Joint Estimation and Inference for Multiple Multi-layered Gaussian Graphical Models

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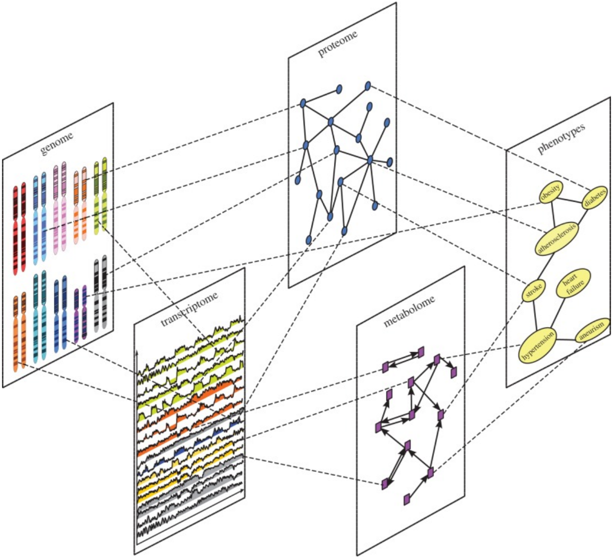
- Biological processes in the body have a natural hierarchical structure, e.g. **Gene > Protein > Metabolite**;
- There are within layer and between-layer connections in this structure;
- These connections can be different inside different organs, experimental conditions, or for different subtypes of the same disease;
- We design a framework for *joint estimation and hypothesis testing* for all the connections in these complex biological structure.

Schematic of data integration

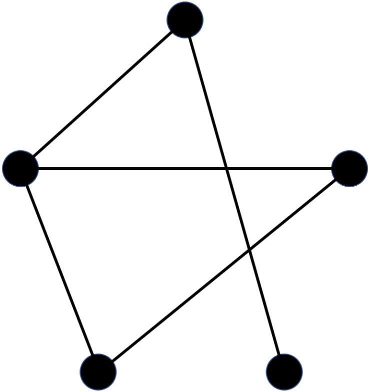


(Source: Gligorijević and Pržulj (2015))

Outline

Gaussian Graphical models

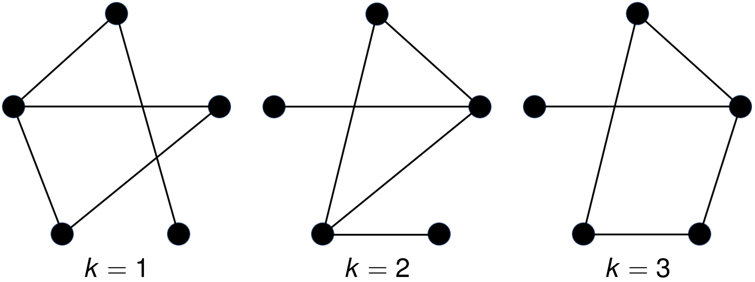
$$\mathbb{X} = (X_1, \ldots, X_p)^T \sim \mathcal{N}_p(0, \Sigma_x); \quad \Omega_x = \Sigma_x^{-1}$$



Sparse estimation of Ω_x: Meinshausen and Bühlmann (2006)
Multiple testing and error control: Drton and Perlman (2007).

Multiple Gaussian Graphical models

$$\mathbb{X}^k = (X_1^k, \ldots, X_p^k)^T \sim \mathcal{N}_p(0, \Sigma_x^k); \quad \Omega_x^k = (\Sigma_x^k)^{-1}$$
$$k = 1, 2, \ldots, K$$

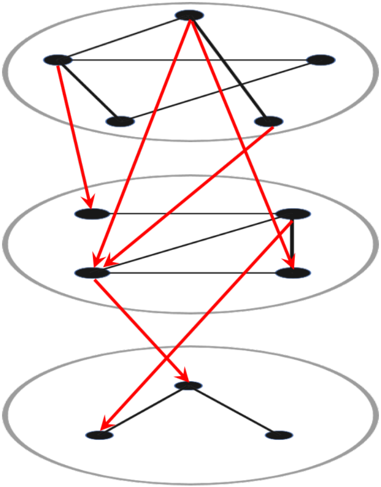


- Joint estimation of $\{\Omega_x^k\}$: Guo et al. (2011); Ma and Michailidis (2016)
- Difference and similarity testing with FDR control: Liu (2017+)

