Streaming Codes Summer 2021

Optimal-Rate Streaming Erasure Codes

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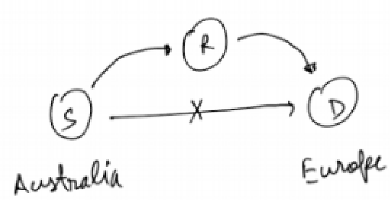
Abstract

We investigate low-latency streaming codes for a three-node relay network. The source transmits a sequence of messages (streaming messages) to the destination through the relay between them, where the first-hop channel from the source to the relay and the second-hop channel from the relay to the destination are subject to random and burst packet erasures. Every source message generated at a time slot must be recovered perfectly at the destination within the subsequent T time slots. In any sliding window of T + 1 time slots, we assume no more than a_1 random or b_1 burst and a_2 random or b_2 burst erasures are introduced by the first-hop channel and second hop channel respectively. Maximum achievable rate is fully characterised in terms of T, a_1, a_2, b_1, b_2. The achievability is proved for the two cases (i)$a_1 = a_2(= a), b_1 = b_2(= b), b|T - a + 1$ and (ii)$a_1 = a_2, b_1 + b_2 = T - a + 1$ by using a SDE-based construction and symbol-wise decode-forward strategy where the source symbols within the same message are decoded by the relay with different delays.

1 Introduction

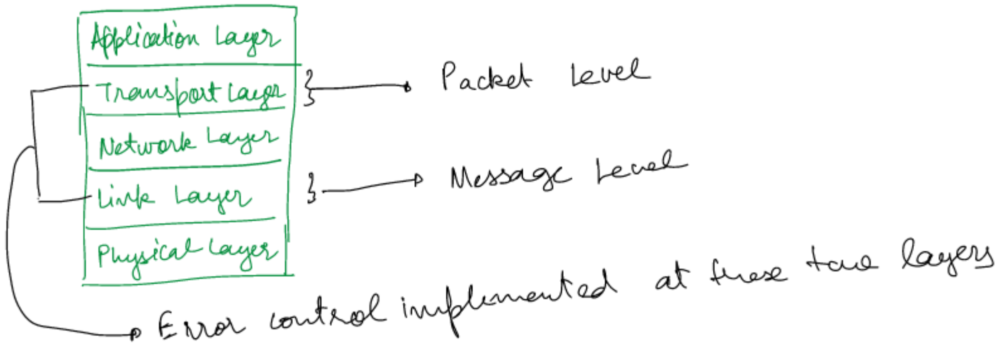
Motivation



Data center network: Direct Link between two nodes might not be available

Consider a simple relaying scenario within a cloud where a data center transmits streaming messages to another data center through an intermediate data center or other network node. The simple relaying scenario can be modeled as a source transmitting streaming messages to a destination over a three-node relay network with no direct link between the source node and the destination node, as illustrated above. In this paper, we focus on the three-node relay network model and investigate the performance of streaming codes over the three-node network subject to random and burst packet erasures.

Internet Stack Layers

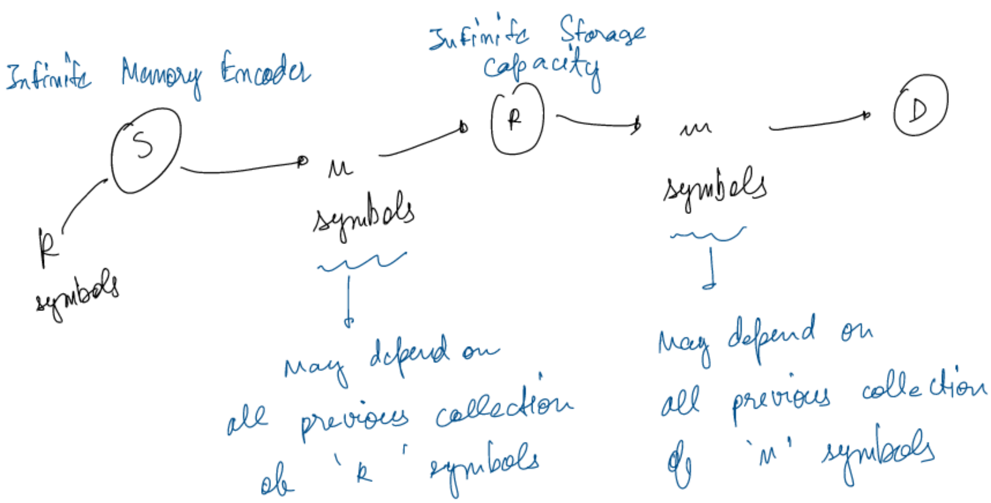


• ARQ cant support the low-delay requirement.
• FEC removes the need of re-transmission in case of packet loss by adding redundancy.

Packet-Extension
Streaming codes represent a packet-level FEC scheme for ensuring reliable, low latency communication. We consider Packet-Extension encoding framework where the parity symbols are appended to the message packets rather than being transmitted as separate packets to prevent network congestion.

• LDPC and fountain codes have large block lengths thus are not suitable for interactive streaming applications.

2 Network Model



Achievable scheme requires only finite memory and storage

$$\text{Overall Coding Rate} = \frac{k}{max\{n_1, n_2\}}$$

Overall coding rate can be interpreted as the reciprocal of the amount of time needed to simultaneously transfer one unit of information over (s, r) and (r, d).
Since every low-latency application is subject to a tight delay constraint, we assume that every message generated in a time slot must be decoded by node d with delay T , i.e., within the future T time slots.

2.1 Sliding Window Model

Every message must be perfectly recovered with delay T as long as the numbers of erasures introduced by (s, r) and (r, d) in every sliding window of size w=T+1 do not exceed b_1 burst or a_1 random and b_2 burst or a_2 random erasures respectively.

3 Optimal-Rate Theorem

The maximum achievable rate of a point-to-point channel subject to random and burst erasures denoted by $C_{T,a,b}$ equals

$$C_{T,a,b} = \frac{T - a + 1}{T - a + 1 + b}$$

The maximum achievable rate of the three node relay network subject to random and burst erasures denoted by $C_{T,a,b}$ equals

$$C_{T,a1,b1,a2,b2} = min\{C_{T-b_2,a_1,b_1}, C_{T-b_1,a_2,b_2}\}$$

where $C_{T,a,b}$ is the point-to-point channel capacity.

4 Converse proof of Optimal-Rate Theorem

5 Symbol-wise Time-division Decode Forward Strategy

A traditional relaying scheme for the three-node relay network is time-division decode-forward where the relay decodes every message with delay T1 before forwarding it to the destination with an additional delay T2.
In symbol-wise DF the relay decodes the symbols in the same source message subject to possibly different delay constraints, and similarly the destination decodes the symbols re-encoded by the relay subject to possibly different delay constraints.

