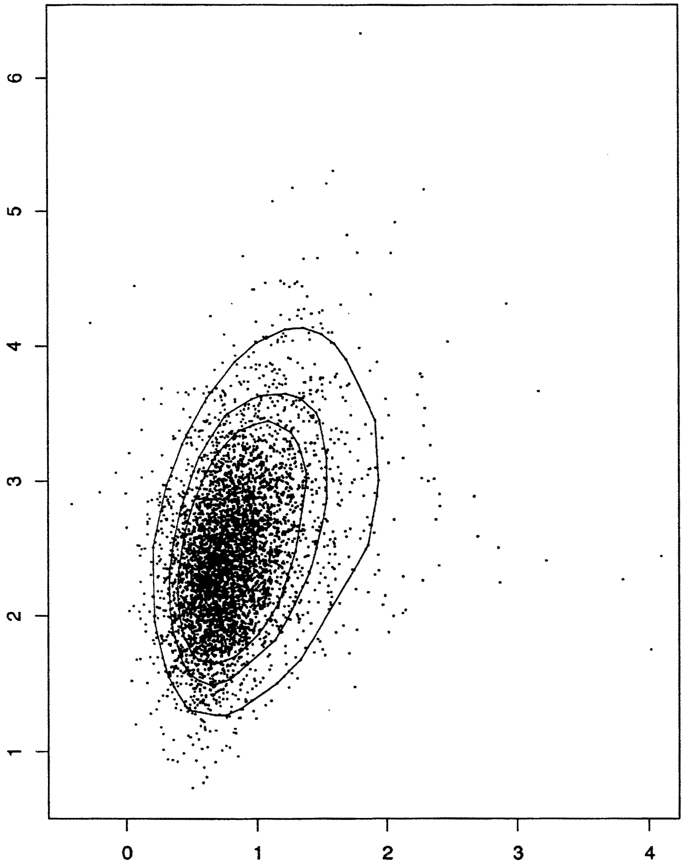

Fig. 4. 75%, 85% and 95% confidence regions produced by the Mahalanobis distance

4. SIMULATION STUDY

Let $X = (Z_1^2, Z_2^2)$, where (Z_1, Z_2) is a bivariate normal distribution with mean 0, $\sigma(Z_1) = 1$, $\sigma(Z_2) = 2$ and $\mathrm{corr}(Z_1, Z_2) = 0.8$. Let $\mu = EX$ and we would like to construct 75%, 85% and 95% confidence regions for μ. The distribution of X, unlike the bivariate normal distribution, is heavily skewed. Fig. 1 displays the scatterplot of a random sample of 5000 observations from the distribution of X. We notice that this distribution has a heavy tail towards the upper right-hand corner of the first quadrant. To construct the bootstrap confidence regions, we start with a random sample of size 20 which is chosen independently from those displayed in Fig. 1. The scatterplot of this random sample of size 20 (Fig. 2) reveals a little information on the skewness of the underlying distribution though not as dramatic as Fig. 1. This skewed distribution is chosen to demonstrate the applicability of our proposed method in dealing with asymmetric distributions.

The bootstrap is based on T_n and the bootstrap iteration is equal to 5000. There are several competitors in the literature. Here we choose two alternatives with which our method would be compared. One is based on the Mahalanobis depth (Mahalanobis,

