Assignment6

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(Dated: Mar.27th, 2020)

Collaborated with Hongyou Lin on 6.14 and Q3, with and Jonnathan Frost on Q3.

1. 6.14 QM SCATTERING, PARTICLE/QUANTUM PROBLEM

a)

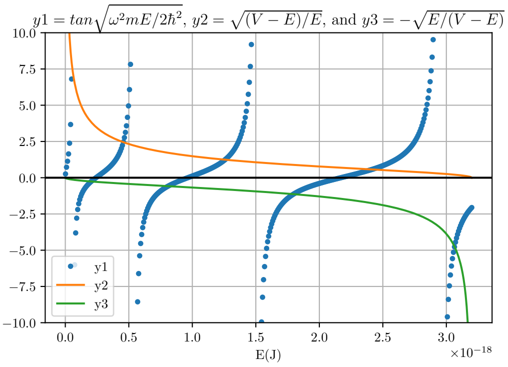

FIG. 1: y_1, y_2 and y_3 in one plot

The first sixth intersections between y_1 and y_2, or between y_1 and y_3 (excluding the intersection on all very sharp vertical lines) are the solutions to energies of the first sixth energy levels.

Based on FIG1, the first-sixth solutions are: 0J, 0.2×10^{-18}J, 0.4×10^{-18}J, 0.8×10^{-18}J, 1.3×10^{-19}J, 1.7×10^{-18}J, and 2.4×10^{-18}J.

b) Rewrite the equations

$$\begin{cases} tan(\sqrt{\frac{\omega^2 m_e E}{2\hbar^2}}) = -\sqrt{E/(V-E)} \\ \quad \text{for the odd numbered states} \\ tan(\sqrt{\frac{\omega^2 m_e E}{2\hbar^2}}) = \sqrt{(V-E)/E} \\ \quad \text{for the even numbered states} \end{cases} \tag{1}$$

as:

$$f(E) = \begin{cases} tan(\sqrt{\frac{\omega^2 m_e E}{2\hbar^2}}) + \sqrt{E/(V-E)} \\ \quad \text{for the odd numbered states} \\ tan(\sqrt{\frac{\omega^2 m_e E}{2\hbar^2}}) - \sqrt{(V-E)/E} \\ \quad \text{for the even numbered states} \end{cases} \tag{2}$$

The solutions to the energy are the value of E such that $f(E) = 0$ for each energy state n.

Binary Search method requires a range that brackets the solution. So I first found the bracket ranges for all sixth solutions. To find the ranges, I define a python function $brakcet(f)$ which goes through E values from 0 to 20eV($20\times 1.6 \times 10^{-19}$J), and records the neighboring pairs of values $[E_0, E_1]$where $f(E)$ changes sign, $f(E_0)f(E_1) < 0$ in a list. To exclude all the intersections in FIG1 at which $y1 = tan(\sqrt{\omega^2 E/2\hbar^2}) \to \infty$, I added an additional constraint: $|f(E_0) - f(E_1)| < 0.001$.

With a list of bracketing ranges, in a loop that iterates all the ranges, I followed Step 6(a)-(f) listed in the running lectures notes to find the roots by bisection method (P54-55). I found the first sixth energy levels are: 1.272, 2.855, 5.057, 7.860, 11.229, 15.088 eV and the ground state 0eV.

To use Newton-Raphson method to solve the equations $y_1 = y_2$ and $y_1 = y_3$, we need to define the derivative of $f(E)$ analytically first:

$$\text{set } C = \frac{\omega^2 m_e}{2\hbar^2} \tag{3}$$

$$f'(E) = \begin{cases} \frac{\sqrt{C}Sec^2(\sqrt{CE})}{2\sqrt{E}} + \frac{\frac{1}{V-E} + \frac{E}{(V-E)^2}}{2\sqrt{E/(V-E)}} \\ \quad \text{for the odd numbered states} \\ \frac{\sqrt{C}Sec^2(\sqrt{CE})}{2\sqrt{E}} + \frac{-\frac{V-E}{E^2} - \frac{1}{E}}{2\sqrt{(V-E)/E}} \\ \quad \text{for the even numbered states} \end{cases} \tag{4}$$

To find all sixth roots numerically, I used Newton-Raphson method based on all sixth guesses one by one. For each solution, the approximate guess $x_i, i = 0$ is the initial guess. And the next better guess x_1 is defined as:

$$x_{i+1} = x_i - f(x_i)/f'(x_i) \tag{5}$$

The solutions are: 1.253, 2.817, 5.000, 7.813, 11.053, 15.000 eV and the ground state 0eV.

Compare to the results from Mathematica, Newtown method gives more accurate results.