5.1 Delay Profile

A SYMBOL-WISE DF STRATEGY WITH DELAY PROFILE ((2, 1), (1, 2)) WHICH CAN CORRECT ONE ERASURE FOR EACH CHANNEL

Time i	0	1	2	3	4
$s_i[0]$	$\boxed{s_0[0]}$	$s_1[0]$	$s_2[0]$	$s_3[0]$	$s_4[0]$
$s_i[1]$	$s_0[1]$	$\boxed{s_1[1]}$	$s_2[1]$	$s_3[1]$	$s_4[1]$
$s_{i-2}[0] + s_{i-1}[1]$	0	$s_0[1]$	$\boxed{s_0[0] + s_1[1]}$	$s_1[0] + s_2[1]$	$s_2[0] + s_3[1]$

(a) Symbols transmitted by node s from time 0 to 4.

Time i	0	1	2	3	4	5
$\hat{s}^{(r)}_{i-1}[1]$	0	$\boxed{\hat{s}^{(r)}_0[1]}$	$\hat{s}^{(r)}_1[1]$	$\hat{s}^{(r)}_2[1]$	$\hat{s}^{(r)}_3[1]$	$\hat{s}^{(r)}_4[1]$
$\hat{s}^{(r)}_{i-2}[0]$	0	0	$\boxed{\hat{s}^{(r)}_0[0]}$	$\hat{s}^{(r)}_1[0]$	$\hat{s}^{(r)}_2[0]$	$\hat{s}^{(r)}_3[0]$
$\hat{s}^{(r)}_{i-3}[0] + \hat{s}^{(r)}_{i-3}[1]$	0	0	0	$\boxed{\hat{s}^{(r)}_0[0] + \hat{s}^{(r)}_0[1]}$	$\hat{s}^{(r)}_1[0] + \hat{s}^{(r)}_1[1]$	$\hat{s}^{(r)}_2[0] + \hat{s}^{(r)}_2[1]$

(b) Symbols transmitted by node r from time 0 to 5.

Time i	0	1	2	3	4	5
$\hat{s}^{(d)}_{i-3}[0]$	0	0	0	$\hat{s}^{(d)}_0[0]$	$\hat{s}^{(d)}_1[0]$	$\hat{s}^{(d)}_2[0]$
$\hat{s}^{(d)}_{i-3}[1]$	0	0	0	$\hat{s}^{(d)}_0[1]$	$\hat{s}^{(d)}_1[1]$	$\hat{s}^{(d)}_2[1]$

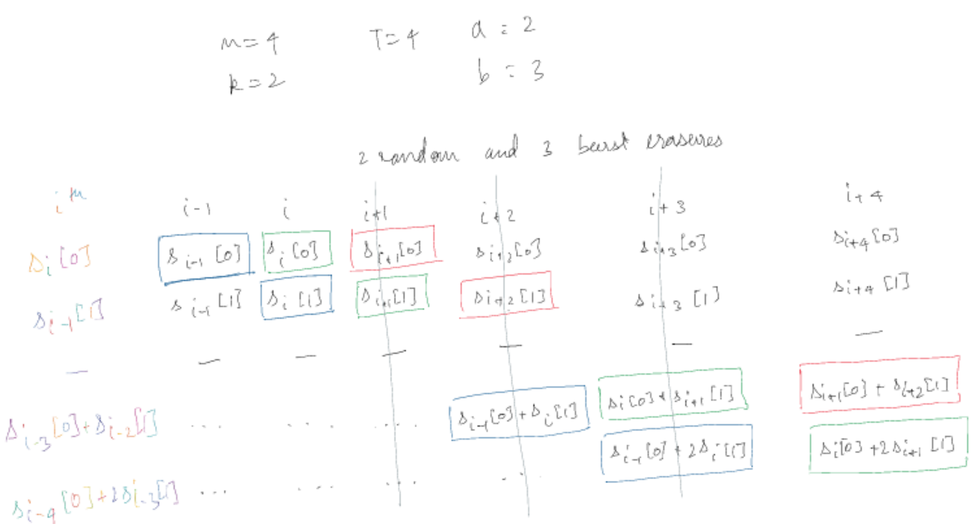
(c) Estimates constructed by node d from time 0 to 5.

Note: Above Example is for the case when there are only random erasures *(Ref: Three-node paper)*

6 Achievability proof of Optimal-Rate Theorem

6.1 SDE-based code constructions for point-to-point channel

Ref.: Simple Streaming Codes Paper



$(b = 3, a = 2)$-achievable $(n = 4, k = 2, ds = (4, 3))_{\mathbb{F}}$-streaming code with optimal rate

6.2 SDE-based code constructions for Three-node network

Given a_1, a_2, b_1, b_2, T three constraints on the values of k, n_1, n_2, N_1, N_2 will need to be satisfied for the SDE-based constructions to be rate-optimal.
Assumption: Base codes are MDS. Therefore $n_1 = k + a_1$ and $n_2 = k + a_2$

Constraint 1:Optimal Rate Constraint

$$
\begin{aligned}
&\frac{k}{max\{n_2, n_1\}} = min\{C_{T-b_2,a_1,b_1}, C_{T-b_1,a_2,b_2}\} \\
\implies &\frac{k}{k + max\{a_1, a_2\}} = min\{\frac{T - b_2 - a_1 + 1}{T - b_2 - a_1 + 1 + b_1}, \frac{T - b_1 - a_2 + 1}{T - b_1 - a_2 + 1 + b_2}\} \\
\implies &\frac{k}{k + max\{a_1, a_2\}} = min\{\frac{1}{1 + \frac{b_1}{T - b_2 - a_1 + 1}}, \frac{1}{1 + \frac{b_2}{T - b_1 - a_2 + 1}}\} \\
\implies &\frac{k}{k + max\{a_1, a_2\}} = \frac{1}{1 + max\{\frac{b_1}{T - b_2 - a_1 + 1}, \frac{b_2}{T - b_1 - a_2 + 1}\}} \\
\implies &\frac{k + max\{a_1, a_2\}}{k} = 1 + max\{\frac{b_1}{T - b_2 - a_1 + 1}, \frac{b_2}{T - b_1 - a_2 + 1}\} \\
\implies &1 + \frac{max\{a_1, a_2\}}{k} = 1 + max\{\frac{b_1}{T - b_2 - a_1 + 1}, \frac{b_2}{T - b_1 - a_2 + 1}\} \\
\implies &k = \frac{max\{a_1, a_2\}}{max\{\frac{b_1}{T - b_2 - a_1 + 1}, \frac{b_2}{T - b_1 - a_2 + 1}\}} \\
\implies &k = max\{a_1, a_2\}.min\{\frac{T - b_2 - a_1 + 1}{b_1}, \frac{T - b_1 - a_2 + 1}{b_2}\}
\end{aligned}
$$

