Optimal Delivery Dep. in ride hailing (Costa-2020) [(a_1, a_2, b_1, b_2)]

Case-(ii): $a_1 = a_2 (= a)$ and $b_1 \neq b_2$

Case 1: $T - a + 1 > b_1 + b_2$

$$\Rightarrow k = a\left(\frac{T-a+1-\max\{b_1,b_2\}}{\min\{b_1,b_2\}}\right)$$

WLOG, assume $b_1 > b_2$

$$k = a\left(\frac{T-a+1-b_2}{b_1}\right), M_1 = M_2 = M = a\left(\frac{T-a+1+b_1-b_2}{b_1}\right) \quad \Big| \; k > a$$

Dispersion-span constraint:

$$T\tilde{b}_1 - (k+1) - (b_1 - a)\left[\frac{M-1}{a}\right] \geq 0 \quad \text{--- (1)}$$

and

$$T - b_1 - (M-1) - (b_2 - a)\left[\frac{M-1}{a}\right] \geq 0 \quad \text{--- (2)}$$

Proof (1)

$$T\tilde{b}_1 - \frac{a}{b_1}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left[\frac{T-a+1+b_1-b_2}{b_1} - \frac{1}{a}\right]$$

$$= T\tilde{b}_1 - \frac{a}{b_1}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left(\frac{T-a+1+b_1-b_2}{b_1} - 1\right)$$

$$= \cancel{T\tilde{b}_1} + \cancel{1} - \cancel{T\tilde{b}_1} + \cancel{a} - \cancel{\frac{a}{b_1}b_1} + \cancel{\frac{a}{b_1}b_2} - \cancel{1} + \cancel{b_1} - \cancel{a}$$

$$= 0$$

$$\Rightarrow (T - b_1) - (M-1) - (b_1 - a)\left[\frac{M-1}{a}\right] \leq 0$$

Therefore for inequality (1) to hold

$$b_1 \mid T - a + 1 - b_2$$

Same Argument that was made in case(i)

Equality holds iff $b_1 \mid T - a + 1 - b_2 + b_1$ or $b_1 \mid T - a + 1 - b_2$

Proof (2)

$$T - b_1 - \frac{a}{b_1}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left[\frac{T-a+1+b_1-b_2}{b_1} - \frac{1}{a}\right]$$

$$= T - b_1 - \frac{a}{b_1}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left(\frac{T-a+1+b_1-b_2}{b_1} - 1\right)$$

$$= \cancel{T} - \cancel{b_1} + \cancel{1} - \cancel{T} + \cancel{a} - \cancel{1} + \cancel{b_1} - \cancel{a} + \cancel{b_2} - \cancel{a}$$

$$= 0$$

$$\Rightarrow (T - b_1) - (M-1) - (b_2 - a)\left[\frac{M-1}{a}\right] \leq 0$$

Therefore for inequality (1) to hold

$$b_1 \mid T - a + 1 - b_1$$

Equality holds since $b_1 \mid T - a + 1 - b_1 + b_1$

Delay Feasible Constraint:

To $b_1 + b_2 \leq (b_1 - a)\left[\frac{k}{a}\right] + (b_2 - a)\left[\frac{k-1-k}{a}\right] + k - 1$

$$\text{R.H.S.} = b_1 + b_2 + b_1\left[\frac{k}{a}\right] + b_2\left[\frac{k-1-k}{a}\right] - a\left(\left[\frac{k}{a}\right] + \left[\frac{k-1-k}{a}\right]\right) + k - 1 \quad \Big| \; a \mid k$$

$$= b_1 + b_2 + b_1\left[\frac{k}{a}\right] + b_2\left[\frac{k-1-k}{a}\right] - a\left(\left[\frac{k}{a}\right] + \left[\frac{-1}{a}\right]\right) + k - 1 \quad \Big| \; \left[\frac{k}{a}\right] + \left[\frac{-1}{a}\right] = -1$$

