ELSEVIER
Statistics & Probability Letters 53 (2001) 235–239
STATISTICS & PROBABILITY LETTERS
www.elsevier.nl/locate/stapro

On inverse moments of nonnegative random variables

Nancy Lopes Garcia[a,*], José Luis Palacios[b]

[a] IMECC/UNICAMP, Caixa Postal 6065, 13.081-970 Campinas, SP, Brazil
[b] Universidad Simón Bolívar - Caracas, Venezuela

Received June 2000; received in revised form November 2000

Abstract

We give sufficient conditions under which

$$\mathbb{E}\left[(1+X_n)^{-\alpha}\right] \approx (1+\mathbb{E}X_n)^{-\alpha},$$

for a sequence of nonnegative random variables and $\alpha > 0$.

MSC: primary: 60E05; secondary: 62E20

Keywords: Inverse moments

1. Introduction

Sometimes there is a need for evaluating moments of the form

$$\mathbb{E}\left[\frac{1}{(1+X)^{\alpha}}\right], \tag{1.1}$$

for a nonnegative random variable X and $\alpha > 0$. Garcia and Palacios (2000), for instance, needed to evaluate such an inverse moment for $X \sim \mathrm{Bin}(n, \frac{1}{2})$ when finding couplings of random walks on n-dimensional cubes, and our original motivation there as well as that of this note is to find the right order of magnitude (that is, upper and lower bounds that converge to the same limit) , as a function of n, of (1.1) when X is a sequence of random variables $\{X_n\}$. We show here that, under certain natural conditions, this order of magnitude is given by the (usually much easier to compute) expression $1/(1+\mathbb{E}X_n)^{\alpha}$, or more precisely, we prove that

$$\lim_{n\to\infty} \frac{\mathbb{E}[(1+X_n)^{-\alpha}]}{(1+\mathbb{E}X_n)^{-\alpha}} = 1. \tag{1.2}$$

The variations on the theme "approximating the inverse moment by the inverse of the moment" are many, and we will give in this introduction a brief discussion without attempting to be exhaustive. An application of Schwarz's inequality to any positive random variable yields $\mathbb{E}(1/X) \geqslant 1/\mathbb{E}X$; a good deal of attention

* Corresponding author.

0167-7152/01/$ - see front matter
PII: S0167-7152(01)00008-6

(Lew, 1976; Zacks, 1980; Jones, 1986) has been placed in obtaining better lower and upper bounds and exact values for some inverse moments of particular variables, as the positive Binomial, positive Poisson, positive Hypergeometric, etc., defined as the appropriate normalizations of the usual variables minus the value 0; these variables are of interest in certain problems of random sampling, reliability and life testing. The works of Wooff (1985) and Pittenger (1990) are the closest to the present note. They develop distribution-free methods to find upper and lower bounds for (1.1) in the case where α is a positive integer with applications in Stein estimation and in classical and Bayesian post-stratification. In fact, their methods differ but the bounds coincide. However, for large n they do not achieve the right order of magnitude claimed by this work given by (1.2). The problem of the existence of the inverse moments $\mathbb{E}X^{-\alpha}$ is addressed in Piegorsch and Casella (1985), one of the few references where α is not necessarily a positive integer. In our work, we do not need to worry about the existence of the moments, since $X \geqslant 0$ implies $0 \leqslant 1/(1+X)^{\alpha} \leqslant 1$ and, therefore, we can always guarantee the existence of the α-moment.

