Designing a Minimum Length Supersonic Nozzle using the Method of Characteristics

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1 Method of Characteristics

1.1 Introduction

The method of characteristics (MoC) is a powerful mathematical technique for solving hyperbolic partial differential equations (PDEs), such as the wave equation or the velocity potential equation. The fundamental basis of the MoC lies in transforming PDEs into simpler ordinary differential equations (ODEs) along special curves called as characteristics. These characteristics represent the path along which information propagates in the solution. Physically, this means that specific boundary condition or initial condition data forms the solution along particular paths. By transforming the PDE into ODEs along these characteristic curves, the method simplifies the problem, reducing the number of independent variables.

1.2 Characteristic Curves for Inviscid Irrotational Flow

Consider a two-dimensional, inviscid, irrotational flow. The velocity potential equation reads:

$$\left(1 - \frac{\phi_x^2}{a^2}\right)\phi_{xx} + \left(1 - \frac{\phi_y^2}{a^2}\right)\phi_{yy} - \frac{2\phi_x\phi_y}{a^2}\phi_{xy} = 0 \tag{1}$$

$\phi(x, y)$ is the velocity potential function and a is the local speed of sound. The velocity potential is related to the local velocity field via

$$u = \phi_x \qquad\qquad v = \phi_y \tag{2}$$

Using the chain rule, we can write the exact differentials of the partial derivatives of ϕ as:

$$d\phi_x = \phi_{xx}dx + \phi_{xy}dy \tag{3}$$

$$d\phi_y = \phi_{xy}dx + \phi_{yy}dy \tag{4}$$

Expressing these using velocity components, we can simplify to

$$\phi_{xx}dx + \phi_{xy}dy = du \tag{5}$$

$$\phi_{xy}dx + \phi_{yy}dy = dv \tag{6}$$

Equation 1 can be rewritten using the velocity components as

$$\left(1-\frac{u^2}{a^2}\right)\phi_{xx}+\left(1-\frac{v^2}{a^2}\right)\phi_{yy}-\frac{2uv}{a^2}\phi_{xy}=0 \tag{7}$$

Equations 5-7 form a system of equations for the second order partial derivatives of ϕ. These can be solved for using the Cramer's rule. For example, we can solve for ϕ_{xy}:

$$\phi_{xy}=\frac{\begin{vmatrix}1-\frac{u^2}{a^2} & 0 & 1-\frac{v^2}{a^2}\\ dx & du & 0\\ 0 & dv & dy\end{vmatrix}}{\begin{vmatrix}1-\frac{u^2}{a^2} & -\frac{2uv}{a^2} & 1-\frac{v^2}{a^2}\\ dx & dy & 0\\ 0 & dx & dy\end{vmatrix}}=\frac{N}{D} \tag{8}$$

At each location in space, ϕ_{xy} has a particular value. For general choices of dx and dy pointing away from this location, the incremantal change in the velocity is given by du and dv. Regardless of the choice of dx and dy, the associated change in velocity will ensure ϕ_{xy} is the same for all choices.

An exception to this occurs for specific values of dx and dy when the denominator vanishes. These lines in space are called as the characterlistic lines. These directions are found be setting $D=0$:

$$\left(1-\frac{u^2}{a^2}\right)(dy)^2+\frac{2uv}{a^2}dxdy+\left(1-\frac{v^2}{a^2}\right)(dx)^2=0 \tag{9}$$

$$\left(1-\frac{u^2}{a^2}\right)(dy)^2+\frac{2uv}{a^2}dxdy+\left(1-\frac{v^2}{a^2}\right)(dx)^2=0 \tag{10}$$

Rearranging and solving the quadratic equation gives:

$$\left(\frac{dy}{dx}\right)_{char}=\frac{-\frac{uv}{a^2}+\sqrt{\frac{u^2+v^2}{a^2}-1}}{1-\frac{u^2}{a^2}} \tag{11}$$

