Chapter 5: Discrete Probability

Learning Outcomes:
After completing this chapter, you should be able to:

1. Understand the probability theory.
2. Calculate and apply probability law and conditional probability.
3. Calculate expected mean and standard deviation for probability distribution.

5.1 Probability Theory

- Sample space – A sample space, S is a set of all possible outcomes of an experiment.
- Event – a set of outcomes which satisfy a given condition. It is a subset of the sample space.
- Probability of an event occurring is the ratio of the number of outcomes in the event to the number of outcomes in the sample space.
  - Formula – P(A) = $\frac{\text{number of outcomes in the events}}{\text{number of outcomes in the sample space}}$

    $$P(A) = \frac{n(A)}{n(S)}$$
  - Must occur between $0 \le P(A) \le 1$
    - P(A) = 1 means that event the is certain to occur.
    - P(A) = 0 means that event will not occur.
  - It can be said that sum of all probability should be equal to 1, and each probability must be ≥0.
- Let $\bar{A}$ be the event 'E does not occur' and S the sample space. Then

  $P(A) = 1 - P(\bar{A})$ **or** $P(A) + P(\bar{A}) = 1$

Example:

1. For one throw of an ordinary die, what is the possible outcomes?

   S={1,2,3,4,5,6}
   n(S)=6

   a) What are possible outcomes for odd numbers?

   n(O) = 3

   b) What are possible outcomes for numbers are less than 4?

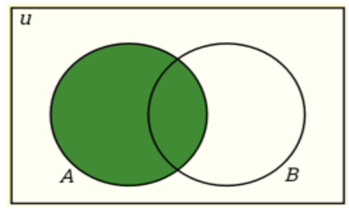
   n(A) = 4

   c) What is the probability for both events in (a) and (b)?

   $P(O) = \frac{3}{6} = \frac{1}{2}$ $P(A) = \frac{3}{6} = \frac{1}{2}$

Venn Diagram

- A pictorial representation of the relationship between sets.
- The set are represented by shapes; usually circles or ovals. The elements of a set are labeled within the circle.



Set *A*

- The set of all elements being considered is called the Universal Set (U) and is represented by a rectangle.

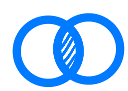
5.2 Probability Laws

5.2.1 Additive Laws

Additive Laws of Mutually inclusive Events

- If A and B are two events from an experiment with the conditions that $P(A) \neq 0$ and $P(B) \neq 0$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

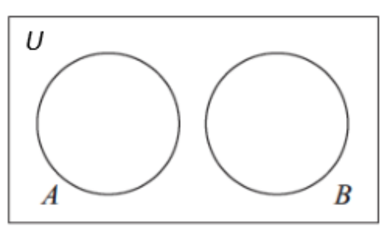


where $A \cup B$ is the event that A occurs or B occurs or both events A and B occur, while $A \cap B$ is the event that both A and B occur together.

Additive Laws of Mutually exclusive Events

- If an event A can occur or event B can occur but not both events A and B can occur, then two events A and B are said to be mutually exclusive.
- For two mutually exclusive events A and B, $A \cap B = \phi, P(A \cap B) = 0$ and

$$P(A \cup B) = P(A) + P(B)$$



Examples:

In a group of 30 students all study at least one of the subject physics and biology. 20 attend the physics class and 21 attend the biology class. Find the probability that a student chosen at random studies both physics and biology.

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\frac{30}{30} = \frac{20}{30} + \frac{21}{30} - P(A\cap B)$

$P(A\cap B) = \frac{11}{30}$

5.2.2 Multiplication Laws

- Multiplication law in probability applies to combination of events.
- When the events have to occur together, then we make use of the multiplication law of probability.
- Now two cases arise: whether the events are independent or dependent.

Independent events	When the occurrence of one event does not change the probability of occurrence of any other event with it. does not affect
Dependent events	When the occurrence of one event changes the probability of occurrence of any other events. affect

- Rules: The probability of occurrence of given two events or in other words the probability of intersection of two given events is equal to the product obtained by finding the product of the probability of occurrence of both events.

