Title

Abstract:

Keywords:

1 Formulation

Consider a random variable $\mathbb{X} \in \mathbb{R}^p$ that has a sparse dependency structure among its features. This graph structure is potentially non-linear, and we want to infer the structure from a data matrix $\mathbf{X} \in \mathbb{M}(n, p)$.

We assume a multi-layer generative model for the structure:

$$
\begin{aligned}
\mathbf{X} &= \varphi(\mathbf{H}_1)\mathbf{B}_1 + \mathbf{E}_x; \quad \mathbb{E} \sim \mathcal{N}_p(\mathbf{0}, \Sigma_x), \\
\mathbf{H}_1 &= \varphi(\mathbf{H}_2)\mathbf{B}_2 + \mathbf{F}_1; \quad \mathbb{F}_1 \sim \mathcal{N}_{p_1}(\mathbf{0}, \Sigma_1), \\
&\cdots \\
\mathbf{H}_{L-1} &= \varphi(\mathbf{H}_L)\mathbf{B}_L + \mathbf{F}_{L-1}; \quad \mathbb{F}_{L-1} \sim \mathcal{N}_{p_{L-1}}(\mathbf{0}, \Sigma_{L-1}), \\
\mathbb{H}_L &\sim \mathcal{N}_{p_L}(\mathbf{0}, \Sigma_L).
\end{aligned}
$$

with L hidden layers, and $\varphi(\cdot)$ being a pointwise known transformation (e.g. ReLU, sigmoid, tanh). When Σ_x and $\Sigma_l, l \in \mathcal{I}_L$ are diagonal, it is the Non-linear Gaussian Belief Network of Frey and Hinton (1999). In our case, we keep Σ_x non-diagonal (but sparse), while others diagonal.

The negative log-likelihood function is

$$
-\ell(\mathbf{X}|\mathcal{H}, \mathcal{B}, \Omega) = \frac{n}{2}\left[\mathrm{Tr}\left(\mathbf{S}_x \Omega_x\right) - \log\det\Omega_x + \sum_{l=1}^{L} \left\{ \mathrm{Tr}\left(\mathbf{S}_l \Omega_l\right) - \log\det\Omega_l \right\} \right]
$$

where $\mathbf{S}_x = \mathbf{E}_x^T \mathbf{E}_x / n, \mathbf{S}_l = \mathbf{F}_l^T \mathbf{F}_l / n$ for $l = 1, \ldots, L-1$ and $\mathbf{S}_L = \mathbf{H}_L^T \mathbf{H}_L / n$. Inferring the distribution of the hidden variables is difficult so we assume pointwise variational approximations:

$$
h_{ij,l} \sim N(\mu_{ijl}, s_{ijl}); \quad i \in \mathcal{I}_n, j \in \mathcal{I}_{p_l}, l \in \mathcal{I}_L.
$$

Collect the variational parameters in $\mathcal{M} := \{\mathbf{M}_1, \ldots, \mathbf{M}_L\}, \mathcal{S} := \{\mathbf{S}_1, \ldots, \mathbf{S}_L\}$. Now we have the variational lower-bound

$$
\ell(\mathbf{X}|\mathcal{H}, \mathcal{B}, \Omega) \geq \mathbb{E}_q \ell(\mathbf{X}, \mathcal{H}|\mathcal{B}, \Omega, \mathcal{M}, \mathcal{S}) - \mathbb{E}_q \log q(\mathcal{H}|\mathbf{X}, \mathcal{B}, \Omega, \mathcal{M}, \mathcal{S}) \tag{1.1}
$$

Denote this lower bound by $\ell_q(\mathbf{X}|\mathcal{B}, \Omega, \mathcal{M}, \mathcal{S})$. Under the simplified model $\Sigma_l = \boldsymbol{\sigma}_l \mathbf{I}$ for $l \in \mathcal{I}_L$, the second term becomes (Frey and Hinton, 1999)

$$\mathbb{E}_q \log q(\mathcal{H}|\mathbf{X}, \mathcal{B}, \Omega, \mathcal{M}, \mathcal{S}) = \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{p_l}\sum_{l=1}^{L} \log\frac{s_{ijl}}{\sigma_{jl}} - \frac{s_{ijl}}{\sigma_{jl}} + n \log\det\Omega_x + \text{constant}\right]. \tag{1.2}$$

