Chapter 1: The Foundations: Logic and Proofs

Learning Objectives

After studying this topic, you should be able to:

1. identify the propositional logic
2. apply the propositional logic in digital logic circuit
3. apply rules of inferences in discrete mathematics.

1.1 Propositional Logic

- Proposition is a declarative sentence that is either true or false but not both.
- The truth value of a proposition is true (T) and the truth value of a proposition is false (F).

Example:

Proposition	Not Proposition
Toronto is the capital of Canada	What time is it?
$2 + 2 = 3$	$x + 1 = 2$

symbol of negation

- Let p be a proposition. The negation of p, denoted by $\neg p (\bar{p})$ is the statement "it is not the case that p". The proposition $\neg p$ is read "not p".
- The truth table for the negation of a proposition p:

p	$\neg p$
T	F
F	T

Example:

Find the negation of the proposition "Michael's PC runs Linux" and express this in simple English.

Answer: Michael's PC is not runs Linux.

- The logical operators that are used to form new propositions from two or more existing propositions. These logical operators are called connectives.

Logical operators

conjunction — and — $\wedge$

disjunction — or — $\vee$

Conjunction	Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$ is the proposition "p and q".
Disjunction	Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$ is the proposition "p or q". Disjunct has two types which are inclusive or and exclusive or.

Truth Table					
Conjunction			Disjunction		
p	q	$p \wedge q$	p	q	$p \vee q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	F	T	T
F	F	F	F	F	F

Example:

Find the conjunction and disjunction of the propositions p and q where p is the proposition "Rebecca's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Rebecca's PC runs faster than 1 GHz."

- Conjunction: (p) and (q)
- Disjunction: (p) or (q)

$p \to q$: if p, then q
hypothesis conclusion

- Conditional statements – Let p and q be propositions. The conditional statement $p \to q$ is the proposition "if p, then q". The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the hypothesis and q is the conclusion.

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	

- The statement $p \to q$ is true when both p and q are true and when p is false.

The truth table for the conditional statement $p \rightarrow q$		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \rightarrow q$ as a statement in English.

Answer: If Maria learns discrete mathematics, then Maria will find a good job.

- There are three related conditional statements which are converse, contrapositive and inverse.

Converse	$q \rightarrow p$
Contrapositive	$\neg q \rightarrow \neg p$
Inverse	$\neg p \rightarrow \neg q$

Example:

What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining?" Original statement : If p, then q
q p

Answer:

Contrapositive: If the home team does not win, then it is not raining.

Converse : If the home team wins, then it is raining.

Inverse : If it is not raining, then the home team does not win.
sunny lose

- Biconditionals – let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition p if and only if q. The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise.

"p is necessary and sufficient for q"
"if p then q, and conversely"
"p iff q."

The truth table for the biconditional statement $p \leftrightarrow q$		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$\neq$ different → false

- Truth Tables of Compound Propositions – We use four important logical connectives to build up complicated compound propositions involving any number of propositional variables.

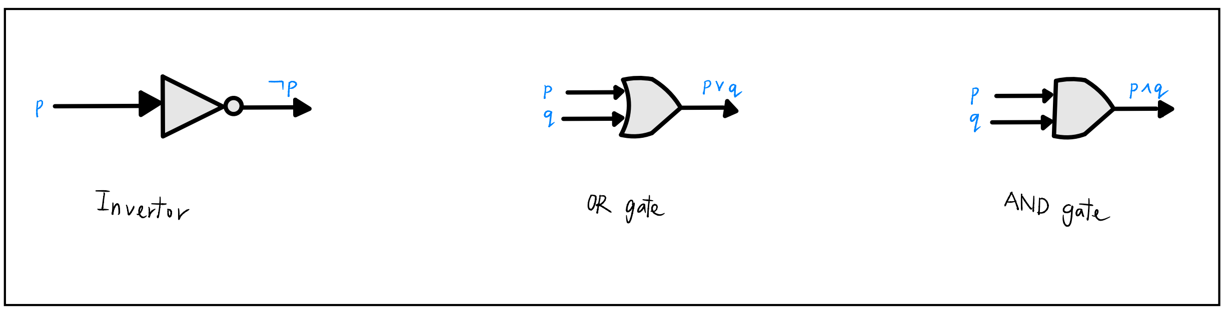
Example:

		negation	disjunction	conjunction	conditional statement
p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Construct the truth table of the compound proposition $(p \vee \neg q) \rightarrow (p \wedge q)$

