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Theory & Methods Section

Negative Moments of Positive Random Variables

M. T. CHAO and W. E. STRAWDERMAN*

We investigate the problem of finding the expected value of functions of a random variable X of the form $f(X)=(X+A)^{-n}$ where $X+A>0$ a.s. and n is a non-negative integer. The technique is to successively integrate the probability generating function and is suggested by the well-known result that successive differentiation leads to the positive moments. The technique is applied to the problem of finding $E[1/(X+A)]$ for the binomial and Poisson distributions.

1. INTRODUCTION

We investigate the problem of finding the expected value of functions of a random variable X, of the form

$$f(X) = (X + A)^{-n}, \tag{1.1}$$

where $X+A>0$ a.s., and n is a non-negative integer. The technique is to successively integrate the probability generating function, and is suggested by the well-known result that successive differentiation leads to the positive moments. We develop the technique in Section 2 and apply it to finding $E[1/(X+A)]$ for the binomial and Poisson distributions in Section 3. Section 4 contains some further observations.

Negative moments are useful in applications in several contexts, notably in life testing problems, and survey sampling problems where ratio estimates are used. See [1, 2, 3, 4, 6] for some applications.

The results in the literature seem to have been confined primarily to the case of the truncated Poisson and binomial distributions [5, 6, 8, 9].

2. THE BASIC RESULTS

Let X be a random variable defined on a probability space $(\mathfrak{X}, \mathcal{A}, P)$ and suppose $X+A>\delta>0$ a.s. $[P]$. Define the probability generating function of $X+A-1$ as

$$g_1(t) = E(t^{X+A-1}) \qquad 0 \leq t \leq 1. \tag{2.1}$$

Now inductively define $g_{k+1}(t)$, for $k=1, 2, \cdots$ as follows:

$$g_{k+1}(t) = t^{-1}\int_0^t g_k(u)du. \qquad 0 \leq t \leq 1. \tag{2.2}$$

Clearly (2.1) and (2.2) exist under the assumption on X. We have the following result.

Theorem 1. For $0 \leq t \leq 1$,

$$E\left[\left(\frac{1}{X+A}\right)^k t^{X+A}\right] = \int_0^t g_k(u)du.$$

Proof:

Since

$$\frac{1}{t}\int_0^t u^{\Delta-1}du = \frac{t^{\Delta-1}}{\Delta} \quad \text{for } \Delta > 0$$

we have

$$\frac{t^{\Delta-1}}{\Delta^k} = \frac{1}{t}\int_0^t \left[\frac{1}{t_k}\int_0^{t_k}\left[\cdots\left[\frac{1}{t_2} \cdot \int_0^{t_2} t_1^{\Delta-1}dt_1\right]dt_2\right]\cdots\right]dt_n\right]dt.$$

Letting $\Delta = X-A$ and taking expectations gives Theorem 1.

We have immediately, the

Corollary:

$$E\left(\frac{1}{X+A}\right)^k = \int_0^1 g_k(u)du.$$

The potential applicability of the corollary is immediately evident. In Section 3 we apply it in the binomial and Poisson cases.

3. APPLICATIONS

3.1 Binomial Distribution

Let X be a binomially distributed random variable with parameters n and p. It is easy to show

$$g_1(t) = t^{A-1}(q + pt)^n, \tag{3.1}$$

and using successive integrations by parts we are led to

$$\int_0^t g_1(u)du = \int_0^t u^{A-1}(q+pu)^n du$$

$$= q^n\left(\frac{q}{p}\right)^A \cdot \left[\sum_{k=1}^r (-1)^{k+1} \frac{(A-1)(A-2)\cdots(A-k+1)}{(n+1)(n+2)\cdots(n+k)} \right. \tag{3.2}$$

$$b^{A-k}(1+b)^{n+k}$$

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* M. T. Chao is member of technical staff, Bell Telephone Laboratories, Holmdel, N. J. 07733. W. E. Strawderman is assistant professor, Department of Statistics, Rutgers—the State University of New Jersey, New Brunswick, N. J.

$$+(-1)^r \frac{(A-1)(A-2)\cdots(A-r)}{(n+1)(n+2)\cdots(n+r)} \cdot \int_0^b v^{A-r-1}(1+v)^{n+r}dv\Bigg],$$

where, by convention, $(A-1)(A-2)\cdots(A-k)=1$ if $k=0$; and where $A-r>0$, and $b=pt/q$.

