Supplement to “Large-Scale Multiple Testing of Correlations”

Tony Cai and Weidong Liu

Abstract

This supplementary material includes the proof of main results.

1 Proofs of main results

In this section we prove the main theorems and the key technical result, Proposition 1. We begin by collecting and proving two additional technical lemmas that will be used in the proofs of the main results.

Let $\xi_1,\dots,\xi_n$ be independent d-dimensional random vectors with mean zero. Let $G(t)=2-2\Phi(t)$ and define $|\cdot|_d$ by $|\boldsymbol{z}|_d=\min\{|z_i|;1\le i\le d\}$ for $\boldsymbol{z}=(z_1,\dots,z_d)'$.

Lemma 1. *Suppose that $p\le n^r$ and $\max_{1\le k\le n}\mathsf{E}\|\xi_k\|^{bdr+2+\epsilon}\le\kappa$ for some $r>0$, $\kappa>0$, $b>0$ and $\epsilon>0$. Assume that*

$$\Big\|a_n^{-1}\mathsf{Cov}(\sum_{k=1}^{n}\xi_k)-I\Big\|\le C(\log p)^{-2-\gamma}$$

for some $\gamma>0$ and some $a_n\asymp n$. Then we have

$$\sup_{0\le t\le b\sqrt{\log p}}\Big|\frac{\mathsf{P}(|\sum_{k=1}^{n}\xi_k|_d\ge t\sqrt{a_n})}{(G(t))^d}-1\Big|\le C(\log p)^{-1-\gamma_1}$$

for $\gamma_1=\min\{\gamma,\frac{1}{2}\}$.

Let $\eta_k=(\eta_{k1},\eta_{k2})'$ are independent 2-dimensional random vectors with mean zero.

Lemma 2. *Suppose that* $p \leq n^r$ *and* $\max_{1\leq k\leq n} \mathsf{E}\|\eta_k\|^{2br+2+\epsilon} \leq \kappa$ *for some* $r > 0$, $\kappa > 0$, $b > 0$ *and* $\epsilon > 0$. *Assume that for some* $a_{n1} \asymp n$, $a_{n2} \asymp n$, $\gamma > 0$ *and* $0 \leq \delta \leq 1$,

$$\Big|a_{n1}^{-1}\mathsf{Var}(\sum_{k=1}^{n}\eta_{k1}) - 1\Big| \leq C(\log p)^{-2-\gamma},$$
$$\Big|a_{n2}^{-1}\mathsf{Var}(\sum_{k=1}^{n}\eta_{k2}) - 1\Big| \leq C(\log p)^{-2-\gamma},$$
$$\Big|\mathsf{Cov}(\sum_{k=1}^{n}\eta_{k1}, \sum_{k=1}^{n}\eta_{k2})\Big|/\sqrt{a_{n1}a_{n2}} \leq \delta$$

when n *is large. Then we have*

$$\mathsf{P}\Big(\Big|\sum_{k=1}^{n}\eta_{k1}\Big| \geq t\sqrt{a_{n1}}, \Big|\sum_{k=1}^{n}\eta_{k2}\Big| \geq t\sqrt{a_{n2}}\Big) \leq C(t+1)^{-2}\exp(-t^2/(1+\delta))$$

uniformly for $0 \leq t \leq b\sqrt{\log p}$, *where* C *is a finite constant.*

Proof of Lemmas 1 and 2. The proof follows exactly from those of Lemmas 6.1 and 6.2 in Liu (2013). Only a few notations should be changed and hence the proof is omitted. $\square$

Without loss of generality, we assume that $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = 0$ and $\sigma_{ii1} = \sigma_{ii2} = 1$ for $1 \leq i \leq p$. Define

$$T^o_{ij} = \frac{\frac{1}{n_1}\sum_{k=1}^{n_1}\boldsymbol{X}_{kij} - \frac{1}{n_2}\sum_{k=1}^{n_2}\boldsymbol{Y}_{kij}}{\sqrt{\frac{\kappa_1}{n_1}(1-\rho_{ij1}^2)^2 + \frac{\kappa_2}{n_2}(1-\rho_{ij2}^2)^2}},$$
$$\hat{T}_{ij} = \frac{\hat{\rho}_{ij1} - \hat{\rho}_{ij2}}{\sqrt{\frac{\hat{\kappa}_1}{n_1}(1-\hat{\rho}_{ij}^2)^2 + \frac{\hat{\kappa}_2}{n_2}(1-\hat{\rho}_{ij}^2)^2}},$$

where

$$\boldsymbol{X}_{kij} = X_{k,i}X_{k,j} - \rho_{ij1} - \frac{1}{2}\rho_{ij1}(X_{k,i}^2 - 1) - \frac{1}{2}\rho_{ij1}(X_{k,j}^2 - 1),$$
$$\boldsymbol{Y}_{kij} = Y_{k,i}Y_{k,j} - \rho_{ij2} - \frac{1}{2}\rho_{ij2}(Y_{k,i}^2 - 1) - \frac{1}{2}\rho_{ij2}(Y_{k,j}^2 - 1).$$

