4. Sheaf of modules

Recall: Separatedness

$$\begin{tikzcd} \operatorname{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R = T \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

$\exists$ at most 1 $T \to X$

Prop: $k = \bar{k}$. The image of the functor $t: \mathrm{Var}(k) \to \mathrm{Sch}(k)$ is exactly the set of quasi-projective integral schemes over k. The image of projective varieties is the set of projective integral schemes. In particular, for any variety V, $t(V)$ is an integral separated scheme of finite type over k. If it is proper over k, we will also say it is complete.

Def: $(X, \mathcal{O}_X)$ a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ on X such that $\forall$ open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with module structure via $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

- A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves st. $\forall$ open $U \subseteq X$, $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homo of $\mathcal{O}_X(U)$-modules.

Remark:
- The ker, coker, Im of a morphism is again $\mathcal{O}_X$-modules.
- The quotient, $\oplus$, $\prod$, $\varinjlim$, $\varprojlim$ (of sheaves) are $\mathcal{O}_X$-modules

Def:
- sheaf $\mathcal{H}om$: $U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf
- tensor product: $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ is a presheaf.

  $\mathcal{F} \otimes \mathcal{G}$ is defined to be the sheafification.

Def: An $\mathcal{O}_X$-module $\mathcal{F}$ is free if $\mathcal{F} \cong \bigoplus \mathcal{O}_X$. It is locally free if X can be covered by open sets $\{U_i\}_{i \in I}$ for which $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module. In that case the rank of $\mathcal{F}$ on U_i is the # of copies of $\mathcal{O}_X|_{U_i}$.

Remark: • If X is connected, the rank of a locally free sheaf is the same everywhere.

• A locally free sheaf of rank 1 is also called an invertible sheaf.

Def: • A sheaf of ideals on X is a sheaf of modules $\mathcal{F}$ which is a subsheaf of $\mathcal{O}_X$ i.e. $\mathcal{F}(U)$ is an ideal of $\mathcal{O}_X(U)$.

• $f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ — pull-back

Remark: $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$ — adjunction

Def: sheaf associated to M on $\mathrm{Spec}\,A$ where M is an A-module, $\widetilde{M}$:

$$\widetilde{M}(U) := \{ s: U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \mid \forall \mathfrak{p} \in U,\ s(\mathfrak{p}) \in M_{\mathfrak{p}} \text{ and } s \text{ is locally a fraction} \}$$

Prop: A: a ring, M: an A-module, $\widetilde{M}$: sheaf associated to M on $\mathrm{Spec}\,A$. Then

(a) $\widetilde{M}$ is an $\mathcal{O}_X$-module

(b) $\forall \mathfrak{p} \in X$, $\widetilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$

(c) $\forall f \in A$, $\widetilde{M}(D(f)) \cong M_f$ as A_f-modules

(d) $\Gamma(X, \widetilde{M}) = M$.

Prop: A, B: rings, $X = \mathrm{Spec}\,A$, $A \longrightarrow B$ ring homo, $f: \mathrm{Spec}\,B \longrightarrow \mathrm{Spec}\,A$ induced

surjective

from $A \longrightarrow B$. Then:

(a) the map $M \longmapsto \widetilde{M}$ gives an exact, fully faithful functor from A-Mod to $\mathcal{O}_X$-Mod

(b) $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$

(c) $\widetilde{\bigoplus_i M_i} \cong \bigoplus_i \widetilde{M_i}$

(d) $f_*\widetilde{N} \cong \widetilde{{}_AN}$ where ${}_AN$ means N considered as A-module via $A \longrightarrow B$.

(e) $f^* \widetilde{M} \cong \widetilde{M \otimes_A B}$ $\quad (f: \operatorname{Spec} B \to \operatorname{Spec} A)$ $\quad \varphi: A \to B$

Def: $(X, \mathcal{O}_X)$ scheme. $\mathcal{F}$: $\mathcal{O}_X$-module. $\mathcal{F}$ is quasi-coherent if $\exists \{U_i\}_{i \in I}$ affine open local of X, $U_i = \operatorname{Spec} A_i$, $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ where M_i is an A_i-module. $\mathcal{F}$ is coherent if M_i is f.g. A_i-module $\forall i$.

