## Chapter 4: Counting

### Learning Objectives

After studying this topic, you should be able to:
1. Describe the concept of basic counting
2. Apply the concept of basic counting in permutations and combinations.
3. Evaluate the binomial coefficients.
4. Find the binomial expansion.

### 4.1 The basic of counting

- Two basic counting principles are the product rule and the sum rule.
- Product rule: Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$, ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

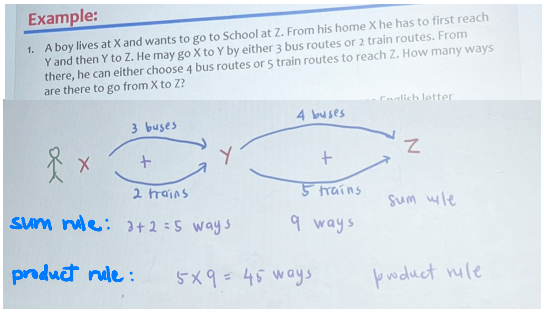
**Example:**

1. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

   A1 . . . . A100 $26 \times 100 = 2600$
   B1 . . . . B100
   ⋮
   Z1 . . . . Z100

2. There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

   $32 \times 24 = 768$ ports to a microcomputer



## 4.2 Permutations and combinations

| Permutations | Combinations |
|---|---|
| - The number of ways to arrange things<br>- Order matters<br>- These are for lists<br>- Arranging people, digits, numbers, alphabets, letters, colours<br>- ${}^{n}P_{r}$ (n: number, r: position) ${}^{5}P_{3} = 5 \times 4 \times 3$ 1, 2, 3, 4, 5 | - The number of ways to choose things<br>- Order doesn't matter<br>- These are for groups<br>- Selection of menu, food, clothes, subjects, teams<br>- ${}^{n}C_{r}$ |

**Example:**

1. In how many different ways can the letters of the word 'LEADING' be arranged in such a way that the vowels always come together?
   ① A D L E I N G: $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = {}^{7}P_{7} = 5040$
   ② AEI: $\_\ \_\ \_\ \_\ \_$ × A E I: ${}^{5}P_{5} \times {}^{3}P_{3} = 720$
2. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?
   3M+2W: $({}^{7}C_{3} \times {}^{6}C_{2}) = 525$ + 4M+1W: $({}^{7}C_{4} \times {}^{6}C_{1}) = 210$ + 5M+0W: $({}^{7}C_{5} \times {}^{6}C_{0}) = 21$ = 756 ways
3. In how many ways a committee, consisting of 5 men and 6 women can be formed from 8 men and 10 women?
   M × W: ${}^{8}C_{5} \times {}^{10}C_{6} = 11760$
4. In how many different ways can the letters of the word 'DETAIL' be arranged in such a way that the vowels occupy only the odd positions?
   V C V C V C: ${}^{3}P_{3} \times {}^{3}P_{3} = 6 \times 6 = 36$ ways
5. How many ways are there to select five players from a 10-member tennis team to make a trip to match at another school?
   ${}^{10}C_{5} = 252$

Exercise:

1. 'MATHEMATICS'
   (A A E I) M T H M A T C S
   1 2 3 4 5 6 7 8
   $\frac{{}^{8}P_{8}}{{}^{2}P_{2} \cdot {}^{2}P_{2}} \times \frac{{}^{4}P_{4}}{{}^{2}P_{2}} = \frac{40320}{4} \times \frac{24}{2}$ (M, T, A)
   $= 10080 \times 12$
   $= 120960$ ways

2. at most 3 men
   0M+5W: $({}^{7}C_{0} \times {}^{6}C_{5})$ + 1M+4W: $({}^{7}C_{1} \times {}^{6}C_{4})$ + 2M+3W: $({}^{7}C_{2} \times {}^{6}C_{3})$ + 3M+2W: $({}^{7}C_{3} \times {}^{6}C_{2})$
   $= 6 + 105 + 420 + 525$
   $= 1056$ ways

## 4.3 Binomial Expression and Binomial Coefficients

expand → apply binomial theorem

- A *binomial* expression is the sum of two terms, such as $x+y$. (More generally, these terms can be products of constants and variables.)
- The number *r*-combinations from a set with n elements is often denoted by $\binom{n}{r}$.
- This number is also called a binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$.
- The binomial theorem can be described as follow:

Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}y^n$$

or, in summation notation,

$$(x+y)^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

* final

where $\binom{n}{j}$ is called a binomial coefficient because of its role in the expansion.