Define

$$v_{ij} = \sqrt{\frac{\kappa_1}{n_1}(1-\rho_{ij1}^2)^2 + \frac{\kappa_2}{n_2}(1-\rho_{ij2}^2)^2},$$
$$\hat{v}_{ij} = \sqrt{\frac{\hat{\kappa}_1}{n_1}(1-\hat{\rho}_{ij1}^2)^2 + \frac{\hat{\kappa}_2}{n_2}(1-\hat{\rho}_{ij2}^2)^2}.$$

By Lemma 5, it is easy to see that, for any $M > 0$, there exists $C > 0$ such that

$$\mathsf{P}\Big(\max_{1\le i\le p}\Big|\frac{1}{n_1}\sum_{k=1}^{n_1}((X_{k,i}-\bar{X}_i)^4-\mathsf{E}X_{k,i}^4)\Big|\ge C\sqrt{\frac{\log p}{n_1}}\Big)=O(p^{-M}).$$

A similar inequality holds for $Y_{k,i}$. Hence we have

$$\mathsf{P}\Big(\Big|\frac{\hat{\kappa}_1}{\kappa_1}-1\Big|\ge C\sqrt{\frac{\log p}{n_1}}\Big)+\mathsf{P}\Big(\Big|\frac{\hat{\kappa}_2}{\kappa_2}-1\Big|\ge C\sqrt{\frac{\log p}{n_2}}\Big)=O(p^{-M}).$$

By Lemma 5, we have

$$\begin{aligned}
&\mathsf{P}\Big(\max_{1\le i,j\le p}|\hat{\sigma}_{ij1}-\sigma_{ij1}|\ge C\sqrt{\frac{\log p}{n_1}}\Big)+\mathsf{P}\Big(\max_{1\le i,j\le p}|\hat{\sigma}_{ij2}-\sigma_{ij2}|\ge C\sqrt{\frac{\log p}{n_2}}\Big)=O(p^{-M}),\\
&\mathsf{P}\Big(\max_{1\le i,j\le p}|\hat{\rho}_{ij1}-\rho_{ij1}|\ge C\sqrt{\frac{\log p}{n_1}}\Big)+\mathsf{P}\Big(\max_{1\le i,j\le p}|\hat{\rho}_{ij2}-\rho_{ij2}|\ge C\sqrt{\frac{\log p}{n_2}}\Big)=O(p^{-M}),
\end{aligned}\tag{1}$$

where $\hat{\sigma}_{ij1}$ and $\hat{\sigma}_{ij2}$ are the sample covariance coefficients of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. Combining the above three inequalities, we obtain that

$$\mathsf{P}\Big(\max_{1\le i,j\le p}|\hat{v}_{ij}/v_{ij}-1|\ge C\sqrt{\frac{\log p}{n}}\Big)=O(p^{-M}).\tag{2}$$

Proof of Proposition 1. Without loss of generality, we assume that $\boldsymbol{\mu}_1=\boldsymbol{\mu}_2=0$ and $\sigma_{ii1}=\sigma_{ii2}=1$ for $1\le i\le p$. Put

$$\xi_k=\boldsymbol{X}_{kij}\quad\text{for } 1\le k\le n_1;\quad \xi_k=-\frac{n_1}{n_2}\boldsymbol{Y}_{kij}\quad\text{for } n_1+1\le k\le n.\tag{3}$$

By (C2),

$$\mathsf{E}X_{k,i}^4=3\kappa_1,\ \mathsf{E}(X_{k,i}X_{k,j})^2=\kappa_1(2\sigma_{ij1}^2+1),\ \mathsf{E}X_{k,i}^3X_{k,j}=\mathsf{E}X_{k,i}X_{k,j}^3=3\kappa_1\sigma_{ij1}.\tag{4}$$

Then by (4) it is easy to prove that $\mathsf{Var}(\sum_{k=1}^n\xi_i)=(n_1v_{ij})^2\asymp n$. Hence, by Lemma 1,

$$\sup_{1\le i<j\le p}\ \sup_{0\le t\le b\sqrt{\log p}}\Big|\frac{\mathsf{P}(|T_{ij}^o|\ge t)}{2-2\Phi(t)}-1\Big|\to 0.\tag{5}$$

This, together with (1), (2) and some tedious calculations, implies that for any $M>0$, there exists $C>0$ such that

$$\mathsf{P}\Big(\max_{(i,j)\in\mathcal{H}_0}|T_{ij}-T_{ij}^o|\ge C\sqrt{\frac{(\log p)^2}{n}}\Big)=O(p^{-M}),$$

$$\mathsf{P}\Big(\max_{1\le i<j\le p}|\hat{T}_{ij}-T^{o}_{ij}|\ge C\sqrt{\frac{(\log p)^2}{n}}\Big)=O(p^{-M}). \tag{6}$$

By (5) and (6), (15) in Proposition 1 is proved.