Lemma: $X = \operatorname{Spec} A$, $\mathcal{F}$ quasi-coherent

(a) $s \in \Gamma(X, \mathcal{F})$ s.t. $s|_{D(f)} = 0 \Rightarrow \exists n > 0$ s.t. $f^n s = 0$

(b) $t \in \mathcal{F}(D(f)) \Rightarrow \exists n > 0$ s.t. $f^n t$ extends to a global section.

Prop: Being quasi-coherent (coherent) is a locally property.

Corollary: $X = \operatorname{Spec} A$. The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories

$$\begin{array}{ccc} A\text{-Mod} & \longrightarrow & \mathcal{O}_X\text{-Mod} \\ M & \longmapsto & \widetilde{M} \\ \Gamma(X, \mathcal{F}) & \longleftarrow\!\shortmid & \mathcal{F} \end{array}$$

Prop: $X = \operatorname{Spec} A$, $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact, $\mathcal{F}'$ is quasi-coherent $\Longrightarrow 0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ exact.

$\mathcal{F}$ sheaf

injective resolution: $0 \to \mathcal{F} \to I^0 \to I^1 \to I^2 \to \cdots$

$\Gamma(X, -)$: $\quad \Gamma(X, I^0) \xrightarrow{d^0} \Gamma(X, I^1) \xrightarrow{d^1} \cdots$ $\quad$ complex

cohomology : $\quad H^n(X, \mathcal{F}) := \ker(d^n) / \operatorname{Im}(d^{n-1})$

$\mathcal{F}$ is flabby $\Rightarrow H^i(X, \mathcal{F}) = 0 \quad i > 0$

$\hookrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ surjective

flasque

$H^0(\{U_i \to U\}_{i\in I}, X)$ is a functor $Cov^{op} \longrightarrow Group$.

5. Sheaves of Modules

Prop: • X : scheme. Ker, Coker, Im of quasi-coherent are quasi-coherent.

• Any extension of quasi-coherent sheaves is quasi-coherent i.e.

$$0 \to \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

• If X is noetherian, then the same is true for coherent sheaf.

proof: $\mathcal{F} = \widetilde{M}$, $\mathcal{F}'' = \widetilde{M''}$, $M = \Gamma(X, \mathcal{F})$, $M'' = \Gamma(X, \mathcal{F}'')$, $M' = \Gamma(X, \mathcal{F}')$

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & \widetilde{M} \arrow[r] \arrow[d, "\wr"] & \widetilde{M'} \arrow[r] \arrow[d] & \widetilde{M''} \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F}'' \arrow[r] & 0 \end{tikzcd}$$

Prop: $f: X \longrightarrow Y$ morphism

(a) If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module.

(b) X, Y noetherian, $\mathcal{G}$ coherent $\Rightarrow f^*\mathcal{G}$ coherent

(c) f quasi-compact and separated, $\mathcal{F}$ quasi-coherent $\mathcal{O}_X$-module, $\Rightarrow f_*\mathcal{F}$ is quasi-coherent $\mathcal{O}_Y$-module.

Def: Y closed subscheme of a scheme X, $i: Y \longrightarrow X$ inclusion. We define the ideal sheaf of Y, denoted $\mathcal{I}_Y$ to be the kernel of

$$i^\# : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y.$$

Prop: $\{\text{closed subschemes}\} \xleftrightarrow{1:1} \{\text{quasi-coherent sheaves of ideals on } X\}$

$$Y \longmapsto \mathcal{I}_Y$$

$$\operatorname{Supp} \mathcal{O}_X/\mathcal{I} \longleftarrow\!\shortmid\ \mathcal{I}$$

$\operatorname{Supp} \mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\}$

Cor: $X = \operatorname{Spec} A$

$\{ \text{ideals} \} \xleftrightarrow{1:1} \{ \text{closed subschemes} \}$

Def: S: graded ring, M: graded S-module.

We define $\widetilde{M}$ on $\operatorname{Proj} S$ as follows:

$$\widetilde{M}(U) := \{ s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in U} M_{(\mathfrak{p})} \mid s \text{ is locally a fraction} \}.$$

Prop: $X = \operatorname{Proj} S$

(a) $(\widetilde{M})_{\mathfrak{p}} = M_{(\mathfrak{p})}$

(b) homogeneous element $f \in S_+$, $\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$

(c) $\widetilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If S is noetherian, M is f.g. then $\widetilde{M}$ is coherent.