For the first term we have

$$\begin{aligned}\mathbb{E}_q \ell(\mathbf{X}, \mathcal{H}|\mathcal{B}, \Omega, \mathcal{M}, \mathcal{S}) &= \frac{n}{2}\mathbb{E}_q\left[\text{Tr}(\mathbf{S}_x\Omega_x) + \sum_{l=1}^{L}\text{Tr}(\mathbf{S}_l\Omega_l)\right] \\ &= \end{aligned}$$

which simplifies to (Frey and Hinton, 1999)

$$-\left[\mathbb{E}_q \,\text{Tr}(\mathbf{E}_x^T\mathbf{E}_x\Omega_x) + \sum_{i=1}^{n}\sum_{j=1}^{p_l}\sum_{l=1}^{L-1}\frac{1}{\sigma_{jl}}\left\{(\mu_{ijl} - b_{ij,l+1}m_{ij,l+1})^2 + b_{ij,l+1}^2 v_{ij,l+1}\right\} + \text{const}\right] \tag{1.3}$$

where $m_{ijl} = \mathbb{E}_q\varphi(h_{ijl}), v_{ijl} = \mathbb{E}_q(\varphi(h_{ijl}) - m_{ijl})^2$.

1.1 Objective function

We shall solve a penalized version of the variational lower bound in (1.1):

$$-\frac{2}{n}\ell_q(\mathbf{X}|\mathcal{B}, \Omega, \mathcal{M}, \mathcal{S}) + \sum_{l=1}^{L}\|\mathbf{B}_l\|_1 + \|\Omega_x\|_{1,\text{off}} + P(\mathcal{M}) + Q(\mathcal{S})$$

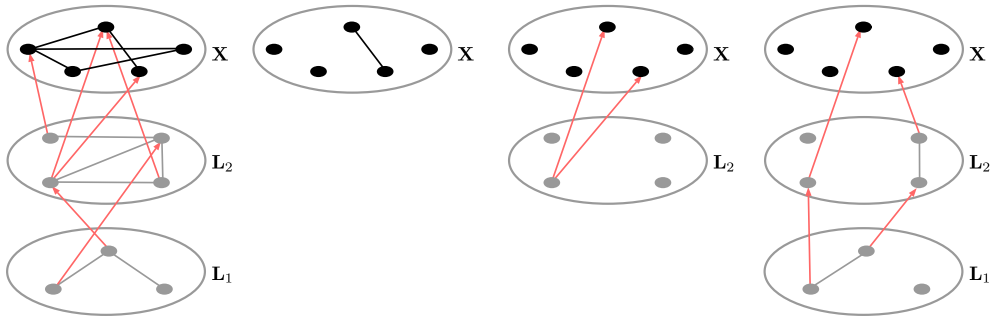
with P, Q being penalties over the variational parameters. We solve this using a variational (monte-carlo?) EM algorithm-
E -step: Given values of $\mathcal{B}, \Omega_x, \boldsymbol{\sigma}_l$, solve for the variational parameters by solving

$$-\frac{2}{n}\ell_q(\mathbf{X}|\mathcal{B}, \Omega, \mathcal{M}, \mathcal{S}) + P(\mathcal{M}) + Q(\mathcal{S})$$

M -step: Given the variational parameters, solve for the model parameters by solving an ℓ_1-penalized version of (1.3).

We take the greedy strategy of solving two-layer problems successively. This means molte-carlo *sequential* EM: first solve for the variational parameters $(\mathbf{M}_1, \mathbf{S}_1) = ((\mu_{ij,1}, s_{ij,1}))$, in the E step, then solve for $(\mathbf{B}_1, \Omega_x)$ in the M step, and continue until convergence. After that only go to the next layer. Similar to Bengio et al. (2007); Hinton and Salakhutdinov (2006). We assume a rank-1 representation for $\mathbf{M} \equiv \mathbf{M}_1$ and $\mathbf{S} \equiv \mathbf{S}_1$:

$$\begin{aligned}\mathbf{M} &= \mathbf{ab}^T, \mathbf{a} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^q, q \equiv p_1, \\ \mathbf{S} &= \mathbf{cd}^T, \mathbf{c} \in \mathbb{R}^n, \mathbf{d} \in \mathbb{R}^q\end{aligned}$$



We further assume generative parameters for $\mathbf{a}$: $a_i \sim N(\mu, \sigma^2)$ for $i \in \mathcal{I}_n$.

