Chaotic Systems

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This work analyzes two chaotic system Damped Driven Pendulum system(DDP) and Hénon-Heiles potential by finding the numerical solution, and looking at the Phase space in 2D and 3D and Poincaré section. Bifurcation behavior of the DDP is also discussed.

1. CHAOTIC SYSTEMS

Chaos is one of the discoveries made in the last few decades. Chaotic system are described by nonlinear equations of motions. These systems do not have analytical solutions. The behavior is very sensitive to initial conditions. Use Newton's second law, the motion of the systems can be written as differential equations. To analyze chaotic systems, we solve the differential equations numerically and utilize many different visualizations to show the behavior chaos.

In this work, I investigate two chaotic systems, the damped driven pendulum system(DDP) and Hénon-Heiles system(HH). Both systems can be mathematically described by an integrable ODE systems with an addition of perturbation term. The Canonical Perturbation Theory suggests when the nonlinear perturbation to the system is small, the system is solvable by adding a "small" term to the mathematical description of the exactly solvable problem; however, when the perturbation strength is big, the system is unpredictable, that is chaotic [1]. I hope the discussion in this work captures the big idea of this theory. For both systems, I used similar procedures and tools to analyze their behavior under different initial conditions. We will see that the pattern is solving the ODE systems with a given initial conditions, looking at its phase plot (or State-Space Orbit [2]) and Poincaré sections. For DDP, I also analyzed the bifurcation of the system over a range of the perturbation term.

2. DAMPED DRIVEN PENDULUM SYSTEM

Damped Driven Pendulum system (DDP) is one of the simplest set-ups of chaotic system. It is not much different from the classical linear pendulum system under small angle approximation, but it provides enough interesting information to analyze chaos extensively. In DDP, the pendulum with mass m and length L experiences a damping force proportional to its tangential velocity $-bv = -bL\dot{\theta}$. The pendulum is also driven by an external sinusoidal force with a driven frequency of ω_d in the tangential direction of the pendulum motion, $F = F_0 cos\omega_d t$ [3],[2]. Recall that the second order differential equation for the classical linear pendulum system is:

$$-mgLsin\theta = mL^2\ddot{\theta} \tag{1}$$

Based on Eq1, including addition of the damping force and the additional driving force, we have the equation of motion of the DDP system:

$$-mgLsin\theta - bL^2\dot{\theta} + LF_0 cos\omega_d t = mL^2\ddot{\theta} \tag{2}$$

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{L}sin\theta = \frac{F_0}{mL}cos\omega_d t \tag{3}$$

Now, I would like to describe the damping and the external perturbation in terms their ratios to the momentum of inertia mL^2 [2]. Following the notation used by Taylor, rewrite $\frac{b}{m}$ as 2β and $\frac{F_0}{mL}$ as $\gamma\omega_0^2$ where

$$\gamma = \frac{F_0}{mL\omega_0^2} = \frac{F_0}{mL}\frac{L}{g} = \frac{F_0}{mg}. \tag{4}$$

$$\text{and } \omega_0 = \sqrt{\frac{L}{g}} \tag{5}$$

Thus, we have the final form of the second order DEQ that describes the motion of damped driven pendulum[2]:

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2 sin\theta = \gamma\omega_0^2 cos\omega_d t \tag{6}$$

2.1. Algorithm for Solving the ODE

In this work, I used *odeint()* from python **Scipy.integrate** package to solve the differential equation Eq6.

This second derivative equation must first be written into a system of coupled first order differential equation:

$$\frac{d\theta}{dt} = \omega \tag{7}$$

$$\frac{d\omega}{dt} = -{\omega_0}^2 sin(\theta) - 2\beta\omega + \gamma{\omega_0}^2\ \omega_d t \tag{8}$$

After defining a specific initial condition, $(\theta(0), \omega(0))$, the program uses odeint() to find the numerical solution of θ and $\dot{\theta}$ from $t = 0$ to its 1000th period.

2.2. Visualization

With the numerical solutions, we can 2D and 3D phase plots, and the Poincaré section.

Phase plot, namely, is plotting the phases of the system on 2 axis in 2D and 3 axies in 3D. For DDP, its 2D phase plot plots $\theta(t)$ against $\dot{\theta}(t)$, and its 3D phase plot adds the "drive phase", $\omega_d t$[3]. 2D and 3D phase plots are generated using the numerical solution from odeint() in each case.

Poincaré section for DDP is to show $(\theta(t), \dot{\theta}(t)$ in 2D phase space at one-cycle intervals. It is as if we measure θ and $\dot{\theta}$ only once per cycle of the driving force and plot the point on plane of the 2D phase space [3] [2]. With that being said if the system is a simple harmonic system, the Poincaré section is just a dot $(\theta, \dot{\theta})$ because the system comes back to the same value of θ and every period. With numerical solution of θ and $\dot{\theta}$ over time, a Poicaré section is easily produced by selecting $(\theta, \dot{\theta})$ once per cycle after the transient state of the system is gone.

