Chapter 2: Basic Structures

Learning Objectives

After studying this topic, you should be able to:

1. Describe about the set theory
2. Distinguish all the set operation.
3. State the sequence and calculate the summation
4. Understand about the matrices
5. Apply some operations in matrices

2.1 Sets

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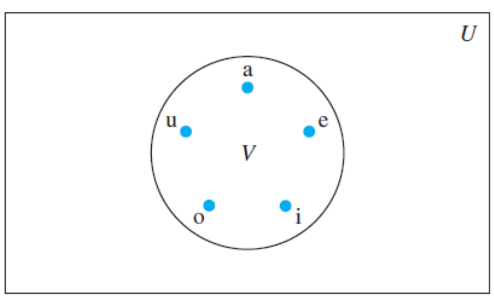
- It is denoted as: -
  - $a \in A$: is an element of the set A.
  - $a \notin A$: is not an element of the set A.

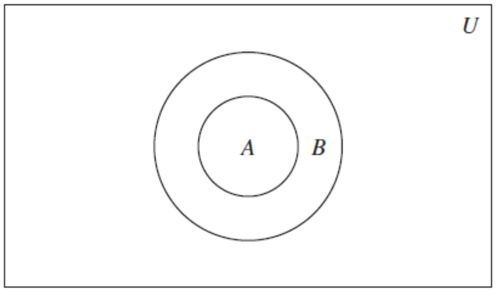
Set Theory	
N	The set of natural numbers {0, 1, 2, … }
Z	The set of integers { …, -2, -1, 0, 1, 2, …}
Z^{+}	The set of positive integers {1, 2, 3, … }
Q	The set of rational numbers $\{\, p/q \mid p \in Z, q \in Z \text{ and } q \neq 0 \,\}$
R	The set of real numbers
R^{+}	The set of positive real numbers
C	The set of complex numbers

- Sets can be represented graphically using Venn Diagrams. In Venn Diagrams, the Universal Sets U, which contains all the objects under consideration, is represented by a rectangle.



Venn Diagram

- Let A and B be the sets. The set A is a subset of B if and only if every element of A also an element of B. It is denoted as $A \subseteq B$: A is a subset of the set B.



A is a subset of the set B

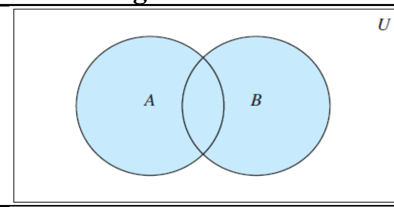
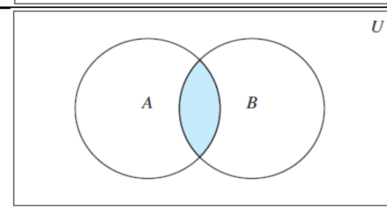
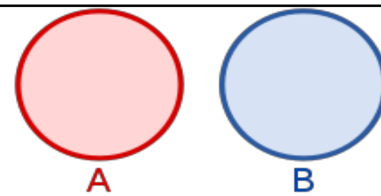
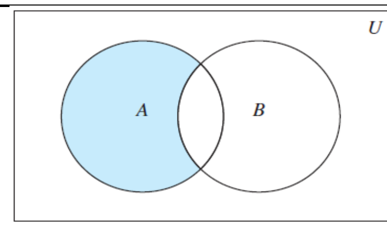
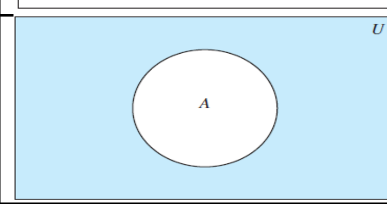
Example:

1. The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10.

2. The set of rational number is a subset of the set of real numbers.

2.2 Set Operation

Let A and B be the sets.

Set Operations	Definition	Venn Diagram
Union	$A \cup B$	
Intersection	$A \cap B$	
Disjoint		
Difference	$A - B$ / $A \setminus B$	
Complement	A' / $\bar{A}$	

2.3 Sequences and summations

SEQUENCES.

- A sequence is a function from a subset of the set of integers.
- We use the notation a_n to denote the image of the integer n.
- We call a_n a term of the sequence.
- There are two types of sequence, which are:
  - Geometric progression – a sequence of the form $a, ar, ar^2, ..., ar^n$ where the initial term a and the common ratio r are real numbers.
  - Arithmetic progression – a sequence of the form $a, a+d, a+2d, ...a+nd$ where the initial term a and the common difference d.

