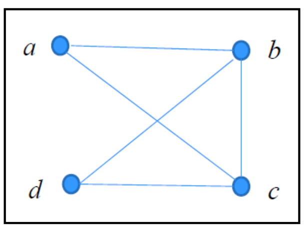
Chapter 7: Graph and Trees

Learning Objectives
After studying this topic, you should be able to:

1. Understand the concept of graph and trees.
2. Distinguish between directed graph and undirected graph.
3. Calculate the minimum spanning trees.
4. Apply Kruskal's Algorithm in trees.

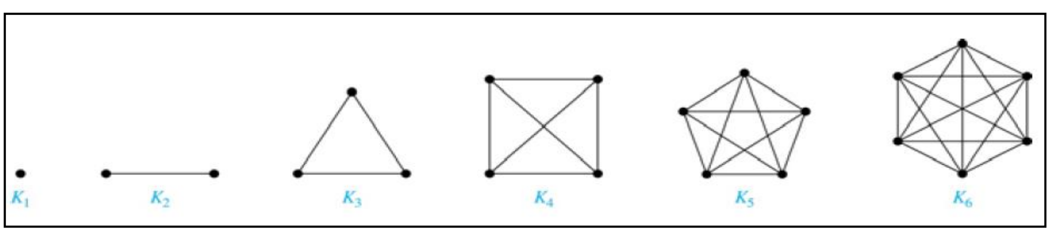
7.1 Introduction to Graph

- Definition: A graph G = (V, E) consists of nonempty set V of vertices (or nodes) and a set E of edges.
- Each edge has either one or two vertices associated with it, called its endpoints.
- An edge is said to connect its endpoints.



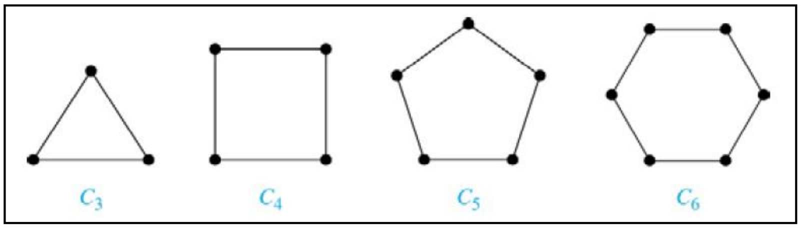
Example of Graph

- In a simple graph edge connects two different vertices and no two edges connect the same pair of vertices.
- Multigraphs may have multiple edges connecting the same two vertices. When m different edges connect the vertices u and v, we say that {u, v} is an edge of multiplicity m.
- An edge that connects a vertex to itself is called a loop.
- A pseudograph may include loops, as well as multiple edges connecting the same pair of vertices.
- A complete graph on n vertices, denoted by K_n is the simple graph that contains exactly one edge between each pair of distinct vertices.



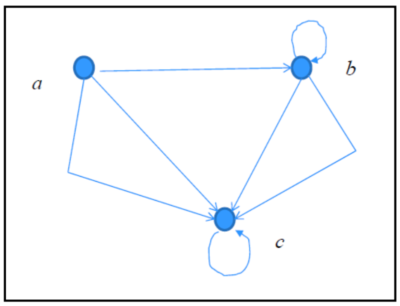
Complete Graph

- A cycle C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

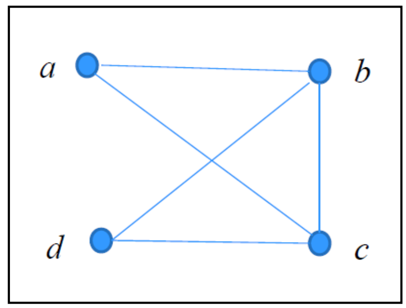


Cycle

- Graph has two basic types which are directed graph and undirected graph.