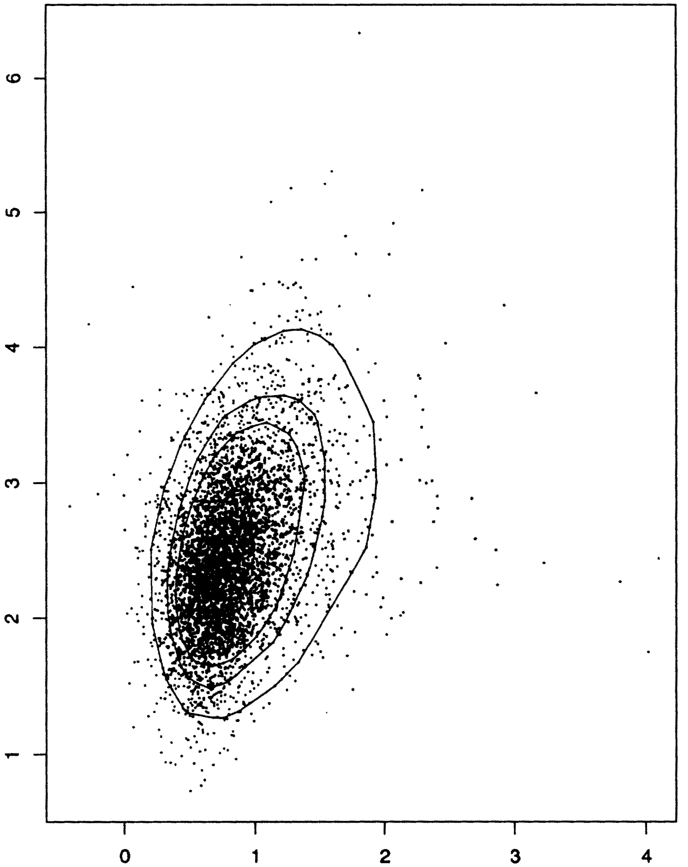

Fig. 5. 75%, 85% and 95% confidence regions produced by Tukey's depth

1936) since it is widely used for its computational simplicity. The other is based on the so-called symmetric method (Hall, 1992) since it is known to have a coverage error of the order $O(n^{-2})$. We construct $A^*_{n,1-\alpha}$s (see equation (1.2)) for $\alpha = 0.05$, 0.15 and 0.25 based on the symmetric method; these are shown in Fig. 3. Next, we construct $A^*_{n,1-\alpha}$s for $\alpha = 0.05$, 0.15 and 0.25 based on the Mahalanobis distance. These $A^*_{n,1-\alpha}$s are plotted in Fig. 4.

The $A^*_{n,1-\alpha}$s based on the symmetric method (Fig. 3) have the smallest coverage errors and are always elliptical. They, however, do not reflect the true underlying distribution which is heavily skewed. This observation also applies to $A^*_{n,1-\alpha}$s constructed on the basis of the Mahalanobis distance (Fig. 4). The elliptically shaped $A^*_{n,1-\alpha}$s under the Mahalanobis distance do not reveal the skewness of the underlying distribution from which the sample is chosen. The story is quite different when we look at the $A^*_{n,1-\alpha}$s shown in Fig. 5 which are constructed on the basis of Tukey's depth, which have the ability to reflect the true shape of the underlying distribution especially when the distribution is asymmetric.

Apparently, there is a trade-off between capturing the first-order departure from symmetry and obtaining a better coverage probability. From a practical point of view, any departure from symmetry could be considered vital to the investigator. Our Tukey depth-based bootstrap confidence regions have the ability to reflect that as can be seen in our simulation study.

5. CONCLUDING REMARKS

There are several notions of data depth in the literature, e.g. simplicial depth and majority depth. Will the confidence regions constructed on the basis of these depths achieve some balancing property in the sense discussed in this paper or in some other sense? A thorough investigation along these lines should be worthwhile.

In our simulation study, bootstrap confidence regions based on the simplicial and majority depths are not included. This is partly due to the difficulties encountered when computing these depths in higher dimensions. In a recent study by Rousseeuw and Ruts (1992), an algorithm for computing bivariate simplicial depth is provided. As for majority depth, a computational algorithm is not yet available for R^p, $p \geqslant 2$. Even for Tukey's depth, a computational algorithm is available only for R^2. Algorithms for computing depths in higher dimensions need to be developed. A study in this direction would be valuable.

ACKNOWLEDGEMENTS

We are grateful to the Editor and two referees for their encouragement and kind suggestions. We wish to thank the referees for their patience and careful reading of the manuscript during the review process. In particular, we thank one referee who pointed out some errors in an earlier version of the paper and made numerous suggestions, which greatly improved the presentation of the paper.

APPENDIX A

A.1. *Proof of Theorem 1*

We shall prove expression (3.5) for the case when $p = 2$. Let us begin with

$$\overline{\lim_{n\to\infty}}\,(W^*_{n,1-\alpha}) \subseteq C_{\gamma_{1-\alpha}} \qquad \text{almost surely.} \tag{A.1}$$

We first observe that, under the assumptions,

$$\sup_{x\in R^2} |K^*_n(x) - \Phi(x)| \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{almost surely} \tag{A.2}$$

and

$$\sup_{x\in R^2} |\mathrm{TD}(K^*_n, x) - \mathrm{TD}(\Phi, x)| \overset{n\to\infty}{\longrightarrow} 0 \qquad \text{almost surely.} \tag{A.3}$$