Can calculate gradients of E-step using chain rule and Appendix A of Frey and Hinton (1999).

Now the objective function for a two-layer model becomes:

$$\operatorname{Tr}\left[\frac{1}{n}(\mathbf{X}-\varphi(\mathbf{H})\mathbf{B})^T(\mathbf{X}-\varphi(\mathbf{H})\mathbf{B})\Omega_x\right] + \log\det\Omega_x + \|\mathbf{B}\|_1$$

We assume the following hierarchical structures for the hidden variables and associated variational parameters:

$$\begin{aligned} h_{ij} &\sim N(\mu_{ij}, \sigma_{ij}^2), \\ \mu_{ij} &\sim \sum_{k=1}^{K} I_{mk} N(\mu_{mk}, \sigma_{mk}^2); \quad I_{mk} = \operatorname{Ber}(\pi_{mk}), \\ \sigma_{ij} &\sim \sum_{k=1}^{K} I_{sk} N(\mu_{sk}, \sigma_{sk}^2); \quad I_{sk} = \operatorname{Ber}(\pi_{sk}), \end{aligned}$$

Thus in total there are $6K$ variational parameters.

E-step: we solve for the variational parameters by minimizing the following (take $\mathbf{H}_\varphi \equiv \varphi(\mathbf{H})$)

$$\begin{aligned} \mathcal{F}(\mathbf{M},\mathbf{S}) &= \mathbb{E}_q \operatorname{Tr}\left[\frac{1}{n}(\mathbf{X}-\mathbf{H}_\varphi\mathbf{B})^T(\mathbf{X}-\mathbf{H}_\varphi\mathbf{B})\Omega_x\right] \\ &= \operatorname{Tr}\left[\left\{\frac{1}{n}(\mathbf{X}-\mathbf{M}_\varphi\mathbf{B})^T(\mathbf{X}-\mathbf{M}_\varphi\mathbf{B}) + \mathbf{B}^T\mathbf{V}_\varphi\mathbf{B}\right\}\Omega_x\right] \\ &= \left[\sum_{j=1}^{p}\sum_{j'=1}^{p}\omega_{jj'}\left\{\frac{1}{n}(\mathbf{X}_j-\mathbf{M}_\varphi\mathbf{B}_j)^T(\mathbf{X}_{j'}-\mathbf{M}_\varphi\mathbf{B}_{j'}) + \mathbf{B}_j^T\mathbf{V}_\varphi\mathbf{B}_{j'}\right\}\right] \\ &= \sum_{j,j'=1}^{p}\omega_{jj'}\left\{-\frac{2}{n}\mathbf{X}_j^T\mathbf{M}_\varphi\mathbf{B}_{j'} + \mathbf{B}_j^T\left(\frac{1}{n}\mathbf{M}_\varphi^T\mathbf{M}_\varphi + \mathbf{V}_\varphi\right)\mathbf{B}_{j'}\right\} + c \end{aligned}$$

where $(\mathbf{M}_\varphi)_{ik} = \mathbb{E}_q \varphi(h_{ik})$ for $i \in \mathcal{I}_n, k \in \mathcal{I}_q$, and $\mathbf{V}_\varphi = \mathbb{E}_q[(\mathbf{H}_\varphi - \mathbf{M}_\varphi)^T(\mathbf{H}_\varphi - \mathbf{M}_\varphi)/n]$. Differentiating with respect to entries of $\mathbf{M}$ we now have

$$\frac{\partial \mathcal{F}}{\partial \mu_{ik}} = \sum_{j,j'=1}^{p} \omega_{jj'} \left[-\frac{2}{n} x_{ij} \frac{dm_{ik}}{d\mu_{ik}} b_{j'k} + \frac{1}{n} \left\{ 2b_{jk} \left(2m_{ik} + \sum_{i' \neq i} m_{i'k} \right) \frac{dm_{ik}}{d\mu_{ik}} b_{j'k} \right\} + \textbf{tbd} \right]$$