Let $\mathcal{X}=\{\boldsymbol{X}_k,\boldsymbol{Y}_l,1\le k\le n_1,1\le l\le n_2\}$. Let $\mathsf{P}^*(\cdot)$ be the conditional probability given $\mathcal{X}$. For any fixed $\alpha>0$, put

$$\begin{aligned}
\boldsymbol{F}_{1\boldsymbol{X}} &= \Big\{\max_{1\le i\le p}\frac{1}{n_1}\sum_{k=1}^{n_1}|X_{k,i}-\bar{X}_i|^{\alpha}\le C_0,\max_{1\le i\le p}\Big|\frac{\sum_{k=1}^{n_1}(X_{k,i}-\bar{X}^i)^2}{n_1}-1\Big|\le\frac{1}{2}\Big\},\\
\boldsymbol{F}_{2\boldsymbol{X}} &= \Big\{\max_{1\le i\le p}\Big|\frac{\sum_{k=1}^{n_1}(X_{k,i}-\bar{X}_i)^4}{n_1}-\mathsf{E}X^4_{1,i}\Big|\le C\sqrt{\frac{\log p}{n_1}},\Big|\frac{\hat{\kappa}_1}{\kappa_1}-1\Big|\le C\sqrt{\frac{\log p}{n_1}}\Big\},\\
\boldsymbol{F}_{3\boldsymbol{X}} &= \Big\{\max_{1\le i,j\le p}\Big|\frac{\sum_{k=1}^{n_1}(X_{k,i}-\bar{X}_i)^2(X_{k,j}-\bar{X}_j)^2}{n_1}-\mathsf{E}X^2_{1,i}X^2_{1,j}\Big|\le C\sqrt{\frac{\log p}{n_1}}\Big\},\\
\boldsymbol{F}_{4\boldsymbol{X}} &= \Big\{\max_{1\le i,j\le p}\Big|\frac{\sum_{k=1}^{n_1}(X_{k,i}-\bar{X}_i)^3(X_{k,j}-\bar{X}_j)}{n_1}-\mathsf{E}X^3_{1,i}X_{1,j}\Big|\le C\sqrt{\frac{\log p}{n_1}}\Big\},\\
\boldsymbol{F}_{5\boldsymbol{X}} &= \Big\{\max_{1\le i,j\le p}\Big|\frac{\sum_{k=1}^{n_1}(X_{k,i}-\bar{X}_i)(X_{k,j}-\bar{X}_j)^3}{n_1}-\mathsf{E}X_{1,i}X^3_{1,j}\Big|\le C\sqrt{\frac{\log p}{n_1}}\Big\},\\
\boldsymbol{F}_{6\boldsymbol{X}} &= \Big\{\max_{1\le i,j\le p}|\hat{\rho}_{ij1}-\rho_{ij1}|\le C\sqrt{\frac{\log p}{n_1}}\Big\},
\end{aligned}$$

where C_0 is a sufficiently large constant. Define $\boldsymbol{F}_{1\boldsymbol{Y}},\ldots,\boldsymbol{F}_{6\boldsymbol{Y}}$ similarly by replacing the notation X with Y. Let $\boldsymbol{F}=\cap_{k=1}^{6}\{\boldsymbol{F}_{k\boldsymbol{X}}\cap\boldsymbol{F}_{k\boldsymbol{Y}}\}$. By Lemma 5, $\mathsf{P}(\boldsymbol{F})\to 1$. On $\boldsymbol{F}$, by Lemma 5 again, we have for any $M>0$, there exists sufficiently large $C>0$ and $\alpha>0$ such that

$$\mathsf{P}^*\Big(\max_{i,j}|\hat{\rho}^*_{ij1}-\hat{\rho}_{ij1}|\ge C\sqrt{\frac{\log p}{n_1}}\Big)+\mathsf{P}^*\Big(\max_{i,j}|\hat{\rho}^*_{ij2}-\hat{\rho}_{ij2}|\ge C\sqrt{\frac{\log p}{n_2}}\Big)=O(p^{-M}),$$

where C, α and $O(1)$ only depend on r, C_0 and M. Put $Z^*_{ki}=X^*_{ki}-\bar{X}_i$. Then we have, on $\boldsymbol{F}$,

$$\begin{aligned}
&\mathsf{E}^*(Z^*_{ki})^4=3\kappa_1+O(\sqrt{\frac{\log p}{n}}),\ \mathsf{E}^*(Z^*_{ki}Z^*_{kj})^2=2\kappa_1\sigma^2_{ij1}+\kappa_1+O(\sqrt{\frac{\log p}{n}}),\\
&\mathsf{E}^*Z^{*3}_{ki}Z^*_{kj}=\mathsf{E}^*Z^*_{ki}Z^{*3}_{kj}=3\kappa_1\sigma_{ij1}+O(\sqrt{\frac{\log p}{n}}).
\end{aligned}$$