For a proof of expression (A.2), see chapter 5 of Hall (1992). As for expression (A.3), see Liu (1990) and Liu and Singh (1993). Suppose that, for some large n, $y \in \cup_{k=n}^{\infty} W^*_{k,1-\alpha}$; then there exists some $m_n \geqslant n$ such that $y \in W^*_{m_n,1-\alpha}$. By expression (A.3), we have

$$|\mathrm{TD}(K^*_{m_n}, y) - \mathrm{TD}(\Phi, y)| \leqslant \delta_{m_n} \qquad \text{almost surely,} \tag{A.4}$$

where δ_{m_n} depends only on n but not on y and $\delta_{m_n} \to 0$ as $n \to \infty$. But this implies that, for large n,

$$\mathrm{TD}(\Phi, y) \geqslant \mathrm{TD}(K^*_{m_n}, y) - \delta_{m_n} \qquad \text{almost surely.} \tag{A.5}$$

Since $W^*_{k,1-\alpha}$ never exceeds the $100(1-\alpha)$% level and, for any fixed y, $\{\mathrm{TD}(K^*_k, y), k \geqslant n\}$ is a Cauchy sequence, $\mathrm{TD}(K^*_{m_n}, y)$ can be bounded from below by $c_0 - \eta_{m_n}$, i.e.

$$\mathrm{TD}(K^*_{m_n}, y) \geqslant c_0 - \eta_{m_n} \qquad \text{almost surely,} \tag{A.6}$$

where $\mathrm{TD}(\Phi, x) = c_0$ for $|x| = \gamma_{1-\alpha}$ and $\eta_{m_n} \to 0$ as $n \to \infty$. Combining inequalities (A.5) and (A.6), we obtain, for large n,

$$\mathrm{TD}(\Phi, y) \geqslant c_0 - \epsilon_{m_n} \qquad \text{almost surely,} \tag{A.7}$$

where ϵ_{m_n} depends only on n and $\epsilon_{m_n} \to 0$ as $n \to \infty$. Inequality (A.7) implies that $y \in C_{\gamma_{1-\alpha}+\gamma_{m_n}}$, where $\mathrm{TD}(\Phi, x) = c_0 - \epsilon_{m_n}$ for $|x| = \gamma_{1-\alpha} + \gamma_{m_n}$ and $\gamma_{m_n} \to 0$ as $n \to \infty$. This, together with the supposition, implies that

$$\bigcup_{k=n}^{\infty} W^*_{k,1-\alpha} \subseteq C_{\gamma_{1-\alpha}+\gamma_{m_n}} \qquad \text{almost surely.}$$

Now, letting $n \to \infty$, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} W^*_{n,1-\alpha} \subseteq \bigcap_{n=1}^{\infty} C_{\gamma_{1-\alpha}+\gamma_{m_n}} \qquad \text{almost surely.}$$

Therefore, expression (A.1) follows.

As for the other direction, we proceed as follows. Let $y \in C^{\mathrm{int}}_{\gamma_{1-\alpha}}$, i.e. $|y| < \gamma_{1-\alpha}$. We claim that $y \in W^*_{n,1-\alpha}$ for all $n \geqslant n_0(\omega)$ almost surely (i.e. n_0 depends on the sample path).

To establish the claim, we show that

$$P_{K^*_n}\{x: \mathrm{TD}(K^*_n, x) \geqslant \mathrm{TD}(K^*_n, y)\} < 1 - \alpha, \tag{A.8}$$

for all $n \geqslant n_0(\omega)$. Note that, for all n sufficiently large, $\mathrm{TD}(K^*_n, y) \geqslant c_0 + \epsilon$ for some $\epsilon > 0$. Thus it suffices to show that, for all large n,

$$P_{K^*_n}\{x: \mathrm{TD}(K^*_n, x) \geqslant c_0 + \epsilon\} < 1 - \alpha.$$

For this, note that, for all large n,

$$\{x: \mathrm{TD}(K^*_n, x) \geqslant c_0 + \epsilon\} \subseteq C_{\gamma_{1-\alpha}-\epsilon_0},$$

for some $\epsilon_0 > 0$. The last claim uses the convergence of $\mathrm{TD}(K^*_n, \cdot)$ to $\mathrm{TD}(\Phi, \cdot)$. Therefore inequality (A.8) follows and the proof is completed.

We now proceed to prove lemmas 1–3, assuming $i = 1$.