$$= b_1 + b_2 + k - 1 + b_2\frac{k}{a} + b_1\left[\frac{k}{a}\right] + b_2\left[\frac{-1-k}{a}\right]$$

$$= b_1 + b_2 + k - 1 + \frac{b_2}{b_1}(T - k - b_2) + b_1\left[\frac{k}{a}\right] + b_2\left[\frac{-1-k}{a}\right]$$

put $k = a - 1$

Let $T = b_1 + b_2 + a - 1 + x \quad , x > 0$

$$= T - x + \frac{b_2}{b_1}(b_1 a) - b_2$$

$$\Rightarrow \underbrace{T - x + \frac{b_2}{b_1}a}_{< 0} < T \quad \Rightarrow \text{R.H.S.} < T$$

$\therefore$ Delay feasible constraint does not hold

→ Similar Argument

Case 2: $T - a + 1 < b_1 + b_2$

$$\Rightarrow k = a\left(\frac{T-a+1-\max\{b_1,b_2\}}{\min\{b_1,b_2\}}\right)$$

WLOG, assume $b_1 > b_2$

$$k = a\left(\frac{T-a+1-b_2}{b_2}\right), M_1 = M_2 = M = a\left(\frac{T-a+1+b_1-b_2}{b_2}\right) \quad \Big| \; k < a$$

Dispersion-span constraint:

$$T\tilde{b}_1 - (k+1) - (b_1 - a)\left[\frac{M-1}{a}\right] \geq 0 \quad \text{--- (1)}$$

and

$$T - b_1 - (M-1) - (b_2 - a)\left[\frac{M-1}{a}\right] \geq 0 \quad \text{--- (2)}$$

Proof (1)

$$T\tilde{b}_1 - \frac{a}{b_2}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left[\frac{T-a+1+b_1-b_2}{b_2} - \frac{1}{a}\right]$$

$$= T\tilde{b}_1 - \frac{a}{b_2}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left(\frac{T-a+1+b_1-b_2}{b_2} - 1\right)$$

$$= \cancel{T\tilde{b}_1} - \cancel{\frac{a}{b_2}T} + \cancel{a} - \cancel{1} + \cancel{\frac{a}{b_2}b_1} - \cancel{\frac{a}{b_2}b_2} + \cancel{1} - \cancel{b_1} + \cancel{a}$$

$$= 0$$

$$\Rightarrow (T - b_1) - (M-1) - (b_2 - a)\left[\frac{M-1}{a}\right] \leq 0$$

Therefore for inequality (1) to hold

$$b_2 \mid T - a + 1 - b_1$$

Same Argument that was made in case(i)

Equality holds iff $b_2 \mid T - a + 1 + b_2 - b_1$ or $b_2 \mid T - a + 1 - b_1$

Proof (2)

$$T - b_2 - \frac{a}{b_2}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left[\frac{T-a+1+b_1-b_2}{b_2} - \frac{1}{a}\right]$$

$$= T - b_2 - \frac{a}{b_2}(T-a+1+b_1-b_2) + 1 - (b_1 - a)\left(\frac{T-a+1+b_1-b_2}{b_2} - 1\right)$$

$$= T - b_2 + 1 - \frac{b_1}{b_2}(T-a+1+b_2-b_1) + b_1 - a$$

$$= (T+1-a)\left(\frac{b_2-b_1}{b_2}\right) + (b_1 - b_2)\left(\frac{b_1+b_2}{b_2}\right)$$

$$= \frac{b_2-b_1}{b_2}\left((T+1-a) - (b_1+b_2)\right) > 0$$

Equality holds since $b_2 \mid T - a + 1 + b_2 - b_1$

Similar Argument

Case-3: $T - a + 1 = b_1 + b_2$

$$k = \left(\frac{T-a+1-b_1}{b_2}\right)a \Rightarrow k = a$$

$$\Rightarrow M = 2a$$

Dispersion-span constraint:

$$T\tilde{b}_1 - (k+1) - (b_1 - a)\left[\frac{M-1}{a}\right] \geq 0 \quad \text{--- (1)}$$

and

$$T - b_1 - (M-1) - (b_2 - a)\left[\frac{M-1}{a}\right] \geq 0 \quad \text{--- (2)}$$