Using chain rule, we get the derivatives with respect to the component vectors:

$$\frac{\partial \mathcal{F}}{\partial \mu} = \mathbf{b}^T \frac{\partial \mathcal{F}}{\partial (\mu \mathbf{b})}; \quad \frac{\partial \mathcal{F}}{\partial \sigma} = \textbf{tbd}; \quad \frac{\partial \mathcal{F}}{\partial b_k} = \mathbf{a}^T \frac{\partial \mathcal{F}}{\partial (b_k \mathbf{a})}.$$

M-step: First generate data $\mathbf{H}_\varphi$ using the variational parameters $(\mathbf{M}, \mathbf{S})$. Then obtain $\mathbf{B}, \Omega_x$ by solving a penalized LS problem:

$$\{\hat{\mathbf{B}}, \hat{\Omega}_x\} = \underset{\mathbf{B}, \Omega_x}{\arg\min}\, \mathrm{Tr}(\mathbf{S}_x^\varphi \Omega_x) + \log\det \Omega_x + \|\mathbf{B}\|_1 + \|\Omega_x\|_{\text{off},1}.$$

2 Theoretical properties

Define equivalence classes, $\boldsymbol{\theta} = \mathrm{vec}(\mathbf{B}, \Omega_{x,off})$, $\boldsymbol{\vartheta}$ denoting the variational parameters, $\boldsymbol{\eta} = (\boldsymbol{\theta}, \boldsymbol{\vartheta})$. Then we are minimizing

$$\mathbb{E}_q \left[l(\mathbf{x}; \mathbf{z}, \boldsymbol{\eta}) + \mathrm{KL}(q(\mathbf{z}|\boldsymbol{\vartheta}_1) \| p(\mathbf{z})) + \mathrm{KL}(r(\boldsymbol{\vartheta}_1|\mathbf{z}; \boldsymbol{\vartheta}) \| q(\boldsymbol{\vartheta}_1; \boldsymbol{\vartheta})) \right] + P(\boldsymbol{\theta}).$$

define the negative hierarchical ELBO by $\bar{l}(\cdot)$. We consider a ℓ_1-penalty

$$P(\boldsymbol{\theta}) = \rho_1 \|\boldsymbol{\beta}\|_1 + \rho_2 \|\boldsymbol{\omega}\|_1 = \lambda P_\alpha(\boldsymbol{\theta})$$

by reparameterizing the penalties: $\lambda = \rho_1 + \rho_2, \alpha = \rho_1/\lambda$.

Conditions 1, 2, 3 same as those in SPINN paper.

Define $V_n(\boldsymbol{\eta}) = \mathbb{E}\bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{X}; \boldsymbol{\eta})$, $\mathcal{E}(\boldsymbol{\eta}|\boldsymbol{\eta}_0), \bar{\mathcal{E}}(\boldsymbol{\eta}|\boldsymbol{\eta}_0)$ as in Städler et al. (2010).

Theorem 2.1. *Define the event*

$$\mathcal{T} = \left\{ \sup_{\boldsymbol{\eta}} \frac{|V_n(\boldsymbol{\eta}_0^{\boldsymbol{\eta}}) - V_n(\boldsymbol{\eta})|}{\lambda_0 \vee (P_\alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0^{\boldsymbol{\eta}}) + \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_0^{\boldsymbol{\eta}}\|_2)} \leq T\lambda_0 \right\}$$

for $T \geq 1, \lambda_0 > 0$. Then for the solution $\hat{\boldsymbol{\eta}}$ defined in **tbd**, *we have*

$$\mathcal{E}(\hat{\boldsymbol{\eta}}) + \frac{\lambda - 2T\lambda_0}{2} \|\hat{\boldsymbol{\theta}}_{S^c}\|_1 \leq \left[(\lambda + 2T\lambda_0)(\alpha\sqrt{s_\beta} + (1-\alpha)\sqrt{s_\omega})C_0 \right]^2$$

Proof of Theorem 2.1. Just prove an equivalent lemma of Städler et al. (2010). Details **tbd**.

Other details similar to Thm 1 of Städler et al. (2010).