Def: • $\forall n \in \mathbb{Z}$ we define $\mathcal{O}_X(n)$ to be $\widetilde{S(n)}$.

• $\mathcal{O}_X(1)$ is called the twisting sheaf of Serre

• $\forall \mathcal{F}$ $\mathcal{O}_X$-module, $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

$S(n)$ is a graded ring s.t. $S(n)_i = S_{i+n}$

$$S = \bigoplus_i S_i$$

$$S(n) = \bigoplus S_{i+n}$$

Prop: S is generated by S_1 as an S_0-algebra

(a) $\mathcal{O}_X(n)$ is an invertible sheaf

(b) $\widetilde{M}(n) = \widetilde{M(n)}$ In particular, $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) = \mathcal{O}_X(m+n)$

(c) $\varphi : S \longrightarrow T$, $U \subseteq Y = \operatorname{Proj} T$, $f : U \longrightarrow X$ determined by φ

Then $f^*(\mathcal{O}_X(n)) \cong \mathcal{O}_Y(n)|_U$ and $f_*(\mathcal{O}_Y(n)|_U) \cong (f_* \mathcal{O}_U)(n)$.

Def: S graded ring, $X = \operatorname{Proj} S$, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We define the graded S-module associated to $\mathcal{F}$ as a group to be

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

S-module structure:

$s \in S_d$, s determines a global section in $\Gamma(X, \mathcal{O}_X(d))$.

we define, $\forall\, t \in \Gamma(X, \mathcal{F}(n))$

$$s \cdot t = s \otimes t \quad \text{using } \mathcal{F}(n) \otimes \mathcal{O}_X(d) \cong \mathcal{F}(n+d).$$

$$\Gamma(X, \mathcal{O}_X(d)) \otimes \Gamma(X, \mathcal{F}(n)) \cong \Gamma(X, \mathcal{F}(n+d)) \subseteq \Gamma_*(\mathcal{F})$$

Lemma: X scheme, $\mathcal{L}$ invertible sheaf on X, $f \in \Gamma(X, \mathcal{L})$,

$X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x\}$, $\mathcal{F}$ quasi-coherent

(a) X quasi-compact, $s \in \Gamma(X, \mathcal{F})$ s.t. $s|_{X_f} = 0 \Rightarrow \exists\, n > 0$ s.t. $f^n s = 0$

where $f^n s$ is considered as a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

(b) X quasi-compact, $\{U_i\}$ a finite covering s.t. $\mathcal{L}|_{U_i}$ is free for each i and s.t. $U_i \cap U_j$ is quasi-compact $\Rightarrow$ given a section $t \in \Gamma(X_f, \mathcal{F})$ $\exists\, n > 0$ s.t. $f^n t \in \Gamma(X_f, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

Prop: S graded ring which is f.g. by S_1 as an S_0-algebra. $X = \operatorname{Proj} S$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}.$$

Def:
- For any scheme Y, we define the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ to be $g^*(\mathcal{O}(1))$ where $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ is the natural map.

$$\mathbb{P}^r_Y = \mathbb{P}^r_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y, \qquad \mathbb{P}^r_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \dots, x_r]$$

- If X is any scheme over Y, an invertible sheaf $\mathcal{L}$ on X is very ample relative to Y if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r s.t. $i^*(\mathcal{O}(1)) \cong \mathcal{L}$.

- X a scheme, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is generated by global sections if $\exists \{s_i\}_{i\in I}$, $s_i \in \Gamma(X,\mathcal{F})$ s.t. $\forall x\in X$, the images of s_i in the stalk $\mathcal{F}_x$ generate the stalk as $\mathcal{O}_x$-module.

($\Leftrightarrow$ $\mathcal{F}$ can be written as a quotient of a free sheaf)

Theorem (Serre): X projective scheme over a noetherian ring A. $\mathcal{F}$ coherent $\Rightarrow \exists\, n_0 \in \mathbb{Z}$ s.t. $\forall n \geq n_0$, $\mathcal{F}(n)$ can be generated by a finite number of global sections.