Multi-Layered Gaussian Graphical models

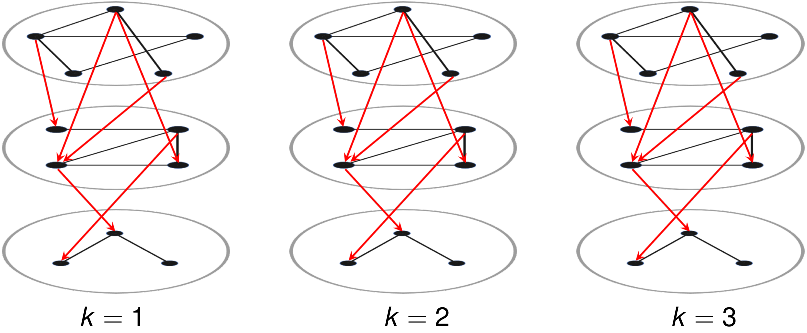
$$\mathbb{E} = (E_1, \ldots, E_q)^T \sim \mathcal{N}_p(0, \Sigma_y);$$
$$\Omega_y = (\Sigma_y)^{-1}$$
$$\mathbb{Y} = \mathbb{X}\mathbf{B} + \mathbb{E}$$

- Ω_x, Ω_y give undirected within-layer edges, while **B** gives directed between-layer edges.
- Sparse estimation of $(\Omega_y, \Omega_x, \mathbf{B})$: Lin et al. (2016).



Multiple Multi-layered Gaussian Graphical models

$$\mathbb{E}^k = (E_1^k, \ldots, E_q^k)^T \sim \mathcal{N}_p(0, \Sigma_y^k); \quad \Omega_y^k = (\Sigma_y^k)^{-1}$$
$$\mathbb{Y}^k = \mathbb{X}^k \mathbf{B}^k + \mathbb{E}^k; \quad k = 1, 2, \ldots, K$$



- We estimate $\{\Omega_x^k, \Omega_y^k, \mathbf{B}^k\}$ jointly for all k from a single model;
- For $K = 2$ and $i \in \{1, 2, \ldots, p\}$, we also provide a global test for $\mathbf{b}_i^1 = \mathbf{b}_i^2$, and do multiple testing for $b_{ij}^1 = b_{ij}^2, j = 1, 2, \ldots, q$.

- $\mathcal{Y} = \{\mathbf{Y}^1, \ldots, \mathbf{Y}^k\}, \mathcal{X} = \{\mathbf{X}^1, \ldots, \mathbf{X}^k\}, \mathcal{B} = \{\mathbf{B}^1, \ldots, \mathbf{B}^k\}$;
- Group structures in X-network is denoted by

$$\mathcal{G}_x = \{\mathcal{G}_{x,ii'} : i \neq i', 1 \leq i, i' \leq p\}$$

  Each $\mathcal{G}_{x,ii'}$ is a partition of $\{1, \ldots, K\}$ denoting grouping over k for the $(i, i')^{\text{th}}$ elements of the X-precision matrices. For example, for $K = 5$,

$$\mathcal{G}_{x,12} = \{(1, 2), (3), (4, 5)\}; \quad \mathcal{G}_{x,13} = \{(1), (2, 3), (4, 5)\}$$

- Define $\mathcal{G}_y = \{\mathcal{G}_{y,jj'} : j \neq j', 1 \leq j, j' \leq q\}$ similarly.
- Group structures in $\mathcal{B}$ is denoted by $\mathcal{H}$, with each $h \in \mathcal{H}$ being a collection of 3-tuples (h_i, h_j, h_k) so that $1 \leq h_i \leq p, 1 \leq h_j \leq q, 1 \leq h_k \leq K$.