For integral A, the formula leads to an exact result. Using relation (3.2) and the Corollary we have for integral A,

$$E\left(\frac{1}{X+A}\right) = q^n\left(\frac{q}{p}\right)^A \cdot\Bigg[\sum_{k=1}^{A-1}(-1)^{k+1}\frac{(A-1)(A-2)\cdots(A-k+1)}{(n+1)(n+2)\cdots(n+k)} \cdot b^{A-k}(1+b)^{n+k} + (-1)^{A-1}\frac{(A-1)!}{(n+1)(n+2)\cdots(n+A-1)}\frac{1}{n+A} \cdot((1+b)^{n+A}-1)\Bigg], \tag{3.3}$$

where $b=p/q$.

In particular,

$$E\left(\frac{1}{X+1}\right) = \frac{1-q^{n+1}}{(n+1)p}\cdot \tag{3.4}$$

and

$$\begin{aligned} E\left(\frac{1}{X+2}\right) &= q^n\left(\frac{q}{p}\right)^2\left[\frac{1}{n+1}\left(\frac{p}{q}\right)^1\left(\frac{1}{q}\right)^{n+1} - \frac{1}{(n+1)(n+2)}\left(\frac{1}{q^{n+2}}-1\right)\right] \\ &= \frac{1}{(n+1)p} - \frac{1}{(n+1)(n+2)p^2}\cdot(1-q^{n+2}) \\ &= \frac{1}{(n+1)p}\left[1-\frac{(1-q^{n+2})}{(n+2)p}\right]. \end{aligned} \tag{3.5}$$

Unfortunately (3.2) does not seem particularly useful for computing purposes when A is non-integral. Also the higher negative moments do not seem to be easily computed using our procedure, although the "negative factorial moments" are more tractable, a fact we will not prove here.

3.2 The Poisson Distribution

Let X be a random variable with the Poisson distribution with parameter λ.

Here

$$g_1(t) = t^{A-1}e^{-\lambda+\lambda t}. \tag{3.6}$$

$$\begin{aligned} \int_0^t g_1(u)du &= \int_0^t u^{A-1}e^{-\lambda+\lambda u}du \\ &= e^{-\lambda}\int_0^t u^{A-1}e^{\lambda u}du, \qquad \text{for } A>0 \\ &= e^{-\lambda}\left[\frac{t^{A-1}e^{\lambda t}}{\lambda} - \frac{A-1}{\lambda}\cdot\int_0^t u^{A-2}e^{\lambda u}du\right] \qquad \text{for } A>1. \end{aligned} \tag{3.7}$$

Letting $t=1$, we have

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda}\left[1-(A-1)E\left(\frac{1}{X+A-1}\right)\right] \qquad \text{for } A>1. \tag{3.8}$$

When $A=1$ we have directly

$$E\left(\frac{1}{X+1}\right) = \frac{e^{-\lambda}}{\lambda}[e^{\lambda}-1] = \frac{1-e^{-\lambda}}{\lambda}\cdot \tag{3.9}$$

(3.8) and (3.9) together allow an inductive calculation of $E[1/(X+A)]$ for any integer $A\geq 1$.

The formula is

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda}\left[1+\sum_{j=1}^{r}\frac{\prod_{i=1}^{j}(A-i)}{\lambda^j}(-1)^j + (-1)^{r+1}\frac{\prod_{i=1}^{r+1}(A-i)}{\lambda^r}E\left(\frac{1}{X+A-r-1}\right)\right]$$