Constraint 2:Delay Profile Constraint

Test case where optimal rate constraint is satisfied but delay profile constraint is violated:

Example: (violation of Delay Constraint)

Code 1: $a_1 = 1$, $b_1 = 2$, $T_1 = 8$, $m = 5$, $R_1 = \frac{4}{5}$

Code 2: $a_2 = 2$, $b_2 = 5$, $T_2 = 11$, $m = 6$, $R_2 = \frac{4}{6}$

$T = 13$, $k = 4$

$R_{opt_1} = \frac{8}{8+2} = \frac{4}{5}$

$R_{opt_2} = \frac{10}{10+5} = \frac{2}{3}$

$$\boxed{R_{cons} = \frac{2}{3}}$$

$$R_{opt\ \text{3-Node}} = \min \left\{ \frac{T-b_2-a_1+1}{T-b_2-a_1+1+b_1}, \frac{T-b_1-a_2+1}{T-b_1-a_2+1+b_2} \right\}$$

optimal Rate

$$\Rightarrow \boxed{R_{opt\ \text{3-Node}} = \min \left\{ \frac{4}{5}, \frac{2}{3} \right\} = \frac{2}{3}}$$

(s,r) - channel

0 1 2 3 4 5 6 7 8

$u_0 - u_1 - u_2 - u_3 - u_4$

Delay spectrum = $(8, 6, 4, 2)$

(r,d) - channel $(11, 10, 6, 5)$

0 1 2 3 4 5 6 7 8 9 10 11

u_0 u_1 – – – u_2 u_3 – – – u_4 u_5

Not optimal violates delay constraint

Delay spectrum of SDE scheme for a point-to-point channel with parameters (T,a,b):

$$(T, T-1, , ..., T-a+1, T-b, T-b-1, ..., T-b-a+1, T-2b, T-2b-1, ..., T-2b-a+1, ...)$$

Therefore,

Decoding delay constraint for t^{th} message symbol:

$$ds_{p-p}^{t} = T - t - (b-a).\left\lfloor \frac{t}{a} \right\rfloor \text{, where } t \in [0, k-1]$$

While re-encoding the message symbols at the relay node the order of the message symbols within a packet is flipped.
Delay profile of SDE scheme for a three-node netwrok with parameters (T, a_1, a_2, b_1, b_2):

$$\left((ds_{s-r}^{0}, ds_{r-d}^{k-1}), (ds_{s-r}^{1}, ds_{r-d}^{k-2}), ..., (ds_{s-r}^{k-1}, ds_{r-d}^{0}) \right)$$

Delay Profile constraint: $\forall t \in [0, k-1]$

$$\begin{aligned} ds_{s-r}^{t} + ds_{r-d}^{k-1-t} &\leq T \\ \implies T - b_2 - t - (b_1 - a_1).\left\lfloor \frac{t}{a_1} \right\rfloor + T - b_1 - (k-t-1) - (b_2 - a_2).\left\lfloor \frac{k-t-1}{a_2} \right\rfloor &\leq T \\ \implies T \leq b_2 + b_1 + (b_1 - a_1).\left\lfloor \frac{t}{a_1} \right\rfloor + (b_2 - a_2).\left\lfloor \frac{k-t-1}{a_2} \right\rfloor &+ k - 1 \end{aligned}$$

Constraint 3:Dispersion Span Constraint

Test case where optimal rate constraint is satisfied but dispersion-span constraint is violated:

Example: (Violation of dispersion-span constraint)

$k = 3$

$T = 11$ $\quad$ $T_1 = T_2 = 7$

$a_1 = 2$ $\quad$ $a_2 = 1$

$b_1 = 4$ $\quad$ $b_2 = 4$

$n_1 = 5$ $\quad$ $n_2 = 4$

(5,4):

$u_1 \; u_2 \; - \; - \; u_3 \; u_4 \; - \; - \; u_5$

$0 \; 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7 \; 8$

$n_1 > T - b_2$ } violation of dispersion-span constraint

$$R_{opt_{3\text{-Node}}} = \min\left\{ \cancel{\frac{7-2+1}{7-2+1+4}}^{\,3/5}, \cancel{\frac{7-1+1}{7-1+1+4}}^{\,7/11} \right\} = \frac{3}{5}$$

optimal rate

$$\frac{k}{\max\{n_1, n_2\}} = \frac{3}{5}$$

$$\begin{aligned}\text{Dispersion span} &= \text{position of the nth symbol of the base code in the dispersed code}\\ &\quad - \text{position of the 1st symbol of the base code in the dispersed code} + 1\end{aligned}$$

Therefore,

$$N_1 = \left((n_1 - 1) + (b_1 - a_1).\left\lfloor \frac{n_1 - 1}{a_1} \right\rfloor\right) - 0 + 1$$
$$N_2 = \left((n_2 - 1) + (b_2 - a_2).\left\lfloor \frac{n_2 - 1}{a_2} \right\rfloor\right) - 0 + 1$$

Ineq-1:

$$N_1 \leq T - b_2 + 1$$
$$\implies T - b_2 - (n_1 - 1) - (b_1 - a_1).\left\lfloor \frac{n_1 - 1}{a_1} \right\rfloor \geq 0$$

Ineq-2:

$$N_2 \leq T - b_1 + 1$$
$$\implies T - b_1 - (n_2 - 1) - (b_2 - a_2).\left\lfloor \frac{n_2 - 1}{a_2} \right\rfloor \geq 0$$

Property 1:

Let $x \in \mathbb{Q}$ and $a \in \mathbb{Z}_+$

$$\left\lfloor x - \frac{1}{a} \right\rfloor = \begin{cases} x - 1 & x = \lfloor x \rfloor \\ \lfloor x \rfloor - 1 & \lfloor x \rfloor < x < \lfloor x \rfloor + \frac{1}{a} \\ \lfloor x \rfloor & \lfloor x \rfloor + \frac{1}{a} < x < \lfloor x \rfloor + 1 \end{cases}$$

For Case 1, $\left\lfloor x - \frac{1}{a} \right\rfloor = x - 1$

For Case 2, $\left\lfloor x - \frac{1}{a} \right\rfloor < x - 1$

For Case 3, $\left\lfloor x - \frac{1}{a} \right\rfloor > x - 1$

If $x = \frac{k}{a}$, where $k \in \mathbb{Z}_+$, then Case 2 never occurs. Since k is an integer it can't be strictly between any two consecutive integers.

Property 2:

Let $x, y \in \mathbb{Z}, y \neq 0$

$$\left\lfloor \frac{x}{y} \right\rfloor = \frac{x - (x\%y)}{y}$$
$$\left\lfloor \frac{(-1 - x)}{y} \right\rfloor = \frac{(-1 - x) - ((-1 - x)\%y)}{y}$$
$$\implies \left\lfloor \frac{x}{y} \right\rfloor + \left\lfloor \frac{(-1 - x)}{y} \right\rfloor = \frac{1 - ((-1)\%y)}{y} = \frac{-1 - y + 1}{y} = -1$$

Case (i) : $a_1 = a_2$ and $b_1 = b_2$

Let $a_1 = a_2 = a$ and $b_1 = b_2 = b$.