Neighborhood selection

- **For single GGM**: Estimate neighboring edges for each node, then refit.

$$\hat{\zeta}_i = \underset{\zeta_i}{\text{argmin}} \left\{ \frac{1}{n} \| \mathbf{X}_i - \mathbf{X}_{-i} \zeta_i \|^2 + \nu_n \sum_{i' \neq i} |\zeta_{ii'}| \right\};$$

$$\hat{\Omega}_x = \underset{\Omega_x \in \cup_i \text{support}(\zeta_i)}{\text{argmin}} \{ \text{Tr}(\mathbf{S}_x \Omega_x) + \log\det(\Omega_x) \}$$

  Can do this because $\zeta_{ii'} = -\omega_{ii'}/\omega_{ii}$, so zeros of the precision matrix and neighborhood matrix are same.

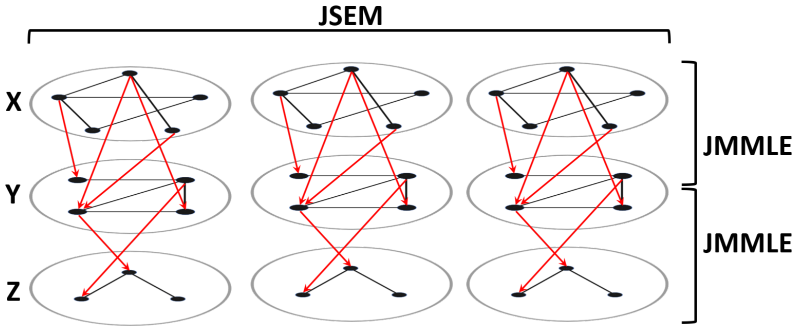
- **For multiple GGM**: Incorporate penalty across different k (JSEM: Ma and Michailidis (2016)).

$$\hat{\zeta}_i = \underset{\zeta_i}{\text{argmin}} \left\{ \sum_{k=1}^{K} \frac{1}{n_k} \| \mathbf{X}_i^k - \mathbf{X}_{-i}^k \zeta_i^k \|^2 + \nu_n \sum_{i' \neq i, g \in \mathcal{G}_{x,ii'}} \| \zeta_{ii'}^{[g]} \| \right\};$$

$$\hat{\Omega}_x^k = \underset{\Omega_x^k \in \cup_i \text{support}(\zeta_i^k)}{\text{argmin}} \left\{ \text{Tr}(\mathbf{S}_x^k \Omega_x^k) + \log\det(\Omega_x^k) \right\};$$

Joint Multiple Multi-Level Estimation (JMMLE)

We decompose the multi-layer problem into a series of two layer problems. For within-layer connections in the topmost layer, we use JSEM. For other connections we use JMMLE.



We combine sparse neighborhood selection in the Y-network with sparse estimation of $\{\mathbf{B}^k\}$.
Take $\Theta_j = (\boldsymbol{\theta}_j^1, \ldots, \boldsymbol{\theta}_j^K), \Theta = \{\Theta_j\}_{j=1}^q$.

$$\{\hat{\mathcal{B}}, \hat{\Theta}\} = \underset{\mathcal{B}, \Theta}{\text{argmin}} \left\{ \sum_{j=1}^q \frac{1}{n_k} \sum_{k=1}^K \left\| \mathbf{Y}_j^k - (\mathbf{Y}_{-j}^k - \mathbf{X}^k \mathbf{B}_{-j}^k) \theta_j^k - \mathbf{X}^k \mathbf{B}_j^k \right\|^2 \right.$$
$$\left. + \lambda_n \sum_{h \in \mathcal{H}} \| \mathbf{B}^{[h]} \| + \gamma_n \sum_{j' \neq j, g \in \mathcal{G}_{jj'}} \| \boldsymbol{\theta}_{jj'}^{[g]} \| \right\}$$
$$\hat{\Omega}_y^k = \underset{\Omega_y^k \in \cup_i \text{support}(\boldsymbol{\theta}_i^k)}{\text{argmin}} \left\{ \text{Tr}(\mathbf{S}_y^k \Omega_y^k) + \log \det(\Omega_y^k) \right\} \quad k = 1, 2, \ldots, K$$

$$\{\hat{\mathcal{B}}, \hat{\Theta}\} = \underset{\mathcal{B},\Theta}{\operatorname{argmin}} \{f(\mathcal{Y}, \mathcal{X}, \mathcal{B}, \Theta) + P(\mathcal{B}) + Q(\Theta)\}$$