Corollary: Any coherent sheaf can be written as a quotient of a sheaf $\mathcal{E}$ where $\mathcal{E}$ is a finite direct sum of $\mathcal{O}(n_i)$, $n_i \in \mathbb{Z}$.

proof: $\mathcal{F}(n)$ is generated by a finite number of global sections, n large enough

$$\Rightarrow \bigoplus_{i=1}^{N} \mathcal{O}_X \longrightarrow \mathcal{F}(n) \longrightarrow 0$$

$$\otimes\, \mathcal{O}_X(-n) \Rightarrow \bigoplus_{i=1}^{N} \mathcal{O}_X(-n) \longrightarrow \mathcal{F} \longrightarrow 0$$

Theorem: k field, A f.g. k-algebra, X projective scheme over A. $\mathcal{F}$ coherent $\Rightarrow \Gamma(X,\mathcal{F})$ is a f.g. A-module. In particular, if $A = k$, $\Gamma(X,\mathcal{F})$ is finite-dim k-vector space.

Corollary: $f: X \to Y$ projective morphism of finite type over a field k.
$\mathcal{F}$ coherent $\Rightarrow f_*\mathcal{F}$ coherent.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\text{closed immersion}"'] & Y \\ \mathbb{P}^r_Y \arrow[ur] & \end{tikzcd}$$

proof: $Y = \operatorname{Spec} A$, A f.g. k-algebra

$f_*\mathcal{F}$ is quasi-coherent $\Rightarrow f_*\mathcal{F} = \widetilde{\Gamma(Y, f_*\mathcal{F})} = \widetilde{\Gamma(X,\mathcal{F})}$

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f^*\mathcal{G})$$

6. Divisors — Weil divisors

Def: A scheme X is regular in codim $=1$ if every local ring $\mathcal{O}_{X,x}$ of x of $\dim=1$ is regular. ($\dim \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$).

(*) X is noetherian integral separated scheme which is regular in codim 1.

Def: • Let X satisfy (*). A prime divisor on X is a closed integral subscheme of codim $=1$.

• A Weil divisor is an element of the free abelian group $\operatorname{Div} X$ generated by prime divisors. We write $D = \sum n_i Y_i$, $n_i \in \mathbb{Z}$. If all the $n_i \geq 0$, we say D is effective.

Consider Y a prime divisor on X and let $\eta \in Y$ be its generic point. Then the local ring $\mathcal{O}_{X,\eta}$ is DVR with fraction field K and so $\mathrm{Frac}\,\mathcal{O}_X(X) \subseteq K$.
We call the corresponding discrete valution ν_Y the valuation of Y.
We call the corresponding discrete valution by its valuation.

Fact: X separated $\Longrightarrow$ Y is uniquely determined by its valuation.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dotted] & Y = \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

$\nu_Y(f') \in \mathbb{N}$, $\forall f' \in \mathcal{O}_{X,\eta}$

$\nu_Y(f) = \nu_Y(\frac{g}{h})$, $g, h \in \mathcal{O}_{X,\eta}$

$= \nu_Y(g) - \nu_Y(h)$

$\mathrm{Frac}\,\mathcal{O}_X(X)$
$\cup \quad \cap$

Let $f \in K^*$ any rational function on X. Then $\nu_Y(f) \in \mathbb{Z}$. If $\nu_Y(f) > 0$, we say f has a zero along Y of order $\nu_Y(f)$; if $\nu_Y(f) < 0$, we say f has a pole along Y of order $-\nu_Y(f)$.

Lemma: Let X satisfy (*) and $f \in K^*$ be a nonzero function on X. Then $\nu_Y(f) = 0$ for all but finitely many prime divisors Y.

proof: $U = \operatorname{Spec} A$ affine open subset of X s.t. f is regular on U

$Z := X \setminus U$ proper closed subset of X

X noetherian $\Longrightarrow$ Z contains at most finitely many prime divisors

It suffices to show that $\nu_Y(f) \neq 0$ for finitely many prime divisors Y of U.

f regular on $U \Rightarrow \nu_Y(f) \geq 0$.

$\nu_Y(f) > 0 \Leftrightarrow Y \subseteq V((f)) \subseteq U$

$f \neq 0 \Rightarrow Y \subseteq V((f)) \subsetneq U \Rightarrow$ finitely many Y.

Def: Let X satisfy $(*)$ and $f \in K^*$. We define the divisor of f denoted by $\operatorname{div} f$ / (f) by

$$\operatorname{div} f = \sum_{\substack{Y \\ \text{prime divisors}}} \nu_Y(f) \cdot Y \quad \text{—— principal divisor.}$$

Note: $\operatorname{div} \frac{f}{g} = \operatorname{div} f - \operatorname{div} g$ as $\nu_Y(\frac{f}{g}) = \nu_Y(f) - \nu_Y(g)$

Therefore, the image of $K^* \longrightarrow \operatorname{Div} X$ gives a subgroup.