2.3. From nonlinear system to chaos

To follow the canonical perturbation theory, I investigated how a nonlinear system diverges from predictable periodicity as the perturbation term, the value of γ increases. For all of the simulations below, $\omega_d = 2\pi$, $\omega_0 = 1.5\pi$ and the damping coefficient is $\beta = \frac{\pi}{4}$[2]. Therefore, the period of driving force is 1 second. For all the cases below, odeint() is used to find the numerical solution of θ and $\dot{\theta}$ from $t = 0$ to 1000s, separated by 0.01s. Different γ values used in this section follows the value used in Ref[2].

Let us start with small perturbation to the system $\gamma = 0.2$. The canonical perturbation theory suggests that the system will evolve like a integrable linear system over time. The initial condition at $t = 0$ is $\theta_0 = 0$ and $\dot{\theta}_0 = 0$. As shown in Fig1, at first the system is in the transient state, after that, *theta* oscillates at a constant amplitude with a period of 1s. The 2D phase plot, Fig2 shows that over time, when the system is stable, $(\theta(t), \dot{\theta}(t))$ stays on the oval orbit over time. The 3D phase plot suggests the attractor of the system under small driving force is helix[3].

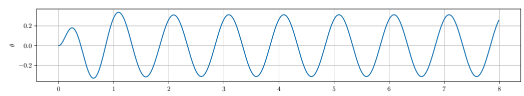
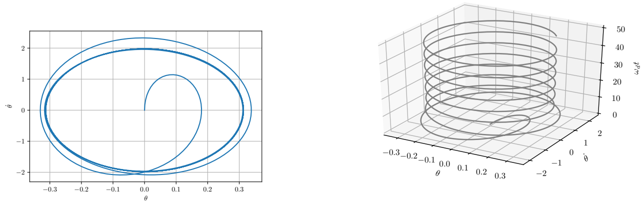


FIG. 1: The numerical solution of $\theta(t)$ when $\gamma = 0.2$ and the initial condition is $\theta_0 = 0$, $\dot{\theta}_0 - 0$

Each cycle is 1 second, with 100 data points. The Poincaré section is plotted by selecting every 100 data points after the transient sate is completed. Fig3 shows the Poincaré section. The two blue dots represent the value when the system is in transient state. Over time, $(\theta(t), \dot{\theta}(t))$ always has the same value of the black dot once every cycle (100 data points apart). Therefore, the periodicity of the system is single. This is the same situation of an undamped and undriven linear pendulum.

Now, let us turn up the driving coefficient slightly higher, but still under 1, $\gamma = 0.9$. The initial condition is still $\theta_0 = 0$ and $\dot{\theta}_0 = 0$. As shown in the numerical solution (Fig4), the pendulum oscillates with the period of the driving force after the initial transient state is gone. The function $\theta(t)$ looks like a pure sine function; however, a closer look at *theta*(t) in Fig5 shows that the numerical solution is flatter than a pure sine solution in the case of a linear system[2].



(a) $\theta - \dot{\theta}$ (b) $\theta - \dot{\theta} - \omega_d t$

FIG. 2: 2D and 3D phase plot when $\gamma = 0.2$

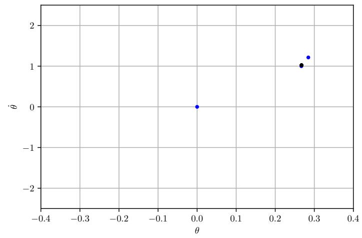


FIG. 3: Poincaré section when $\gamma = 0.2$

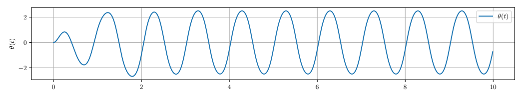


FIG. 4: The numerical solution of $\theta(t)$ when $\gamma = 0.9$ and the initial condition is $\theta_0 = 0$, $\dot{\theta}_0 - 0$

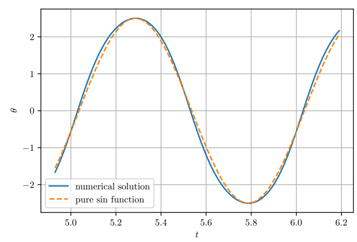
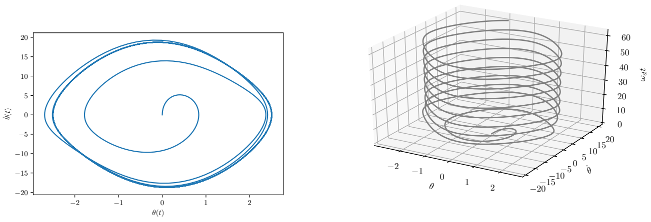


FIG. 5: An enlargement of $\theta(t)$ from $t = 5$ to 6s. The solid blue curve is the actual motion of the DDP, and the dashed orange line show a pure sine function with same period and phase.

