(d) $Var(Y) = E(Y^2) - E(Y)^2$

$Var(Y|X) = E([Y - E(Y|X)]^2 | X)$

$E\,Var(Y|X) = E[E([Y-E(Y|X)]^2|X)]$

$= E(Y - E(Y|X))^2$

$= E(Y^2) - E(E(Y|X)^2)$

$Var(Y) = E\,Var(Y|X) + E[E(Y|X)^2] - E(Y)^2$

$Var\,E(Y|X) = E(E[Y|X]^2) - E[E[Y|X]]^2 = E[[E(Y|X)]^2] - E(Y)^2$

$\Rightarrow Var(Y) = E\,Var(Y|X) + Var\,E(Y|X)$

(a) $\min_{\hat\beta} (y - X\hat\beta)^2$

$\Rightarrow \min_{\hat\beta} y^Ty - y^TX\hat\beta - \hat\beta^TX^Ty + \hat\beta^TX^TX\hat\beta$

$[\hat\beta]. \quad -2X^Ty + 2X^TX\hat\beta = 0$

$\Rightarrow X'X\hat\beta = X'y$

$\Rightarrow \hat\beta = (X^TX)^{-1}X^Ty$

$\hat y = X\hat\beta = X(X^TX)^{-1}X^Ty$

$H = X(X^TX)^{-1}X^T$

thus. $\hat y = Hy = X\hat\beta$ are the least square estimates

(b) $HX = X(X^TX)^{-1}X^TX$

$= X(X^TX)^{-1}(X^TX)$

$= X$

(c) $H^T = [X(X^TX)^{-1}X^T]^T = (X^T)^T[(X^TX)^{-1}]^T \cdot X^T$

$= X[(X^TX)^{-1}]^T \cdot X^T$

$= X[(X^TX)^T]^{-1} X^T$

since the transpose of inverse equals the inverse of transpose.

$H^T = X[(X^TX)]^{-1}X^T = H$

(d) $H^2 = X(X^TX)^{-1}X^T X(X^TX)^{-1}X^T$

$= X(X^TX)^{-1}(X^TX)(X^TX)^{-1}X^T$

$= X(X^TX)^{-1}X^T = H.$

(e) the residual is $e = y - \hat y = y - Hy = My$

where $M = I - H$

$X^Te = 0$ by first order condition. e is orthogonal to X.