Mathematical Derivation of Bayes Filter

$p(x_t \mid z_{1:t}, u_{1:t})$

$$= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})}$$

$$= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})$$

We now exploit the assumption that our state is complete
(Markov Assumption)

$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

Therefore,

$p(x_t \mid z_{1:t}, u_{1:t})$

$$= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t})$$

and hence

$$\boxed{bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)}$$

implemented in line 4 of Bayes Filter Algorithm

$$\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$$

Using total probability theorem

$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx$$

Once again, we exploit that our state is complete (Markov Assumption)

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Here we retain u_t, since it does not predate the state x_{t-1}.

Finally, we make an independence assumption

$$p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) = p(x_{t-1} \mid z_{1:t-1}, u_{t-1})$$
$$= bel(x_{t-1})$$

$$\therefore \overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx$$

~~implimented in line 3 of~~

~~Bayes Filter Algorithm~~

To summarize,

the Bayes filter Algorithm calculates the posterior over the state x_t conditioned on the measurement and control data up to time t.

The derivation assumes that the world is Markov, that is, the state is complete.

Mathematical Derivation of the KF

Part I: PREDICTION

$$\overline{bel}(x_t) = \int \underbrace{p(x_t \mid x_{t-1}, u_t)}_{\sim N(x_t;\, A_t x_{t-1} + B_t u_t;\, R_t)} \; \underbrace{bel(x_{t-1})}_{\sim N(x_{t-1};\, \mu_{t-1}, \Sigma_{t-1})} \, dx_{t-1}$$

The "prior" belief $bel(x_{t-1})$ is represented by the mean μ_{t-1} and the covariance Σ_{t-1}

The state transition probability was given as a normal distribution over x_t and with mean $A_t x_{t-1} + B_t u_t$ and covariance R_t

Writing in Gaussian form:

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\} \exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

in short we have

$$\overline{bel}(x_t) = \eta \int \exp\{-L_t\}\, dx_{t-1}$$

- quadratic in x_{t-1} and quadratic in x_t

where

$$\underset{\sim}{L_t} = \frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)$$

$$+ \frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})$$

To solve this integral in closed form, we will decompose L_t into two functions,

$$L_t(x_{t-1}, x_t) \text{ and } L_t(x_t):$$

$$L_t = L_t(x_{t-1}, x_t) + \underbrace{L_t(x_t)}$$

Independent of x_{t-1}

Therefore,

$$\overline{bel}(x_t) = \eta \exp\{-L_t(x_t)\} \int \exp\{-L_t(x_{t-1}, x_t)\}\, dx_{t-1}$$

we will decompose such that the value of $\int \exp\{L_t(x_{t-1}, x_t)\} dx_{t-1}$ does not depend on x_t

Are not same

$$\Rightarrow \overline{bel}(x_t) = \eta \exp\{-L_t(x_t)\}$$

Let us now perform this decomposition.

We are seeking a function $L_t(x_{t-1}, x_t)$ quadratic in x_{t-1}.

To determine the coefficients of this quadratic, we calculate the first two derivatives of L_t:

$$\frac{\partial L_t}{\partial x_{t-1}} = -A_t^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t) + \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})$$

$$\frac{\partial^2 L_t}{\partial x_{t-1}} = A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1} \triangleq \Psi_t^{-1}$$

Ψ_t defines the curvature of $L_t(x_{t-1}, x_t)$

Setting the first derivative to 0 gives us the mean:

$$A_t^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t) = \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})$$

Solving for x_{t-1}

$\Leftrightarrow A_t^T R_t^{-1}(x_t - B_t u_t) - A_t^T R_t^{-1} A_t x_{t-1} = \Sigma_{t-1}^{-1} x_{t-1} - \Sigma_{t-1}^{-1} \mu_{t-1}$

$\Leftrightarrow A_t^T R_t^{-1} A_t x_{t-1} + \Sigma_{t-1}^{-1} x_{t-1} = A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}$