By definition we now have that

$$\bar{l}(\mathbf{X}; \hat{\boldsymbol{\eta}}) + \lambda P_\alpha(\hat{\boldsymbol{\theta}}) \leq \bar{l}(\mathbf{X}; \boldsymbol{\eta}_0) + \lambda P_\alpha(\boldsymbol{\theta}_0)$$

for any $\boldsymbol{\eta}_0 \in \mathcal{Q}_0$. Adding $\mathcal{E}(\hat{\boldsymbol{\eta}}) = \mathbb{E}\bar{l}(\mathbf{x}; \hat{\boldsymbol{\eta}}) - \mathbb{E}\bar{l}(\mathbf{x}; \boldsymbol{\eta}_0)$ on both sides, we get

$$\begin{aligned} \mathcal{E}(\hat{\boldsymbol{\eta}}) + \lambda P_\alpha(\hat{\boldsymbol{\theta}}) &\leq |V_n(\boldsymbol{\eta}_0) - V_n(\hat{\boldsymbol{\eta}})| + \lambda P_\alpha(\boldsymbol{\theta}_0) \\ &\leq T\lambda_0 \left(\lambda_0 \vee (P_\alpha(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + \|\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0\|_2) \right) + \lambda P_\alpha(\boldsymbol{\theta}_0) \end{aligned} \tag{2.1}$$

on the set $\mathcal{T}$. There are three cases now.

Case I. Suppose $\lambda_0 \geq P_\alpha(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + \|\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0\|_2$. Then rearranging the terms in (2.1) we have

$$\mathcal{E}(\hat{\boldsymbol{\eta}}) + \lambda P_\alpha(\hat{\boldsymbol{\theta}}_{S^c}) \leq T\lambda_0^2 + \lambda P_\alpha(\hat{\boldsymbol{\theta}}_S - \boldsymbol{\theta}_{0,S}) \leq T\lambda_0^2 + \lambda\lambda_0$$

since $\lambda_0 \geq P_\alpha(\hat{\boldsymbol{\theta}}_S - \boldsymbol{\theta}_{0,S})$. □

Case II. Suppose $\lambda_0 < P_\alpha(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + \|\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0\|_2$. Then after some rearrangement we get

$$\begin{aligned}\mathcal{E}(\hat{\boldsymbol{\eta}}) + (\lambda - T\lambda_0) P_\alpha(\hat{\boldsymbol{\theta}}_{S^c}) &\leq T\lambda_0 \|\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0\|_2 + T\lambda_0 P_\alpha(\hat{\boldsymbol{\theta}}_S - \boldsymbol{\theta}_{0,S}) + \lambda(P_\alpha(\boldsymbol{\theta}_{0,S}) - P_\alpha(\hat{\boldsymbol{\theta}}_S)) \\ &\leq T\lambda_0 \|\hat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0\|_2 + (\lambda + T\lambda_0) P_\alpha(\hat{\boldsymbol{\theta}}_S - \boldsymbol{\theta}_{0,S})\end{aligned}$$

Condition 4. The gradient of $\bar{l}(\cdot)$ with respect to the model parameters is bounded above:

$$\left\| \nabla_{\boldsymbol{\eta}} \bar{l}(\mathbf{x}; \boldsymbol{\eta}) \right\|_\infty \leq G(\mathbf{x})$$

for some function $G : \mathbb{R}^p \mapsto \mathbb{R}^+$. Further, there exists $c' > 0$ such that

$$|\bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}, \boldsymbol{\eta}')| \mathbb{I}(G(\mathbf{x}) \leq M)) \leq c'$$

for any $M \geq 0$ and $\boldsymbol{\eta}, \boldsymbol{\eta}'$.

Theorem 2.2. *For the choice of λ_0:*

$$\lambda_0 = \textbf{tbd},$$

and any $T \geq 1$, the event $\mathcal{T}$ happens with probability $\geq$

tbd

Proof of Theorem 2.2. We follow an approach similar to Städler et al. (2010) and Feng and Simon (2017) to obtain probability bounds for truncated versions and tails of the quantity $|V_n(\boldsymbol{\eta}_0^{\boldsymbol{\eta}}) - V_n(\boldsymbol{\eta})|$ after proper scaling.