Example:

$1, -1, 1, -1, 1, -1, \ldots$
$b_n = (-1)^n$
$b_0 = (-1)^0 = 1$
$b_1 = (-1)^1 = -1$
$b_2 = (-1)^2 = 1$
$b_3 = (-1)^3 = -1$
$b_4 = (-1)^4 = 1$
$b_5 = (-1)^5 = -1$

1. The sequence $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot \left(\frac{1}{3}\right)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1; 2 and 5; and 6 and 1/3, respectively, if we start n = 0. Find the sequences for n = 0 until n = 5.

2. The sequence $\{s_n\}$ with $s_n = -1 + 4n$, $\{t_n\}$ and $t_n = 7 - 3n$, are arithmetic progressions with initial term and common difference equal to -1 and 4, and 7 and -3, respectively, if we start n = 0. Find the sequences for n = 0 until n = 5.

SUMMATION. $\sum_{n=1 \to \text{initial value}}^{n=5 \to \text{last value}}$ sigma notation

- We consider the addition of the terms of a sequence. Hence, we introduce summation notation.
- Let $a_m, a_{m+1}, ..., a_n$ be the terms, we express the sum of this terms in the notation

$$\sum_{j=m}^{n} a_j, \sum\nolimits_{j=m}^{n} a_j, \quad or \quad \sum\nolimits_{m \le j \le n} a_j$$

- Double summations arise in many contexts (as in the analysis of nested loops in computer programs.

$$\sum_{i=m}^{n} \sum_{j=r}^{s} ij$$

- Summation notation also is used to add all values of the function, or terms of an indexed set, where the index of summation runs over all values in a set.

$$\sum_{s \in S} f(s)$$

- Some formulae are formulated and useful in summation.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Some Useful Summation Formulae

Example:

1. Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = \frac{1}{j}$ for $j = 1, 2, 3, \ldots$

$\sum_{j=1}^{100} \frac{1}{j}$

2. What is the value of $\sum_{j=1}^{5} j^2$?
3. What is the value of $\sum_{s \in \{0,2,4\}} s$?
4. Find $\sum_{k=50}^{100} k^2$.

2. $\sum_{j=1}^{5} j^2$

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 55$

3. $\sum_{s \in \{0,2,4\}} s$

$s = 0 + 2 + 4$

$= 6$

4. $\sum_{k=50}^{100} k^2$

$= \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$

$= \frac{100(100+1)(200+1)}{6} - \frac{49(49+1)(48+1)}{6}$

$= 297925$

2.4 Cardinality of sets

- The cardinality of a set states the number of elements a set contains. The cardinality is expressed as |A| where A is the given set.

Example:

Given set A = {1, 2, 3}. State the cardinality of this set.

|A| = 3. The cardinality of A is 3.

2.5 Matrices

column

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow$ row

row x column

2 x 2

square matrix : row = column

- Matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an m x n matrix.
- A matrix with the same number of rows as columns is called square matrix.
- Two matrices are equal if they have the same number of rows and the same number of columns.
- Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, the ith row of A is the 1 x n matrix $\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$, the jth column of A is the n x 1 matrix $\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ a_{mj} \end{bmatrix}$. The $(i, j)^{th}$ element or entry of A is the element a_{ij}.
- Element: each value in matrix.
- Dimension: A matrix that has the same number of rows and columns.
- Square matrix: Number of rows by number of columns of a matrix.

Example:

Find the dimension of each matrix.

a. $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 3 & -2 \end{bmatrix}$ 3 x 2 ✓

b. $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ 4 x 1

c. $C = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 5 & 4 \\ 2 & 1 & 3 \end{bmatrix}$ 3 x 3

Matrix operation

A × B

= 4×2 . 2×3

= 4×3

1. Matrix addition and subtraction.
   - Only matrices of same dimension can be added and subtracted.

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

B × A

= 2×3 . 4×2

B can't multiply with A because the no. of columns in B is not equal to the no. of row in A.

Find the value of addition and subtraction matrix A and B.

2. Matrix multiplication.
   - Multiply rows times columns.
   - Only can multiply if the number of columns in 1st matrix is equal to number of rows in the 2nd matrix.

1. Find the product of this matrices : $\begin{bmatrix} -3 & 2 & 5 \\ 7 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} -8 & 2 \\ 1 & 5 \\ 0 & -3 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, does AB = BA?

$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ and $BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$. $\therefore$ $AB \neq BA$ because matrix multiplication is not commutative.

4×2 $\qquad\qquad$ 2×3

1. $A = \begin{bmatrix} 7 & 3 \\ 2 & 5 \\ 6 & 8 \\ 9 & 0 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 7 & 4 & 9 \\ 8 & 1 & 5 \end{bmatrix}$

4×3

$A \times B$

$$= \begin{bmatrix} 49+24 & 28+3 & 63+15 \\ 14+40 & 8+5 & 18+25 \\ 42+64 & 24+8 & 54+40 \\ 63+0 & 36+0 & 81+0 \end{bmatrix}$$

$$= \begin{bmatrix} 73 & 31 & 78 \\ 54 & 13 & 43 \\ 106 & 32 & 94 \\ 63 & 36 & 81 \end{bmatrix}$$

Tutorial Questions.