As shown in the 2D and 3D phase plot in Fig6, although the state orbit is not circular, the system still oscillate in a single periodic orbit, and the attractor of motion is a helix. Therefore, the system is in the linear regime.

The non linearity of the system gets more obvious as the driving coefficient or the perturbation term, γ, is increased over 1. Now let us increase γ to 1.07 while



(a) $\theta - \dot{\theta}$ (b) $\theta - \dot{\theta} - \omega_d t$

FIG. 6: 2D and 3D phase plot when $\gamma = 0.9$

keeping ω_d, ω_0 and β the same. The initial condition is $\theta_0 = \dot{\theta}_0 = 0$. The numerical solution from $t = 30$ to 40 is shown in Fig**??**. The orange line in Fig**??** shows that not all troughs have the same height. The motion has two periods, and the amplitude of the two periods are not the same. The 2D phase plot and Poincaré section in Fig8 help us see the effect of period doubling better. The system circulate around the two oval-shaped state orbits. And it alternates between the two points on the Poincaré section from one period to the subsequent one. We say the periodicity of the system is 2[2].

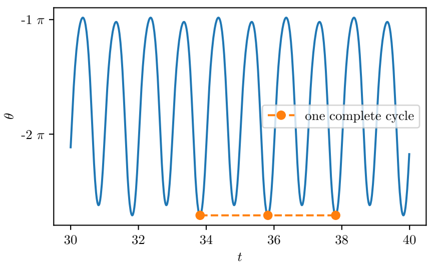
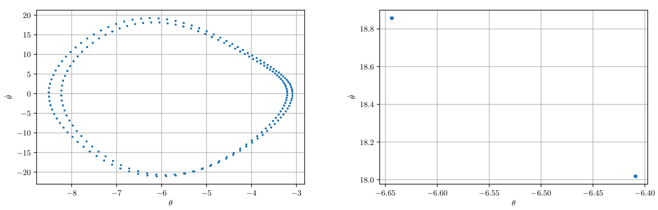


FIG. 7: $theta(t)$ from $t = 30$ to 40 when $\gamma = 1.073$. The horizontal orange line is drawn from the bottom the lowest trough



(a) $\theta - \dot{\theta}$ (b) Poincaré section of $\theta - \dot{\theta}$

FIG. 8: 2D phase plot and Poincaré section when $\gamma = 1.073$. The numerical solution in the transient state is not included.

When $\gamma = 1.081$ while keeping all other parameters and the initial conditions unchanged, the period of the system doubles again. As shown in Fig9, the system repeats fully once every four oscillations. The system alternates the minima between four different values. The state orbit has four loops and the value of $(\theta, \dot{\theta})$ alternates between the four points from period to period on the Poincaré section. Therefore, the periodic attractor is 4 in when $\gamma = 1.081$[2].

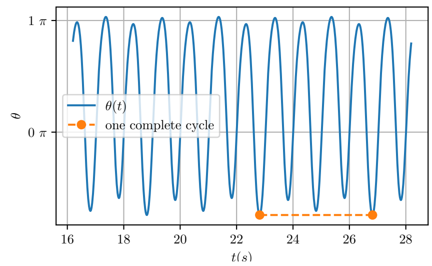
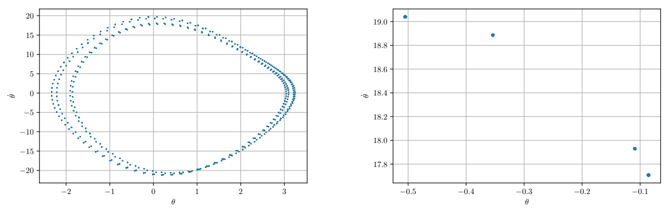


FIG. 9: $theta(t)$ from $t = 16$ to 28 when $\gamma = 1.081$. The horizontal orange line is drawn from the bottom the lowest trough



(a) $\theta - \dot{\theta}$ (b) Poincaré section of $\theta - \dot{\theta}-$

FIG. 10: 2D and 3D phase plot for $\gamma = 1.081$. The numerical solution in the transient state is not included.

As the driving strength becomes even larger, the motion of the system suddenly changes its character. When $\gamma = 1.40$, the system undergoes a rolling motion. Fig11 shows the phase orbit for 6 periods since $t = 0$. The change in θ is approximately 2π in the one direction each period. Therefore, the pendulum is rolling over time.