Proof (1)

$$T - b_2 - (2a-1) - (b_1 - a)\left[\frac{2a-1}{a}\right]$$

$$= T - b_2 - 2a + 1 - (b_1 - a)\left(2 + \cancel{\left[\frac{-1}{a}\right]}^{-1}\right)$$

$$= T - b_2 - 2a + 1 - b_1 + a$$

$$= \underbrace{T - a + 1 - (b_1 + b_2)}_{\text{zero}}$$

$$= 0$$

$$\Rightarrow T - b_2 - (2a-1) - (b_1 - a)\left[\frac{2a-1}{a}\right] = 0$$

Therefore inequality (1) holds

Proof (2)

$$T - b_1 - (2a-1) - (b_2 - a)\left[\frac{2a-1}{a}\right]$$

$$= T - b_1 - 2a + 1 - (b_2 - a)\left(2 + \cancel{\left[\frac{-1}{a}\right]}^{-1}\right)$$

$$= T - b_1 - 2a + 1 - b_2 + a$$

$$= \underbrace{T - a + 1 - (b_1 + b_2)}_{\text{zero}}$$

$$= 0$$

$$\Rightarrow T - b_1 - (2a-1) - (b_2 - a)\left[\frac{2a-1}{a}\right] = 0$$

Therefore inequality (2) holds

Therefore for dispersion span constraint to hold

$$\boxed{T - a + 1 = b_1 + b_2}$$

Delay Feasible Constraint: $k = a\left(\frac{T-a+1-b_2}{b_1}\right) = a \quad \Big| \; b_1 + b_2 = T - a + 1 \Rightarrow T - a + 1 - b_2 = b_1$

To $b_1 + b_2 \leq (b_1 - a)\left[\frac{k}{a}\right] + (b_2 - a)\left[\frac{k-1-k}{a}\right] + k - 1$

$$b_1 + b_2 + b_1\left[\frac{k}{a}\right] + b_2\left[\frac{k-1-k}{a}\right] - a\left(\left[\frac{k}{a}\right] + \left[\frac{k-1-k}{a}\right]\right) + a - 1$$

$$= b_1 + b_2 + b_1\left[\frac{k}{a}\right] + b_2\left(1 + \left[\frac{-1-k}{a}\right]\right) - a\left(\left[\frac{k}{a}\right] + \left[\frac{-1-k}{a}\right] + 1\right) + a - 1 \quad \Big| \; \left[\frac{k}{a}\right] + \left[\frac{-1-k}{a}\right] = -1$$

$$= b_1 + b_2 + a - 1 + b_1\left[\frac{k}{a}\right] + b_2\left[\frac{-1-k}{a}\right] + b_2$$

$$= T + b_1 \cancel{\left[\frac{k}{a}\right]}^{0} + b_2 \cancel{\left[\frac{-1-k}{a}\right]}^{-1} + b_2$$

$$\Rightarrow T$$

$\because k = a$

$0 \leq k < a \Rightarrow 0 \leq \frac{k}{a} < 1$

$1 \leq 1 + k \leq a$

$\Rightarrow -\frac{1}{a} \geq -\frac{1+k}{a} \geq -1$

Constraints on parameters:

- $a_1 = a_2 (= a)$
- $k = a$
- $T - a + 1 = b_1 + b_2$
- $M_1 = M_2 = 2a$
- $N_1 = T - b_2 + 1$
- $N_2 = T - b_1 + 1$

$$R_1 = \frac{T - b_2 - a + 1}{T - b_2 - a + 1 + b_1} = R_{opt} = R_2 = \frac{T - b_1 - a + 1 + b_2}{T - b_1 - a + 1 + b_2} = R_{opt_{2\text{-Node}}}$$