✓ $P(A\cap B) = P(A)\times P(B)$

Example:

A bag contains 3 pink candies and 7 green candies. Two candies are taken out from the bag with replacement. Find the probability that both candies are pink?

$P(A)$ = first is pink $= \frac{3}{10}$

$P(B)$ = second is pink $= \frac{3}{10}$

$P(A\cap B) = \frac{3}{10}\times\frac{3}{10} = \frac{9}{100}$

If without replacement:

$P(A) = \frac{3}{10}$

$P(B) = \frac{2}{9}$

$P(A\cap B) = \frac{3}{10}\times\frac{2}{9} = \frac{1}{15}$

5.2.3 Conditional Probability

- The conditional probability of event B is the probability that the event will occur given the knowledge that an event A has already occurred.
- If A and B are two events and P(A) ≠ 0 and P(B) ≠ 0, then probability of A, given that B has already occurred written P(A|B) and

$$P(A\mid B) = \frac{P(A\cap B)}{P(B)}$$

second event   given that   first event

Example:

When a die is thrown, an odd number occurs. What is the probability that the number is prime?

first event   second event

Odd = 1, 3, 5

Prime = 2, 3, 5

$O\cap P = \{3, 5\}$

$P(O) = \frac{3}{6}$

$P(O\cap P) = \frac{2}{6}$

$P(P\mid O) = \frac{P(O\cap P)}{P(O)} = \frac{2/6}{3/6} = \frac{2}{3}$

5.3 Probability Distribution.

- A listing of the possible values of the variable and the corresponding probabilities.
- Probability distribution is equal to 1. $\sum P(X = x) = 1$

Example:

A turn consists of throwing a dice. Tabulate the probability distribution of the score of the dice.

$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

5.3.1 Expected mean and standard deviation.

- Expected mean value of a discrete random variable

$$\mu = E(x) = \sum xp$$

score

probability

*final

- Standard deviation of a discrete random variable

$$\sigma = \sqrt{\sum p(x-\mu)^2}$$

expected mean

Example:

1. Find the mean and standard deviation which have the following probability distribution.

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

x	$P(X=x)$	xp	$P(x-\mu)^2$
0	$\frac{1}{8}$	0	$\frac{1}{8}(0-1.875)^2 = 0.439$
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}(1-1.875)^2 = 0.287$
2	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}(2-1.875)^2 = 0.002$
3	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}(3-1.875)^2 = 0.316$
4	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}(4-1.875)^2 = 0.564$

expected mean

$$\mu = \sum xp = 0 + \frac{3}{8} + \frac{1}{4} + \frac{3}{4} + \frac{1}{2} = 1.875$$

$$\sigma = \sqrt{\sum p(x-\mu)^2} = \sqrt{0.439 + 0.287 + 0.002 + 0.316 + 0.564} = \sqrt{1.608} = 1.268$$

Tutorial Questions.

1. An ordinary die is thrown. Find the probability that the number obtained

    a) is a multiple of 3
    b) is less than 7
    c) is a factor of 6

2. A counter is drawn from a box containing 10 red, 15 blacks, 5 green and 10 yellow counters. Find the probability that the counter is

    a) black
    b) not green or yellow
    c) not yellow
    d) red or black or green
    e) not blue

3. An ordinary die is thrown. Find the probability that the number obtained is

    a) even
    b) prime
    c) even or prime

4. For events A and B it is known that $P(A) = \frac{2}{3}, P(A \cup B) = \frac{7}{12}$ and $P(A \cap B) = \frac{5}{12}$. Find *P(B)*.

5. A bag has 4 white cards and 5 blue cards. We draw two cards from the bag one by one without replacement. Find the probability of getting both cards white.