This implies that, on $\boldsymbol{F}$, $\mathsf{Var}^*(\sum_{k=1}^{n}\xi^*_i)=(n_1v_{ij})^2+O(\sqrt{n\log p})\asymp n$, where ξ^*_i is the bootstrap version of ξ_i defined in (3), with X_{ki} being replaced by Z^*_{ki}. Let $G^*_{ij}(t)=\mathsf{P}^*(|T^*_{ij,k}|\ge t)$. By the proof of (15) in Proposition 1, for any $b>0$, we have on $\boldsymbol{F}$,

$$\max_{1\le i<j\le p}\sup_{0\le t\le b\sqrt{\log p}}\Big|\frac{G^*_{ij}(t)}{2-2\Phi(t)}-1\Big|=o(1) \tag{7}$$

for sufficiently large $\alpha > 0$. We claim that, on $\boldsymbol{F}$, for any $\varepsilon > 0$,

$$\int_0^{b_p} \mathsf{P}^*\Big(\frac{2}{N(p^2-p)G(t)}\Big|\sum_{k=1}^{N}\sum_{1\le i<j\le p}[I\{|T^*_{ij,k}|\ge t\}-G^*_{ij}(t)]\Big|\ge\varepsilon\Big)dt = o(v_p) \tag{8}$$

and

$$\sup_{0\le t\le b_p} \mathsf{P}^*\Big(\frac{2}{N(p^2-p)G(t)}\Big|\sum_{k=1}^{N}\sum_{1\le i<j\le p}[I\{|T^*_{ij,k}|\ge t\}-G^*_{ij}(t)]\Big|\ge\varepsilon\Big) = o(1), \tag{9}$$

where $v_p = 1/\sqrt{(\log p)c_p^2}$ and $c_p = \log\log\log\log p$. Note that the correlation matrices of $\boldsymbol{X}^*_k$ and $\boldsymbol{Y}^*_k$ given $\mathcal{X}$ are $(\hat{\rho}_{ij1})_{1\le i,j\le p}$ and $(\hat{\rho}_{ij2})_{1\le i,j\le p}$ respectively. On $\boldsymbol{F}_{6,\boldsymbol{X}} \cap \boldsymbol{F}_{6,\boldsymbol{Y}}$, the correlation matrices of $\boldsymbol{X}^*_k$ and $\boldsymbol{Y}^*_k$ satisfy (C1) with k_p being replaced by $2k_p$ and θ being replaced by some θ' which is sufficiently close to θ. So, by (7) and the proof of (11) and (12) below, we have (8) and (9) on $\boldsymbol{F}$. By (8) and (9) and the proof of Lemma 3, on $\boldsymbol{F}$,

$$\mathsf{P}^*\Big(\sup_{0\le t\le b_p}\frac{2}{N(p^2-p)G(t)}\Big|\sum_{k=1}^{N}\sum_{1\le i<j\le p}[I\{|T^*_{ij,k}|\ge t\}-G(t)]\Big|\ge\varepsilon\Big) = o(1),$$

where $G(t) = 2-2\Phi(t)$. This, together with $\mathsf{P}(\boldsymbol{F}) \to 1$ implies that

$$\mathsf{P}\Big(\sup_{0\le t\le b_p}\frac{2}{N(p^2-p)G(t)}\Big|\sum_{k=1}^{N}\sum_{1\le i<j\le p}[I\{|T^*_{ij,k}|\ge t\}-G(t)]\Big|\ge\varepsilon\Big) = o(1).$$

Hence (16) in Proposition 1 holds.

It remains to prove (17) in Proposition 1. We only need to show that

$$\sup_{0\le t\le b_p}\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0} I\{|T_{ij}|\ge t\}}{2q_0(1-\Phi(t))}-1\Big| \to 0 \tag{10}$$

in probability.

Lemma 3. *Suppose that for any $\varepsilon > 0$,*

$$\sup_{0\le t\le b_p} P\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t\}-P(|T_{ij}|\ge t)]}{2q_0(1-\Phi(t))}\Big|\ge\varepsilon\Big) = o(1), \tag{11}$$

and

$$\int_0^{b_p} P\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t\}-P(|T_{ij}|\ge t)]}{2q_0(1-\Phi(t))}\Big|\ge\varepsilon\Big)dt = o(v_p), \tag{12}$$

where $v_p = 1/\sqrt{(\log p)c_p^2}$. Then (10) holds.

Let's first finish the proof of (17) in Proposition 1. The proof of Lemma 3 will be given in Section 6.1. We only need to prove (11) and (12). Define

$$\mathcal{A}_i = \{j : |\rho_{ij1}| + |\rho_{ij2}| \geq 2(\log p)^{-2-\gamma}\},$$
$$\mathcal{S} = \{(i,j) : 1 \leq i \leq p, j \in \mathcal{A}_i\},$$
$$\mathcal{H}_{01} = \mathcal{H}_0 \cap \mathcal{S}, \quad \mathcal{H}_{02} = \mathcal{H}_0 \cap \mathcal{S}^c.$$

By (C1), we have $\text{Card}(\mathcal{H}_{01}) \leq Cp^{\xi}$ with $\xi < 1/3$. Recall that $q_1 \leq cq$ for some $0 < c < 1$. Thus, by Proposition 1,

$$\mathsf{E}\Big|\frac{\sum_{(i,j)\in\mathcal{H}_{01}}[I\{|T_{ij}| \geq t\} - \mathsf{P}(|T_{ij}| \geq t)]}{qG(t)}\Big| \leq C\frac{p^{1+\xi}G(t)}{qG(t)} = O(p^{-1+\xi}). \tag{13}$$