For supersonic flows where $\frac{u^2+v^2}{a^2}=M^2>1$, there are two real characteristic lines. Hence, the MoC is most suited for finding solutions to hyperbolic PDEs.
Since $u=V\cos\theta$ and $v=V\sin\theta$, Equation 11 can be simplified to

$$\left(\frac{dy}{dx}\right)_{char}=tan(\theta\pm\mu) \tag{12}$$

where θ is the angle made by the local velocity and $\mu=sin^{-1}\left(\frac{1}{M}\right)$ is the Mach angle.
Thus, the two characteristic lines passing through any given point are also the Mach lines. The left running $\theta+\mu$ line is called as the C_+ characteristic, while the right running $\theta-\mu$ line is called as the C_- characteristic.

1.3 Compatibility relations

Next, we need to find the evolution equation for the flow variables along the characteristic curves. Recall that along the characteristic curves, $D = 0$. However, an infinite value of ϕ_{xy} is physically inconsistent. To keep it finite, we also need $N = 0$. That is $\phi_{xy} = \frac{N}{D}$ is indeterminate. Setting $N = 0$ yields

$$\frac{dv}{du} = \frac{-(1 - \frac{u^2}{a^2})}{-(1 - \frac{v^2}{a^2})} \frac{dy}{dx} \tag{13}$$

Equation 13 holds true only along the characteristic lines. Hence, $\frac{dy}{dx}$ can be substituted for using Equation 12. On simplifying, we get

$$d\theta = \mp\sqrt{M^2 - 1}\frac{dV}{V} \tag{14}$$

Equation 14 is called as the compatibility relation. Recall that a similar equation was derived in class for prandtl-meyer flows.
Comparing with Equation 12, we note that

$$d\theta = -\sqrt{M^2 - 1}\frac{dV}{V} \qquad \text{holds along the } C_- \text{ characteristic} \tag{15}$$

$$d\theta = +\sqrt{M^2 - 1}\frac{dV}{V} \qquad \text{holds along the } C_+ \text{ characteristic} \tag{16}$$

Equations 15, 16 can be expressed in their integral forms using $\nu(M)$ - the Prandtl Meyer function. The compatibility equations thus become algebraic equations.

$$\theta + \nu(M) = K_- \qquad \text{along the } C_- \text{ characteristic} \tag{17}$$

$$\theta - \nu(M) = K_+ \qquad \text{along the } C_+ \text{ characteristic} \tag{18}$$

1.4 Unit Processes

In order to numerically apply the MoC, we need to perform a series of specific computations called unit processes. Consider a point P formed by the intersection of two characteristic curves 1 and 2. Note that two characteristics of the same nature do not intersect in a shock-free flow. From our previous discussion, we have the following relations.

$$\theta_1 + \nu_1 = (K_-)_1 \qquad \text{along the } C_1 \text{ characteristic} \tag{19}$$

$$\theta_2 - \nu_2 = (K_+)_2 \qquad \text{along the } C_2 \text{ characteristic} \tag{20}$$

At the intersection point P :

$$\theta_P + \nu_P = (K_-)_P = (K_-)_1 \tag{21}$$

$$\theta_P - \nu_P = (K_+)_P = (K_+)_2 \tag{22}$$

Solving Equations 21 and 22 gives us :

$$\theta_P = \frac{1}{2}((K_-)_1 + (K_+)_2) \tag{23}$$

$$\nu_P = \frac{1}{2}((K_-)_1 - (K_+)_2) \tag{24}$$

In order to find the coordinate of P, we assume that the characteristics are straight line segments between 1-P and 2-P, with slopes that are average values. For example, the C_- characteristic through point 1 is considered a straight line with slope

$$\frac{1}{2}[(\theta_1 + \theta_P) - (\mu_1 + \mu_P)] \tag{25}$$

Boundary points need to be treated separately, since here the second characteristic is outside the flow domain. At the walls, flow tangency condition ensures that the local velocity is along the boundary. Hence, θ_P is already know, which can be appropriately used in 21 or 22 depending on the nature of the characteristic intersecting with the wall. The intersection point at the wall can be found using methods similar to that discussed earlier.

