Optimal Solution for a sub problem Given $\{(a_1,b_1),(a_2,b_2)\}$

Case (iv) : $a_1 \neq a_2$ and $b_1 \neq b_2$

$$k = \max\{0, a_1, a_2\} + \min\left\{ \frac{T-b_1-a_1+1}{b_1} , \frac{T-b_1-a_2+1}{b_2} \right\}$$

W.L.O.G. assume $b_1 > b_2$

Case (1) : $a_2 > a_1$ $\quad k = a_2 + \min\left\{ \frac{T-b_1-a_2+1}{b_1} , \frac{T-b_2-a_2+1}{b_2} \right\}$

Case (iv) : $k = \frac{a_2}{b_1}\left(T-b_1-a_2+1\right)$

Dispersion Upon Constraint :

Step ①

$$T-b_2-(b_1+1)-(b_2+1)\left[\frac{b_1-1}{b_2}\right] \geq 0 \quad \Big| \quad m_1 = k+a_1$$

$$L.H.S = T-b_2-k-a_1+1-(b_1+1)\left[\frac{b_1-1}{b_2}\right] = T-b_2-b_1-k+1-(b_1+1)\left[\frac{b_1-2}{b_2}\right]$$

Same Argument that was made in case(i) $\left[\frac{b_1-2}{b_2}\right] > \left(\frac{b_1}{b_2}-1\right)$

$$\Rightarrow L.H.S \leq T-b_2-b_1-k+1-b_1\left(\frac{b_1}{b_2}-1\right) = T-b_2-b_1-k+1-\frac{b_1^2}{b_2}+b_1$$

$$= T-b_2-b_1+1-\frac{b_1}{b_2}(T-b_1-a_2+1)$$

$$= T-b_2-b_1+1-\underbrace{\frac{b_1}{b_2}}_{\geq 1}\underbrace{(T-b_1-a_2+1)}_{\geq 1} < 0$$

$\Rightarrow$ L.H.S. < 0

Case (v) : $k = \frac{a_2}{b_2}\left(T-b_1-a_2+1\right)$

Dispersion Upon Constraint :

Step ①

$$T-b_2-(b_1+1)-(b_2+1)\left[\frac{b_1-1}{b_2}\right] \geq 0 \quad \Big| \quad m_1 = k+a_1$$

$$L.H.S = T-b_2-k-a_1+1-(b_1+1)\left[\frac{b_1-1}{b_2}\right] = T-b_2-b_1-k+1-(b_1+1)\left[\frac{b_1-2}{b_2}\right]$$

Case (2) $a_2 > a_1$ $\left[\frac{b_1-1}{b_2}\right] = \frac{b_1}{b_2}-1$

Case (3) $a_2 = a_1$ $\left[\frac{b_1-1}{b_2}\right] = \frac{b_1}{b_2}$

Case (4) $a_2 < a_1$ $\left[\frac{b_1-1}{b_2}\right] > \left(\frac{b_1}{b_2}-1\right)$

Case (2) : $k = \frac{p}{q}a_1 \quad \frac{p}{q} \notin \mathbb{Z}_+$

$$L.H.S = T-b_2-b_1-b_2(1+\frac{p}{q})k \quad \Big| \quad k = \frac{a_2}{b_2}\left(T-b_1-a_2+1\right) \Rightarrow T = \frac{b_2 k}{a_2}+b_1+a_2-1$$

$$= T-b_2-a_1+1-\frac{b_2}{a_2}k-b_1(1+\frac{p}{q})k \quad \Big| \quad k = \frac{p}{q}a_1$$

$$= \frac{p\left(b_2 k - \frac{q b_2}{a_2}\right)}{q} + b_1 a_2 - b_2 - a_1 \quad \Big| \quad k = \frac{p}{q}a_1$$

$$= \frac{p\left(a_2 b_1 - a_1 b_2\right)}{a_2} + b_1 a_2 - b_2 - a_1$$

$$= \frac{b_1 a_2(1+p) + b_2(-a_1 p) - b_2 a_2 - a_1 a_2}{a_2}$$

$$= \frac{b_1 a_2(1+p) + b_2(-a_1 p) - b_2 a_2(1+1) - a_1 a_2}{a_2}$$

$$= \frac{\overbrace{(p+1)}^{\text{Non-zero}}\overbrace{(b_1 a_2 - b_2 a_1)}^{\text{non-zero}} + \overbrace{(a_1 - a_2)}^{\text{non-zero}}\overbrace{(b_2 - b_2)}^{\text{non-zero}}}{a_2} < 0$$

$\Rightarrow$ L.H.S. < 0

Case (3) : $k = \frac{p}{q}a_1 \quad \frac{p}{q} \in \mathbb{Z}_+$

$$L.H.S = T-b_2-b_1-b_2\left[\frac{b_1}{b_2}\right] \Rightarrow \left[\frac{b_1-1}{b_2}\right] = \left[\frac{b_1}{b_2}\right] \quad \Big| \quad k = \frac{a_2}{b_2}\left(T-b_1-a_2+1\right) \Rightarrow T = \frac{b_2 k}{a_2}+b_1+a_2-1$$

$$\Rightarrow T-b_2-b_1-b_2\left[\frac{b_1}{b_2}\right] \quad \Big| \quad k = \frac{p}{q}a_1$$

$$= \frac{p}{q}\left(b_2 a_1\right) - \left(b_1 a_2 - a_2\right) - b_1 \quad \Big| \quad k = \frac{p}{q}a_1$$

$$= \frac{p}{q}\left(b_2 a_1\right) - \left(b_1 a_2 - a_2\right) - b_1 < 0$$

$\Rightarrow$ L.H.S. < 0

$b_1 < b_2 \Rightarrow \frac{b_1}{a_2} < \frac{b_2}{a_1}$

$\Rightarrow b_1 a_1 < b_2 a_2$

$\Rightarrow \frac{b_1}{a_2} < \frac{b_2}{a_1} \Rightarrow \left[\frac{b_1}{a_2}\right] < \left[\frac{b_2}{a_1}\right]$