Optimal Rate Constraint:

$$k = a.\frac{T+1-b-a}{b}$$
$$n_1 = n_2 = k + a = a.\frac{T+1-a}{b}$$

Dispersion span Constraint:

$$T - b - (n-1) - (b-a)\left\lfloor \frac{n-1}{a} \right\rfloor \geq 0$$
$$\text{L.H.S.} = T - b - (a.\frac{T+1-a}{b} - 1) - (b-a)\left\lfloor \frac{n}{a} - \frac{1}{a} \right\rfloor$$

By Property 1,

$$\begin{aligned}
\text{L.H.S.} &\leq T - b - a.\frac{T+1-a}{b} + 1 - (b-a)\Big(\frac{n}{a} - 1\Big) \\
&= T - b - a.\frac{T+1-a}{b} + 1 - (b-a)\Big(\frac{T+1-a}{b} - 1\Big) \\
&= T - b - a.\frac{T+1-a}{b} + 1 - T + a - 1 + a.\frac{T+1-a}{b} + b - a \\
&= 0
\end{aligned}$$

Therefore,

$$\text{L.H.S.} \leq 0$$

Therefore for dispersion span constraint to hold, $a|n \implies a|k \implies b|(T-a+1)$

Delay Profile Constraint: $\forall t \in [0, k-1]$

$$2b + (b-a).\left(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{k-1-t}{a} \right\rfloor\right) + k - 1 \geq T$$
$$\text{L.H.S.} = 2b + (b-a).\left(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{k}{a} + \frac{-1-t}{a} \right\rfloor\right) + k - 1$$

Since $a|k$,

$$\text{L.H.S.} = 2b + (b-a).\left(\frac{k}{a} + \left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{-1-t}{a} \right\rfloor\right) + k - 1$$

By Property 2,

$$\begin{aligned}
\text{L.H.S.} &= 2b + (b-a).\left(\frac{k}{a} - 1\right) + k - 1 \\
&= 2b + (b-a).\frac{T-b-a+1}{b} - b + a + a.\frac{T-b-a+1}{b} - 1 \\
&= 2b + T - b - a + 1 - a.\frac{T-b-a+1}{b} - b + a + a.\frac{T-b-a+1}{b} - 1 \\
\text{L.H.S.} &= T
\end{aligned}$$

Therefore delay profile constraint holds.

Whenever the following constraints are satisfied by a_1, a_2, b_1, b_2, T :

$$\begin{aligned} a_1 &= a_2 (= a) \\ b_1 &= b_2 (= b) \\ b &| (T - a + 1) \end{aligned}$$

there exists an SDE-based optimal-rate construction with parameters :

$$\begin{aligned} k &= a.\frac{T - b - a + 1}{b} \\ n_1 = n_2 &= a.\frac{T - a + 1}{b} \\ N_1 = N_2 &= T - b + 1 \end{aligned}$$

Example:
$T = 7, k = 2, a_1 = a_2 = 2, b_1 = b_2 = 3, k = 2, n_1 = n_2 = 4, N_1 = N_2 = 5$

$$D_1 = (\overset{0^{th}}{4}, \overset{1^{st}}{3})$$

At Node s :

j^{th} time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
$s_j[0]$	$\boxed{s_{i-1}[0]}$	$\boxed{s_i[0]}$	$s_{i+1}[0]$	$s_{i+2}[0]$	$s_{i+3}[0]$	$s_{i+4}[0]$
$s_j[1]$	$s_{i-1}[1]$	$\boxed{s_i[1]}$	$\boxed{s_{i+1}[1]}$	$s_{i+2}[1]$	$s_{i+3}[1]$	$s_{i+4}[1]$
—	—	—	—	—	—	—
$s_{j-2}[0] + s_{j-2}[1]$	...	...	...	$\boxed{s_{i-1}[0] + s_i[1]}$	$\boxed{s_i[0] + s_{i+1}[1]}$	$s_{i+1}[0] + s_{i+2}[1]$
$s_{j-4}[0] + 2s_{j-3}[1]$	...	...	...	...	$\boxed{s_{i-1}[0] + 2s_i[1]}$	$\boxed{s_i[0] + 2s_{i+1}[1]}$

At Node r :

$$\begin{array}{cc} 0^{th} & 1^{st} \end{array}$$
$$s_2 = (3, 4)$$

j^{th} time	$i+2$	$i+3$	$i+4$	$i+5$	$i+6$	$i+7$
$s_{j-3}[1]$	$\boxed{s_{i-1}[1]}$	$\boxed{s_i[1]}$	$s_{i+1}[1]$	$s_{i+2}[1]$		
$s_{j-4}[0]$	$s_{i-1}[0]$	$\boxed{s_{i-1}[0]}$	$\boxed{s_i[0]}$	$s_{i+1}[0]$		
–	–	–	–	–	–	–
$s_{j-6}[1]+s_{j-6}[0]$	...	...	...	$\boxed{s_{i-1}[1]+s_{i-1}[0]}$	$\boxed{s_i[1]+s_i[0]}$	$s_{i+1}[1]+s_{i+1}[0]$
$s_{j-7}[1]+2s_{j-7}[0]$	...	...	...	...	$\boxed{s_{i-1}[1]+2s_{i-1}[0]}$	$\boxed{s_i[1]+2s_i[0]}$

At Node d :

j^{th} time	$i+5$	$i+6$	$i+7$	$i+8$	$i+9$	$i+10$
$s_{j-7}[0]$	$s_{i-2}[0]$	$s_{i-1}[0]$	$s_i[0]$	$s_{i+1}[0]$	$s_{i+2}[0]$	$s_{i+3}[0]$
$s_{j-7}[1]$	$s_{i-2}[1]$	$s_{i-1}[1]$	$s_i[1]$	$s_{i+1}[1]$	$s_{i+2}[1]$	$s_{i+3}[1]$

Case (ii) : $a_1 = a_2$ and $b_1 \neq b_2$

Let $a_1 = a_2 = a$

Case 1: $T - a + 1 > b_1 + b_2$

$$\text{If } b_2 > b_1,$$
$$\frac{T - b_2 - a + 1}{b_1} > \frac{T - b_1 - a + 1}{b_2}$$
$$\text{If } b_2 < b_1,$$
$$\frac{T - b_2 - a + 1}{b_1} < \frac{T - b_1 - a + 1}{b_2}$$

Therefore,

$$k = a.\frac{T - a + 1 - min\{b_1, b_2\}}{max\{b_1, b_2\}}$$

W.L.O.G. Assume $b_1 > b_2$. Therefore,

$$k = a.\frac{T - a + 1 - b_2}{b_1}$$
$$n_1 = n_2 = n = a.\frac{T - a + 1 - b_2 + b_1}{b_1}$$

