Electronic Journal of Statistics
Vol. 0 (0000)
ISSN: 1935-7524
DOI: 10.1214/154957804100000000

On Weighted Multivariate Sign Functions: Supplementary Material

Subhabrata Majumdar* **and Snigdhansu Chatterjee†**

School of Statistics,
University of Minnesota,
313 Ford Hall,
224 Church Street S.E.,
Minneapolis 55455, USA
e-mail: subho@research.att.com; chatt019@umn.edu

Appendix A: Form of V_W

First observe that for F having covariance matrix $\Sigma = \Gamma\Lambda\Gamma^T$,

$$V_W = (\Gamma \otimes \Gamma) V_{W,\Lambda} (\Gamma \otimes \Gamma)^T \tag{1}$$

where $V_{W,\Lambda}$ is the covariance matrix of $\mathbb{F}_\Lambda$, the elliptic distribution with mean μ and covariance matrix Λ. Now,

$$\begin{aligned} V_{W,\Lambda} &= \mathbb{E}\left[\text{vec}\left\{ \frac{W^2(Z,\mathbb{F}_Z)\Lambda^{1/2}ZZ^T\Lambda^{1/2}}{Z^T\Lambda Z} - \tilde\Lambda \right\} \text{vec}^T \left\{ \frac{W^2(Z,\mathbb{F}_Z)\Lambda^{1/2}ZZ^T\Lambda^{1/2}}{Z^T\Lambda Z} - \tilde\Lambda \right\} \right] \\ &= \mathbb{E}\left[\text{vec}\left\{ W^2(Z,\mathbb{F}_Z)\mathbb{S}(\Lambda^{1/2}z;\mathbf{0}) \right\} \text{vec}^T\left\{ W^2(Z,\mathbb{F}_Z)\mathbb{S}(\Lambda^{1/2}Z;\mathbf{0}) \right\} \right] \\ &- \quad \text{vec}(\tilde\Lambda)\text{vec}^T(\tilde\Lambda) \end{aligned}$$

The matrix $\text{vec}(\tilde\Lambda)\,\text{vec}^T(\tilde\Lambda)$ consists of elements $\lambda_i\lambda_j$ at $(i,j)^{\text{th}}$ position of the $(i,j)^{\text{th}}$ block, and 0 otherwise. These positions correspond to variance and covariance components of on-diagonal elements. For the expectation matrix, all its elements are of the form $\mathbb{E}[\sqrt{\lambda_a\lambda_b\lambda_c\lambda_d} Z_a Z_b Z_c Z_d . W^4(Z,\mathbb{F}_Z)/(Z^T\Lambda Z)^2]$, with $1 \leq a,b,c,d \leq p$. Since $W^4(Z,\mathbb{F}_Z)/(Z^T\Lambda Z)^2$ is even in Z, which has a spherically symmetric distribution, all such expectations will be 0 unless a,b,c,d are all equal or pairwise equal. Following a similar derivation for spatial sign covariance matrices in [2], we collect the non-zero elements and write the matrix of expectations:

$$(\mathbb{I}_{p^2} + \mathbb{K}_{p,p}) \left\{ \sum_{a=1}^p \sum_{b=1}^p \tilde\gamma_{ab}(e_a e_a^T \otimes e_b e_b^T) - \sum_{a=1}^p \tilde\gamma_{aa}(e_a e_a^T \otimes e_a e_a^T) \right\} + \sum_{a=1}^p \sum_{b=1}^p \tilde\gamma_{ab}(e_a e_b^T \otimes e_a e_b^T)$$

where $\mathbb{I}_k = (e_1,...,e_k), \mathbb{K}_{m,n} = \sum_{i=1}^m \sum_{j=1}^n \mathbb{J}_{ij} \otimes \mathbb{J}_{ij}^T$ with $\mathbb{J}_{ij} \in \mathbb{R}^{m\times n}$ having 1 as (i,j)-th element and 0 elsewhere, and $\tilde\gamma_{mn} = \mathbb{E}[\lambda_m\lambda_n Z_m^2 Z_n^2 . W^4(Z,\mathbb{F}_Z)/(Z^T\Lambda Z)^2]; 1 \leq m,n \leq p$.

*Currently at AT&T Labs Research
†Corresponding author

Putting everything together, denote by $\hat{\tilde{\Lambda}}$ the sample version of $\tilde{\Lambda}$, the weighted covariance matrix obtained from $\mathbb{F}_\Lambda$, i.e. $\hat{\tilde{\Lambda}} = \sum_{i=1}^n W_n^2(Z_i, \mathbb{F}_{Z,n}) \mathbb{S}(\Lambda^{1/2} Z_i; \hat{\mu}_n)/n$. Then the different types of elements in the matrix $\hat{\tilde{\Lambda}}$ are as given below ($1 \leq a, b, c, d \leq p$):

- Variance of on-diagonal elements

$$A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(a,a)) = \mathbb{E}\left[\frac{W^4(Z, \mathbb{F}_Z)\lambda_a^2 Z_a^4}{(Z^T \Lambda Z)^2}\right] - \tilde{\lambda}_a^2$$