Note that for $(i,j) \in \mathcal{H}_{02}$ and $(k,l) \in \mathcal{H}_{02}$,

$$Corr(X_iX_j, X_kX_l) = \rho_{ik1}\rho_{jl1} + \rho_{il1}\rho_{kj1} + O((\log p)^{-2-\gamma}).$$

For some large constant $C > 0$, define

$$\begin{aligned}
\mathcal{H}_{3X} &= \{(i,j,k,l) :\ (i,j) \in \mathcal{H}_{02},\ (k,l) \in \mathcal{H}_{02},\ |Corr(X_iX_j, X_kX_l)| \leq C(\log p)^{-2-\gamma}\},\\
\mathcal{H}_{4X} &= \{(i,j,k,l) \notin \mathcal{H}_{3X} :\ (i,j) \in \mathcal{H}_{02},\ (k,l) \in \mathcal{H}_{02},\ |Corr(X_iX_j, X_kX_l)| \leq \theta + C(\log p)^{-2-\gamma}\},\\
\mathcal{H}_{5X} &= \{(i,j,k,l) \notin \mathcal{H}_{3X} \cup \mathcal{H}_{4X} :\ (i,j) \in \mathcal{H}_{02},\ (k,l) \in \mathcal{H}_{02}\}.
\end{aligned}$$

The $\mathcal{H}_{3Y}, \mathcal{H}_{4Y}, \mathcal{H}_{5Y}$ are defined in the similar way. Let $\mathcal{H}_3 = \mathcal{H}_{3X} \cap \mathcal{H}_{3Y}$, $\mathcal{H}_5 = \mathcal{H}_{5X} \cup \mathcal{H}_{5Y}$ and $\mathcal{H}_4 = \mathcal{H}_{4X} \cup \mathcal{H}_{4Y} \setminus \mathcal{H}_5$. It is easy to show that $Card(\mathcal{H}_4) = O(p^{2+2\xi})$ and $Card(\mathcal{H}_5) = O(p^{1+3\xi} + p^2)$.

Lemma 4. *We have*

$$\max_{(i,j,k,l)\in\mathcal{H}_3}\Big|\frac{P\Big(|T_{ij}| \geq t, |T_{kl}| \geq t\Big)}{G^2(t)} - 1\Big| \leq A_n \tag{14}$$

and

$$\max_{(i,j,k,l)\in\mathcal{H}_4} P\Big(|T_{ij}| \geq t, |T_{kl}| \geq t\Big) \leq C(t+1)^{-2}\exp(-t^2/(1+\theta)) \tag{15}$$

uniformly in $0 \leq t \leq b_p$, where $A_n \leq C(\log p)^{-1-\gamma_2}$ for some $\gamma_2 > 0$.

Proof. By (6), it suffices to prove that T^o_{ij} satisfies (14) and (15). It can be proved that, uniformly for $(i,j,k,l) \in \mathcal{H}_3$,

$$\Big\|\mathsf{Cov}((\boldsymbol{X}_{1ij}, \boldsymbol{X}_{1kl})) - I_2\Big\| + \Big\|\mathsf{Cov}((\boldsymbol{Y}_{1ij}, \boldsymbol{Y}_{1kl})) - I_2\Big\| = O((\log p)^{-2-\gamma}),$$

and uniformly for $(i,j,k,l)\in\mathcal{H}_4$,

$$|\mathsf{Cov}(\boldsymbol{X}_{1ij},\boldsymbol{X}_{1kl})|\le\kappa_1\theta+O((\log p)^{-2-\gamma}),$$
$$|\mathsf{Cov}(\boldsymbol{Y}_{1ij},\boldsymbol{Y}_{1kl})|\le\kappa_2\theta+O((\log p)^{-2-\gamma}).$$

By Lemmas 1 and 2, we prove Lemma 4. □

Set $f_{ijkl}(t)=\mathsf{P}(T_{ij}\ge t,T_{kl}\ge t)-\mathsf{P}(T_{ij}\ge t)\mathsf{P}(T_{kl}\ge t)$. Then

$$\begin{aligned}&\mathsf{E}\Big[\frac{\sum_{(i,j)\in\mathcal{H}_{02}}[I\{|T_{ij}|\ge t\}-\mathsf{P}(|T_{ij}|\ge t)]}{qG(t)}\Big]^2\\&=\frac{\sum_{(i,j,k,l)\in\mathcal{H}_3}f_{ijkl}(t)}{q^2G^2(t)}+\frac{\sum_{(i,j,k,l)\in\mathcal{H}_4}f_{ijkl}(t)}{q^2G^2(t)}+\frac{\sum_{(i,j,k,l)\in\mathcal{H}_5}f_{ijkl}(t)}{q^2G^2(t)}.\end{aligned}$$