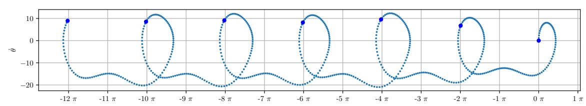
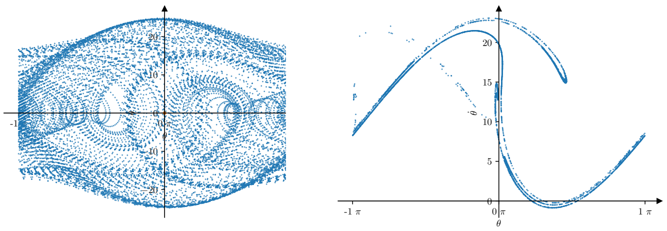


FIG. 11: The phase plot for 6 periods since t=0 when $\gamma = 1.4$. The 7 highlighted blue points are $(\theta, \dot{\theta})$ when $t = 0, 1, 2, 3, 4, 5$ and 6.

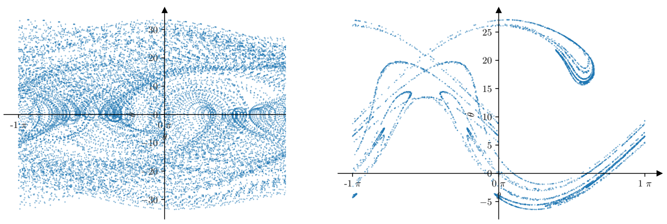
With that being said, as γ is increased larger than 1.4, the system all undergoes a rolling motion. To analyze the phase plot and Poincaré section when the system has large γ value, I corrected θ so that it is within the range from $-\pi$ to π. The phase plot and Poincaré section when $\gamma = 1.5$ is shown in Fig12.

If instead the damping coefficient is reduced to $\frac{\omega_0}{8}$ and $\gamma = 1.5$, the rolling motion is more erratic[2]. The phase plot and Poincaré section in Fig13 are more complex and tangled. Therefore, the motion of the system is more chaotic under these parameters. However, even though the system is chaotic, the Poincaré section still has a lot of white space. The value of $(\theta, \dot{\theta})$ once every period is not arbitrary. It follows an elaborated orbit. Therefore, the motion of chaotic DDP system is not completely random.



(a) (b)

FIG. 12: a) 2D phase plot when $\gamma = 1.5$; b) the corresponding Poincaré section. For both plots, x-axis and y-axis are θ and $\dot{\theta}$ respectively. The numerical solution in the transient state is not included.



(a) (b)

FIG. 13: a) 2D phase plot; b) The Poincaré section when $\gamma = 1.5$ and $\beta = \frac{\omega_0}{8}$. For both plots, x-axis and y-axis are θ and $\dot{\theta}$ respectively. The numerical solution in the transient state is not included.

2.4. Sensitivity to Initial Conditions

A chaotic system also means the behavior of the system is unpredictable even if all the parameters are the same (in DDP, the parameters are the damping and driving coefficients, the natural and driven frequencies), but the initial conditions are just slightly different[2]. Let us examine a comparison when $gamma$ is small (the system is still in the linear regime) and when $gamma$ is big (larger than 1,the system is nonlinear) will. Define $theta_1(t)$ and $\theta_2(t)$ be the solutions to the motion of the system under two slightly different initial conditions while all the parameters are the same, and $\Delta\theta(t) = |theta_2(t) - theta_1(t)|$. Fig 14 shows the logarithm scale of $\Delta\theta(t)$ when $\gamma = 0.1$. The initial conditions of the two cases are $\theta_0 = 0$ and 0.1 (rad), while $\omega_0 = 0$ for both. For a linear oscillator, $ln|\delta\theta(t)|$ follows a relationship with t in this form [2]:

$$ln|\Delta\theta(t)| = lnD - \beta t + ln|cos(\omega_l t - \delta) \tag{9}$$

Indeed, in Fig14 the successive maxima of $ln|\Delta\theta(t)|$ decrease linearly over time, that is $|\Delta\theta(t)|$ decreases exponentially. Taylor points out an important practical consequence of this result. When we try to predict the future motion of the DDP with an initial condition, but our experimental measurement may differ from the true initial conditions slightly, maybe due to uncertainty in measurement, the initial error will reduce quickly to zero, so we can achieve prediction with an prescribed accuracy by ascertaining the same accuracy for the initial condition [2]. That is the linear system is insensitive to initial conditions.

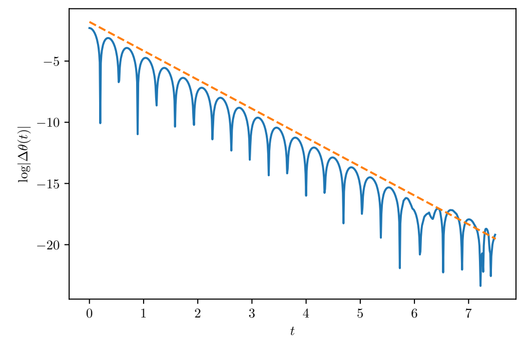


FIG. 14: Logarithmic plot of $\Delta\theta(t)$ from t=0 to 8 for two identical DDPs with a weak driving strength $\gamma = 0.1$, and the initial positions are differed by 0.1 rad. The orange dashed line is a linear line approximated graphically to fits the successive maxima of $\Delta\theta(t)$.