$\Leftrightarrow \underbrace{(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})}_{\Psi_t^{-1}} x_{t-1} = A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}$

$\Leftrightarrow \Psi_t^{-1} x_{t-1} = A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}$

$\therefore$ Mean $= \Psi_t^{-1}\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]$

Thus, we now have a quadratic function defined as follows:

$$\{L_t(x_{t-1}, x_t)\} = \frac{1}{2}\left(x_{t-1} - \Psi_t[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}]\right)^T \Psi^{-1} \left(x_{t-1} - \Psi_t[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}]\right)$$

- common quadratic form of the -ve exponent of normal distribution. In fact the function

$\det(2\pi\Psi)^{-1/2} \exp\{-L_t(x_{t-1}, x_t)\}$

is a valid PDF for the variable x_{t-1}. As the PDF is of the form

$$p(x) = \det(2\pi\Sigma)^{-1/2} \exp\{-\tfrac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}$$

As the PDF integrates to 1

it follows that

$$\int \exp\{-L_t(x_{t-1}, x_t)\}\, dx_{t-1} = \underbrace{\det(2\pi\Psi)^{1/2}}$$

The value of the integral is independent of x_t, this integral is constant.

$$\overline{bel}(x_t) = \eta \exp\{-L_t(x_t)\} \int \exp\{-L_t(x_{t-1}, x_t)\}\, dx_{t-1}$$

Therefore, η are not same

$$\boxed{\overline{bel}(x_t) = \eta \exp\{-L_t(x_t)\}}$$

$$L_t(x_t) = L_t - L_t(x_{t-1}, x_t)$$

$$= \frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)$$

$$+ \frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})$$

$$- \frac{1}{2}\left(x_{t-1} - \Psi_t\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]\right)^T \Psi^{-1}$$

$$\left(x_{t-1} - \Psi_t\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]\right)$$

Upon simplification, we observe that

$L_t(x_t)$ indeed does not depend on x_{t-1}

All the terms containing x_{t-1} cancel out

Therefore, quadratic in x_t

$$L_t(x_t) = +\frac{1}{2}(x_t - B_t u_t)^T R_t^{-1}(x_t - B_t u_t)$$

$$+ \frac{1}{2}\mu_{t-1}^T \Sigma_{t-1}^T \mu_{t-1}$$

$$- \frac{1}{2}\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]^T \left(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1}\right)^{-1}$$

$$\left[A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1}\mu_{t-1}\right]$$

The observation that $L_t(x_t)$ is quadratic in x_t means that $\overline{bel}(x_t)$ is indeed normal (gaussian) distributed.

The mean and covariance of this distribution are given by the minimum and curvature of $L_t(x_t)$ with respect to x_t :

$$\frac{\partial L_t(x_t)}{\partial x_t} = R_t^{-1}(x_t - B_t u_t) - R_t^{-1} A_t (A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} [A_t^T R_t^{-1}(x_t - B_t u_t) + \Sigma_{t-1}^{-1} \mu_{t-1}]$$

$$= [R_t^{-1} - R_t^{-1} A_t (A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} A_t^T R_t^{-1}](x_t - B_t u_t)$$

$$- R_t^{-1} A_t (A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1} \mu_{t-1}$$

Page No.:
Date:

Using the inversion Lemma for the first factor

for any invertible quadratic matrices R and Q and any matrix P with appropriate dimension, the following holds true

$$(R + PQP^T)^{-1} = R^{-1} - R^{-1}P(Q^{-1} + P^TR^{-1}P)^{-1}P^TR^{-1}$$

we get,

$$\frac{\partial L_t(x_t)}{\partial x_t} = (R_t + A_t\Sigma_{t-1}A_t^T)^{-1}(x_t - B_t u_t)$$
$$- R_t^{-1}A_t(A_t^TR_t^{-1}A_t + \Sigma_{t-1}^{-1})^{-1}\Sigma_{t-1}^{-1}\mu_{t-1}$$