By Proposition 1 and Lemma 4, we have

$$\Big|\frac{\sum_{(i,j,k,l)\in\mathcal{H}_3}f_{ijkl}(t)}{q^2G^2(t)}\Big|\le CA_n, \tag{16}$$

$$\Big|\frac{\sum_{(i,j,k,l)\in\mathcal{H}_4}f_{ijkl}(t)}{q^2G^2(t)}\Big|\le\frac{C(t+1)^{2/(1+\theta)-2}}{p^{2-2\xi}[G(t)]^{(2\theta)/(1+\theta)}} \tag{17}$$

and

$$\Big|\frac{\sum_{(i,j,k,l)\in\mathcal{H}_5}f_{ijkl}(t)}{q^2G^2(t)}\Big|\le\frac{C}{p^{3-3\xi}G(t)}+\frac{C}{p^2G(t)}. \tag{18}$$

Combining (13), (16), (17), (18) and the inequality

$$\int_0^{b_p}\Big[p^{-1+\xi}+A_n+\frac{C(t+1)^{2/(1+\theta)-2}}{p^{2-2\xi}[G(t)]^{(2\theta)/(1+\theta)}}+\frac{C}{p^2G(t)}\Big]dt=o(v_p),$$

we prove (12). The proof of (11) is exactly the same with that of (12) and hence is omitted. □

Let Y_{ki}, $1\le k\le n$, be independent random variables for each i. Suppose that $\mathsf{E}Y_{ki}=0$ and $\max_{1\le i\le n^\beta}\mathsf{E}|Y_{1i}/\sqrt{\mathsf{Var}(Y_{1i})}|^\alpha\le K$ for some $\beta>0$, $\alpha>1$ and $K>0$.

Lemma 5. *We have for any $M>0$, there exists $C_0>0$ such that*

$$P\Big(\max_{1\le i\le n^\beta}\Big|\frac{1}{n\sqrt{Var(Y_{1i})}}\sum_{k=1}^nY_{ki}\Big|\ge C_0\sqrt{\frac{\log p}{n}}\Big)=O(p^{-M}+n^{-\alpha/2+\beta+1}(\log p)^{\alpha/2}),$$

where $O(1)$ depends only on r,α,β,C_0,M and K.

Proof. We can assume that $\mathsf{Var}(Y_{1i}) = 1$. Let

$$\hat{Y}_{ki} = Y_{ki} I\Big\{|Y_{ki}| \leq \sqrt{\frac{n}{\log p}}\Big\} - \mathsf{E} Y_{ki} I\Big\{|Y_{ki}| \leq \sqrt{\frac{n}{\log p}}\Big\},$$
$$\breve{Y}_{ki} = Y_{ki} - \hat{Y}_{ki}$$

By Bernstein's inequality, it is easy to prove that for any $M > 0$, there exists $C_1 > 0$ such that

$$\mathsf{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^{n} \hat{Y}_{ki}\Big| \geq 2^{-1} C_0 \sqrt{\frac{\log p}{n}}\Big) \leq C_1 p^{-M},$$

where C_1 depends only on C_2 and M. Note that, by $p \leq n^r$ and the moment condition on Y_{1i},

$$\Big|\mathsf{E} Y_{ki} I\Big\{|Y_{ki}| \leq \sqrt{\frac{n}{\log p}}\Big\}\Big| \leq K n^{-(\alpha-1)/2} (r \log n)^{(\alpha-1)/2}.$$

So we have

$$\begin{aligned} \mathsf{P}\Big(\frac{1}{n}\Big|\sum_{k=1}^{n} \breve{Y}_{ki}\Big| \geq 2^{-1} C_0 \sqrt{\frac{\log p}{n}}\Big) &\leq n \mathsf{P}(|Y_{ki}| \geq \sqrt{n/\log p}) \\ &\leq K n^{-\alpha/2+1} (\log p)^{\alpha/2}. \end{aligned}$$

This proves the lemma. $\square$

1.1 Proof of Lemma 3

Let $0 = t_0 < t_1 < \cdots < t_m = b_p$ satisfy $t_i - t_{i-1} = v_p$ for $1 \leq i \leq m-1$ and $t_m - t_{m-1} \leq v_p$. So $m \sim b_p/v_p$. Recall that $G(t) = 2 - 2\Phi(t)$. For any $t_{j-1} \leq t \leq t_j$, we have

$$\frac{\sum_{(i,j)\in\mathcal{H}_0} I\{|T_{ij}| \geq t\}}{q_0 G(t)} \leq \frac{\sum_{(i,j)\in\mathcal{H}_0} I\{|T_{ij}| \geq t_{j-1}\}}{q_0 G(t_{j-1})} \frac{G(t_{j-1})}{G(t_j)} \tag{19}$$

and

$$\frac{\sum_{(i,j)\in\mathcal{H}_0} I\{|T_{ij}| \geq t\}}{q_0 G(t)} \geq \frac{\sum_{(i,j)\in\mathcal{H}_0} I\{|T_{ij}| \geq t_j\}}{q_0 G(t_j)} \frac{G(t_j)}{G(t_{j-1})}. \tag{20}$$