- Variance of off-diagonal elements ($a \neq b$)

$$A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(a,b)) = \mathbb{E}\left[\frac{W^4(Z, \mathbb{F}_Z)\lambda_a \lambda_b Z_a^2 Z_b^2}{(Z^T \Lambda Z)^2}\right]$$

- Covariance of two on-diagonal elements ($a \neq b$)

$$A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(a,a), \sqrt{n}\hat{\tilde{\Lambda}}(b,b)) = \mathbb{E}\left[\frac{W^4(Z, \mathbb{F}_Z)\lambda_a \lambda_b Z_a^2 Z_b^2}{(Z^T \Lambda Z)^2}\right] - \tilde{\lambda}_a \tilde{\lambda}_b$$

- Covariance of two off-diagonal elements ($a \neq b, c \neq d$)

$$A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(a,b), \sqrt{n}\hat{\tilde{\Lambda}}(c,d)) = 0$$

- Covariance of one off-diagonal and one on-diagonal element ($a \neq b \neq c$)

$$A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(a,b), \sqrt{n}\hat{\tilde{\Lambda}}(c,c)) = 0$$

The above give all the elements of $V_{W,\Lambda}$. We plug these in (1) to recover V_W.

Appendix B: Proofs

Proof sketch of Corollary 2.4. We start with slightly modified versions of Lemmas A.4 and A.5 in [4]:

Lemma B.1. *Given that condition 1 in Corollary 2.4 holds, we have* $\lambda_{\max}(\Psi_{1W}) \leq 2W_{\max} \frac{\lambda_{\max}(\Sigma)}{Tr(\Sigma)}(1 + o(1))$.

Lemma B.2. *Define* $\Psi_{3W} = \mathbb{E}\left[\frac{W^2(\epsilon, \mathbb{F}_\epsilon)}{|\epsilon|^2}(\mathbb{I}_p - S(\epsilon)S(\epsilon)^T)\right]$. *Then* $\lambda_{\max}(\Psi_{2W}) \leq W_{\max}\mathbb{E}(|\epsilon|^{-1})$ *and* $\lambda_{\max}(\Psi_{3W}) \leq (W_{\max})^2 \mathbb{E}(|\epsilon|^{-2})$. *Further, if conditions 1 and 2 of Corollary 2.4 hold then* $\lambda_{\min}(\Psi_{2W}) \geq \mathbb{E}(W(\epsilon, \mathbb{F}_\epsilon)/|\epsilon|)(1 + o(1))/\sqrt{3}$.

The lemmas can be proved using similar steps as the proofs of the original lemmas in [4] and using the upper bound on the weight function. Corollary 2.4 is now proved by applying Corollary 2.2, plugging in upper bound of $\lambda_{\max}(\Psi_{1W})$ from Lemma B.1 and lower bound of $\lambda_{\max}(\Psi_{1W})$ from Lemma B.2 into the ARE expression. □

Proof of Theorem 3.4. We suppose $G_n = (g_1, \ldots, g_p), L_n = \text{diag}(l_1, \ldots, l_p)$. In spirit, this corollary is similar to Theorem 13.5.1 in [1]. We start with the following result, due to [3], allows us to obtain asymptotic joint distributions of eigenvectors and eigenvalues of $\hat{\tilde{\Sigma}}$, provided we know the limiting distribution of $\hat{\tilde{\Sigma}}$ itself:

Theorem B.3. *Let $\mathbb{F}_\Lambda$ be defined as before, and $\hat{C}$ be any positive definite symmetric $p \times p$ matrix such that at F_Λ the limiting distribution of $\sqrt{n}\,\text{vec}(\hat{C} - \Lambda)$ is a p^2-variate (singular) normal distribution with mean zero. Write the spectral decomposition of $\hat{C}$ as $\hat{C} = \hat{P}\hat{\Lambda}\hat{P}^T$. Then the limiting distributions of $\sqrt{n}\,\text{vec}(\hat{P} - \mathbb{I}_p)$ and $\sqrt{n}\,\text{vec}(\hat{\Lambda} - \Lambda)$ are multivariate (singular) normal and*

$$\sqrt{n}\,\text{vec}(\hat{C} - \Lambda) = [(\Lambda \otimes \mathbb{I}_p) - (\mathbb{I}_p \otimes \Lambda)]\sqrt{n}\,\text{vec}(\hat{P} - \mathbb{I}_p) + \sqrt{n}\,\text{vec}(\hat{\Lambda} - \Lambda) + o_P(1) \tag{2}$$

The first matrix picks only off-diagonal elements of the left-hand side and the second one only diagonal elements. We shall now use this as well as the form of the asymptotic covariance matrix of the vectorized $\hat{\tilde{\Sigma}}$, i.e. V_W to obtain limiting variance and covariances of eigenvalues and eigenvectors.