2. 6.16 THE LAGRANGE POINT

According to the definition of the Lagrange point, at the Lagrange point at a distance of r from the center of the Earth:

$$\sum F = F_{in} - F_{out} = F_{cent} \tag{6}$$

$$G\frac{M_{Earth} m_{satellite}}{r^2} - G\frac{m m_{satellite}}{(R-r)^2} = m_{satellite}\frac{v^2}{r} \tag{7}$$

$$\tag{8}$$

, where $v = \omega r$.
Therefore, the distance r from the center of the Earth

and the L_1 point satisfies:

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r \tag{9}$$

b) To use Newton-Raphson method to solve equation 9, rewrite the equation as a function of r, $f(r)$:

$$f(r) = \frac{GM}{r^2} - \frac{Gm}{(R-r)^2} - \omega^2 r \tag{10}$$

$$f'(r) = -2\frac{GM}{r^3} - 2\frac{Gm}{(R-r)^3} - \omega^2 \tag{11}$$

The solution to Equation 9 is the value of r_0 at which $f(r_0) = 0$. I plotted $f(r)$ to make an approximate guess to the solution r_0:

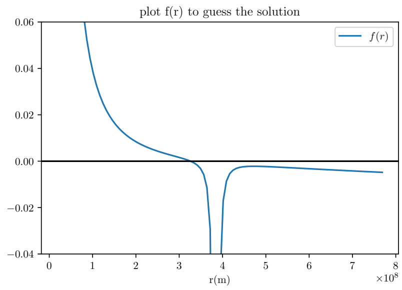

FIG. 2: $f(r) = \frac{GM}{r^2} - \frac{Gm}{(R-r)^2} - \omega^2 r$ as a function of r from $r = 0$ to $1.5R$

The intersection of $f(r)$ and $y = 0$ is the solution. Based on FIG2, the solution in the range of $0 \le r \le R$ is around $r = 3.3 \times 10^{-18}$m.

Now I can use the same Newton-Raphson algorithm defined in Question 6.14 based on this single guess to solve the equation. The solution is $r = 3.2605 \times 10^8$m.

3. INTERPOLATION/CMB

a) See FIG 3, FIG 4 and 5

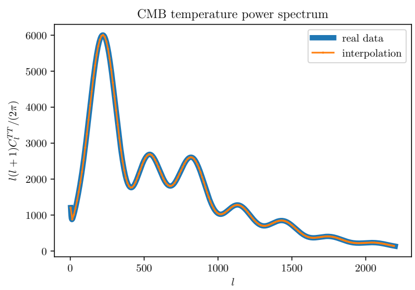

FIG. 3: The blue line shows the real CMB power spectrum, and the orange dots are the result of spline interpolation from **SCIPY** library using using 200 points from the data file.

The actual fractional error from the data and the result of interpolation is much lower then $3/l$ and decreases as l increase, so the spine line does well enough.

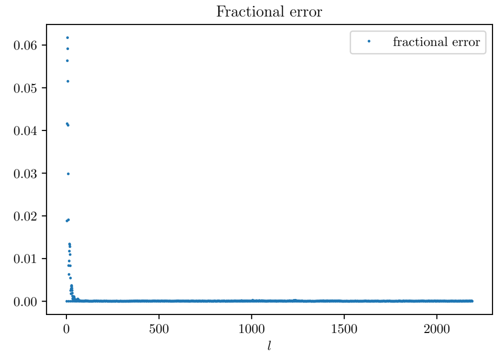

FIG. 4: Fractional error of the interpolation using **SCIPY** library.

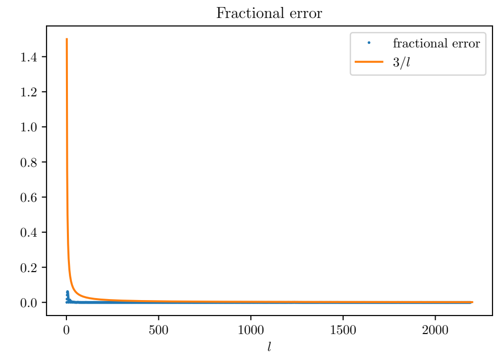

FIG. 5: Fractional error of the interpolation compared with $3/l$.