The objective function is biconvex, so we solve the above by the following alternating iterative algorithm:

1. Start with initial estimates of $\mathcal{B}$ and Θ, say $\mathcal{B}^{(0)}, \Theta^{(0)}$.
2. Iterate:

$$\mathcal{B}^{(t+1)} = \underset{\mathcal{B}}{\operatorname{argmin}} \left\{ f(\mathcal{Y}, \mathcal{X}, \mathcal{B}, \Theta^{(t)}) + Q(\mathcal{B}) \right\}$$
$$\Theta^{(t+1)} = \underset{\Theta}{\operatorname{argmin}} \left\{ f(\mathcal{Y}, \mathcal{X}, \mathcal{B}^{(t+1)}, \Theta) + P(\Theta) \right\}$$

3. Continue till convergence.

The two subproblems

Non-asymptotic error bounds for $\hat{\mathcal{B}}$

For $\lambda_n \geq 4\sqrt{|h_{\max}|}\mathbb{R}_0\sqrt{\frac{\log(pq)}{n}}$, the following hold with probability approaching 1 as $n \to \infty$,

$$\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|_1 \leq \frac{48\sqrt{|h_{\max}|}s_\beta \lambda_n}{\psi^*}$$

$$\|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\| \leq \frac{12\sqrt{s}_\beta \lambda_n}{\psi^*}$$

$$\sum_{h \in \mathcal{H}} \|\boldsymbol{\beta}^{[h]} - \boldsymbol{\beta}_0^{[h]}\| \leq \frac{48 s_\beta \lambda_n}{\psi^*}$$

with $\psi^*, \mathbb{R}_0$ being constants, and $\boldsymbol{\beta} = (\text{vec}(\mathbf{B}^1)^T, \ldots, \text{vec}(\mathbf{B}^K)^T)^T$, $|h_{\max}|$ the maximum group size in $\boldsymbol{\beta}_0$ and s_β the sparsity of $\boldsymbol{\beta}_0$.

Error bounds for $\hat{\Theta}, \hat{\Omega}$

For $\gamma_n = 4\sqrt{|g_{\max}|}\mathbb{Q}_0\sqrt{\frac{\log(pq)}{n}}$, the following hold with probability approaching 1 as $n \to \infty$,

$$
\begin{aligned}
\|\hat{\Theta}_j - \Theta_{0,j}\|_F &\leq \frac{12\sqrt{s_j}\gamma_n}{\psi} \\
\sum_{j \neq j', g \in \mathcal{G}_y^{jj'}} \|\hat{\boldsymbol{\theta}}_{jj'}^{[g]} - \boldsymbol{\theta}_{0,jj'}^{[g]}\| &\leq \frac{48 s_j \gamma_n}{\psi} \\
\left|\text{support}(\hat{\Theta}_j)\right| &\leq \frac{128 s_j}{\psi} \\
\frac{1}{K}\sum_{k=1}^{K} \|\hat{\Omega}_y^k - \Omega_y^k\|_F &\leq O\left(\frac{\sqrt{S}\gamma_n}{\sqrt{K}}\right)
\end{aligned}
$$

with $\psi, \mathbb{Q}_0$ being constants, $|g_{\max}|$ the maximum group size in Θ_0, s_j the sparsity of Θ_j and $S = \sum_j s_j$.