Directed Graph



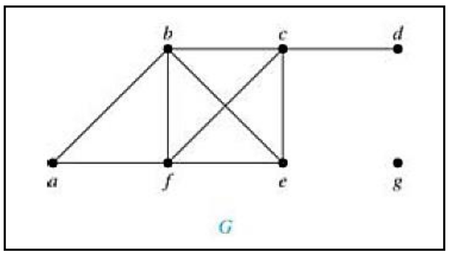
Undirected Graph

7.2 Undirected Graph

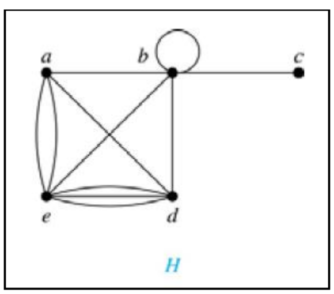
- Two vertices u, v in an undirected graph G are called adjacent (or neighbours) in G if there is an edge e between u and v. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.
- The set of all neighbours of a vertex v of $G = (V, E)$, denoted by N(v) is called the neighbourhood of v. If A is a subset of V, we denote by N(A) the set of all vertices in G that are adjacent to at least one vertex in A.
- The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

Example:

1. What are the degrees and neighbourhoods of the vertices in the graph G?



2. What are the degrees and neighbourhoods of the vertices in the graph H?

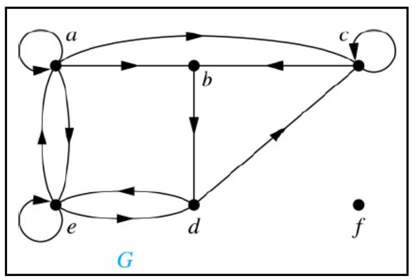


7.3 Directed Graph

- A directed graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes), and E, a set of directed edges or arcs. Each edge is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.
- Let (u, v) be an edge in G. Then u is the initial vertex of this edge and is adjacent to v and v is the terminal (or end) vertex of this edge and is adjacent from u. The initial and terminal vertices of a loop are the same.
- The in-degree if a vertex v, denoted $\deg^{-}(v)$ is the number of edges which terminate at v. The out-degree of v, is denoted $\deg^{+}(v)$, is the number of edges with v as their initial vertex. Note that a loop at vertex contributes 1 to both in-degree and the out-degree of the vertex.

Example:

Assume graph G:

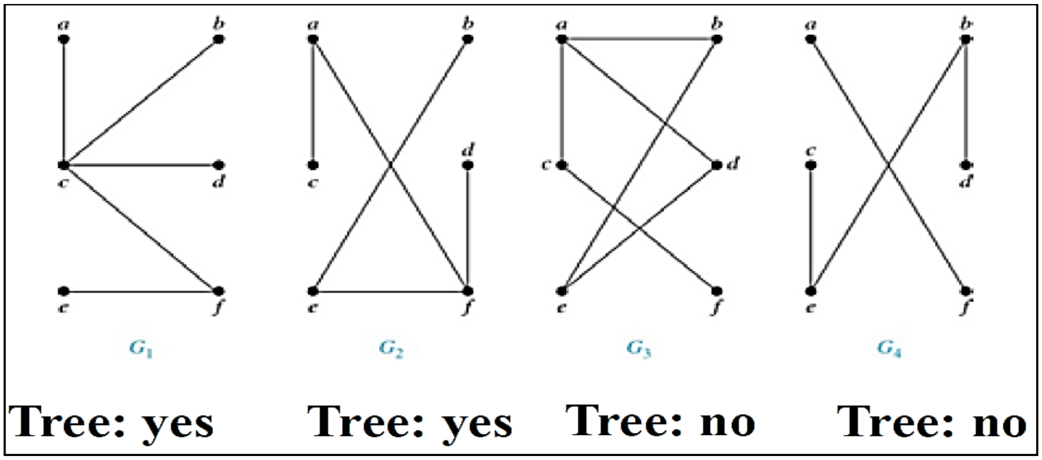


a) What are in-degrees of vertices?

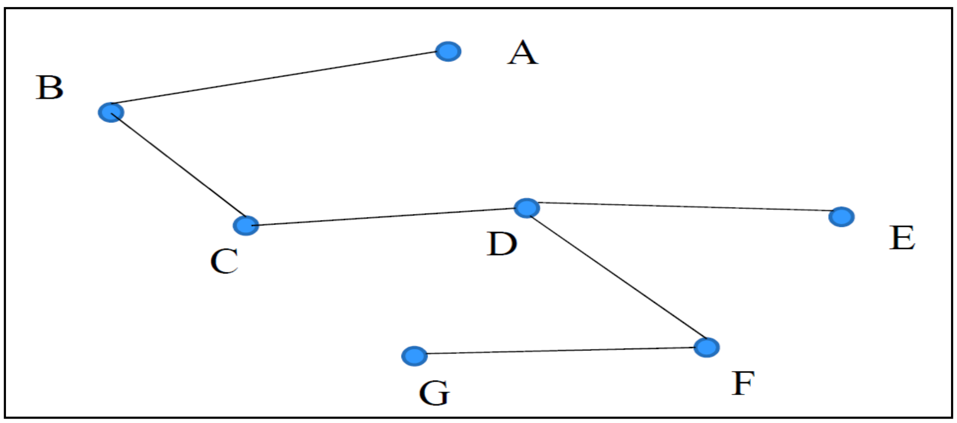
b) What are out-degrees of vertices?