<u>Dispersion span Constraint:</u>

Ineq-1:

$$T - b_2 - (n-1) - (b_1 - a)\left\lfloor \frac{n-1}{a} \right\rfloor \geq 0$$
$$\text{L.H.S.} = T - b_2 - (a.\frac{T + 1 - a - b_2 + b_1}{b_1} - 1) - (b_1 - a)\left\lfloor \frac{n}{a} - \frac{1}{a} \right\rfloor$$

By Property 1,

$$\begin{aligned}
\text{L.H.S.} &\leq T - b_2 - a.\frac{T + 1 - a - b_2 + b_1}{b_1} + 1 - (b_1 - a)\Big(\frac{n}{a} - 1\Big) \\
&= T - b_2 - a.\frac{T + 1 - a - b_2 + b_1}{b_1} + 1 - (b_1 - a)\Big(\frac{T + 1 - a - b_2 + b_1}{b_1} - 1\Big) \\
&= T - b_2 - a.\frac{T + 1 - a - b_2 + b_1}{b_1} + 1 - T + a - 1 - b_2 + b_1 + a.\frac{T + 1 - a - b_2 + b_1}{b_1} + b_1 - a \\
&= 0
\end{aligned}$$

Therefore,

$$\text{L.H.S.} \leq 0$$

Therefore for Ineq-1 to hold, $a|n \implies a|k \implies b_1|(T - a + 1 - b_2)$

Ineq-2:

$$T - b_1 - (n-1) - (b_2 - a)\left\lfloor \frac{n-1}{a} \right\rfloor \geq 0$$
$$\text{L.H.S.} = T - b_1 - (a.\frac{T + 1 - a - b_2 + b_1}{b_1} - 1) - (b_2 - a)\left\lfloor \frac{n}{a} - \frac{1}{a} \right\rfloor$$

Since $b_1|(T - a + 1 - b_2 + b_1)$,

$$\begin{aligned}
\text{L.H.S.} &= T - b_1 - a.\frac{T + 1 - a - b_2 + b_1}{b_1} + 1 - (b_2 - a)\Big(\frac{n}{a} - 1\Big) \\
&= T - b_1 - a.\frac{T + 1 - a - b_2 + b_1}{b_1} + 1 - (b_2 - a)\Big(\frac{T + 1 - a - b_2 + b_1}{b_1} - 1\Big) \\
&= T - b_1 + 1 - b_2.\frac{T + 1 - a - b_2 + b_1}{b_1} + b_2 - a \\
&= (T + 1 - a).\frac{b_1 - b_2}{b_1} + (b_2 - b_1).\frac{b_1 + b_2}{b_1} \\
&= \frac{b_1 - b_2}{b_1}\Big((T + 1 - a) - (b_1 + b_2)\Big) > 0
\end{aligned}$$

Therefore,

$$\text{L.H.S.} > 0$$

Therefore Ineq-2 holds.

Therefore Dispersion-Span Constraint holds when $a|k$.

Delay Profile Constraint: $\forall t \in [0, k-1]$

$$b_1 + b_2 + (b_1 - a).\left\lfloor \frac{t}{a} \right\rfloor + (b_2 - a)\left\lfloor \frac{k-1-t}{a} \right\rfloor + k - 1 \geq T$$

$$\text{L.H.S.} = b_1 + b_2 + b_1.\left\lfloor \frac{t}{a} \right\rfloor + b_2.\left\lfloor \frac{-1-t}{a} + \frac{k}{a} \right\rfloor - a.\left(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{-1-t}{a} + \frac{k}{a} \right\rfloor \right) + k - 1$$

Since $a|k$,

$$\text{L.H.S.} = b_1 + b_2 + b_2.\frac{k}{a} + b_1.\left\lfloor \frac{t}{a} \right\rfloor + b_2.\left\lfloor \frac{-1-t}{a} \right\rfloor - a.\left(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{-1-t}{a} \right\rfloor \right) - 1$$

By Property 2,

$$\text{L.H.S.} = b_1 + b_2 + b_2.\frac{k}{a} + b_1.\left\lfloor \frac{t}{a} \right\rfloor + b_2.\left\lfloor \frac{-1-t}{a} \right\rfloor + a - 1$$

$a|k, a \neq k \implies a < k$, Checking for $t = a - 1$

$$\text{L.H.S.} = b_1 + b_2 + b_2.\frac{k}{a} - b_2 + a - 1$$

$$= b_1 + b_2 + b_2.\frac{T - a + 1 - b_2}{b_1} - b_2 + a - 1$$

Let $T = b_1 + b_2 + a - 1 + \alpha, \alpha > 0$

$$\text{L.H.S.} = T - \alpha + \frac{b_2}{b_1}.(b_1 + \alpha) - b_2$$

$$= T - \alpha + \frac{b_2}{b_1}\alpha$$

Since $b_2 < b_1$,

$$\text{L.H.S.} < T$$

Therefore delay profile constraint does not hold.

Case 2: $T - a + 1 < b_1 + b_2$

If $b_2 > b_1$,

$$\frac{T - b_2 - a + 1}{b_1} < \frac{T - b_1 - a + 1}{b_2}$$

If $b_2 < b_1$,

$$\frac{T - b_2 - a + 1}{b_1} > \frac{T - b_1 - a + 1}{b_2}$$

Therefore,

$$k = a.\frac{T - a + 1 - max\{b_1, b_2\}}{min\{b_1, b_2\}}$$

W.L.O.G. Assume $b_2 > b_1$. Therefore,

$$k = a.\frac{T-a+1-b_2}{b_1}$$
$$n_1 = n_2 = n = a.\frac{T-a+1-b_2+b_1}{b_1}$$

Therefore, similar argument made in Case 1 follows.

Case 3: $T-a+1 = b_1+b_2$

$$\frac{T-b_2-a+1}{b_1} = \frac{T-b_1-a+1}{b_2}$$

Therefore,

$$k = a, n = 2a$$

<u>Dispersion span Constraint:</u>
Ineq-1:

$$\begin{aligned} T-b_2-(n-1)-(b_1-a)\left\lfloor\frac{n-1}{a}\right\rfloor &\geq 0 \\ \text{L.H.S.} &= T-b_2-(2a-1)-(b_1-a)\left\lfloor\frac{2a-1}{a}\right\rfloor \\ &= T-b_2-(2a-1)-(b_1-a).1 \\ &= T-a+1-(b_1+b_2) \\ &= 0 \\ \text{Therefore,} & \\ \text{L.H.S.} &= 0 \end{aligned}$$

Therefore Ineq-1 holds.