2 Design of 2D Minimum Length Nozzle

2.1 Background

Supersonic nozzles are widely used in engineering to expand a flow to desired supersonic conditions. These nozzles can be classified into two types: gradual-expansion nozzles and minimum-length nozzles. Gradual-expansion nozzles are typically preferred in applications where maintaining a high-quality flow at specific exit conditions is critical, such as in supersonic wind tunnels. In contrast, applications like rocket nozzles face substantial penalties in weight and length with gradual-expansion designs, making them impractical. Therefore, minimum-length nozzles, which initiate expansion through a sharp corner, are commonly used in these cases.

In this project, a two-dimensional minimum length rocket is designed to ensure shock-free expansion of the flow from sonic conditions to an exit mach number of 2.1. Note that two-dimensional design implies that the cross-sectional area is rectangular with sufficiently large aspect ratio. The governing equations for solving for axisymmetric flow involves the solution of ODEs at each unit process, and can be found in [1].

The sonic line is assumed to be vertical at the throat. The incoming sonic flow is suddenly expanded through an expansion fan, and the emanating characteristics are absorbed at the opposite wall to avoid reflections, which would result in further expansion or shocks depending on the contour.

The wall angle at the starting line is

$$\theta_w = \frac{\nu(M_e)}{2} \tag{26}$$

This result is obtained from the compatibility relations for the furthest characteristic line at the nozzle inlet and exit points, and making the exit flow angle zero.

The nozzle walls are approximated as straight lines between the characteristic curves, with slope equal to the average of the flow angles across the two characteristics.

2.2 Results

Figure 1 shows the computed characteristics for the shock-free minimum length nozzle. The expansion at the throat area has been discretized using 30 characteristics for this particular result. As can be seen in the figure, the characteristics originating at the lower end of the throat are absorbed at the upper wall of the nozzle.