A.2. *Proof of Lemma 1*

Note that

$$\mathrm{TD}(K_n, y_1) = \inf_A \left\{ \int_A \mathrm{d}K_n(x) \right\},$$

where A is a half-closed space containing y_1. But this can be rewritten as, by equation (3.3),

$$\inf_A \left\{ \int_A \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_A s_1(x)\,\phi(x)\,\mathrm{d}x + O(n^{-1}) \right\} = \inf_A \left\{ \int_A \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_A s_1(x)\,\phi(x)\,\mathrm{d}x \right\} + O(n^{-1}).$$

Let us suppose that A' is such a half-closed space, i.e.

$$\begin{aligned} \inf_A \left\{ \int_A \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_A s_1(x)\,\phi(x)\,\mathrm{d}x \right\} &= \int_{A'} \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_{A'} s_1(x)\,\phi(x)\,\mathrm{d}x \\ &\leqslant \int_{\{y_1\}^\perp} \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_{\{y_1\}^\perp} s_1(x)\,\phi(x)\,\mathrm{d}x \\ &\leqslant \int_{A'} \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_{\{y_1\}^\perp} s_1(x)\,\phi(x)\,\mathrm{d}x. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\left| \int_{\{y_1\}^\perp} \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_{\{y_1\}^\perp} s_1(x)\,\phi(x)\,\mathrm{d}x - \inf_A \left\{ \int_A \phi(x)\,\mathrm{d}x + \frac{1}{\sqrt{n}} \int_A s_1(x)\,\phi(x)\,\mathrm{d}x \right\} \right| \\ &\qquad \leqslant \frac{1}{\sqrt{n}} \left| \int_{A'} s_1(x)\,\phi(x)\,\mathrm{d}x - \int_{\{y_1\}^\perp} s_1(x)\,\phi(x)\,\mathrm{d}x \right| \\ &\qquad = o(n^{-1/2}). \end{aligned}$$

The last bound arises from the fact that K_n converges to Φ in distribution. Therefore, equation (3.7) follows from the above bound and equation (3.3). As for equation (3.8), the proof is similar.

A.3. *Proof of Lemma 2*

It can easily be seen from lemma 1 and equation (3.9) that

$$\int_{E_1} \mathrm{d}K_n - \int_{\{y_1\}^\perp} \mathrm{d}K_n = \int_{E_1} \phi(x)\,\mathrm{d}x - \int_{\{y_1\}^\perp} \phi(x)\,\mathrm{d}x + o(n^{-1/2}).$$

Note that the radii of $\{y_1\}^\perp$ and E_1 only differ by the order of $o(n^{-1/2})$. To see this, suppose that the radii of $\{y_1\}^\perp$ and E_1 differ by $c/\sqrt{n}$. Consider the point at the boundary of E_1 which has the shortest distance from the centre, and call this point $\tilde{y}_1$. By equation (3.7) (lemma 1),

$$\mathrm{TD}(K_n, \tilde{y}_1) = \mathrm{TD}(K_n, y_1) + \frac{c'}{\sqrt{n}} + o(n^{-1/2}),$$

with some $c' > 0$. This implies that the point $\tilde{y}_1$ is in the interior of $W_{n,1-\alpha}$, which contradicts the definition of E_1. Therefore, equation (3.11) follows. As for equation (3.12), the proof is similar.

A.4. *Proof of Lemma 3*

From lemma 2 and the assumptions, one can show that

$$\int_{E_1^*} \mathrm{d}K_n - \int_{E_1} \mathrm{d}K_n = \int_{\{y_1^*\}^\perp} \mathrm{d}K_n - \int_{\{y_1\}^\perp} \mathrm{d}K_n + o(n^{-1/2}) \qquad \text{almost surely}$$

$$= \int_{\{y_1^*\}^\perp} \phi(x)\,\mathrm{d}x - \int_{\{y_1\}^\perp} \phi(x)\,\mathrm{d}x + o(n^{-1/2}) \qquad \text{almost surely.}$$

Note that $\hat{v}_1 - v_1 = o_p(1)$ ($\hat{v}_1$ and v_1 typically differ by at least second order although $o_p(1)$ is all that we need here). Thus, equation (3.13) follows from equations (3.9), (3.10) and the above observation.

A.5. *Proof of Theorem 2*

First note that we can write, for $i = 1, 2$,

$$\int_{E_i^*} \mathrm{d}K_n - \mathrm{TD}(K_n, y_i) = \left(\int_{E_i^*} \mathrm{d}K_n - \int_{E_i} \mathrm{d}K_n\right) + \left(\int_{E_i} \mathrm{d}K_n - \int_{\{y_i\}^\perp} \mathrm{d}K_n\right)$$

$$+ \left\{\int_{\{y_i\}^\perp} \mathrm{d}K_n - \mathrm{TD}(K_n, y_i)\right\}.$$

We can easily conclude assertion (3.6) by lemmas 1–3, and noting that $\mathrm{TD}(K_n, y_1) = \mathrm{TD}(K_n, y_2)$.

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