The minimum of $L_t(x_t)$ is attained when the first derivative is zero

$$(R_t + A_t\Sigma_{t-1}A_t^T)^{-1}(x_t - B_t u_t) = R_t^{-1}A_t(A_t^TR_t^{-1}A_t + \Sigma_{t-1}^{-1})^{-1}\Sigma_{t-1}^{-1}\mu_{t-1}$$

Page No.:
Date:

Solving for x_t

$$x_t = B_t u_t + \overbrace{(R_t + A_t \Sigma_{t-1} A_t^T) R_t^{-1} A_t}^{A_t + A_t \Sigma_{t-1} A_t^T R_t^{-1} A_t} \underbrace{(A_t^T R_t^{-1} A_t + \Sigma_{t-1}^{-1})^{-1} \Sigma_{t-1}^{-1}}_{(\Sigma_{t-1} A_t^T R_t^{-1} A_t + I)^{-1}} \mu_{t-1}$$

$$= B_t u_t + A_t \underbrace{(I + \Sigma_{t-1} A_t^T R_t^{-1} A_t)(\Sigma_{t-1} A_t^T R_t^{-1} A_t + I)^{-1}}_{I} \mu_{t-1}$$

Therefore

Mean $= B_t u_t + A_t \mu_{t-1}$

$\therefore \bar{\mu}_t =$ Mean of $\overline{bel}(x_t) = B_t u_t + A_t \mu_{t-1}$

After incoporating control command u_t

This proves the correctness of Line 2 of the Kalman filter Algorithm.

$$\frac{\partial^2 L_t(x_t)}{\partial x_t^2} = \left(R_t + A_t \Sigma_{t-1} A_t^T\right)^{-1}$$

$\downarrow$

Inverse of this is the covariance of $\overline{bel}(x_t)$

$$\therefore \boxed{\bar{\Sigma}_t = R_t + A_t \Sigma_{t-1} A_t^T}$$

This proves the correctness of line 3

Summary of Prediction Step

We showed that lines 2 and 3 of the Kalman filter Algorithm indeed implement the Bayes filter prediction step.

To do so,

Step 1 : $L_t \xrightarrow{\text{Decompose}} L_t(x_{t-1}, x_t) + L_t(x_t)$

Step 2 : $L_t(x_{t-1}, x_t)$ changes the predicted belief $\overline{bel}(x_t)$ only by a constant factor.

Step 3 : we finally determined $L_t(x_t)$ and showed that it results in mean $\bar{\mu}_t$ and covariance $\bar{\Sigma}_t$ of the Kalman Filter prediction $\overline{bel}(x_t)$

Part - II : MEASUREMENT UPDATE

$$bel(x_t) = \eta \underbrace{p(z_t|x_t)}_{\sim N(z_t;\, C_t x_t,\, Q_t)} \underbrace{\overline{bel}(x_t)}_{\sim N(x_t;\, \bar{\mu}_t,\, \bar{\Sigma}_t)}$$

Thus, the product is given by an exponential

$$bel(x_t) = \eta \exp\{-J_t\}$$

with

$$J_t = \frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)$$

$$+ \frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)$$

This function is quadratic in x_t,

hence $bel(x_t)$ is a gaussian.

To calculate parameters we again calculate the first and second derivative of J_t w.r.t. x_t:

Date:

$$\frac{\partial J}{\partial x_t} = -C_t^T Q_t^{-1}(z_t - C_t x_t) + \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)$$

$$\frac{\partial^2 J}{\partial x_t^2} = C_t^T Q_t^{-1} C_t + \bar{\Sigma}_t^{-1}$$

Inverse of this is the covariance of bel (x_t)

$$\therefore \boxed{\Sigma_t = \left(C_t^T Q_t^{-1} C_t + \bar{\Sigma}_t^{-1}\right)^{-1}}$$