Exact computations for the inverse moments usually follow the method found in Chao and Strawderman (1972) and Cressie et al. (1981), when α is a nonnegative integer, which consists of successively integrating the moment generating function $M_X(t)$ of X, namely:

$$\mathbb{E}\left[\frac{1}{(1+X)^k}\right] = \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty M_X(-t_k)\mathrm{e}^{-t_k}\,\mathrm{d}t_k \cdots \mathrm{d}t_2\,\mathrm{d}t_1. \tag{1.3}$$

Performing the changes of variables $\mathrm{e}^{-t_i} = s_i, 1 \leqslant i \leqslant k$, renders the alternative recursive expression for (1.3) found in Chao and Strawderman (1972). Also, by exchanging successively the orders of integration in (1.3) there is yet another expression:

$$\mathbb{E}\left[\frac{1}{(1+X)^k}\right] = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}\mathrm{e}^{-t}M_X(-t)\,\mathrm{d}t. \tag{1.4}$$

The more recent reference Adell et al. (1996) gives still another method, closely related to the previous one: for $a > 0$, k a positive integer and X a nonnegative random variable,

$$\mathbb{E}\left(\frac{a}{a+X}\right)^k = \int_0^1 \mathbb{E}s^X \frac{a^k}{(k-1)!}s^{a-1}\log^{k-1}(1/s)\,\mathrm{d}s. \tag{1.5}$$

Whichever alternative expression is chosen, carrying out the integrations may not be easy; thus, for instance, when $X \sim \mathrm{Bin}(n,p)$, $M_X(t) = [1-p+p\mathrm{e}^t]^n$ and neither (1.3) nor (1.4) nor (1.5) yield closed-form expressions except when $k=1$, as was observed in Chao and Strawderman (1972), in which case we get

$$\mathbb{E}\left[\frac{1}{(1+X)}\right] = \frac{1}{p(n+1)}[1-(1-p)^{n+1}].$$

Lacking such closed-form expressions for (1.1), even when α is an integer, the next best thing to have is a good approximation. As early as Stephan (1945) and as recent as Marciniak and Wesolowski (1999), there has been interest in expanding inverse integer moments as an infinite series. Our interest in the next section, as mentioned before, will be in "approximating the inverse moment by the inverse of the moment" for any real $\alpha > 0$.

2. Approximating $\mathbb{E}[1/(1+X)^{\alpha}]$

We will consider in this note two different prototypes for X: either it behaves like a sum of i.i.d.'s, or like an average of i.i.d.'s. For the first prototype we have the following:

Theorem 2.1. *Let X_n be a sequence of nonnegative random variables and let μ_n and σ_n^2 be constants such that*

(i) $\dfrac{X_n-\mu_n}{\sigma_n} \Rightarrow \mathrm{N}(0,1)$ *as* $n\to\infty$,

(ii) $\mathbb{E}X_n-\mu_n \to 0$ *and* $\mu_n\to\infty$ *as* $n\to\infty$,

(iii) *there is* $\varepsilon<1$ *such that* $\mu_n^\varepsilon/\sigma_n>1$, *for n sufficiently large.*

Then

$$\frac{1/(1+\mu_n)^\alpha}{\mathbb{E}\,[1/(1+X_n)^\alpha]} \to 1, \tag{2.1}$$

as $n\to\infty$ *for any* $\alpha>0$.

Proof. On account of the function $f(x)=1/(1+x)^\alpha$ being convex, Jensen's inequality (see, for example, Chung, 1974, p. 47), gives the lower bound:

$$\mathbb{E}\left[\frac{1}{(1+X_n)^\alpha}\right] \geqslant \frac{1}{(1+\mathbb{E}X_n)^\alpha} \approx \frac{1}{(1+\mu_n)^\alpha}. \tag{2.2}$$

For the upper bound we take $\delta=(1+\varepsilon)/2$ and argue:

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{(1+X_n)^\alpha}\right] &= \mathbb{E}\left[\frac{1}{(1+X_n)^\alpha}\mathbf{1}_{[X_n<\mu_n-\mu_n^\delta]}\right] + \mathbb{E}\left[\frac{1}{(1+X_n)^\alpha}\mathbf{1}_{[X_n\geqslant\mu_n-\mu_n^\delta]}\right]\\
&\leqslant \mathbb{P}[X_n<\mu_n-\mu_n^\delta] + \frac{1}{(1+\mu_n-\mu_n^\delta)^\alpha}\mathbb{P}[X_n\geqslant\mu_n-\mu_n^\delta]\\
&\leqslant \Phi\left(-\frac{\mu_n^\delta}{\sigma_n}\right) + \frac{1}{(1+\mu_n-\mu_n^\delta)^\alpha}\\
&\approx \mathrm{e}^{-\mu_n^{1+\varepsilon}/\sigma_n^2} + \frac{1}{(1+\mu_n-\mu_n^\delta)^\alpha}\\
&< \mathrm{e}^{-\mu_n^{1-\varepsilon}} + \frac{1}{(1+\mu_n-\mu_n^\delta)^\alpha}
\end{aligned} \tag{2.3}$$