Ineq-2:

$$\begin{aligned} T-b_1-(n-1)-(b_2-a)\left\lfloor\frac{n-1}{a}\right\rfloor &\geq 0 \\ \text{L.H.S.} &= T-b_1-(2a-1)-(b_2-a)\left\lfloor\frac{2a-1}{a}\right\rfloor \\ &= T-b_1-(2a-1)-(b_2-a).1 \\ &= T-a+1-(b_1+b_2) \\ &= 0 \\ \text{Therefore,} & \\ \text{L.H.S.} &= 0 \end{aligned}$$

Therefore Ineq-2 holds.

Therefore Dispersion-Span Constraint holds.

Delay Profile Constraint: $\forall t \in [0, k-1]$

$$b_1 + b_2 + (b_1 - a).\left\lfloor \frac{t}{a} \right\rfloor + (b_2 - a)\left\lfloor \frac{k-1-t}{a} \right\rfloor + k - 1 \geq T$$

$$\text{L.H.S.} = b_1 + b_2 + b_1.\left\lfloor \frac{t}{a} \right\rfloor + b_2.\left\lfloor \frac{-1-t}{a} + \frac{k}{a} \right\rfloor - a.\left(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{-1-t}{a} + \frac{k}{a} \right\rfloor \right) + k - 1$$

Since $k = a$,

$$\text{L.H.S.} = b_1 + b_2 + b_2 + b_1.\left\lfloor \frac{t}{a} \right\rfloor + b_2.\left\lfloor \frac{-1-t}{a} \right\rfloor - a.\left(\left\lfloor \frac{t}{a} \right\rfloor + \left\lfloor \frac{-1-t}{a} \right\rfloor \right) - 1$$

By Property 2,

$$\text{L.H.S.} = b_1 + 2b_2 + b_1.\left\lfloor \frac{t}{a} \right\rfloor + b_2.\left\lfloor \frac{-1-t}{a} \right\rfloor + a - 1$$

$a = k \implies t < a$

$$\begin{aligned} \text{L.H.S.} &= b_1 + 2b_2 - b_2 + a - 1 \\ &= b_1 + b_2 + a - 1 = T \\ \text{L.H.S.} &= T \end{aligned}$$

Therefore, delay profile constraint holds.

Whenever the following constraints are satisfied by a_1, a_2, b_1, b_2, T :

$$\begin{aligned} a_1 &= a_2 (= a) \\ b_1 + b_2 &= (T - a + 1) \end{aligned}$$

there exists an SDE-based optimal-rate construction with parameters :

$$\begin{aligned} k &= a \\ n_1 &= n_2 = 2a \\ N_1 &= T - b_2 + 1 \\ N_2 &= T - b_1 + 1 \end{aligned}$$

Example:
$T = 8, k = 2, a_1 = a_2 = 2, b_1 = 3, b_2 = 4, k = 2, n_1 = n_2 = 4, N_1 = 5, N_2 = 6$

$$s_1 = (\overset{0^{th}}{4}, \overset{1^{st}}{3})$$

At Node s:

j^{th} time	$i-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
$s_j[0]$	$\boxed{s_{i-1}[0]}$	$\boxed{s_i[0]}$	$s_{i+1}[0]$	$s_{i+2}[0]$	$s_{i+3}[0]$	$s_{i+4}[0]$
$s_j[1]$	$s_{i-1}[1]$	$\boxed{s_i[1]}$	$\boxed{s_{i+1}[1]}$	$s_{i+2}[1]$	$s_{i+3}[1]$	$s_{i+4}[1]$
—	—	—	—	—	—	—
$s_{j-2}[0]+s_{j-2}[1]$	...	...	...	$\boxed{s_{i-1}[0]+s_i[1]}$	$\boxed{s_i[0]+s_{i+1}[1]}$	$s_{i+1}[0]+s_{i+2}[1]$
$s_{j-4}[0]+2s_{j-3}[1]$	...	...	...	...	$\boxed{s_{i-1}[0]+2s_i[1]}$	$\boxed{s_i[0]+2s_{i+1}[1]}$

$$s_2 = (\overset{0^{th}}{4}, \overset{1^{st}}{5})$$

At Node r:

j^{th} time	$i+2$	$i+3$	$i+4$	$i+5$	$i+6$	$i+7$	$i+8$
$s_{j-3}[1]$	$\boxed{s_{i-1}[1]}$	$\boxed{s_i[1]}$	$s_{i+1}[1]$	$s_{i+2}[1]$	$s_{i+3}[1]$	$s_{i+4}[1]$	$s_{i+5}[1]$
$s_{j-4}[0]$	$s_{i-1}[0]$	$\boxed{s_{i-1}[0]}$	$\boxed{s_i[0]}$	$s_{i+1}[0]$	$s_{i+2}[0]$	$s_{i+3}[0]$	$s_{i+4}[0]$
—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—
$s_{j-7}[1]+s_{j-7}[0]$	...	...	...	...	$\boxed{s_{i-1}[1]+s_{i-1}[0]}$	$\boxed{s_i[1]+s_i[0]}$	$s_{i+1}[1]+s_{i+1}[0]$
$s_{j-8}[1]+2s_{j-8}[0]$	...	...	...	...	.	$\boxed{s_{i-1}[1]+2s_{i-1}[0]}$	$\boxed{s_i[1]+2s_i[0]}$

At Node d :

j^{th} time	$i+6$	$i+7$	$i+8$	$i+9$	$i+10$
$s_{j-8}[0]$	$s_{i-2}[0]$	$s_{i-1}[0]$	$s_i[0]$	$s_{i+1}[0]$	$s_{i+2}[0]$
$s_{j-8}[1]$	$s_{i-2}[1]$	$s_{i-1}[1]$	$s_i[1]$	$s_{i+1}[1]$	$s_{i+2}[1]$

Case (iii) : $a_1 \neq a_2$ and $b_1 = b_2$

Let $b_1 = b_2 = b$

Optimal Rate Constraint:

$$k = max\{a_1, a_2\}.min\left\{\frac{T+1-b-a_1}{b}, \frac{T+1-b-a_2}{b}\right\}$$
$$\implies k = max\{a_1, a_2\}.\frac{T+1-b-max\{a_1,a_2\}}{b}$$

W.L.O.G. Assume $a_1 > a_2$. Therefore,

$$k = a_1.\frac{T+1-b-a_1}{b}$$
$$n_1 = k + a_1 = a_1.\frac{T+1-a_1}{b}$$
$$n_2 = k + a_2 = a_2 + a_1.\frac{T-b+1-a_1}{b}$$

Dispersion span Constraint:

Ineq-1:

$$T - b - (n_1 - 1) - (b - a_1)\left\lfloor \frac{n_1 - 1}{a_1} \right\rfloor \geq 0$$
$$\text{L.H.S.} = T - b - n_1 + 1 - (b - a_1)\left\lfloor \frac{n_1}{a_1} - \frac{1}{a_1} \right\rfloor$$