6. A die is thrown twice. Find the probability of obtaining a number less than 3 on both throws.

7. Two men fire at a target. The probability that Alan hits the target is $\frac{1}{2}$ and the probability that Bob does not hit the target is $\frac{1}{3}$. Alan fires at the target first, then Bob fires at the target. Find the probability that

    a) both Alan and Bob hit the target.

    b) only one hits the targets

    c) neither hits the target

conditional probability

8. A card is picked at random from a pack of 20 cards numbered 1, 2, 3, …, 20. Given that the card shows an even number, find the probability that is a multiple of 4.

first event — second event

$P(E) = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} = \frac{10}{20} = \frac{1}{2}$

$P(M) = \{4, 8, 12, 16, 20\}$

$P(E \cap M) = \frac{5}{20} = \frac{1}{4}$

$P(M|E) = \frac{P(M \cap E)}{P(E)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

9. Two sets of cards with a letter on each card as follows are placed into separated bags.

Bag 1:	**I**	**L**	**J**	**A**	**U**	
Bag 2:	**L**	**R**	**H**	**E**	**C**	**A**

Sara randomly picked one card from each bag. Find the probability that:

a) She picked the letter 'J' and 'R'.

b) Both letters are 'L'

c) Both letters are vowels.

10. I travel to work by route A or route B. The probability that I choose routed A is $\frac{1}{4}$. The probability that I am late for work if I go via route A is $\frac{2}{3}$ and the corresponding if I go via route B is $\frac{1}{3}$.

a) What is the probability that I am late for work on Monday?

b) Given that I am late for work, what is the probability that I went via route B?

11. A coin is biased so that the probability that it lands showing heads is $\frac{2}{3}$. The coin is tossed three times. Find the probability that

a) no heads are obtained

b) more heads than tails are obtained.

12. The table shows the probability distribution of the random variable T, find the value of c.

t	1	2	3	4	5
$P(T=t)$	c	2c	2c	2c	c

13. Find the mean and standard deviation which have the following probability distribution.

x	0	1	2	3	4
$P(X=x)$	0.15	0.25	0.30	0.05	0.25

Answer:

1. a) $\frac{1}{3}$ b) 1 c) $\frac{2}{3}$

2. a) $\frac{3}{8}$ b) $\frac{5}{8}$ c) $\frac{3}{4}$ d) $\frac{3}{4}$
   e) 1

3. a) $\frac{1}{2}$ b) $\frac{1}{2}$ c) $\frac{5}{6}$

4. $\frac{1}{3}$

5. $\frac{1}{6}$

6. $\frac{1}{9}$

7. a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{1}{6}$

8. $\frac{1}{2}$

9. a) $\frac{1}{30}$ b) $\frac{1}{30}$ c) $\frac{1}{5}$

10. a) $\frac{5}{12}$ b) $\frac{3}{5}$

11. a) $\frac{1}{27}$ b) $\frac{20}{27}$

12. $\frac{1}{8}$

13. mean = 2 standard deviation = 1.378

10. I travel to work by route A or route B. The probability that I choose routed A is $\frac{1}{4}$. The probability that I am late for work if I go via route A is $\frac{2}{3}$ and the corresponding if I go via route B is $\frac{1}{3}$.

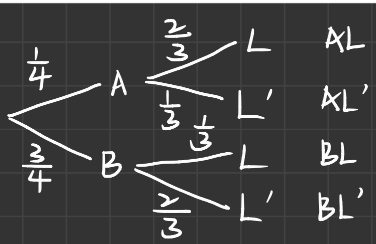
a) What is the probability that I am late for work on Monday?

Conditional probability

b) Given that I am late for work, what is the probability that I went via route ~~B~~ A?

first event

second event



a) $P(AL) + P(BL)$

$= \left(\frac{1}{4} \times \frac{2}{3}\right) + \left(\frac{3}{4} \times \frac{1}{3}\right)$

$= \frac{5}{12}$

b) $P(A|L) = \dfrac{P(A \cap L)}{P(L)}$

$= \dfrac{\frac{1}{4}\left(\frac{2}{3}\right)}{\frac{5}{12}}$

$= \frac{2}{5}$

12. The table shows the probability distribution of the random variable T, find the value of c.

t	1	2	3	4	5
$P(T=t)$	c	2c	2c	2c	c

13. Find the mean and standard deviation which have the following probability distribution

$c + 2c + 2c + 2c + c = 1$

$8c = 1$

$c = \frac{1}{8}$