Chapter 6: Relations

Learning Objectives
After studying this topic, you should be able to:
1. Understand the concept of binary relation
2. Classify the relation and function
3. Identify the properties of relations.
4. Understand the combination of relations.

6.1 Binary relation

- **Definition –** Let A and B be two sets. A binary relation from A to B is a subset of Cartesian product $A\ x\ B$.
- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$.
- The notation used is a**R**b to denote (a,b) $\in R$. a**R**b is said as a is related to b by **R**.

Example:

Let A = {a,b,c} and B = {1,2,3}.

- Is R = {(a,1),(b,2),(c,2)} a relation from A to B?
- Is Q = {(1,a),(2,b)} a relation from A to B?
- Is P = {(a,a),(b,c),(b,a)} a relation from A to A?

- Relations can be represented graphically by using arrows and table to represent ordered pairs.

Example:

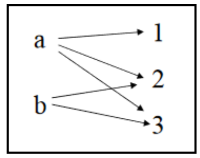
Let A = {0,1,2}, B={u,v} and R={(0,u),(0,v),(1,v),(2,u)}. Represent this relation by drawing the arrow and construct the table.

Arrow	Table

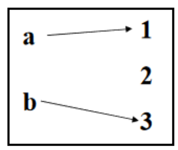
6.2 Relations and functions

- Relations represent one to many relationships between elements in A and B.

Example:



- What is the difference between a relation and a function from A to B?
- A function defined on sets A, B.
- $A \to B$ assigns to each element in the domain set A exactly one element from B. So it is a special relation.



Function

- A relation on the set A is a relation from A to itself.

Example:

1. Let A = $\{1,2,3,4\}$ and $R_{div} = \{(a,b) \mid a\, divides\, b\}$. List all the relations and represent the relations in the table.

2. Let A = $\{1,2,3,4\}$. Define $aR_{\neq}b$ if and only if $a \neq b$ and represent the relations in the table.

6.3 Properties of Relations

Reflexive

- A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$.

Example:

Let A = $\{1,2,3,4\}$ and $R_{div} = \{(a,b) \mid a\, divides\, b\}$. Is R_{div} reflexive?

Irreflexive

- A relation R on a set A is called irreflexive if $(a,a) \notin R$ for every element $a \in A$.

Example:

Let A = $\{1,2,3,4\}$. Define $aR_{\neq}b$ if and only if $a \neq b$. Is $R_{\neq}$ irreflexive?

Symmetric

- A relation R on a set A is called symmetric if $\forall a,b \in A, (a,b) \in R \rightarrow (b,a) \in R$.

Example:

1. Let A = $\{1,2,3,4\}$ and $R_{div} = \{(a,b) \mid a\, divides\, b\}$. Is R_{div} symmetric?

2. Let A = $\{1,2,3,4\}$. Define $aR_{\neq}b$ if and only if $a \neq b$. Is $R_{\neq}$ symmetric?

Transitive

- A relation R on a set A is called transitive if $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a,b,c \in A$.

Example:

1. Let A = $\{1,2,3,4\}$ and $R_{div} = \{(a,b) \mid a\, divides\, b\}$. Is R_{div} transitive?

2. Let A = $\{1,2,3,4\}$. Define $aR_{\neq}b$ if and only if $a \neq b$. Is $R_{\neq}$ transitive?

6.4 Combining Relation

- Two relations from A to B can be combined in any way because relations from A to B are subsets of $A \times B$.
- Relations can be combined via set operations of union, intersection, difference and symmetric difference.

Example:

Let A = $\{1,2,3\}$ and B= $\{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$. Find $R_1 \cup R_2, R_1 \cap R_2, R_1 - R_2$, and $R_2 - R_1$.

Tutorial Questions.

1. List the ordered pairs in the relation R from A = $\{0,1,2,3,4\}$ to B=$\{0,1,2,3\}$, where $(a,b) \in R$ if and only if

   a) a = b
   b) a + b = 4
   c) a > b
   d) a | b
   e) gcd (a,b) = 1
   f) lcm (a,b) = 2.

2. Represent each relation in Question 1 by drawing the arrow and construct the table.

3. For each of these relations on the set $\{1,2,3,4\}$, decide whether it is reflexive, whether it is symmetric, and whether it is transitive.

   a) $\{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$
   b) $\{(1,1),(1,2),(2,1)(2,2),(3,3),(4,4)\}$
   c) $\{(2,4),(4,2)\}$
   d) $\{(1,2),(2,3),(3,4)\}$
   e) $\{(1,1),(2,2),(3,3),(4,4)\}$
   f) $\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$

4. Let $R_1 = \{(1,2),(2,3),(3,4)\}$ and $R_2 = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\}$ be relations from $\{1,2,3\}$ to $\{1,2,3,4\}$. Find

   a) $R_1 \cup R_2$
   b) $R_1 \cap R_2$
   c) $R_1 - R_2$
   d) $R_2 - R_1$