By Property 1,

$$\text{L.H.S.} \leq T - b - n_1 + 1 - (b - a_1)\left(\frac{n_1}{a_1} - 1\right)$$
$$= T - b + 1 - b.\frac{n_1}{a_1} + b - a_1$$
$$= T - b + 1 - T - 1 + a_1 + b - a_1$$
$$= 0$$

Therefore,

$$\text{L.H.S.} \leq 0$$

Therefore for Ineq-1 to hold, $a_1 | n_1 \implies a_1 | k \implies b | T + 1 - a_1$

$$k > 0 \implies T + 1 - a_1 - b > 0 \implies \frac{T + 1 - a1}{b} > 1$$

$$b | T + 1 - a_1 \implies \frac{T + 1 - a_1}{b} \geq 2 \implies T + 1 - a_1 - b - a_2 \geq b - a_2 > 0$$

Ineq-2:

$$\begin{aligned}
T - b - (n_2 - 1) - (b - a_2)\left\lfloor \frac{n_2 - 1}{a_2} \right\rfloor &\geq 0 \\
\text{L.H.S.} &= T - b - a_2 - k + 1 - (b - a_2)\left\lfloor \frac{n_2}{a_2} - \frac{1}{a_2} \right\rfloor \\
\text{By Property 1,} & \\
\text{L.H.S.} &\leq T - b - a_2 - k + 1 - (b - a_2)\left(\frac{n_2}{a_2} - 1\right) \\
&= T - b - a_2 - k + 1 - (b - a_2)\left(\frac{k}{a_2}\right) \\
&= T - b - a_2 + 1 - b\left(\frac{k}{a_2}\right) \\
&= T - b - a_2 + 1 - a_1\left(\frac{T - b - a_1 + 1}{a_2}\right) \\
&= \frac{(a_2 - a_1)(T - b + 1) - (a_2 - a_1)(a_2 + a_1)}{a_2} \\
&= \frac{(a_2 - a_1)(T - b + 1 - a_1 - a_2)}{a_2} \\
\text{Since } a_1 &> a2 \text{ and } T - b + 1 - a_1 - a_2 > 0 \\
\text{Therefore,} & \\
\text{L.H.S.} &< 0
\end{aligned}$$

Therefore Ineq-2 does not hold.
Therefore Dispersion-Span Constraint does not hold.
Thus, whenever $a_1 \neq a_2$ and $b_1 = b_2$ no valid SDE based construction is Rate-Optimal.

Case (iv) : $a_1 \neq a_2$ and $b_1 \neq b_2$

Optimal Rate Constraint:

$$k = max\{a_1, a_2\}.min\left\{\frac{T + 1 - b_2 - a_1}{b_1}, \frac{T + 1 - b_1 - a_2}{b_2}\right\}$$

W.L.O.G assume $b_1 > b_2$
Case (iv).(i): $a_2 > a_1$

$$k = a_2.min\left\{\frac{T + 1 - b_2 - a_1}{b_1}, \frac{T + 1 - b_1 - a_2}{b_2}\right\}$$

$$n_1 = k + a_1$$
$$n_2 = k + a_2$$

Case (iv).(i).(i): $k = a_2 \dfrac{T+1-b_2-a_1}{b_1}$

Dispersion span Constraint:

Ineq-1:

$$T - b_2 - (n_1 - 1) - (b_1 - a_1)\left\lfloor \frac{n_1 - 1}{a_1} \right\rfloor \geq 0$$

$$\text{L.H.S.} = T - b_2 - n_1 + 1 - (b_1 - a_1)\left\lfloor \frac{n_1}{a_1} - \frac{1}{a_1} \right\rfloor$$

By Property 1,

$$\begin{aligned}
\text{L.H.S.} &\leq T - b_2 - n_1 + 1 - (b_1 - a_1)\left(\frac{n_1}{a_1} - 1\right) \\
&= T - b_2 + 1 - b_1\frac{n_1}{a_1} + b_1 - a_1 \\
&= T - b_2 + 1 - b_1\frac{k}{a_1} - a_1 \\
&= T - b_2 + 1 - a_2\frac{T+1-b_2-a_1}{a_1} - a_1 \\
&= \frac{a_1 - a_2}{a_1}(T - b_2 + 1 - a_1)
\end{aligned}$$

Since $a_2 > a_1$ and $T - b_2 + 1 - a_1 > 0$

L.H.S. < 0

Therefore, Ineq-1 does not hold.
Therefore, dispersion span constraint does not hold.

Case (iv).(i).(ii): $k = a_2 \dfrac{T+1-b_1-a_2}{b_2}$

Therefore, $T = \dfrac{b_2 k}{a_2} + b_1 + a_2 - 1$

Dispersion span Constraint:

Ineq-1:

$$T - b_2 - (n_1 - 1) - (b_1 - a_1)\left\lfloor \frac{n_1 - 1}{a_1} \right\rfloor \geq 0$$

$$\begin{aligned}
\text{L.H.S.} &= T - b_2 - n_1 + 1 - (b_1 - a_1)\left\lfloor \frac{k}{a_1} - \frac{1}{a_1} \right\rfloor - b_1 + a_1 \\
&= T - b_2 - b_1 - k + 1 - (b_1 - a_1)\left\lfloor \frac{k}{a_1} - \frac{1}{a_1} \right\rfloor
\end{aligned}$$

Case I: $\dfrac{k}{a_1} = \left\lfloor \dfrac{k}{a_1} \right\rfloor$

Threrefore, $a_1|k \implies k = p.a_1$ where $p \in \mathbb{Z}_+$

$$\begin{aligned}
\text{L.H.S.} &= T - b_2 - b_1 - k + 1 - (b_1 - a_1)\left(\frac{k}{a_1} - 1\right) \\
&= T - b_2 - b_1 - a_1 + 1 - b_1\frac{k}{a_1} \\
&= \frac{b_2 k}{a_2} + b_1 + a_2 - 1 - b_2 - b_1 - a_1 + 1 - b_1\frac{k}{a_1} \\
&= p\left(\frac{a_1 b_2 - a_2 b_1}{a_2}\right) + b_1 + a_2 - b_2 - a_1 \\
&= p\left(\frac{a_1 b_2 - a_2 b_1}{a_2}\right) + b_1 + a_2 - b_2 - a_1 \\
&= \frac{p(a_1 b_2 - a_2 b_1) + b_1 a_2 - b_2 a_2 + a_2(a_2 - a_1)}{a_2} \\
&= \frac{b_1 a_2(1-p) + b_2(a_1 p - a_2) + a_2(a_2 - a_1)}{a_2} \\
&= \frac{b_1 a_2(1-p) + b_2(a_1 - a_2) + b_2 a_1(p-1) + a_2(a_2 - a_1)}{a_2} \\
&= \frac{(p-1)(b_2 a_1 - b_1 a_2) + (a_2 - a_1)(a_2 - b_2)}{a_2}
\end{aligned}$$

Since $b_1 > b_2, a_2 > a_1, b_2 > a_2, p \geq 1$

L.H.S. < 0

Therefore, Ineq-1 does not hold.
Therefore, dispersion span constraint does not hold.