In view of (15) in Proposition 1, (19) and (20), we only need to prove

$$\max_{0\leq j\leq m} \Big|\frac{\sum_{(i,j)\in\mathcal{H}_0} [I\{|T_{ij}| \geq t_j\} - G(t_j)]}{q_0 G(t_j)}\Big| \to 0$$

in probability. We have

$$\begin{aligned}
&\mathsf{P}\Big(\max_{1\le j\le m}\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t_j\}-G(t_j)]}{q_0G(t_j)}\Big|\ge\varepsilon\Big)\\
&\le\sum_{j=1}^{m}\mathsf{P}\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t_j\}-G(t_j)]}{q_0G(t_j)}\Big|\ge\varepsilon\Big)\\
&\le\frac{1}{v_p}\int_0^{b_p}\mathsf{P}\Big(\frac{\sum_{(i,j)\in\mathcal{H}_0}I\{|T_{ij}|\ge t\}}{q_0G(t)}\ge 1+\varepsilon/2\Big)dt\\
&\quad+\frac{1}{v_p}\int_0^{b_p}\mathsf{P}\Big(\frac{\sum_{(i,j)\in\mathcal{H}_0}I\{|T_{ij}|\ge t\}}{q_0G(t)}\le 1-\varepsilon/2\Big)dt\\
&\quad+\sum_{j=m-1}^{m}\mathsf{P}\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t_j\}-G(t_j)]}{q_0G(t_j)}\Big|\ge\varepsilon\Big).
\end{aligned}$$

So it suffices to prove

$$\int_0^{b_p}\mathsf{P}\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t\}-G(t)]}{q_0G(t)}\Big|\ge\varepsilon\Big)dt=o(v_p)$$

and

$$\sum_{k=m-1}^{m}\mathsf{P}\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t_k\}-G(t_k)]}{q_0G(t_k)}\Big|\ge\varepsilon\Big)=o(1).$$

The proof is complete. $\square$

1.2 Proof of Theorem 1

By Proposition 1, we have

$$\begin{aligned}
\mathsf{P}\Big(\sum_{(i,j)\in\mathcal{H}_0}I\{|T_{ij}|\ge 2\sqrt{\log p}\}\ge 1\Big) &\le |\mathcal{H}_0|\max_{(i,j)\in\mathcal{H}_0}\mathsf{P}\Big(|T_{ij}|\ge 2\sqrt{\log p}\Big)\\
&\le C(\log p)^{-1/2}=o(1).
\end{aligned}$$

Set the event $F=\{\hat{t}\text{ exists}\}$ and let I_F be the indicator function of F. By Proposition 1, the continuity of $\Phi(t)$ and the definition of $\hat{t}$, we can see that for any $\varepsilon>0$, with probability tending to one,

$$G^*_{N,n}(\hat{t})I_F\ge\frac{(\alpha-\varepsilon)\max\{\sum_{1\le i<j\le p}I\{|T_{ij}|\ge\hat{t}\},1\}}{q}I_F.$$

Thus,

$$\Big|\frac{qG^*_{N,n}(\hat{t})}{\max\{\sum_{1\le i<j\le p}I\{|T_{ij}|\ge\hat{t}\},1\}}-\alpha\Big|I_F\to 0 \tag{21}$$

in probability. On F, we have $0 \le \hat{t} \le b_p$. To prove Theorem 1, it is enough to prove that

$$\sup_{0\le t\le b_p}\Big|\frac{G^*_{N,n}(t)}{\frac{1}{q_0}\sum_{(i,j)\in\mathcal{H}_0} I\{|T_{ij}|\ge t\}}-1\Big|\to 0$$

in probability. This follows from Proposition 1. $\square$

1.3 Proof of Theorem 2

Proposition 1 together with (19) in Theorem 2 implies that

$$\mathsf{P}\Big(Card\{(i,j):\ \ |T_{ij}|\ge 2\sqrt{\log p}\}\ge\Big(\frac{1}{\sqrt{8\pi}\alpha}+\varepsilon\Big)\sqrt{\log\log p}\Big)\to 1.$$

Note that

$$G^*_{n,N}(b_p)\sim 2(1-\Phi(b_p))\sim\frac{2}{\sqrt{2\pi}b_p}\exp(-b_p^2/2)\sim\frac{\sqrt{\log\log p}}{\sqrt{2\pi}}p^{-2}$$

in probability. Hence, by Proposition 1 and the definition of $\hat{t}$, we have $\mathsf{P}(0\le\hat{t}\le b_p)\to 1$. That is, $\mathsf{P}(I_F=1)\to 1$. The theorem follows from (17) in Proposition 1 and (21). $\square$

1.4 Proof of Theorems 3 and 4

We assume that $\mathsf{E}\boldsymbol{X}=0$. Note that for $(i,j)\in\mathcal{H}_0$ and $(k,l)\in\mathcal{H}_0$, by (C2),

$$\frac{\mathsf{Cov}(X_iX_j,X_kX_l)}{\sqrt{\theta_{ij}\theta_{kl}}}=\rho_{ik}\rho_{jl}+\rho_{il}\rho_{kj}.$$