$\Rightarrow \frac{p}{q}\left(b_1 a_2\right) < \frac{p}{q}\left(b_2 a_1\right)$

Therefore, dispersion-upon constraint does not hold when $a_1 < a_2$

Case (1) : $a_1 > a_2$ $\quad k = a_1 + \min\left\{ \frac{T-b_1-a_1+1}{b_1} , \frac{T-b_1-a_2+1}{b_2} \right\}$

Case (1) : $k = \frac{a_1}{b_1}\left(T-b_1-a_1+1\right)$

Dispersion-Upon Constraint:

Step ①

$$T-b_1-(b_1+1)-(b_2+1)\left[\frac{b_1-1}{a_2}\right] \geq 0 \quad \Big| \quad m_2 = k+a_2$$

$$L.H.S = T-b_1-k-a_1+1-(b_2+1)\left[\frac{b_1-1}{a_2}\right] = T-b_1-b_2-k+1-(b_2+1)\left[\frac{b_1-2}{a_2}\right]$$

Same Argument that was made in case(i) $\left[\frac{b_1-2}{a_2}\right] > \left(\frac{b_1}{a_2}-1\right)$

$$\leq T-b_1-b_2-k+1-(b_2+1)\left(\frac{b_1}{a_2}-1\right)$$

$$= T-b_1-b_2-k+1-\frac{b_1 b_2}{a_2}+b_2$$

$$= T-b_1-a_1+1-\frac{b_1 b_2}{a_2}$$

$$= \left(T-b_1-a_1+1\right)-\frac{b_1}{a_2}\left(T-b_1-a_1+1\right) < 0 \quad \Big| \quad a_1 > a_2$$

$\Rightarrow$ L.H.S. < 0

Case (2) : $k = \frac{a_1}{b_1}\left(T-b_1-a_1+1\right)$

Dispersion-Upon Constraint:

Step ①

$$T-b_1-(b_1+1)-(b_2+1)\left[\frac{b_1-1}{a_2}\right] \geq 0 \quad \Big| \quad m_1 = k+a_1$$

$$L.H.S = T-b_1-k-a_1+1-(b_2+1)\left[\frac{b_1-1}{a_2}\right] = T-b_1-b_2-k+1-(b_2+1)\left[\frac{b_1-2}{a_2}\right]$$

Same Argument that was made in case(i) $\left[\frac{b_1-2}{a_2}\right] > \left(\frac{b_1}{a_2}-1\right)$

$$\leq T-b_1-b_2-k+1-(b_2+1)\left(\frac{b_1}{a_2}-1\right)$$

$$= T-b_1-b_2-k+1-\frac{b_1 b_2}{a_2}+b_2$$

$$= T-b_1-a_1+1-\frac{b_1}{a_2}\left(T-b_1-a_1+1\right)$$

$$= 0$$

Equality holds when $a_1 = a_2$

$\Rightarrow$ L.H.S. ≤ 0

Therefore, Step ① holds when $a_1 > a_2$, $a_1 = a_2$ and $k = \frac{a_1}{b_1}\left(T-b_1-a_1+1\right)$

Step ②:

$$T-b_1-(b_1+1)-(b_2+1)\left[\frac{b_1-1}{a_2}\right] \geq 0 \quad \Big| \quad m_2 = k+a_2$$

$$\Rightarrow T-b_1-k-a_1+1-(b_2+1)\left[\frac{b_1-1}{a_2}\right] \geq 0$$

$$\Rightarrow T-b_1-b_2-k+1-(b_2+1)\left[\frac{b_1-1}{a_2}\right] \geq 0$$

$$\Rightarrow \boxed{T \geq b_1+b_2+k+(b_2+1)\left[\frac{b_1-1}{a_2}\right]+(b_2+1)} \quad \text{— (i)}$$

$\left[\frac{a}{b}\right] + \frac{a}{b}$

Delay Constraint

$$T \leq b_1+b_2+k+(b_2+1)\left[\frac{b_1-1}{a_2}\right]+b_2+1$$

put $k = a_1$

$$T \leq b_1+b_2+a_1+(b_2+1)\left[\frac{b_1-1}{a_2}\right]+b_2+1$$

$\Rightarrow$ $\boxed{T \leq b_1+b_2+a_1+(b_2+1)\left[\frac{b_1-1}{a_2}\right]+(b_2+1)}$ — (ii)

$\neq \in \{a, b\}$

$\Rightarrow \left[\frac{a}{b}\right] = \frac{a}{b}$

$a_1 \in \{a, b\}$ $\quad a_2 \in \{a, b\}$

(i) & (ii) can't hold simultaneously since $b_2 > a_2$

Therefore, DPLDS constraints do not hold simultaneously when $a_1 > a_2$

Whenever $a_1 \neq a_2$ and $b_1 \neq b_2$ no valid SDE based solution is Rate-optimal