Due to the decomposition (2) we have, for $\mathbb{F}_\Lambda$, the following relation between any off-diagonal element of $\hat{\tilde{\Lambda}}$ and the corresponding element in the estimate of eigenvectors, say $\hat{\tilde{\Gamma}}_\Lambda$:

$$\sqrt{n}\hat{\tilde{\gamma}}_{\Lambda,ij} = \sqrt{n}\frac{\hat{\tilde{\Lambda}}(i,j)}{\tilde{\lambda}_i - \tilde{\lambda}_j}; \quad i \neq j$$

So that for eigenvector estimates of the original $\mathbb{F}$ we have

$$\sqrt{n}(\hat{\tilde{\gamma}}_i - \gamma_i) = \sqrt{n}\Gamma(\hat{\tilde{\gamma}}_{\Lambda,i} - e_i) = \sqrt{n}\left[\sum_{k=1;k\neq i}^{p} \hat{\tilde{\gamma}}_{\Lambda,i,k}\gamma_k + (\hat{\tilde{\gamma}}_{\Lambda,i,i} - 1)\gamma_i\right] \tag{3}$$

Now $\sqrt{n}(\hat{\tilde{\gamma}}_{\Lambda,i,i} - 1) - 1) = o_P(1)$ and $A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(i,k), \sqrt{n}\hat{\tilde{\Lambda}}(i,l)) = 0$ for $k \neq l$, so the above equation implies

$$A\mathbb{V}(g_i) = A\mathbb{V}(\sqrt{n}(\hat{\tilde{\gamma}}_i - \gamma_i)) = \sum_{k=1;k\neq i}^{p} \frac{A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(i,k))}{(\tilde{\lambda}_i - \tilde{\lambda}_k)^2}\Gamma_k\Gamma_k^T$$

For the covariance terms, from (3) we get, for $i \neq j$,

$$\begin{aligned} A\mathbb{V}(g_i, g_j) &= A\mathbb{V}(\sqrt{n}(\hat{\tilde{\gamma}}_i - \gamma_i), \sqrt{n}(\hat{\tilde{\gamma}}_j - \gamma_j)) \\ &= A\mathbb{V}\left(\sum_{k=1;k\neq i}^{p} \sqrt{n}\hat{\tilde{\gamma}}_{\Lambda,ik}\gamma_k, \sum_{k=1;k\neq j}^{p} \sqrt{n}\hat{\tilde{\gamma}}_{\Lambda,jk}\gamma_k \right) \\ &= A\mathbb{V}\left(\sqrt{n}\hat{\tilde{\gamma}}_{\Lambda,ik}\gamma_j, \sqrt{n}\hat{\tilde{\gamma}}_{\Lambda,ik}\gamma_i \right) \\ &= -\frac{A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(i,j))}{(\tilde{\lambda}_i - \tilde{\lambda}_j)^2}\gamma_j\gamma_i^T \end{aligned}$$

The exact forms given in the statement of the corollary now follows from the Form of V_W in Section A.

For the on-diagonal elements of $\hat{\tilde{\Lambda}}$, using Theorem B.3 we have for the i^{th} eigenvalue of $\hat{\tilde{\Lambda}}$, say $\lambda_{\Lambda,i}$,

$$\sqrt{n}\hat{\tilde{\lambda}}_{\Lambda,i} = \sqrt{n}\hat{\tilde{\Lambda}}(i,i),$$

for $i = 1, ..., p$. Hence

$$\begin{aligned} A\mathbb{V}(l_i) &= A\mathbb{V}(\sqrt{n}(\hat{\tilde{\lambda}}_{\Lambda,i} - \tilde{\lambda}_i)) \\ &= A\mathbb{V}(\sqrt{n}(\hat{\tilde{\lambda}}_{\Lambda,i} - \tilde{\lambda}_{\Lambda,i})) \\ &= A\mathbb{V}(\sqrt{n}\hat{\tilde{\Lambda}}(i,i)) \end{aligned}$$

A similar derivation gives the expression for $A\mathbb{V}(l_i, l_j); i \neq j$. Finally, since the asymptotic covariance between an on-diagonal and an off-diagonal element of $\hat{\tilde{\Lambda}}$, it follows that the elements of G_n and diagonal elements of L_n are independent. $\square$

References

[1] Anderson, T. (2009). *An Introduction to Multivariate Statistical Analysis*. Wiley, Hoboken, NJ, 3 edition.

[2] Magyar, A. F. and Tyler, D. E. (2014). The asymptotic inadmissibility of the spatial sign covariance matrix for elliptically symmetric distributions. *Biometrika*, 101(3):673–688.

[3] Taskinen, S., Koch, I., and Oja, H. (2012). Robustifying principal component analysis with spatial sign vectors. *Statistics & Probability Letters*, 82(4):765 – 774.

[4] Wang, L., Peng, B., and Li, R. (2015). A high-dimensional nonparametric multivariate test for mean vector. *Journal of the American Statistical Association*, 110(512):1658 – 1669.