(e) $X^TX = V\Sigma^T U^T \cdot U\Sigma V^T$

$= V\Sigma^T\Sigma V^T$

$(X^TX)^{-1} = V(\Sigma^T\Sigma)^{-1}V^T$

$H = U\Sigma V^T V(\Sigma^T\Sigma)^{-1}V^T \cdot V\Sigma^T U^T$

$= U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T U^T$

Since X^TX invertible, there must be $r = \min(m,n)$,

thus $\Sigma^T\Sigma$ invertible too.

$H = U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T U^T$

(f) Since $\Sigma^{(k)}$ has only k non-zero columns, we can throw away u_i's and v_i's for $i > k$. Define $U^{(k)}$ to be the $m \times k$ matrix with columns $u_1, \ldots, u_k$ and $V^{(k)}$ similarly the $n \times k$ matrix with columns $v_1, \ldots, v_k$. Let $\hat{\Sigma}^{(k)}$ be the $k \times k$ matrix that truncates the all-zero columns of $\Sigma^{(k)}$

$$H^{(k)} = U^{(k)}\hat{\Sigma}^{(k)}(\hat{\Sigma}^{(k)T}\hat{\Sigma}^{(k)})^{-1}\hat{\Sigma}^{(k)}U^{(k)T}$$