However, this is not the case when our DDP system is in the nonlinear regime. Fig 14 shows the logarithm scale of $\Delta\theta(t)$ when $\gamma = 1.015$. The initial positions are $\theta_0 = 0$ and 10^{-5} (rad), while $\omega_0 = 0$ for both. Even though, the difference in the initial positions are much smaller, overtime $ln|\Delta\theta(t)|$ increases almost linearly. Noticeably, after around $t = 11s$ (11 cycles), $ln|\Delta\theta(t)|$ can hit 1 occasionally. This means that $\Delta\theta(t)$ increases exponentially and will be as great as about 10. Therefore, a very small uncertainty in initial condition, as tiny as 10^-5rad will lead to a significant error in just a few cycles. As Taylor has suggested, if $\theta(t)$ has an uncertainty of $\pm\pi$, we have no idea where the pendulum is[2]. In other words, our DDP system is extremely sensitive to initial conditions when it is in chaos.

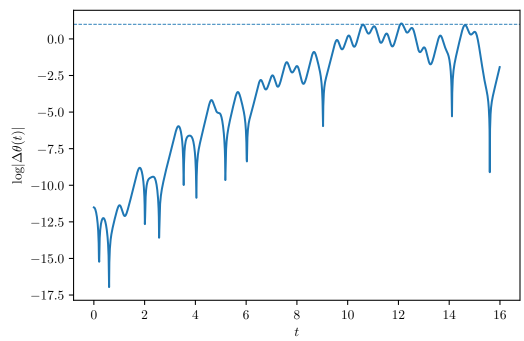


FIG. 15: Logarithmic plot of $\Delta\theta(t)$ from t=0 to 8 for two identical DDPs with a weak driving strength $\gamma = 0.1$, and the initial positions are differed by 10^{-4} rad. The dashed horizontal line is y=1.

2.5. Bifurcation

In the previous discussion, we have briefly seen the effect of period doubling as γ increases. In this part, I will show the whole story in one plot. The goal is to construct a plot that shows how the motion, specifically periodicity and chaos of the DDP changes as γ varies. The idea is to check periodicity by plotting the value of $\theta(t)$ against γ, where $\theta(t)$ is its value at an arbitrary time after the transient state is gone, t_x. And for the same γ_i, repeatedly plot the values at many numbers of driving periods from t_0. In our set up, $\omega_d = 2\pi$, so the driving period is $1s$. That is, for each γ_i, plotting $(\gamma_i, \theta(t))$ every 1s after t_x. And repeat this process for a range of γ value. This plot is called a **Bifurcation diagram** [2].

How could this diagram shows periodicity or chaos? It is very similar to the idea of making a Poincaré section. If the system has exactly one period at $gamma_i$, the value of $\theta(t_x)$ will repeat every second after t_x, so there is only one dot for $(\gamma_i, \theta(t))$. If the periodicity is doubled at γ_i, we will observe two different $\theta(t)$ values periodically, and we should see two dots at $x = \gamma_i$. If periodicity doubles twice, the value of $\theta(t)$ repeats at four different values, and so on. If the system is chaotic at γ_i, the value of $\theta(t)$ will not repeat periodically. Thus, there will be many points on the vertical line $x = \gamma_i$.

Following the procedures in Ref[2], in my simulation, the parameters used are the same as before $\omega_d = 2\pi$, $_0 = 3\pi$, $\beta = \frac{pi}{4}$. The initial condition is $\theta(0) = \frac{\pi}{2}$ and $\dot{\theta}(0) = 0$. γ is varied in a range of value from 1.060 to 1.087 separated by 0.01:

$$\gamma = 1.06, 1.07, ..., 1.086, 1.087 \tag{10}$$

After 500 periods, the transient state of the system must has already gone. The program picks up 100 $\theta(t)$ values from the numerical solution of θ to Eq6 for each γ at:

$$t = 500, 600, 700...1500 \tag{11}$$

We have 271 number of γ values,and for each value, the program loops through 100 periods. That is 271×100 operations in total. This program with nested loops is fairly time-consuming, but I was not able to find a way to make it faster. The result is shown in Fig16

As shown in Fig16, the periodicity is singular when the driving perturbation term, γ, is small. As γ increases, the function diverges into two branches, then to four, then to eight. Beyond 8 periodicity, the system enters chaotic case as many $\theta(t)$ values lie on the same vertical line of $x = \gamma_i$. The program also computes the periodicity by counting the number of different $\theta(t)$ value for each γ_i value, then picks up the critical value of γ where the period diverges into 2, 4, 8 and 16.