and the first summand of (2.3) goes to zero faster than the second summand, whose order of magnitude is precisely $1/(1+\mu_n)^\alpha$, finishing the proof. □

Clearly, if $X_n=\sum_{i=1}^n Y_i$, where the Y_i's are nonnegative i.i.d.'s, with mean μ and variance σ^2, the hypotheses of Theorem 2.1 are fulfilled for $\mu_n=n\mu, \sigma_n^2=n\sigma^2$ and any $\frac{1}{2}<\varepsilon<1$. Notice that in this case, the upper bound given by Wooff (1985) and Pittenger (1990) is

$$\frac{\sigma^2}{\sigma^2+n\mu^2} + \frac{n\mu^2}{\sigma^2+n\mu^2}\left(1+n\mu+\frac{\sigma^2}{\mu}\right)^{-\alpha} \tag{2.4}$$

which is larger than (2.3). In fact, we can rewrite (2.4) as $S_1(n)+S_2(n)$, where

$$S_1(n)=\frac{\sigma^2}{\sigma^2+n\mu^2} \quad\text{and}\quad S_2(n)=\frac{n\mu^2}{\sigma^2+n\mu^2}\left(1+n\mu+\frac{\sigma^2}{\mu}\right)^{-\alpha}.$$

Our bound (2.3) can be written as $T_1(n)+T_2(n)$, where

$$T_1(n)=\mathrm{e}^{-\mu_n^{1-\varepsilon}} \quad\text{and}\quad T_2(n)=\frac{1}{(1+n\mu-(n\mu)^\delta)^\alpha}$$

or

$$T_1(n) = \frac{1}{e^{n^\beta}} \quad \text{with } \beta > 0.$$

Notice that the second summands of both bounds have the same order: $S_2(n)/T_2(n) \to 1$ as $n \to \infty$, but our first summand goes to zero much faster than their first summand, that is $T_1(n) \ll S_1(n)$. Therefore, (2.4) does not achieve the right magnitude order for $\alpha > 1$ (that is the upper bound and the lower bound do not converge to the same limit); this follows from the fact that when $\alpha > 1$, their order of magnitude is given by $S_1(n)$ and our order of magnitude is always $T_2(n)$.

For example, in the case that $X \sim \mathrm{Bin}(n, p)$, our bound (2.3) gives

$$\mathbb{E}\left[\frac{1}{(1+X)^\alpha}\right] \approx \frac{1}{(1+np)^\alpha},$$

whereas (2.4) gives

$$\frac{(1-p)}{(1-p)+np} + \frac{np}{(1-p)+np}(1+np+(1-p))^{-\alpha}.$$

Remark. Condition (i) of Theorem 2.1 can be replaced by a large deviation-type result that makes $\mathbb{P}(X_n < \mu_n - \mu_n^\delta)$ go to zero faster than the second summand of (2.3). Also, notice that the condition $\mathbb{E}X_n - \mu_n \to 0$ as $n \to \infty$ is not necessary for the upper bound.