Case II: $\left\lfloor \frac{k}{a_1} \right\rfloor < \frac{k}{a_1} < \left\lfloor \frac{k}{a_1} \right\rfloor + \frac{1}{a_1}$

Not possible (Since k is an integer)

Case III: $\left\lfloor \frac{k}{a_1} \right\rfloor + \frac{1}{a_1} < \frac{k}{a_1} < \left\lfloor \frac{k}{a_1} \right\rfloor + 1$

Therfore, $\left\lfloor \frac{k}{a_1} - 1 \right\rfloor = \left\lfloor \frac{k}{a_1} \right\rfloor$

$$\begin{aligned}
\text{L.H.S.} &= T - b_2 - b_1 - k + 1 - (b_1 - a_1)\left\lfloor \frac{k}{a_1} \right\rfloor \\
&= \frac{b_2 k}{a_2} + b_1 + a_2 - 1 - b_2 - b_1 - k + 1 - (b_1 - a_1)\left\lfloor \frac{k}{a_1} \right\rfloor \\
&= \left(\frac{k}{a_2} - 1\right)(b_2 - a_2) - \left\lfloor \frac{k}{a_1} \right\rfloor (b_1 - a_1)
\end{aligned}$$

$a1 < a2, b1 > b2 \implies b_2 - a_2 < b_1 - a_1$ and $a_1 < a_2 \implies \frac{k}{a_2} - 1 < \frac{k}{a_1} - 1 < \left\lfloor \frac{k}{a_1} \right\rfloor$

L.H.S. < 0

Therefore, Ineq-1 does not hold.
Therefore, dispersion span constraint does not hold.

<u>**Case (iv).(ii)**</u>: $a_1 > a_2$

$$k = a_1.min\left\{\frac{T+1-b_2-a_1}{b_1}, \frac{T+1-b_1-a_2}{b_2}\right\}$$

$$n_1 = k + a_1$$
$$n_2 = k + a_2$$

Case (iv).(ii).(i): $k = a_1 \dfrac{T+1-b_1-a_2}{b_2}$

<u>Dispersion span Constraint:</u>

Ineq-2:

$$T - b_1 - (n_2 - 1) - (b_2 - a_2)\left\lfloor \frac{n_2 - 1}{a_2} \right\rfloor \geq 0$$

$$\text{L.H.S.} = T - b_1 - n_2 + 1 - (b_2 - a_2)\left\lfloor \frac{n_2}{a_2} - \frac{1}{a_2} \right\rfloor$$

By Property 1,

$$\begin{aligned}
\text{L.H.S.} &\leq T - b_1 - n_2 + 1 - (b_2 - a_2)\left(\frac{n_2}{a_2} - 1\right) \\
&= T - b_1 + 1 - b_2\frac{n_2}{a_2} + b_2 - a_2 \\
&= T - b_1 + 1 - b_2\frac{k}{a_2} - a_2 \\
&= T - b_1 + 1 - a_1\frac{T+1-b_1-a_2}{a_2} - a_2 \\
&= \frac{a_2 - a_1}{a_2}(T - b_1 + 1 - a_2)
\end{aligned}$$

Since $a_1 > a_2$ and $T - b_1 + 1 - a_2 > 0$

$\text{L.H.S.} < 0$

Therefore, Ineq-2 does not hold.
Therefore, dispersion span constraint does not hold.

Case (iv).(ii).(ii): $k = a_1 \dfrac{T+1-b_2-a_1}{b_1}$

<u>Dispersion span Constraint:</u>

Ineq-1:

$$T - b_2 - (n_1 - 1) - (b_1 - a_1)\left\lfloor \frac{n_1 - 1}{a_1} \right\rfloor \geq 0$$

$$\text{L.H.S.} = T - b_2 - n_1 + 1 - (b_1 - a_1)\left\lfloor \frac{n_1}{a_1} - \frac{1}{a_1} \right\rfloor$$

By Property 1,

$$\begin{aligned}
\text{L.H.S.} &\leq T - b_2 - n_1 + 1 - (b_1 - a_1)\left(\frac{n_1}{a_1} - 1\right) \\
&= T - b_2 + 1 - b_1\frac{n_1}{a_1} + b_1 - a_1 \\
&= T - b_2 + 1 - b_1\frac{k}{a_1} - a_1 \\
&= T - b_2 + 1 - (T + 1 - b_2 - a_1) - a_1 \\
&= 0 \\
\text{L.H.S.} &\leq 0
\end{aligned}$$

Therefore for Ineq-1 to hold, $a_1|n_1 \implies a_1|k \implies b_1|T - b_2 - a_1 + 1$

Ineq-2:

$$\begin{aligned}
&T - b_1 - (n_2 - 1) - (b_2 - a_2)\left\lfloor \frac{n_2 - 1}{a_2} \right\rfloor \geq 0 \\
\implies &T - b_1 - k - a_2 + 1 - (b_2 - a_2)\left\lfloor \frac{k-1}{a_2} - 1 \right\rfloor \geq 0 \\
\implies &T - b_1 - b_2 - k + 1 - (b_2 - a_2)\left\lfloor \frac{k-1}{a_2} \right\rfloor \geq 0 \\
\implies &T \geq b_1 + b_2 + (b_2 - a_2)\left\lfloor \frac{k-1}{a_2} \right\rfloor + (k-1)
\end{aligned}$$

<u>Delay Profile Constraint:</u> $\forall t \in [0, k-1]$

$a_1|k \implies a_1 \leq k$

Therefore, $a_2 < k$

$$T \leq b_1 + b_2 + (b_1 - a_1).\left\lfloor \frac{t}{a_1} \right\rfloor + (b_2 - a_2)\left\lfloor \frac{k-1-t}{a_2} \right\rfloor + k - 1$$

Put $t = a_2$

$$T \leq b_1 + b_2 + (b_1 - a_1).\left\lfloor \frac{a_2}{a_1} \right\rfloor + (b_2 - a_2)\left\lfloor \frac{k-1}{a_2} - 1 \right\rfloor + k - 1$$

$$a2 < a1 \implies \frac{a_2}{a_1} = 0$$

$$\implies T \leq b_1 + a_2 + (b_2 - a_2)\left\lfloor \frac{k-1}{a_2} \right\rfloor + k - 1$$

Since $b_2 > a_2$ Delay profile constraint and Ineq-2 cannot hold simultaneously.

Thus, whenever $a_1 \neq a_2$ and $b_1 \neq b_2$ no valid SDE based construction is Rate-Optimal.