With the threshold values (or bifurcation points) above, we can find the **Feigenbaum number** δ which is defined as:

$$(\gamma_{n+1} - \gamma_n) \approx \frac{1}{\delta}(\gamma_n - \gamma_{n-1}) \tag{12}$$

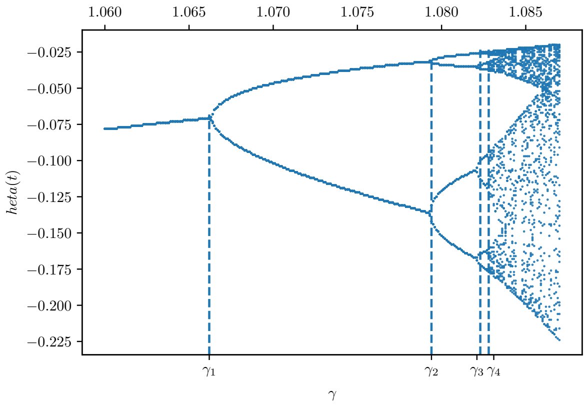


FIG. 16: Bifurcation diagram for the DDP with $\omega_d = 2\pi$, $_0 = 3\pi$, $\beta = \frac{\pi}{4}$, and $\theta_0 = -\frac{\pi}{2}, \dot{\theta}_0 = 0$ and the varying drive strength from $\gamma = 1.060$ to 1.087.

n	periodicity	γ_n	interval
1	$1 \to 2$	1.0661	
2	$2 \to 4$	1.0793	0.0133
3	$4 \to 8$	1.0823	0.003
4	$8 \to 16$	1.0828	0.0006

TABLE I: The first four threshold values γ_n at which the period bifurcates from 1 to 2, 2 to 4, 4 to 8 and 8 to 16 for a DDP system with $\omega_d = 2\pi$, $\omega_0 = 3\pi$, $\beta = \frac{\pi}{4}$, and $\theta_0 = -\frac{\pi}{2}, \dot{\theta}_0 = 0$

where γ_n is the bifurcation values of γ[2]. The physicist Mitchell Feigenbaum showed that period-doubling is universal for many different nonlinear systems and the intervals between the threshold values of the control parameter is related by a constant value δ [2]. Based on the result in TableI, two δ values using 4 threshold values are computed, and their average is the Feigenbaum number δ of this DDP system:

$$\delta = 4.7167 \tag{13}$$

The discrepancy between my result and the result in Ref[2] is 1% [2]. This is very likely due to rounding of $\theta(t)$ in counting how many different values there are for 100 cycles to determine the periodicity.

3. THE HÉNON-HEILES SYSTEM

The Hénon-Heiles system is a Hamiltonian system displaying chaos. The system is introduced by Hénon and Heiles in 1964 to find the third integral galactic motion in a time-independent and axisymmetric potential [4]. They chose to study this potential:

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + \lambda(x^2 y - \frac{y^3}{3}) \tag{14}$$

This potential is symmetric as shown in Fig17.

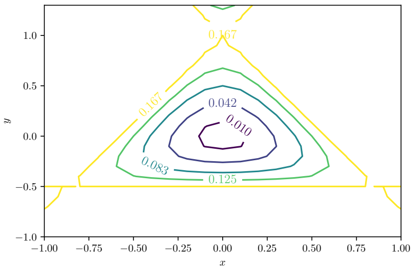


FIG. 17: Contour plot of the Hénon–Heiles potential

The Hénon-Helies Hamiltonian is:

$$H(x, \dot{x}, y, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y) \tag{15}$$

$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2) + \lambda(x^2 y - \frac{y^3}{3}) \tag{16}$$

Applying the Hamiltonian equations:

$$\ddot{y} = \frac{\partial H}{p_x} \tag{17}$$

$$\ddot{x} = -\frac{\partial H}{p_y} \tag{18}$$

the Hénon-Heiles system is described by:

$$\ddot{x} = -\frac{\partial V}{\partial x} = -\lambda(x + 2xy) \tag{19}$$

$$\ddot{y} = -\frac{\partial V}{\partial y} = -y - \lambda(x^2 - y^2) \tag{20}$$

Ozaki and Kurosak suggest there is no essential difference in the result for arbitrary choices of the value of λ [5]. Hénon and Heiles and most other literature I have seen use 1 for λ [1][6][7]. In this work, I also use $\lambda = 1$

The initial condition of the system is defined by a set of $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ values. Most literature ([4][5][6][7]) analyze the motion of the system by grouping the system based the total energy:

$$E = V(x, y) + \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = H \tag{21}$$

Set $x_0 = 0$, use $y_0, \dot{y}_0$ and E as parameters, we can find the corresponding $\dot{x}_0$ value for the initial conditions by solving Eq21 after plugging in x_0, y_0, $\dot{y}_0$ and E. With four initial values of $x_0, \dot{x}_0, y_0$ and $\dot{y}_0$, the numerical solution to the motion of the system can be found. The reason for looking at total energy of the system is that the actual deviation from the integrable limit depends on the energy level considered: if $E << 1$ the nonlinear deviations from the linear system limit are very small, while they become stronger and stronger as E increases[7]. Therefore, no matter what the exact initial condition specified by $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ is, systems in the same energy level will have similar properties on periodicity and chaos.