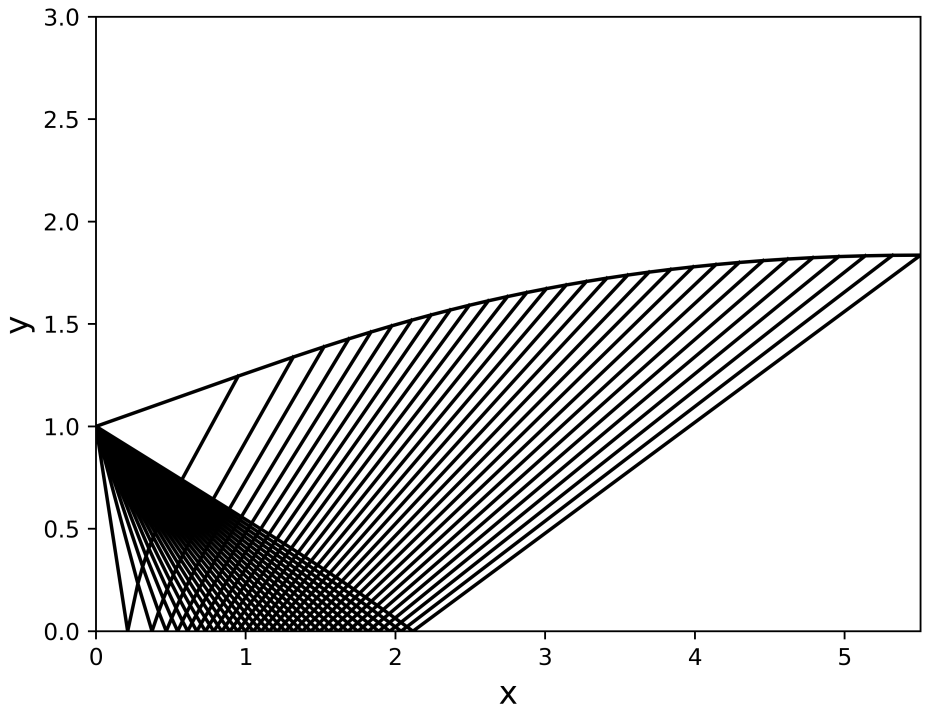


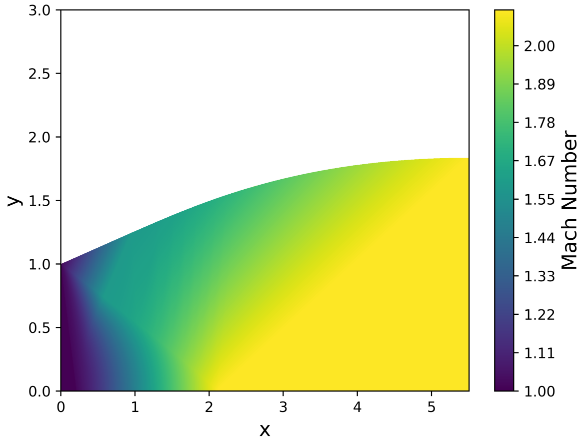
Figure 1: Minimum length nozzle design using 30 characteristics

The variation of flow properties inside the nozzle is shown in figure 2. It can be seen that the properties, especially the mach number, vary in both dimensions, unlike the quasi-one-dimensional approximation.

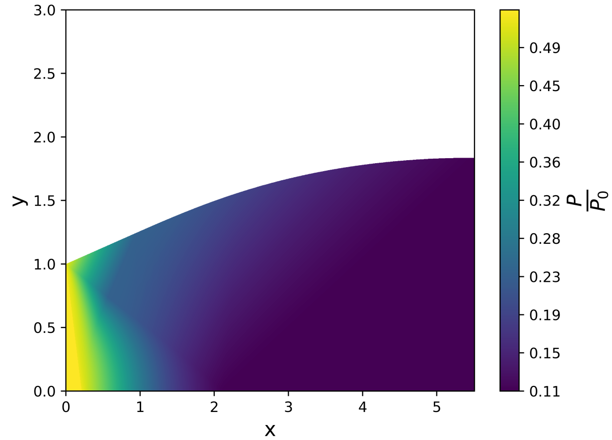
The length of the minimum length nozzle found is 5.50 m. A nozzle smaller than this would result in shocks. The computed area ratio at the nozzle exit A_e/A^* is 1.834, which is within 0.04% of the value obtained from quasi-1D-flow theory. This difference further goes down to 0.02% when the number of characteristics is increased from 30 to 100.

2.3 Conclusions

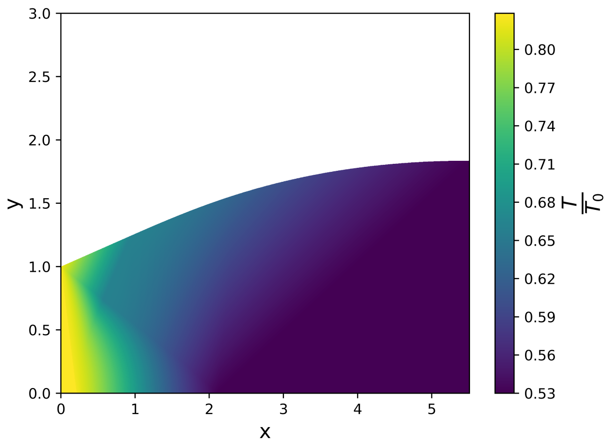
This project applied the method of characteristics to design a 2D supersonic minimum-length nozzle. By controlling the cancellation of the characteristic lines of flow expansion, we achieved the desired supersonic exit conditions with a compact nozzle profile. The results showed good agreement with quasi-1D flow theory, confirming the accuracy of the design approach. The resulting nozzle is efficient and well-suited for applications requiring minimal weight and length,



(a) Mach Number



(b) Pressure Ratio



(c) Temperature Ratio

Figure 2: Contour plots of flow properties inside the nozzle

such as rocket propulsion. This approach can also be extended to design axisymmetric or 3D flow conditions, broadening its applicability in various engineering fields.

References

[1] Anderson JD. Modern Compressible Flow with Historical Perspective. Fourth edition. McGraw-Hill Education; 2020.