7.4 Introduction to Trees

- Definition: A tree is a connected undirected graph with no cycles.



- An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.



- One of the applications of trees is the organization of a computer file system into directories, subdirectories and files are naturally represented as a tree.

7.5 Application of Trees (Minimum Spanning Trees)

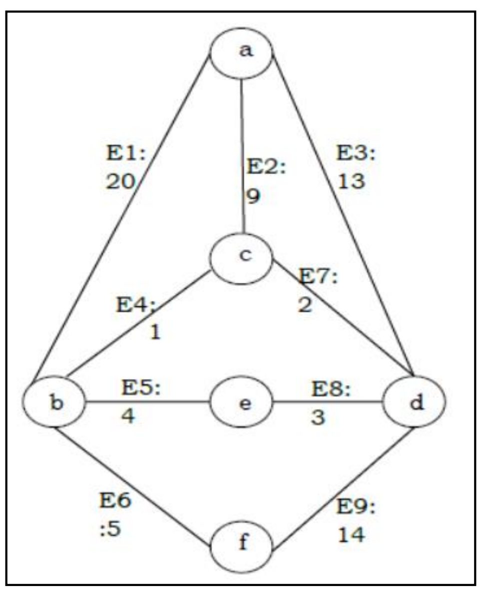
- A spanning tree of a connected undirected graph G is a tree that minimally includes all of the vertices of G. A graph may have many spanning trees.
- Definition: Minimum Spanning Tree (MST) is a spanning tree with assigned weight less than or equal to the weight of every possible spanning tree of a weighted, connected and undirected graph G.
- The weight of a spanning tree is the sum of all the weights assigned to each edge of the spanning tree.
- One of the method to calculate MST is by applying Kruskal's Algorithm.

<u>Kruskal's Algorithm</u>

- Definition: A greedy algorithm that finds a minimum spanning tress for a connected weighted graph.
- It finds a tress of that graph which includes every vertex and the total weight of all edges in the tree is less than or equal to every possible spanning tree.
- Algorithm:
  1. Arrange all the edges of the given graph G(V, E) in non-decreasing order as per their edge weight.
  2. Choose the smallest weighted edge from the graph and check if it forms a cycle with spanning tree formed so far.
  3. If there is no cycle, include this edge to the spanning tree else discard it.
  4. Repeat Step 2 and Step 3 until (V – 1) number of edges are left in the spanning tree.

Example:

Find the minimum spanning tree for the following G using Kruskal's algorithm.



Solution:

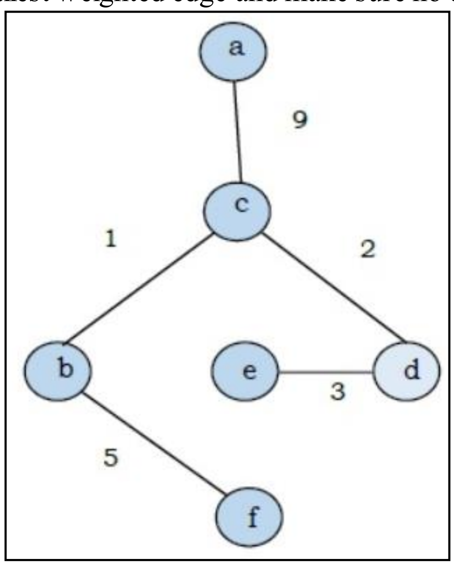
Construct the following table

Edge No.	Vertex Pair	Edge Weight
E1	(a, b)	20
E2	(a, c)	9
E3	(a, d)	13
E4	(b, c)	1
E5	(b, e)	4
E6	(b, f)	5
E7	(c, d)	2
E8	(d, e)	3
E9	(d, f)	14

Rearrange the table in ascending order with respect to edge weight:

Edge No.	Vertex Pair	Edge Weight
E4	(b, c)	1
E7	(c, d)	2
E8	(d, e)	3
E5	(b, e)	4
E6	(b, f)	5
E2	(a, c)	9
E3	(a, d)	13
E9	(d, f)	14
E1	(a, b)	20

Choose the smallest weighted edge and make sure no cycle is formed.