For the second prototype for X we shall use the so-called delta method (see van der Vaart, 1998, Theorem 3.1, p. 26), which states that if

$$\sqrt{n}(X_n - \mu) \Rightarrow \mathrm{N}(0, \sigma^2), \tag{2.5}$$

then for any $g : \mathbb{R} \to \mathbb{R}$ such that $g'(\mu) \neq 0$ we have

$$\sqrt{n}(g(X_n) - g(\mu)) \Rightarrow \mathrm{N}(0, (g'(\mu))^2\sigma^2). \tag{2.6}$$

Then we can prove the following

Theorem 2.2. *Let X_n be a sequence of nonnegative random variables, and μ, $\sigma^2 > 0$ two constants such that* (2.5) holds. *Then the conclusion in* (2.1) *holds with $\mu_n = \mu$.*

Proof. Take $g(x) = (1+x)^{-\alpha}$. Since (2.5) holds, then by (2.6) we have that

$$\sqrt{n}((1+X_n)^{-\alpha} - (1+\mu)^{-\alpha}) \Rightarrow \mathrm{N}(0, \alpha^2(1+\mu)^{-2(\alpha+1)}\sigma^2). \tag{2.7}$$

Now (2.7) implies that $(1+X_n)^{-\alpha}$ converges in probability to $(1+\mu)^{-\alpha}$, and this together with the fact that $(1+X_n)^{-\alpha}$ is bounded above by 1, implies (2.1), with $\mu_n = \mu$, via the dominated convergence theorem (see, for example, Chung, 1974, p. 42). □

Clearly, Theorem 2.7 applies when $X_n = (1/n)\sum_{i=1}^n Y_i$, where the Y_i's are i.i.d. variables with mean μ and variance σ^2.

As a final comment, it is easy to provide examples where (2.1) fails to hold for $\mu_n = \mathbb{E}X_n$. In fact, if $X_n \sim U(0, n)$, one can compute directly

$$\mathbb{E}\left[\frac{1}{(1+X_n)^\alpha}\right] = \frac{1}{\alpha n}[1 - (1+n)^{-\alpha+1}],$$

whereas

$$\frac{1}{(1+\mathbb{E}X_n)^\alpha} = \left(\frac{2}{2+n}\right)^\alpha.$$

References

Adell, J.A., de la Cal, J., Pérez-Palomares, A., 1996. On the Cheney and Sharma operator. J. Math. Anal. Appl. 200, 663–679.

Chao, M.T., Strawderman, W.E., 1972. Negative moments of positive random variables. J. Amer. Statist. Soc. 67, 429–431.

Chung, K.L., 1974. A Course in Probability Theory, 2nd Edition. Academic Press. New York.

Cressie, N., Davis, A.S., Folks, J.L., Policello, G.E., 1981. The moment-generating function and negative integer moments. The Amer. Statist. 35, 148–150.

Garcia, N.L., Palacios, J.L., 2000. On mixing times for stratified walks on the d-cube. Preprint in Los Alamos Archives. http://arXiv.org/abs/physics/0003006.

Jones, M.C., 1986. Inverse moments of negative-binomial distributions. J. Statist. Comput. Simulation 23, 241–242.

Lew, R.A., 1976. Bounds on negative moments. SIAM J. Appl. Math. 30, 728–731.

Marciniak, E., Wesolowski, J., 1999. Asymptotic Eulerian expansions for binomial and negative binomial reciprocals. Proc. Amer. Math. Soc. 127, 3329–3338.

Piegorsch, W.W., Casella, G., 1985. The existence of the 1st negative moment. The Amer. Statist. 39, 60–62.

Pittenger, A.O., 1990. Sharp mean-variance bounds for Jensen-type inequalities. Statist. Probab. Lett. 10 (1), 91–94.

Stephan, F.F., 1945. The expected value and variance of the reciprocal and other negative values of a positive Bernoullian variate. Ann. Math. Statist. 16, 50–61.

Wooff, D.A., 1985. Bounds on reciprocal moments with applications and developments in Stein estimation and post-stratification. J. Roy. Statist. Soc. Ser. B 47, 362–371.

van der Vaart, A.W., 1998. Asymptotic Statistics. Cambridge University Press, New York, NY.

Zacks, S., 1980. On some inverse moments of negative-binomial distributions and their application in estimation. J. Statist. Comput. Simulation 10, 163–165.