2.1 Sets.

1. List the members of these sets.
   a) $\{x|x \text{ is a real number such that } x^2 = 1\}$
   b) $\{x|x \text{ is a positive integer less than } 12\}$
   c) $\{x|x \text{ is the square of an integer and } x < 100\}$
   d) $\{x|x \text{ is an integer such that } x^2 = 2\}$

2. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

   a) the set of airline flights from New York to New Delhi, the set of nonstop airline flight from New York to New Delhi.
   b) the set of people who speak English, the set of people who speak Chinese
   c) the set of flying squirrels, the set of living creatures that can fly
   d) the set of people who speak English, the set of people who speak English with an Australian accent
   e) the set of fruits, the set of citrus fruits
   f) the set of students studying discrete mathematics, the set of students studying data structures

3. Suppose that $A = \{2,4,6\}$, $B = \{2,6\}$, $C = \{4,6\}$ and $D = \{4,6,8\}$. Determine which of these sets are subsets of which other of these sets.

4. Use a Venn Diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all month of the year.

5. Use a Venn Diagram to illustrate the relationship $A \subset B$ and $B \subset C$.

2.2 Set Operations

1. Let A be the set of students who live within one mile of school and let B be the students who walk to classes. Describe the students in each of these sets.

   a) $A \cap B$ b) $A \cup B$
   c) $A - B$ d) $B - A$

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

   a) the set of sophomores taking discrete mathematics in your school.
   b) the set of sophomores at your school who are not taking discrete mathematics.
   c) the set of students at your school who either are sophomores or are taking discrete mathematics
   d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1,2,3,4,5\}$ and $B = \{0,3,6\}$. Find

a) $A \cap B$ b) $A \cup B$
c) $A - B$ d) $B - A$

4. Let $A = \{0,2,4,6,8,10\}$, $B = \{0,1,2,3,4,5,6\}$ and $C = \{4,5,6,7,8,9,10\}$. Find
a) $A \cap B \cap C$ b) $A \cup B \cup C$
c) $(A \cup B) \cap C$ d) $(A \cap B) \cup C$

5. Draw the Venn diagrams for each of these combinations of the sets $A, B\ and\ C$.

a) $A \cap (B - C)$ b) $(A \cap B) \cup (A \cap C)$
c) $(A \cap \bar{B}) \cup (A \cap \bar{C})$

2.3 Sequence and summation

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.
a) a_0 b) a_1 c) a_4 d) a_5

2. What are the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$, where a_n equals
a) $2^n + 1$? b) $(n+1)^{n+1}$?
c) $[n/2]$? d) $[n/2] + [n/2]$?

3. List the first 10 terms of each of these sequences.
a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term
b) the sequence that lists each positive integer three times, in increasing order
c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice
d) the sequence whose nth term is $n! - 2^n$
e) the sequence that begins with 3, where each succeeding term is twice the preceding term
f) the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of two preceding terms

4. What are the values of these sums?
a) $\sum_{k=1}^{5}(k+1)$ b) $\sum_{j=0}^{4}(-2)^j$
c) $\sum_{i=1}^{10} 3i$ d) $\sum_{j=0}^{8}(2^{j+1} - 2^j)$

5. What is the value of each of these sums of terms of a geometric progression?
a) $\sum_{j=0}^{8} 3 \cdot 2^j$ b) $\sum_{j=1}^{8} 2^j$
c) $\sum_{j=2}^{8}(-3)^j$ d) $\sum_{j=0}^{8} 2 \cdot (-3)^j$

6. Compute each of these double sums.
a) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$ b) $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$
c) $\sum_{i=0}^{2}\sum_{j=1}^{3} ij$

2.4 Cardinality of sets

1. Given set A is an odd integer between 0 until 20. What is the cardinality of set A?

2. Set A consists of vowel alphabet in word CAMBRIDGE. State the cardinality of set A.

3. State the cardinality of set of letters in the ENGLISH alphabet.

4. What is the cardinality of elements T in word CRESCENDO.

2.5 Matrix

1. Let A= $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$.
   a) What size is A?
   b) What is the third column of A?
   c) What is the second row of A?
   d) What is the element of A in the (3, 2)*th* position?

2. Find A + B, where

   a) $A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}, B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$

   b) $A = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix}, B = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$

3. Find AB if

   a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

   b) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

c) $A=\begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}, B=\begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$

4. Find the product AB, where

a) $A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}, B=\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b) $A=\begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}, B=\begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$

c) $A=\begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix}, B=\begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$