for $A>r+1$. When $A=r+2$,

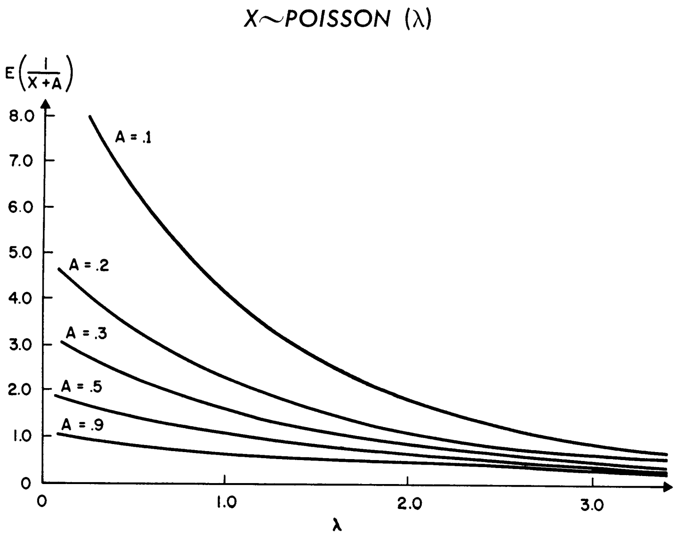
$$\begin{aligned} E\left(\frac{1}{X+A}\right) &= \frac{1}{\lambda}\left[1+\sum_{j=1}^{A-2}\frac{\prod_{i=1}^{j}(A-i)}{\lambda^j}(-1)^j + (-1)^{A-1}\frac{\prod_{i=1}^{A-1}(A-i)}{\lambda^{A-2}}\left(\frac{1}{\lambda}-\frac{e^{-\lambda}}{\lambda}\right)\right] \\ &= \frac{1}{\lambda}\left[1+\sum_{j=1}^{A-1}\frac{\prod_{i=1}^{j}(A-i)}{\lambda^j}(-1)^j + (-1)^A\frac{(A-1)!e^{-\lambda}}{\lambda^{A-1}}\right]. \end{aligned} \tag{3.10}$$

For non-integral A the first equation in (3.10) will serve to reduce the problem to one of finding $E[1/(X+A)]$ $0<A<1$. We include a graph of $E[1/(X+A)]$ for such values of A. We also indicate an asymptotic expression for $E[1/(X+A)]$ for small values of A.

$$\begin{aligned} E\left(\frac{1}{X+A}\right) &= \sum_{k=0}^{\infty}\frac{1}{k+A}\frac{e^{-\lambda}\lambda^k}{k!} \\ &= \frac{1}{A}e^{-\lambda} + \left[\sum_{k=1}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!(k+A)}\right]. \end{aligned} \tag{3.11}$$

If we let

$$F(\lambda, A) = \sum_{k=1}^{\infty}\frac{1}{k!k}\lambda^k e^{-\lambda} + \frac{1}{A}e^{-\lambda}, \tag{3.12}$$



we have

$$\left| E\left(\frac{1}{X+A}\right) - F(\lambda, A) \right|$$

$$\begin{aligned} &= A \textstyle\sum_{k=1}^{\infty} \dfrac{\lambda^k e^{-\lambda}}{k!k(k+A)} \\ &\leq A\left[\frac{\lambda e^{-\lambda}}{A+1} + \textstyle\sum_{k=2}^{\infty} \dfrac{\lambda^k e^{-\lambda}}{(k+1)!}\right] \qquad (3.13) \\ &\leq A\left[\frac{1}{2}\lambda e^{-\lambda} + \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda}\right] \\ &= Af(\lambda) \qquad \text{for } A \leq 1. \end{aligned}$$

Note that $f(\lambda) \to 0$ as $\lambda \to 0$ or $\lambda \to \infty$. Also max $f(\lambda) \doteq .464$ (at $\lambda \doteq 1.35$), and hence with small error we may approximate $E[1/(X+A)]$ by $F(\lambda, A)$ for small A.

But $F(\lambda, A)$, although it doesn't have a simple closed form, can be put in the form of well tabulated functions. In particular, we have

$$F(\lambda, A) = \frac{1}{A} e^{-\lambda} + (1 - e^{-\lambda}) E\left(\frac{1}{X} \mid X > 0\right). \qquad (3.14)$$

Now for a random variable Y with the truncated Poisson distribution (with $A = 0$)

$$g_1(t) = \frac{e^{-\lambda}(e^{\lambda t} - 1)}{(1 - e^{-\lambda})t} = E(t^{Y-1}).$$

Hence,

$$E\left(\frac{1}{X} \mid X > 0\right) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \int_0^1 \frac{e^{\lambda t} - 1}{t} dt. \qquad (3.15)$$

This integral may be expressed in terms of Ei, the exponential integral [7, NBS Tables, pp. 228–51]. Specifically,

$$\int_0^1 \frac{e^{\lambda t} - 1}{t} dt = Ei(\lambda) - \log \lambda - \gamma, \qquad (3.16)$$

where γ is Euler's Constant (.5772 · · ·). Hence $F(\lambda, A)$ may be evaluated with reasonable simplicity. Also we have a simple derivation, as a bonus, of the expression for the expectation of the inverse of the truncated Poisson used by Grab and Savage [5] in calculating their tables.

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