b) I used central difference method to find $\frac{dC_l^{TT}}{dl}$. The value given by the data file is $l(l+1)C_l^{TT}/2\pi$. Therefore, I first computed C_l^{TT} at each l using python *lambda* function. Then I computed $\frac{dC_l^{TT}}{dl}$ by:

$$\frac{dC_l^{TT}}{dl} = \frac{C_{l+1}^{TT} - C_{l-1}^{TT}}{2} \tag{12}$$

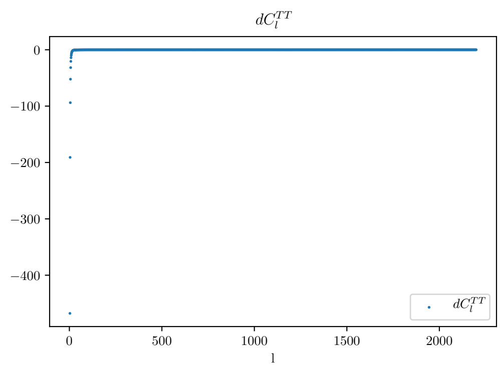

FIG. 6: $\frac{dC_l^{TT}}{dl}$

The theoretical expression for spline errors is defined as:

$$\epsilon = \frac{h^4}{4!} max_{[l_{min}, l_{max}]} f''''(l) \quad (13)$$

where the step size used for interpolation h, is 11, and $f(l) = l(l+1)C_l^{TT}/2\pi$.

The CMB data gives a list of $f(l)$ value from $l = 2$ to 2200. To get the fourth derivative, $f''''(l)$, I calculated $f'(l)$ by:

$$f'(l) = \frac{f(l-1) - f(l+1)}{2} \quad (14)$$

From this array of $f'(l)$ values, using Eq 14, I computed $f''(l)$, then $f'''(l)$ and finally $f''''(l)$. The theoretical absolute and fractional error is then evaluated an plotted:

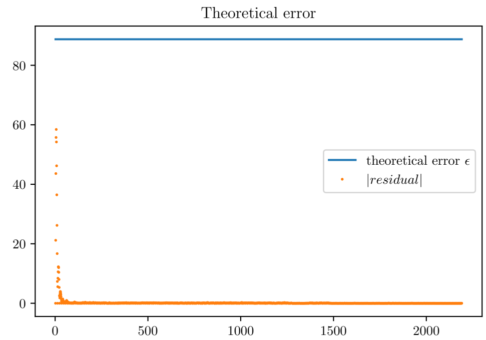

FIG. 7: The maximum theoretical value spline error

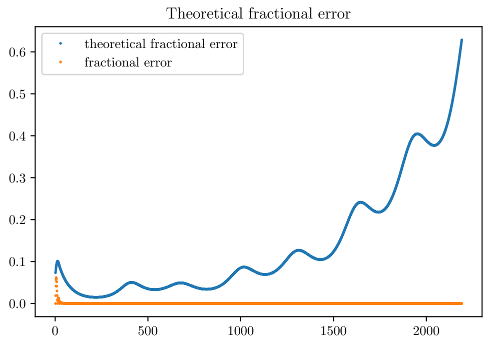

FIG. 8: Theoretical values for fractional spline error

From Figure 7 and 8,the fractional error in part (a) is close to the theoretical expression for spline errors when l is small but much lower for bigger value of l.

c) To compute $\int_1^\infty \frac{ldl}{2\pi} C_l^{TT}$ requires integration by substitution with $z = \frac{l}{l+1}$:

$$\int_1^\infty \frac{ldl}{2\pi} C_l^{TT} = \int_{z_{min}}^1 f(l(z))(1+l)^2 dz, \quad (15)$$

Let's find the integrand $f_z = f(l(z))(1+l)^2$ first:

$$f(l(z)) = \frac{l}{2\pi} C_l^{TT} = \frac{z}{(1-z)2\pi} C_l^{TT} \quad (16)$$

$$(1+l)^2 = \frac{1}{(1-z)^2} \quad (17)$$

$$f_Z = \frac{z}{(1-z)^3 2\pi} C_l^{TT} \quad (18)$$

The data given is $y = l(l+1)C_l^{TT}/2\pi$. Thus we can obtain $f_z(z)$ by:

$$l(l+1) = \frac{z}{(1-z)^2} \quad (19)$$

$$f_z(z) = \frac{y}{l(l+1)} \frac{z}{(1-z^3)} \quad (20)$$

$$= \frac{y(1-z)^2}{z} \frac{z}{(1-z)^3} \quad (21)$$

$$= \frac{y}{1-z} \quad (22)$$

The lower bound of the integration with respect to z is $z_{min} = \frac{l}{1+l}$ when $l = 1$, which is $z_{min} = \frac{1}{2}$. l_{max} of the data is 2200. $z_{max} = \frac{l_max}{1+l_max} = 0.9995$. In this case, the upper bound of the integration is $z_{max} = 0.9995$. After generating a list of $f_z = \frac{y}{1-z}$ values based on the CMB data, I can use trapezoid integration method to find $\int_{0.5}^{z_{max}} \frac{y}{1-z} dz$.

The result is 12844.

4. SURVEY QUESTIONS

The homework took me 15 hours. I learned to implement binary search and Newton method to solve nonlinear equations. I also learned to use **SCIPY** library for interpolation. The potential well question was the most interesting one. This problem set is a little bit too long.