- Each monomial is a length –n list of $x$'s and $y$'s.
- Coefficient of $x^{n-k}y^k$
  - Number of list having $y$ in $k$ places.
  - Number of ways to choose $k$ places from $n$ places $= \binom{n}{k}$.
- Therefore, the coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$

Example:

1. What is the expansion of $(x+y)^4$ ?

$$\sum_{j=0}^{4}\binom{4}{j}x^{4-j}y^j = \binom{4}{0}x^{4-0}y^0 + \binom{4}{1}x^{4-1}y^1 + \binom{4}{2}x^{4-2}y^2 + \binom{4}{3}x^{4-3}y^3 + \binom{4}{4}x^{4-4}y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$ ?

use power of _ at the back

$$\binom{25}{13} = \frac{25!}{12!\,13!} = 5200300$$

3. What is the coefficient $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$$\binom{25}{13}(2x)^{25-13}(-3y)^{13} = \binom{25}{13}2^{12}(-13)^{13}x^{12}y^{13} = {}^{25}C_{13}\cdot 2^{12}\cdot 13^{13}x^{12}y^{13}$$



4. $(x-y^2)^4 = \sum_{j=0}^{4}\binom{4}{j}x^{4-j}(-y^2)^j$
$= \binom{4}{0}x^{4-0}(-y^2)^0 + \binom{4}{1}x^{4-1}(-y^2)^1 + \binom{4}{2}x^{4-2}(-y^2)^2 + \binom{4}{3}x^{4-3}(-y^2)^3 + \binom{4}{4}x^{4-4}(-y^2)^4$
$= x^4 + 4x^3(-y^2) + 6x^2(y^4) + 4x^1(-y^6) + y^8$
$= x^4 - 4x^3y^2 + 6x^2y^4 - 4xy^6 + y^8$

13. Coefficient $x^2y^{20} \rightarrow (x-2y^4)^7$

$= \binom{7}{5}(x)^{7-5}(-2y^4)^5$

$= {}^7C_5(1)^2(-2)^5x^2y^{20}$

$= 21(-32)x^2y^{20}$

$= -672x^2y^{20}$

17. Coefficient of $x^2y^3$ in expansion of $(x^3-3y)^4$

$= \binom{4}{3}(x^3)^{4-3}(-3y)^3$

$= 4C_3(1)^2(-3)^3x^2y^3$

$= 4 \cdot 1 \cdot (-27)x^2y^3$

$= -108x^2y^3$

**Tutorial Questions.**

1. There are 18 mathematics majors and 325 computer science majors at a college.
   a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?
   b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.
   a) In how many ways can a student answer the questions on the test if the student answers every question?
   b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick airline for the flight to Denver and an airline for the continuation flight to San Francisco?

6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

10. Find the value of each of these quantities.
    a) P(6,3)
    b) P(6,5)
    c) P(8,1)
    d) P(8,5)
    e) P(8,8)
    f) P(10,9)

11. Find the value of each of these quantities.
    a) $C(5,1) = {}^5C_1 = 5$
    b) C(5,3)
    c) C(8,4)
    d) C(8,8)
    e) C(8,0)
    e) C(12,6)

12. Find the number of 5-permutations of a set with nine elements.

13. In how many different orders can five runners finish a race if no ties are allowed?

14. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

15. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

16. In how many ways can be a set of five letters be selected from the English alphabet?

17. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [Hint: First position the men and then consider possible positions for the women.]

18. A club has 25 members.
    a. How many ways are there to choose four members of the club to serve on an executive committee?

    b. How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

19. Find the expansion of $(x+y)^4$ using the binomial theorem.

20. Find the expansion of $(x+y)^5$ using the binomial theorem.

21. Find the expansion of $(x+y)^6$ using the binomial theorem.

22. Find the coefficient of $x^5y^8$ in $(x+y)^{13}$.

23. What is the coefficient of $x^8y^9$ in the expansion of $(3x+2y)^{17}$ ?

24. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$ ?