An optimal SDE based construction exists under above constraints

Example: $T = 8 \quad a_1 = a_2 = 1 \quad b_1 = 3 \quad , b_2 = 9$

$k = 2 \quad M_1 = M_2 = 4 \quad N_1 = 5 \quad , N_2 = 6$

$$D_1 = (\overset{M_1}{4}, \overset{M_2}{3})$$

At Node 1:

j^{th} time	$i=-1$	i	$i+1$	$i+2$	$i+3$	$i+4$
$\Delta_j^1[0]$	$\boxed{\Delta_{-1}[0]}$	$\boxed{\Delta_0^1[0]}$	$\Delta_{1_1}[0]$	$\Delta_{1_2}[0]$	$\Delta_{1_3}[0]$	$\Delta_{1_4}[0]$
$\Delta_j^2[1]$	$\Delta_{-1}[1]$	$\Delta_0^1[1]$	$\Delta_{1_1}[1]$	$\Delta_{1_2}[1]$	$\Delta_{1_3}[1]$	$\Delta_{1_4}[1]$
—	—	—	—	—	—	—
$\Delta_{j-2}[0] + \Delta_{j-2}[1]$	...	...	...	$\Delta_{1_2}[0] + \Delta_{1_2}[1]$	$\Delta_1^1[0] + \Delta_{1_3}[1]$	$N_{1_4}[0] + N_{1_4}[1]$
$\Delta_{j-2}[0] + 2\Delta_{j-2}[1]$	...	...	...	...	$\Delta_1^1[0] + 2\Delta_1^1[1]$	$\Delta_1^1[0] + 2\Delta_{1_4}[1]$

At Node 2:

$$D_2 = (\overset{M_1}{4}, \overset{M_2}{5})$$

j^{th} time	$i+2$	$i+3$	$i+4$	$i+5$	$i+6$	$i+7$	$i+8$
$\Delta_{j-3}[0]$	$\boxed{\Delta_{2_1}[0]}$	$\Delta_1^2[0]$	$\Delta_2^2[0]$	$\Delta_{2_1}[0]$	$\Delta_{2_2}[0]$	$\Delta_{2_3}[0]$	$\Delta_{2_4}[0]$
$\Delta_{j-4}[0]$	$\Delta_{2_1}[0]$	$\boxed{\Delta_1^2[0]}$	$\boxed{\Delta_2^2[0]}$	—	—	—	—
—	—	—	—	—	—	—	—
$\Delta_{j-7}[1] + \Delta_{j-7}[0]$	...	...	...	...	$\Delta_{2_2}[1] + \Delta_{2_2}[0]$	$\Delta_2^2[1] + \Delta_2^2[0]$	$N_{2_4}[1] + \Delta_{2_4}[0]$
$\Delta_{j-8}[1] + 2\Delta_{j-8}[0]$	...	...	...	...	...	$\boxed{\Delta_2^2[1] + 2\Delta_2^2[0]}$	$\boxed{\Delta_2^2[1] + 2\Delta_{2_4}[0]}$

At Node 3:

j^{th} time	$i+6$	$i+7$	$i+8$	$i+9$	$i+10$
$\Delta_{j-5}[0]$	$\Delta_{3_1}[0]$	$\Delta_{3_2}[0]$	$\Delta_{3_3}[0]$	$\Delta_{3_4}[0]$	$\Delta_{3_5}[0]$
$\Delta_{j-6}[0]$	$\Delta_{3_1}[0]$	$\Delta_{3_2}[0]$	$\Delta_{3_3}[0]$	$\Delta_{3_4}[0]$	$\Delta_{3_5}[0]$
$\Delta_{j-9}[1]$	$\Delta_{3_1}[1]$	$\Delta_{3_2}[1]$	$\Delta_{3_3}[1]$	$\Delta_{3_4}[1]$	$\Delta_{3_5}[1]$
$\Delta_{j-8}[1] + 2\Delta_{j-8}[0]$	$\Delta_{3_1}[1]$	$\Delta_{3_2}[1]$	$\Delta_{3_3}[1]$	$\Delta_{3_4}[1]$	$\Delta_{3_5}[1]$