Define

$$\begin{aligned}\mathcal{N}_1 &= \{(i,j,k,l):\ (i,j)\neq(k,l),\ |\rho_{ik}\rho_{jl}+\rho_{il}\rho_{kj}|\ge(\log p)^{-2-\gamma}\},\\ \mathcal{N} &= \{(i,j):\ 1\le i<j\le p,\quad |\rho_{ij}|\ge(\log p)^{-2-\gamma}/2\}.\end{aligned}$$

Then $|\mathcal{N}|\ge\min(\sqrt{|\mathcal{N}_1|},|\mathcal{N}_1|/p)$. For (i,j,k,l) satisfying $|\rho_{ik}\rho_{jl}+\rho_{il}\rho_{kj}|\le(\log p)^{-2-\gamma}$, as the proof of (14), we have

$$\Big|\frac{\mathsf{P}\Big(|T_{ij}|\ge t,|T_{kl}|\ge t\Big)}{G^2(t)}-1\Big|\le A_n$$

uniformly for $0<t<b_p$. By the proof of Theorem 2, it is easy to prove that $\mathsf{P}(0\le\hat{t}\le\min\{b_p,G^{-1}(\alpha|\mathcal{N}|/p^2)\})\to 1$. Set $\tilde{b}_p=\min\{b_p,G^{-1}(\alpha|\mathcal{N}|/p^2)\}$. We have

$$\int_0^{\tilde{b}_p}\mathsf{P}\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\ge t\}-G(t)]}{q_0G(t)}\Big|\ge\varepsilon\Big)dt$$

$$
\begin{aligned}
&\leq C\int_0^{\tilde{b}_p}\Big(\frac{|\mathcal{N}_1|}{p^4G(t)}+\frac{1}{p^2G(t)}+A_n\Big)dt\\
&= o(v_p),
\end{aligned}
$$

where we used

$$
\begin{aligned}
\int_0^{\tilde{b}_p}\frac{|\mathcal{N}_1|}{p^4G(t)}dt &\leq \int_0^{\tilde{b}_p}\frac{|\mathcal{N}|}{p^3G(t)}dt+\int_0^{\tilde{b}_p}\frac{|\mathcal{N}|^2}{p^4G(t)}dt\\
&\leq C\int_0^{\tilde{b}_p}\Big(\frac{1}{p}+\frac{|\mathcal{N}|}{p^2}\Big)dt\\
&= o(v_p).
\end{aligned}
$$

Similarly,

$$
\sup_{0\leq t\leq \tilde{b}_p}\mathsf{P}\Big(\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\geq t\}-G(t)]}{q_0G(t)}\Big|\geq\varepsilon\Big)=o(1).
$$

So it follows from Lemma 3 that

$$
\sup_{0\leq t\leq \tilde{b}_p}\Big|\frac{\sum_{(i,j)\in\mathcal{H}_0}[I\{|T_{ij}|\geq t\}-G(t)]}{q_0G(t)}\Big|\to 0
$$

in probability.

It is enough to prove that (16) in Proposition 1 holds for one sample test statistic in (24) in Section 4. Note that, $\mathcal{X}_j^*$, $1\leq j\leq p$, are independent given the observations. So we have $T^*_{ij,k}$ and $T^*_{lm,k}$ are independent when $i\neq j$ and $k\neq m$, given the observations. Note that

$$
\begin{aligned}
&\mathsf{E}^*\Big(\sum_{k=1}^N\sum_{1\leq i<j\leq p}[I\{|T^*_{ij,k}|\geq t\}-G^*_{ij}(t)]\Big)^2\\
&\quad\leq\sum_{k=1}^N\sum_{1\leq i<j\leq p}\mathsf{E}^*[I\{|T^*_{ij,k}|\geq t\}-G^*_{ij}(t)]^2\\
&\qquad+\sum_{k=1}^N\sum_{1\leq i<j\leq p}\sum_{l\neq i,j}^p\mathsf{P}^*\Big(|T^*_{ij,k}|\geq t,|T^*_{il,k}|\geq t\Big).
\end{aligned}
$$

For $l\neq i,j$ and $i\neq j$,

$$
\mathsf{Cov}\Big[(X^*_{ki}-\bar{X}_i)(X^*_{kj}-\bar{X}_j),(X^*_{ki}-\bar{X}_i)(X^*_{kl}-\bar{X}_l)\Big|\mathcal{X}_j,1\leq j\leq p\Big]=0.
$$

By the proofs of (7) and (14), on $\boldsymbol{F}$,

$$
\max_{1\leq k\leq N}\max_{1\leq i<j\leq p,l\neq i,j}\Big|\frac{\mathsf{P}^*\Big(|T^*_{ij,k}|\geq t,|T^*_{il,k}|\geq t\Big)}{G^2(t)}-1\Big|\leq C(\log p)^{-1-\delta}
$$

for some $\delta > 0$, uniformly in $0 \leq t \leq 2\sqrt{\log p}$. Also, (7) holds for one sample $T^{*}_{ij,k}$. Hence, we can show that (8) and (9) hold for one sample case. The rest proof is similar to that of (16) in Proposition 1 and Theorems 1 and 2. $\square$

References

[1] Liu, W.D. (2013). Gaussian graphical model estimation with false discovery rate control. *Annals of Statistics*, 41, 2948-2978.