Def: Let X satisfy $(*)$. • Two divisors D' and D are linearly independent, written $D \sim D'$ if $D - D'$ is a principal divisor.

• The divisor class group of X $Cl(X)$ is defined to be the quotient

$$\operatorname{Div} X / \langle \text{principal divisors} \rangle$$

Prop: A: noetherian domain. Then A is an UFD iff $X = \operatorname{Spec} A$ is normal and $Cl(X) = 0$.

proof: Facts: • UFD $\Rightarrow$ integral closed i.e. X is normal

• UFD $\Leftrightarrow$ every prime ideal of height 1 is principal $\Leftrightarrow Cl(X) = 0$

Show: A integral closed, every prime ideal of height 1 is principal

$\Rightarrow Y \subseteq \operatorname{Spec} A$ a prime divisor $\rightsquigarrow \mathfrak{p}$ of height 1.

Suppose $\mathfrak{p} = (f) \Rightarrow \operatorname{div} f = 1 \cdot Y \Rightarrow Y$ principal $\Rightarrow Cl(X) = 0$.

$\Leftarrow$: $\mathfrak{p}$ of height 1 $\rightsquigarrow Y$ prime divisor

$Cl(X) = 0 \Rightarrow Y = \operatorname{div} f$, $f \in K^*$

Want to show: $f \in A$, $(f) = \mathfrak{p}$

$\operatorname{div} f = Y \Rightarrow \nu_Y(f) = 1 \Rightarrow f \in A_\mathfrak{p}$ and f generates $\mathfrak{p}A_\mathfrak{p}$.

$\nu_Y(f)=1 \Rightarrow f=\frac{g}{h}$, $\nu_Y(g)=1$, $\nu_Y(h)=0$ i.e. $h\notin \mathfrak{p} \Rightarrow f\in A_{\mathfrak{p}}$, $(g)=\mathfrak{p}\cdot\mathfrak{q}$

$$\Rightarrow \frac{(g)}{S}=\frac{\mathfrak{p}\cdot\mathfrak{q}}{S}=\frac{\mathfrak{p}}{S}=\frac{\mathfrak{p}A}{S}=\mathfrak{p}A_{\mathfrak{p}}$$

If $\mathfrak{p}'\subseteq A$ is another prime ideal of height 1 $\rightsquigarrow Y'$ prime divisor
$\nu_{Y'}(f)=0$

$$\Rightarrow f\in A_{\mathfrak{p}'} \Rightarrow f\in \bigcap_{\substack{\text{prime ideal}\\ \text{of height 1}}} A_{\mathfrak{p}'}=A$$

$f\in A$, $f\in \mathfrak{p}A_{\mathfrak{p}} \Rightarrow f\in A\cap \mathfrak{p}A_{\mathfrak{p}}=\mathfrak{p}$

$\forall g\in\mathfrak{p} \Rightarrow \nu_Y(g)\geq 1$, $\nu_{Y'}(g)\geq 0$ for $Y'\neq Y$

$$\Rightarrow \left.\begin{array}{l}\nu_{Y'}(\frac{g}{f})=\nu_{Y'}(g)-\nu_{Y'}(f)\geq 0\\ \nu_Y(\frac{g}{g})=\nu_Y(g)-\nu_Y(f)\geq 0\end{array}\right\}\ \nu_{Y'}(\frac{g}{f})\geq 0 \quad \forall Y' \text{ prime divisor}$$

$$\Rightarrow \frac{g}{f}\in A_{\mathfrak{p}'} \Rightarrow \frac{g}{f}\in \bigcap A_{\mathfrak{p}'}=A \Rightarrow g\in Af=(f).$$

Prop: $X=\mathbb{P}^n_k$, k a field. For any divisor $D=\sum n_iY_i$, define the degree of D by $\deg D:=\sum n_i\deg Y_i$, where $\deg Y_i$ is the degree of the hypersurface Y_i. Let H be the hyperplane $x_0=0$. Then

(a) if D is any divisor of $\deg = d$, then $D\sim dH$;

(b) $\forall f\in K^*$, $\deg(\operatorname{div} f)=0$

(c) $\deg: Cl(X)\longrightarrow \mathbb{Z}$ is an isomorphism.