1.2 Application of Propositional Logic (Digital Logic Circuit)

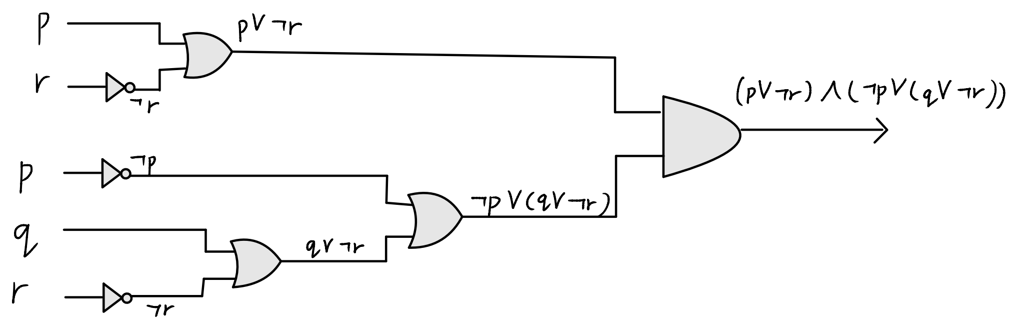
- Logic is used in the specification of software and hardware, because these specifications need to be precise before development begins.
- Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems.
- A logic circuit (or digital circuit) receives input signal $p_1, p_2, \ldots, p_n$, each a bit [either 0 (off) or 1 (on)], and produces output signals $s_1, s_2, \ldots, s_n$, each a bit.
- Complicated digital circuits can be constructed from three basic circuits, called gates.



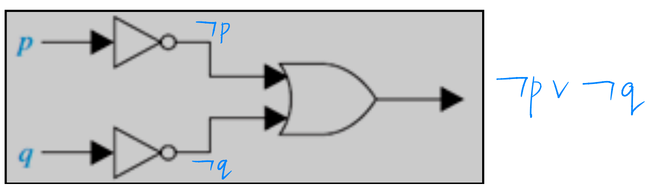
Basic logic gates

Example:

1. Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$ when given input bits p, q and r.



2. Find the output of this combinatorial circuits.



1.3 Propositional Equivalence

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.

Tautology T T	A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it.
Contradiction F F	A compound proposition that is always false.
Contingency T F	A compound proposition that is neither a tautology nor a contradiction.

- The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology . The notation $p \equiv q$ denotes that p and q are logically equivalent.

Example:

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

- There have some important equivalences in propositional equivalence. In these equivalences, T denotes the compound proposition that is always true and F denotes the compound proposition that is always false.

Showing logical equivalent

applying truth table

develop a series of logical equivalence/ logic law

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalence

Example: Prove

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalence.

Propositions	Logic laws
$\neg(p \vee (\neg p \wedge q))$	
$\equiv \neg p \wedge \neg(\neg p \wedge q)$	De Morgan's Law
$\equiv \neg p \wedge (\neg\neg p \vee \neg q)$	De Morgan's Law
$\equiv \neg p \wedge (p \vee \neg q)$	Double Negation
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive Law
$\equiv F \vee (\neg p \wedge \neg q)$	Negation Law
$\equiv (\neg p \wedge \neg q) \vee F$	Commutative Law $\leftrightarrow$
$\equiv \neg p \wedge \neg q$	Identity Law

1.4 Predicates and Quantifiers

- **Predicate** – a property that the subject of the statement can have e.g: $x > 3$, $x = y + 3$, $x + y = z$
- **Quantifier** - an expression (e.g. all, some) that indicates the scope of a term to which it is attached
  - Two types of quantifier:
    - i) universal quantifier
    - ii) existential quantifier

Universal quantifier	$P(x)$ for all values of x in the domain
Existential quantifier	There exists an element x in the domain such that $P(x)$.

1.5 Nested Quantifiers

- Nested quantifiers commonly occur in mathematics and computer science.

– Two quantifiers are interested if one is within the scope of the other

$$\underbrace{\forall x\ \overbrace{\exists y\ (x+y=0)}^{P(x,y)}}_{Q(x)}$$

Example:

1. Translate into the English statement $\overset{\text{Universal quantifier, all}}{\forall x \forall y}((x>0) \overset{\text{and}}{\wedge} (y<0) \overset{\text{if, then}}{\rightarrow} (xy<0))$, where the domain for both variables consists of all real numbers.