The mean of bel (x_t) is the minimum of this quadratic function, which can be calculated by setting the derivative to zero

$$C_t^T Q_t^{-1}(z_t - C_t x_t) = \bar{\Sigma}_t^{-1}(\mu_t - \bar{\mu}_t)$$

$$C_t^T Q_t^{-1}(z_t - C_t \bar{\mu}_t) - C_t^T Q_t^{-1} C_t(\mu_t - \bar{\mu}_t)$$

$$\Rightarrow C_t^T Q_t^{-1}(z_t - C_t \bar{\mu}_t) = (\mu_t - \bar{\mu}_t)\underbrace{\left(C_t^T Q_t^{-1} C_t + \bar{\Sigma}_t^{-1}\right)}_{\Sigma_t^{-1}}$$

and hence we have

$$\Sigma_t C_t^T Q_t^{-1} (z_t - C_t \bar{\mu}_t) = \mu_t - \bar{\mu}_t$$

We define {Kalman gain} as

$$K = \Sigma_t C_t^T Q_t^{-1}$$

and obtain

$$\boxed{\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)}$$

This proves the correctness of line 5 in the Kalman filter algorithm

K_t is a function of Σ_t

and this is at odds with the fact that we use K_t to calculate Σ_t in line 6 of the Kalman filter Algorithm

So we need to express K_t in terms of covariances other than Σ_t.

$$K_t = \Sigma_t C_t^T Q_t^{-1}$$

$$= \Sigma_t C_t^T Q_t^{-1} \underbrace{(C_t \bar{\Sigma}_t C_t^T + Q_t)\overbrace{(C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}}^{N \text{ (for convenience)}}}_{=I}$$

$$= \Sigma_t \left(C_t^T Q_t^{-1} C_t \bar{\Sigma}_t C_t^T + C_t^T \underbrace{Q_t^{-1} Q_t}_{=I} \right) (N)$$

$$= \Sigma_t \left(C_t^T Q_t^{-1} C_t \bar{\Sigma}_t C_t^T + \underbrace{\bar{\Sigma}_t^{-1} \bar{\Sigma}_t}_{=I} C_t^T \right) (N)$$

$$= \Sigma_t \underbrace{\left(C_t^T Q_t^{-1} C_t + \bar{\Sigma}_t^{-1} \right)}_{=\Sigma_t^{-1}} \bar{\Sigma}_t C_t^T (N)$$

$$= \underbrace{\Sigma_t \Sigma_t^{-1}}_{=I} \bar{\Sigma}_t \bar{C}_t (N)$$

Therefore,

$$\boxed{K = \bar{\Sigma}_t \bar{C}_t \left(C_t \bar{\Sigma}_t \bar{C}_t + Q_t \right)^{-1}}$$

This expression proves the correctness of line 4 of the Kalman Filter Algorithm

Line 6 is obtained by expressing the covariance (Σ_t) using Kalman Gain K_t

The advantage of using the equation mentioned in line 6 over using the equation derived earlier for covariance is that we can avoid inverting the state covariance matrix.

$$\Sigma_t = (\bar{\Sigma}_t^{-1} + C_t^T Q_t^{-1} C_t)^{-1}$$

Using Inversion Lemma

$$= \bar{\Sigma}_t - \bar{\Sigma}_t C_t^T (Q_t + C_t \bar{\Sigma}_t C_t^T)^{-1} C_t \bar{\Sigma}_t$$

$$= [I - \underbrace{\bar{\Sigma}_t C_t^T (Q_t + C_t \bar{\Sigma}_t C_t^T)^{-1}}_{K_t \text{ (Kalman Gain}} C_t] \bar{\Sigma}_t$$

$$\therefore \boxed{\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t}$$

This proves the correctness of line 6 of the Kalman Filter Algorithm

Page No.:
Date:

Mathematical Derivation of EKF

The Mathematical derivation of EKF parallels that of Kalman filter