Proof: $S=k[x_0,\cdots,x_n]$ coordinate ring of X.

$g\in S$ homogeneous of $\deg = d \Rightarrow g=g_1^{n_1}\cdots g_r^{n_r}$, g_i irr. polys

$\Rightarrow g_i$ defines a hypersurface Y_i of $\deg$ $d_i=\deg g_i$

$\Rightarrow \operatorname{div} g=\sum n_iY_i \Rightarrow \deg(\operatorname{div} g)=d$

$\mathcal{O}_{X,\eta}=S_{(\eta)}=\{\frac{g}{h} \mid g\in S, h\notin \mathfrak{p}_\eta, \deg g=\deg h\}$

$\forall f\in K^*$, $f=\frac{g}{h}$, g,h are of the same degree

$\Rightarrow \operatorname{div} f=\operatorname{div} g-\operatorname{div} h \Rightarrow \deg(\operatorname{div} f)=0. \Rightarrow$ (b) ✓

Suppose that D is a divisor of $\deg = d$, we write $D = D_1 - D_2$ of effective divisors of degrees d_1, d_2 with $d_1 - d_2 = d$.

Let $D_1 = \operatorname{div} g_1$, $D_2 = \operatorname{div} g_2$.

irr. hyperplane in $\mathbb{P}^n$ $\leftrightsquigarrow$ homogeneous prime ideal of height 1.

$$D_1 = \sum n_i Y_i \rightsquigarrow Y \Rightarrow \ell_i \Rightarrow g_1 = \prod_i \ell_i^{n_i}$$

consider $f = \dfrac{g_1}{x_0^d g_2}$ $\Rightarrow \operatorname{div} f = \operatorname{div} g_1 - dH - \operatorname{div} g_2$

i.e. $\operatorname{div} f = D - dH$

$\Rightarrow D \sim dH$. (a) ✓

7. Divisors — Cartier divisors

Recall: • $\mathrm{Div}(X)$ = abelian group generated by $\underset{\sim\sim\sim}{\text{prime divisors}}$ ↳ closed integral subscheme

• $D_1 \sim D_2$ linear equivalent if $D_1 - D_2 = \mathrm{div}(f)$, $f \in K^*$, $K = \mathrm{Frac}\,\mathcal{O}_{X,\eta}$, η is the generic point

• $\mathrm{div}(f) := \sum\limits_{\substack{Y \\ \text{prime divisors}}} \nu_Y(f) \cdot Y$

Prop: Let X satisfy $(*)$, let Z be a proper closed subset of X and $U = X \setminus Z$.

Then

(a) $\exists$ surjective morphism $Cl(X) \longrightarrow Cl(U)$ defined by

$$\sum n_i Y_i \longmapsto \sum n_i (Y_i \cap U)$$

where we ignore those empty $Y_i \cap U$

(b) if $\mathrm{codim}\, Z \geq 2$, then $Cl(X) \longrightarrow Cl(U)$ is an isomorphism

(c) if Z is an irreducible subset of codim $=1$, then $\exists$ exact sequence

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & Cl(X) & \longrightarrow & Cl(U) & \longrightarrow & 0 \\ 1 & \longmapsto & 1 \cdot Z & & \| & & \\ & & & & Cl(\mathbb{P}^2_k \setminus Y) & & \end{array}$$

e.g. Y: irreducible curve of degree d in $\mathbb{P}^2_k$. Then $Cl(\mathbb{P}^2_k \setminus Y) \cong \mathbb{Z}/d\mathbb{Z}$.

Prop: Let X satisfy $(*)$. Then $X \times \mathbb{A}^1$ also satisfy $(*)$ and $Cl(X) \cong Cl(X \times \mathbb{A}^1)$.

Cartier Divisors

Def: X: a scheme, $U = \mathrm{Spec}\, A$ an open affine subset. $S := \{\text{nonzero divisors}\}$. We define

• $K(U) := S^{-1}A$. We call $K(U)$ the total quotient ring of A.

• $\forall$ open subset U, let $S(U)$ denote the set of elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for $x \in U$. Then $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ is a presheaf, whose associated sheaf of rings $\mathcal{K}$ we call the sheaf of total quotient rings of $\mathcal{O}$

• We denote by $\mathcal{K}^*$ the sheaf of invertible elements in the sheaf of ring $\mathcal{K}$.