In this section, I will study the motion of this chaotic system by solving the Heénon-Helies DEQ system in Eq20, looking at its 3D orbits and Poincaré sections.

3.1. Algorithm

Similar to the algorithm for solving the DDP system, I rewrite the second order derivatives in Eq20 into a system of coupled first order derivative equations, then use *()* from **Scipy.integrate** to find the solutions under specific initial conditions. The system of coupled first order DEQs is:

$$\begin{cases} \frac{dx}{dt} = \dot{x} \\ \frac{dy}{dt} = \dot{y} \\ \frac{d\dot{x}}{dt} = -x - 2xy \\ \frac{d\dot{y}}{dt} = -y - x^2 + y^2 \end{cases} \tag{22}$$

As stated before, I am interested in solving the system under initial conditions defined by E. To solve Eq22, we need to initial values of $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ that corresponds to the system's total energy E. For all the different situations analyzed in this work, for a value of E, I set two of the four variables $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ zero, and a random value for another variable. Then I used fsolve() from *Scipy.optimize* package on this function, $f = V(x_0, y_0) + \frac{1}{2}(\dot{x}_0^2 + \dot{y}_0^2) - E$, to find the initial value of the fourth variable.

3.2. Visualization

For a system under a specified initial condition $(x_0, \dot{x}_0, y_0, \dot{y}_0)$, with the numerical solution using the algorithm above, we can plot the 3D state orbit, $x - y - \dot{y}$.

The Poincaré section of the Hénon-Heiles system is obtained by finding where the trajectory in 3D state orbit above goes through the x-plane and have positive momentum in x direction because Poincaré sections only are interested in one of the intersections[1]. The Poincaré section is in $y - \dot{y}$ space. It is as if looking at the plane $x = 0$ in the 3D state orbit and only keeping where $\dot{x} > 0$. The algorithm for producing a Poincaré section is quite straightforward. The program iterates through every element in the numerical solution of x_n, and if $x_n < 0$ and $x_{n+1} > 0$, it plots the average of y and $\dot{y}$ at n and $n+1$[1].

3.3. Results

In this section, we will look at the 3D state orbit and Poincaré sections of the system with different values of total energy side by side. Following the procedures laid out in Ref[6],[4] and [1], I considered the energy level starting from much smaller than 1 and gradually increases its value.

Let's start with $E = \frac{1}{12}$. The initial condition is $x_0 = 0, y_0 = 0$ and $\dot{y}_0 = 0$. $\dot{x}_0 = 0.408$ satisfies $E = \frac{1}{12}$. Fig 18 and 20c show the 3D state orbit and the Poincaré section of the system with the initial condition above. When $E = \frac{1}{12}$, the trajectory of the system in 3D is confined into a torus. The energy is still high enough that each turn of the orbits move further away from the previous point in Fig19[1], opposite to what we see the Poincaré section for DDP in linear regime has only one dot (Fig3). Still, the points lie exactly on a curve, meaning the state orbit is periodic. Therefore, the system is in the non-chaotic region. In fact, most literature ([7],[6],[4] and [1]) plot a collection of Poincaré sections of systems with same level of total energy in one plot. Fig 20a shows the collection of Poincaré seciونts of 8 systems all with $E = \frac{1}{12}$ but different values for $(x_0, \dot{x}_0, y_0, \dot{y}_0)$. The trajectory of each system is marked with a different color. All 8 trajectories are still bounded to a closed loop of its own, no intersecting with any other trajectories.