Outline

- Consider the case $K = 2$, and suppose we are interested in testing if the effect of variable i in the X-data is different across the two populations.
- For this we use the i^{th} rows of the estimates $\hat{\mathbf{B}}^1$ and $\hat{\mathbf{B}}^2$.
- We debias these row vectors using the neighborhood coefficients in the X-network computed previously using JSEM: In this setup, define the desparsified estimate of $\mathbf{b}_i^k$ as

$$\hat{\mathbf{c}}_i^k = \hat{\mathbf{b}}_i^k + \frac{1}{n t_i^k} \left(\mathbf{X}_i^k - \mathbf{X}_{-i}^k \hat{\zeta}_i^k \right)^T (\mathbf{Y}^k - \mathbf{X}^k \hat{\mathbf{B}}^k)$$

for $k = 1, 2$, where $t_i^k := (\mathbf{X}_i^k - \mathbf{X}_{-i}^k \hat{\zeta}_i^k)^T \mathbf{X}_{-i}^k / n$.

Result

Assume we have 'good enough' estimators:

$$\|\hat{\zeta}^k - \zeta_0^k\|_1 = O\left(\sqrt{\frac{\log p}{n}}\right); \quad \|\hat{\mathbf{B}}^k - \mathbf{B}_0^k\|_1 = O\left(\sqrt{\frac{\log(pq)}{n}}\right)$$

$$\left\|(\hat{\Omega}_y^k)^{1/2} - (\Omega_y^k)^{1/2}\right\|_\infty = O\left(\sqrt{\frac{\log q}{n}}\right)$$

Also define

$$\hat{s}_i^k := \sqrt{\|\mathbf{X}_i^k - \mathbf{X}_{-i}^k \hat{\zeta}_i^k\|^2 / n}; \quad m_i^k := \sqrt{n} t_i^k / \hat{s}_i^k$$

Then for sample size satisfying $n \gtrsim \log(pq), \log p = o(n^{1/2}), \log q = o(n^{1/2})$ we have

$$\begin{bmatrix} \hat{\Omega}_y^1 & \\ & \hat{\Omega}_y^2 \end{bmatrix}^{1/2} \begin{bmatrix} m_i^1(\hat{\mathbf{c}}_i^1 - \mathbf{b}_i^1) & \\ & m_i^2(\hat{\mathbf{c}}_i^2 - \mathbf{b}_i^2) \end{bmatrix} \sim \mathcal{N}_{2q}(\mathbf{0}, \mathbf{I}) + o_P(1)$$

Global test for $H_0 : \mathbf{b}_{0i}^1 = \mathbf{b}_{0i}^2$ at level $\alpha, 0 < \alpha < 1$

1. Obtain the debiased estimators $\hat{\mathbf{c}}_i^1, \hat{\mathbf{c}}_i^2$;
2. Calculate the test statistic

$$D_i = \left\| m_i^1 (\hat{\Omega}_y^1)^{1/2} \hat{\mathbf{c}}_i^1 - m_i^2 (\hat{\Omega}_y^2)^{1/2} \hat{\mathbf{c}}_i^2 \right\|^2$$

3. Reject H_0 if $D_i \geq \chi^2_{2q, 1-\alpha}$.

1. Calculate the pairwise test statistics d_{ij} for $j = 1, \ldots, q$:

$$d_{ij} = \frac{\tau_{ij}^1 \hat{\mathbf{c}}_{ij}^1 - \tau_{ij}^2 \hat{\mathbf{c}}_{ij}^2}{1/m_i^1 + 1/m_i^2}$$

   where τ_{ij}^k is the $(i,j)^{\text{th}}$ element of $(\hat{\Omega}_y^k)^{1/2}, k = 1, 2$.