Def: A Cartier divisor on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$.

A Cartier divisor is principal if it is the image of the natural map $\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Two Cartier divisors are linear equivalent if their difference is principal.

Remark: A Cartier divisor on X can be described by giving an open cover $\{U_i\}_{i\in I}$ of X and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$ s.t. $\forall\, i,j,\ \frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{K}^*/\mathcal{O}^*)$.

$$0 \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*) \longrightarrow \prod_{i\in I} \Gamma(U_i, \mathcal{K}^*/\mathcal{O}^*) \rightrightarrows \prod_{i,j\in I} \Gamma(U_i\cap U_j, \mathcal{K}^*/\mathcal{O}^*)$$

Prop: Let X be an integral, separated noetherian scheme, all of whose local rings are UFD. Then $\mathrm{Div}(X) \cong \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$. Furthermore, the principal Weil divisors correspond to the principal Cartier divisors.

Proof: local rings are UFD $\Rightarrow X$ is normal $\Rightarrow X$ is regular in codim $=1$
$\Rightarrow$ Weil divisors make sense.

X is integral $\Rightarrow \mathcal{K}$ is just the constant sheaf corresponding to the function field K of X

$U \longmapsto \mathrm{Frac}\,\Gamma(U, \mathcal{O}_X) \rightsquigarrow$ constant sheaf K

Cartier divisor: $\{(U_i, f_i)\}_{i\in I}$, $f_i \in \Gamma(U_i, \mathcal{K}^*) = K^*$, $\frac{f_i}{f_j} \in \Gamma(U_i\cap U_j, \mathcal{K}^*/\mathcal{O}^*)$

We define a Weil divisor by
$$D = \sum_Y \nu_Y(f_i)\, Y$$

Suppose that $\nu_Y(f_j)$ is another coefficient. $\nu_Y\left(\frac{f_i}{f_j}\right) = \nu_Y(f_i) - \nu_Y(f_j) = 0$

$\Rightarrow D$ is well-defined.

$$\boxed{\begin{array}{l} \nu : R \longrightarrow \mathbb{N} \\ \quad\big\downarrow \\ \nu : \mathrm{Frac}\,R \longrightarrow \mathbb{Z} \end{array}}$$

Conversely, if D is a Weil divisor on X, let $x\in X$ be a point.

$D \rightsquigarrow$ a Weil divisor D_x on the scheme $\mathrm{Spec}\,\mathcal{O}_{X,x}$.

$\mathcal{O}_{X,x}$ is a UFD $\Rightarrow \mathrm{Cl}(\mathrm{Spec}\,\mathcal{O}_{X,x}) = 0$ i.e. D_x is principal

$\Rightarrow \exists\, f_x \in K$ s.t. $D_x = \mathrm{div}(f_x)$ on $\mathrm{Spec}\,\mathcal{O}_{X,x}$

$\therefore \mathrm{div}(f_x)$ on X has the same restriction to $\mathrm{Spec}\,\mathcal{O}_{X,x}$ as D

$\therefore \{(\operatorname{Spec} \mathcal{O}_{X,x}, f_x)\}_{x\in X}$ gives a Cartier divisor on X.

$\frac{f_x}{f_{x'}} \in \Gamma(\operatorname{Spec}\mathcal{O}_x \cap \operatorname{Spec}\mathcal{O}_{x'}, \mathcal{K}^*/\mathcal{O}^*)$

If f'_x, f_x give the same Weil divisor D_x on $\operatorname{Spec}\mathcal{O}_{X,x} =: U$, then $\operatorname{div}\left(\frac{f_x}{f'_x}\right) = 0$ i.e. $\frac{f_x}{f'_x}$ is invertible $\Rightarrow \frac{f_x}{f'_x} \in \Gamma(U, \mathcal{O}^*)$

$\Rightarrow$ the Cartier divisor is well-defined.

Recall: An invertible sheaf is a locally free $\mathcal{O}_X$-module of rank 1.

Prop: If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, so is $\mathcal{L}\otimes\mathcal{M}$.
If $\mathcal{L}$ is any invertible, then $\exists\ \mathcal{L}^{-1}$ on X s.t. $\mathcal{L}\otimes\mathcal{L}^{-1}\cong\mathcal{O}_X$.