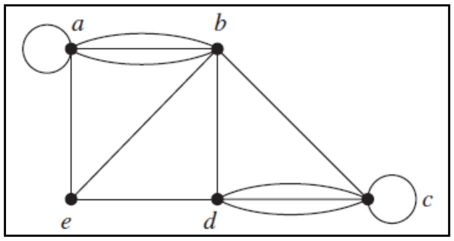
Thus this is the minimal spanning tree.

Total weight = 1 + 2 + 3 + 5 + 9 = 20

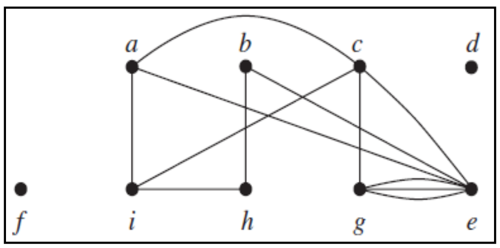
Tutorial Questions.

1. Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph.

a)

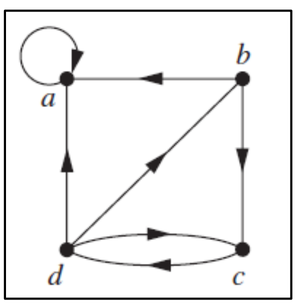


b)

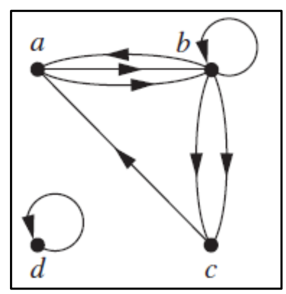


2. Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

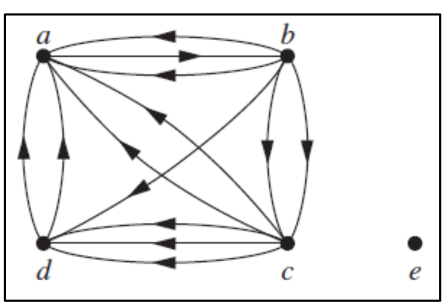
a)



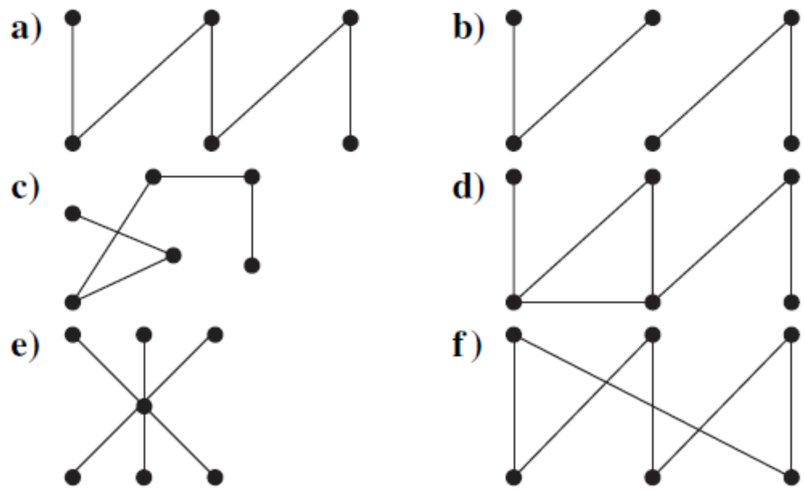
c)



b)

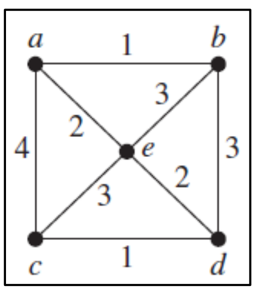


3. Determine either these graphs are trees or not.



4. Find a minimum spanning tress for the given weighted graph by using Kruskal's algorithm.

a)



b)