2. Obtain the threshold

$$\hat{\tau} = \inf \left\{ \tau \in \mathbb{R} : 1 - \Phi(\tau) \leq \frac{\alpha}{2q} \max \left(\sum_{j \in \mathcal{I}_q} \mathbb{I}(|d_{ij}| \geq \tau), 1 \right) \right\}$$

3. For $j \in \mathcal{I}_q$, reject H_0^j if $|d_{ij}| \geq \hat{\tau}$.

Outline

Simulation setup

- Number of categories (K) = 5;
- Structured $\{\Omega_x\}, \{\Omega_y\}, \mathcal{B}$;
- Groups in $\mathcal{B}, \Omega_x$ are non-zero with probability $5/p$, and their elements come from Unif$(-1, -0.5) \cup (0.5, 1)$;
- Groups in Ω_y are non-zero with probability $5/q$, and their elements come from Unif$(-1, -0.5) \cup (0.5, 1)$;
- We generate size-n i.i.d. samples $\mathbf{X}^k$ from $\mathcal{N}_p(0, \Sigma_x^k)$, and $\mathbf{E}^k$ from $\mathcal{N}_p(0, \Sigma_y^k)$, then obtain $\mathbf{Y}^k = \mathbf{X}^k \mathbf{B}^k + \mathbf{E}^k$;
- 100 Replications.

Evaluation metrics

1. True positives-

$$\mathrm{TP}(\hat{\mathcal{B}}) = \frac{\sum_k |\mathrm{supp}(\hat{\mathbf{B}}^k) \cup \mathrm{supp}(\mathbf{B}_0^k)|}{\sum_k |\mathrm{supp}(\mathbf{B}_0^k)|}$$

2. True negatives-

$$\mathrm{TN}(\hat{\mathcal{B}}) = \frac{\sum_k |\mathrm{supp}^c(\hat{\mathbf{B}}^k) \cup \mathrm{supp}^c(\mathbf{B}_0^k)|}{\sum_k |\mathrm{supp}^c(\mathbf{B}_0^k)|}$$

3. Relative error in Frobenius norm-

$$\mathrm{rel.Frob}(\hat{\mathcal{B}}) = \sum_{k=1}^{K} \frac{\|\hat{\mathbf{B}}^k - \mathbf{B}_0^k\|_F}{\|\mathbf{B}_0^k\|_F}$$

Same metrics are used for $\hat{\Theta}$.

Results

Setting 1: $n = 100, p = 60, q = 30, K = 5$						
Method	TP($\hat{\mathcal{B}}$)	TN($\hat{\mathcal{B}}$)	rel.Frob($\hat{\mathcal{B}}$)	TP($\hat{\Theta}$)	TN($\hat{\Theta}$)	rel.Frob($\hat{\Theta}$)
Joint (JMMLE)	0.999 (2e-3)	0.99 (0.01)	0.19 (0.02)	0.66 (0.06)	0.95 (0.01)	0.33 (0.02)
Separate	0.95 (0.02)	0.99 (2e-3)	0.27 (0.03)	0.89 (0.02)	0.63 (0.01)	0.77 (0.04)
Setting 2: $n = 100, p = 30, q = 60, K = 5$						
Method	TP($\hat{\mathcal{B}}$)	TN($\hat{\mathcal{B}}$)	rel.Frob($\hat{\mathcal{B}}$)	TP($\hat{\Theta}$)	TN($\hat{\Theta}$)	rel.Frob($\hat{\Theta}$)
Joint (JMMLE)	0.996 (4e-3)	0.99 (6e-3)	0.21 (0.01)	0.58 (0.04)	0.98 (3e-3)	0.32 (8e-3)
Separate	0.66 (0.04)	0.994 (1e-3)	0.59 (0.03)	0.62 (0.03)	0.81 (7e-3)	0.43 (0.01)
Setting 3: $n = 150, p = 200, q = 200, K = 5$						
Method	TP($\hat{\mathcal{B}}$)	TN($\hat{\mathcal{B}}$)	rel.Frob($\hat{\mathcal{B}}$)	TP($\hat{\Theta}$)	TN($\hat{\Theta}$)	rel.Frob($\hat{\Theta}$)
Joint (JMMLE)	1.00 (0)	1.00 (0)	0.12 (5e-3)	0.39 (0.04)	0.996 (2e-3)	0.30 (7e-3)
Separate	0.95 (0.02)	0.99 (2e-3)	0.27 (0.03)	0.89 (0.02)	0.63 (0.01)	0.77 (0.04)

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THANK YOU!

Questions?