Proof: locally $\mathcal{L}\otimes\mathcal{M}\cong\mathcal{O}_X\otimes\mathcal{O}_X\cong\mathcal{O}_X$

$\mathcal{L}^{-1} := \mathcal{H}om(\mathcal{L},\mathcal{O}_X) \Rightarrow \mathcal{L}^{-1}\otimes\mathcal{L} = \mathcal{H}om(\mathcal{L},\mathcal{L}) = \mathcal{O}_X$

Def: $\operatorname{Pic}(X) :=$ the group of isomorphism classes of invertible sheaves on X under the operation $\otimes$.

Def: D a Cartier divisor represented by $\{(U_i, f_i)\}$. We define a subsheaf of the sheaf of total quotient rings $\mathcal{K}$ by taking $\mathcal{L}(D)$ to be the $\mathcal{O}_X$-submodule of $\mathcal{K}$ generated by f_i^{-1} on U_i. This is well defined since $\frac{f_i}{f_j}$ is invertible on $U_i\cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. We call $\mathcal{L}(D)$ the sheaf associated to D.

$\mathcal{K}(U_i) = S(U_i)^{-1}\Gamma(U_i,\mathcal{O}_X)$ is a $\Gamma(U_i,\mathcal{O}_X)$-module $\quad \Gamma(U_i,\mathcal{O}_X) = \mathcal{O}_X(U_i)$

$\mathcal{L}(D)(U_i) :=$ submodule generated by f_i^{-1}

on $U_i\cap U_j$, $\frac{f_i}{f_j}$ is invertible $\Rightarrow f_i^{-1}, f_j^{-1}$ generate the same module

Prop: X a scheme. Then
(a) $\forall$ Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D\mapsto\mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$.

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

(c) $D_1 \sim D_2$ iff $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$

Proof: (a) $f_i \in \Gamma(U_i, \mathcal{K}^*) \Rightarrow \mathcal{O}_{U_i} \longrightarrow \mathcal{L}(D)|_{U_i}$, $1 \longmapsto f_i^{-1}$ is an isomorphism

$\Rightarrow \mathcal{L}(D)$ invertible

D can be recovered from $\mathcal{L}(D)$ together with its embedding in $\mathcal{K}$ by taking f_i on U_i to be the inverse of a local generator of $\mathcal{L}(D)$. For any subsheaf of $\mathcal{K}$, this construction gives a Cartier divisor.

(b). D_1 locally defined by f_i, D_2 locally defined by g_i, then $\mathcal{L}(D_1 - D_2)$ is locally generated by $f_i^{-1} g_i$ so $\mathcal{L}(D_1 - D_2)(U_i) = \mathcal{L}(D_1)(U_i)\,\mathcal{L}(D_2)(U_i)^{-1}$

$\cong \mathcal{L}(D_1)(U_i) \otimes \mathcal{L}(D_2)(U_i)^{-1}$

$\Rightarrow \mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

(c) (b) $\Rightarrow$ suffices to show $D = D_1 - D_2$ is principal iff $\mathcal{L}(D) \cong \mathcal{O}_X$

D principal defined by $f \in \Gamma(X, \mathcal{K}^*) \Leftrightarrow \mathcal{L}(D)$ is globally generated by f^{-1}.

$(\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*))$

So $\mathcal{O}_X \cong \mathcal{L}(D)$, $1 \longmapsto f^{-1}$

Corollary: On any scheme X, the map $D \longmapsto \mathcal{L}(D)$ gives an injective hom.

$\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}\,X$

↳ Cartier class group

Prop: If X is an integral scheme, then $\mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$, $D \longmapsto \mathcal{L}(D)$

Proof: show: every invertible sheaf is isomorphic to a subsheaf of $\mathcal{K}$.

$\mathcal{L}$ invertible. Consider $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$.

On any open U, $\mathcal{L} \cong \mathcal{O}_X \Rightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K} \cong \mathcal{K}$ is a constant sheaf on U

X irreducible $\Rightarrow$ any sheaf whose restriction to each open set of a covering of X is constant is constant

$\Rightarrow \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$ on X

$\Rightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$

Corollary: If X is a noetherian, integral, separated locally factorial scheme, then

$$Cl(X) \cong \mathrm{Pic}\, X \cong \mathrm{CaCl}(X).$$

Corollary: If $X = \mathbb{P}^n_k$, then every invertible sheaf on $X \cong \mathcal{O}(\ell)$ for some $\ell \in \mathbb{Z}$.