   For all x, y are real numbers, if $x > 0$ and $y < 0$, then $xy < 0$.
   x is greater than 0

2. Translate the statement "The sum of two positive integers is always positive" into a logical expression.

   $\forall a \forall b ((a>0) \wedge (b>0) \rightarrow (a+b>0))$

1.6 Rules of Inference

Rule of Inference	*Tautology*	*Name*
p $p \rightarrow q$ $\therefore q$	$(p \land (p \rightarrow q)) \rightarrow q$	Modus ponens
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \land (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \rightarrow (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \rightarrow p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

Rules of Inference

Example:

1. Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

2. Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

1. p – It is sunny this afternoon
q – It is colder than yesterday
r – We will go swimming
s – We will take a canoe trip
t – We will be home by sunset

Premise 1 : $\neg p \wedge q$
2 : $r \rightarrow p$
3 : $\neg r \rightarrow s$
4 : $s \rightarrow t$
Conclusion : t

	Rule of inference
$\neg p \wedge q$	
$r \rightarrow p$	
$\neg r \rightarrow s$	
$s \rightarrow t$	
$\neg r \rightarrow t$	Hypothetical syllogism of (3) and (4)
$\neg p$	Simplification of (1)
$\neg r$	Modus tollens of (6) and (2)
t	Modus ponens of (7) and (5)

2. p - You send me an e-mail message
q – I will finish writing the program
r – I will go to sleep early
s – I will wake up feeling refreshed

Premise 1 : $p \rightarrow q$
2 : $\neg p \rightarrow r$
3 : $r \rightarrow s$
Conclusion : $\neg q \rightarrow s$

	Rule of inference
$p \rightarrow q$	
$\neg p \rightarrow r$	
$r \rightarrow s$	
$\neg p \rightarrow s$	Hypothetical syllogism of (2) and (3)
$\neg q \rightarrow \neg p$	Contrapositive of (1)
$\neg q \rightarrow s$	Hypothetical syllogism of (5) and (4)

Tutorial Questions.

1.1 Propositional Logic

1. Which of these sentences are propositions? What are the truth values of those that are propositions?
a) Boston is the capital of Massachusetts. Yes, truth
b) Miami is the capital of Florida. Yes
c) $2 + 3 = 5$. Yes, true
d) $5 + 7 = 10$. Yes, false
e) $x + 2 = 11$. No
f) Answer this question. No

2. What is the negation of each of these propositions?
a) Mei has an MP3 player.
b) There is no pollution in New Jersey.
c) $2 + 1 = 3$
d) The summer in Maine is hot and sunny.
e) Jennifer and Teja are friends.
f) 121 is a perfect square.

3. Suppose that during the most recent fiscal year, the annual revenue of Acme Computer was 138 billion dollars, and its net profit was 8 billion dollars, the annual revenue of Nadir software was 87 billion dollars and its net profit was 5 billion dollars, and the annual revenue of Quixote Media was 111 billion dollars and its net profit was 13 billion dollars. Determine the truth value of each propositions for the most recent fiscal year.
a) Quixote Media has the largest annual revenue.
b) Nadir software has the lowest net profit and Acme Computer had the largest annual revenue.
c) Acme Computer had the largest net profit or Quixote Media has the largest annual revenue.
d) If Quixote Media has the smallest net profit, then Acme Computer had the largest annual revenue.
e) Nadir Software had the smallest net profit if and only if Acme Computer had the largest annual revenue.

4. Let p and q be the propositions "Swimming at the New Jersey shore is allowed" and "Sharks have been spotted near the shore," respectively. Express each of these compound propositions as an English sentence.
a) $\neg q$
b) $p \wedge q$
c) $\neg p \vee q$
d) $p \rightarrow \neg q$
e) $\neg q \rightarrow p$
f) $\neg p \rightarrow \neg q$
g) $p \leftrightarrow \neg q$
h) $\neg p \wedge (p \vee \neg q)$

5. Let p and q be the propositions.
p : It is below freezing.
q : It is snowing.

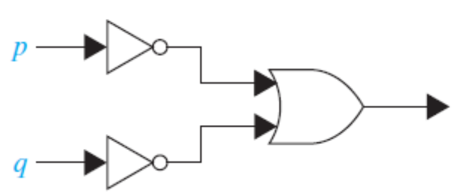
Write the propositions using p and q and logical connectives (including negations).
a) It is below freezing and snowing.
b) It is below freezing and not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both)
e) If it is below freezing, it is also snowing.
f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