Part I: Bounding truncated parts. Define the following:

$$\bar{V}_n(\boldsymbol{\eta}) := \mathbb{E}[\bar{l}(\mathbf{x}; \boldsymbol{\eta}) \mathbb{I}(G(\mathbf{x}) \leq M_n)] - \frac{1}{n} \sum_{i=1}^n \bar{l}(\mathbf{x}_i; \boldsymbol{\eta}) \mathbb{I}(G(\mathbf{x}_i) \leq M_n)$$

so that

$$\begin{aligned}|\bar{V}_n(\boldsymbol{\eta}) - \bar{V}_n(\boldsymbol{\eta}_0)| \leq \mathbb{E}[|\bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}; \boldsymbol{\eta}_0)| \mathbb{I}(G(\mathbf{x}) \leq M_n)] - \\ \frac{1}{n} \sum_{i=1}^n |\bar{l}(\mathbf{x}_i; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}_i; \boldsymbol{\eta}_0)| \mathbb{I}(G(\mathbf{x}_i) \leq M_n)\end{aligned} \tag{2.2}$$

To get an upper bound on the right hand side of (2.2), we start by bounding the entropy of the functional class $\mathcal{E}_r, r > 0$:

$$\mathcal{E}_r := \left\{ \bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}; \boldsymbol{\eta}_0) \mathbb{I}(G(\mathbf{x}) \leq M_n) : P_\alpha(\boldsymbol{\theta} - \boldsymbol{\theta}_0) + \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_0\|_2 \leq r \right\}$$

with respect to the empirical norm $\|h\|_{P_n} = \sqrt{\sum_{i=1}^n h^2(\mathbf{x}_i)/n}$.

Lemma 2.3. *For a collection of functions $\mathcal{H}$ taking values in $\mathcal{X}$, denote its metric entropy by $H(\cdot, \mathcal{H}, \|.\|_{P_n})$. Then for any $u, r, M_n > 0$ the following holds:*

$$H(u, \mathcal{E}_r, \|.\|_{P_n}) \leq \textbf{tbd}$$

Proof of Lemma 2.3. For any $\boldsymbol{\eta}, \boldsymbol{\eta}' \in \Theta$, due to the mean value theorem there exists $\boldsymbol{\eta}''$ so that

$$\left\| \nabla_{\boldsymbol{\eta}} \bar{l}(\mathbf{x}; \boldsymbol{\eta})|_{\boldsymbol{\eta}-\boldsymbol{\eta}''} \right\|_\infty = \frac{|\bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}; \boldsymbol{\eta}')|}{\|\boldsymbol{\eta} - \boldsymbol{\eta}'\|_1}. \tag{2.3}$$

Define $e_{\boldsymbol{\eta}}(\mathbf{x}) = |\bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}; \boldsymbol{\eta}_0)|\mathbb{I}(G(\mathbf{x}) \leq M_n)$. Then, combining (2.3) with Condition (4) we get

$$\begin{aligned}
|e_{\boldsymbol{\eta}}(\mathbf{x}) - e_{\boldsymbol{\eta}'}(\mathbf{x})| &\leq |\bar{l}(\mathbf{x}; \boldsymbol{\eta}) - \bar{l}(\mathbf{x}; \boldsymbol{\eta}')|\mathbb{I}(G(\mathbf{x}) \leq M_n) \\
&\leq M_n(\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|_1 + \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}'\|_1) \\
&\leq M_n(\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|_1 + \sqrt{6K}\|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}'\|_2)
\end{aligned}$$

so that for $u > 0$,

$$\begin{aligned}
H(u, \mathcal{E}_r, \|.\|_{P_n}) \leq H&\left(u, \left\{\boldsymbol{\vartheta} : \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_0\|_2 \leq \frac{r}{\sqrt{6K}}\right\}, \|.\|_{P_n}\right) + \\
&H\left(u, \{\boldsymbol{\theta} : \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|_1 \leq r\}, \|.\|_{P_n}\right)
\end{aligned} \tag{2.4}$$

The first term is bounded above by $6K \log(5r/(\sqrt{6K}u))$ (Städler et al., 2010). $\square$

Pat II: Bounding the tails. $\square$

References

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