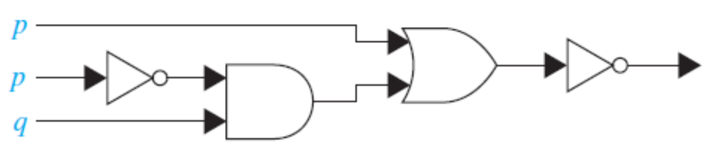
6. Write each of these statements in the form "if p, then q" in English. [Hint: Refer to the list of common ways to express conditional statements provided in this section].
a) It is necessary to wash the boss's car to get promoted.
b) Winds from the south imply a spring thaw.
c) A sufficient condition for the warranty to be good is that you bought the computer less than a year ago.
d) Willy gets caught whenever he cheats.
e) You can access the website only if you pay a subscription fee.
f) Getting elected follows from knowing the right people.
g) Carol gets seasick whenever she is on a boat.

7. Construct a truth table for each of these compound propositions.
a) $p \wedge \neg q$
b) $p \vee \neg q$
c) $(p \vee \neg q) \rightarrow q$
d) $(p \vee q) \rightarrow (p \wedge q)$
e) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
f) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

1.2 Application of Propositional Logic (Digital Logic Circuit)

1. Find the output of each of these combinatorial circuit.

a)


b)


2. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $((\neg p \vee \neg r) \wedge \neg q) \vee (\neg p \wedge (q \vee r))$ from input bits p, q and r.

1.3 Propositional Equivalences.

1. Use De Morgan's Law to find the negation of each of the following statements.
a) Jan is rich and happy.
b) Carlos will bicycle or run tomorrow.
c) Mei walks or takes the bus to class.
d) Ibrahim is smart and hard working.
e) James is young and strong.

2. Show that each of these conditional statements is tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$
b) $p \rightarrow (p \vee q)$
c) $\neg p \rightarrow (p \rightarrow q)$
d) $(p \wedge q) \rightarrow (p \rightarrow q)$
e) $\neg(p \rightarrow q) \rightarrow p$
f) $\neg(p \rightarrow q) \rightarrow \neg q$

3. Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

4. Show that $\neg p \leftrightarrow q$ and $p \leftrightarrow \neg q$ are logically equivalent.

5. Show that $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ are logically equivalent.

6. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.

1.4 Predicates and Quantifiers

1. Let P(x) be the statement "x spends more than five hours every weekday in class," where the domain x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$
b) $\forall x P(x)$
c) $\exists x \neg P(x)$
d) $\forall x \neg P(x)$

2. Translate these statements into English, where C(x) is "x is a comedian" and F(x) is "x is funny" and the domain consists of all people.

a) $\forall x (C(x) \rightarrow F(x))$
b) $\forall x (C(x) \wedge F(x))$
c) $\exists x (C(x) \rightarrow F(x))$
d) $\exists x (C(x) \wedge F(x))$

3. Let P(x) be the statement "x can speak Russian" and let Q(x) be the statement "x knows the computer language C++". Express each of these sentences in terms of P(x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.
b) There is a student at your school who can speak Russian but who doesn't know C++.
c) Every student at your school either can speak Russian or knows C++.
d) No students at your school can speak Russian or knows C++.

4. Translate in two ways each of these statements into logical expressions using predicates, quantifiers and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.
a) Someone in your class can speak Hindi.
b) Everyone in your class is friendly.
c) There is a person in your class who was not born in California.
d) A student in your class has been in a movie.
e) No student in your class has taken a course in logic programming.

1.5 Nested Quantifiers.

1. Let Q(x, y) be the statement “x has sent an e-mail message to y,” where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

a) $\exists x \exists y Q(x, y)$ There is some(z) who(a) some(z)
b) $\exists x \forall y Q(x, y)$
c) $\forall x \exists y Q(x, y)$
d) $\exists y \forall x Q(x, y)$ There is a(z) who has been (a) some
e) $\forall y \exists x Q(x, y)$
f) $\forall x \wedge y Q(x, y)$

1.6 Rules of Inferences.

1. What rule of inference is used of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

2. Use rule of inference to show that the hypotheses “Randy works hard,” “If Randy works hard, then he is a dull boy,” and “If Randy is a dull boy, then he will not get the job” imply the conclusion “Randy will not get the job.”