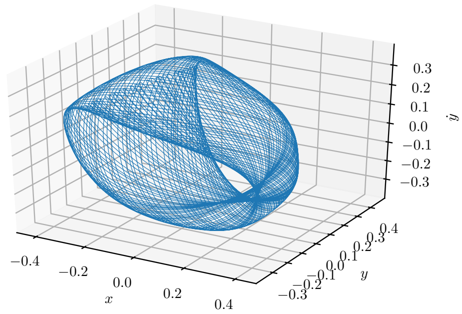


FIG. 18: 3D state orbit when $E = \frac{1}{12}$ and $(x_0, \dot{x}_0, y_0, \dot{y}_0) = (0, 0.408, 0, 0)$

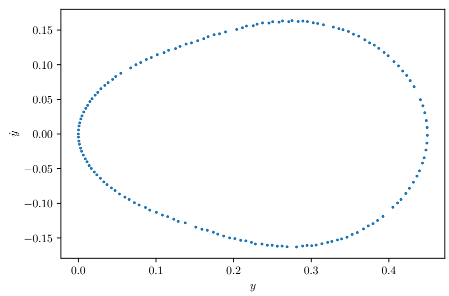
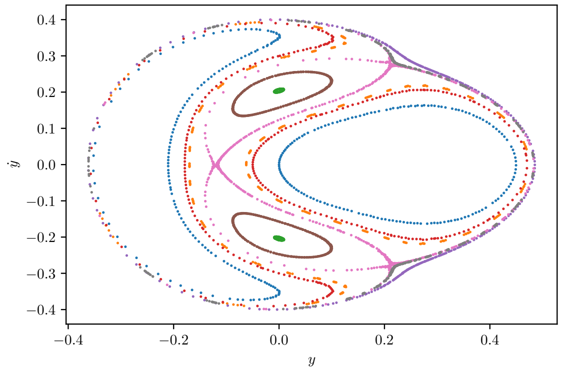
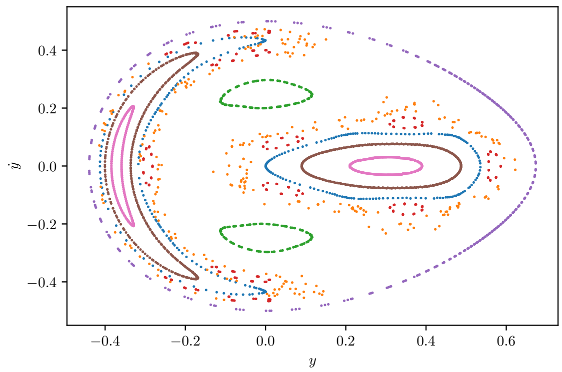


FIG. 19: Poincaré section when $E = \frac{1}{12}$ and $(x_0, \dot{x}_0, y_0, \dot{y}_0) = (0, 0.408, 0, 0)$

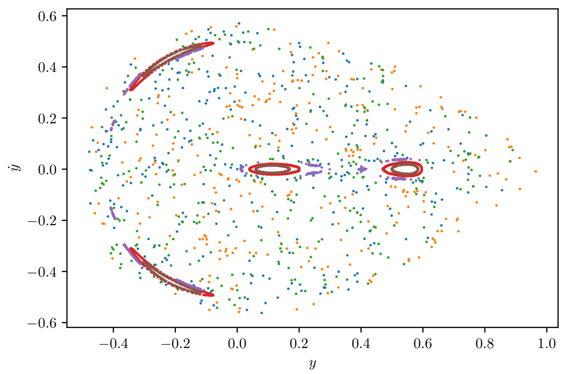
Let us then look at the system with a larger nonlinear perturbation when $E = \frac{1}{8}$ and $E = \frac{1}{6}$. The results of Poincaré sections are shown below. As energy level increases to $\frac{1}{8}$, we see some bounded orbits but also isolated points in the region between the closed orbit. Isolated points with the same color belong to the same trajectory. They seem to be distribute randomly in the area left free between the bounded orbits, but all the trajectories are confined in a nutshell-shaped region on the $y - \dot{y}$ space[1][4]. Besides, there seems to have some dividing line within the nutshell where there is a sudden change in behavior, isolated random points or bounded orbit[4]. For a higher energy level, $E = \frac{1}{6}$, there is another drastic change. Although there are few regular bounded loops,



(a) Collections of Poincaré sections for 8 systems with $E = \frac{1}{12}$



(b) Collections of Poincaré sections for 7 systems with $E = \frac{1}{8}$

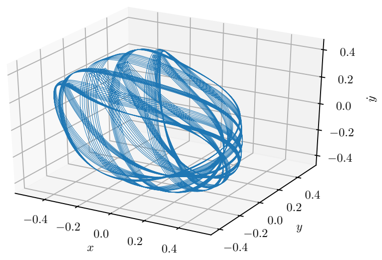


(c) Collections of Poincaré sections for 6 systems with $E = \frac{1}{6}$

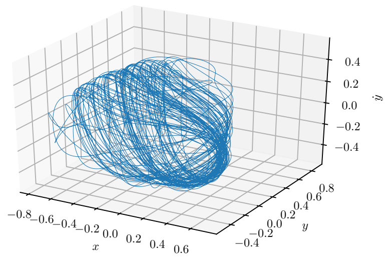
FIG. 20: Poincaré sections created for three different energies: a)$E = \frac{1}{12}$, b)$E = \frac{1}{8}$, c)$E = \frac{1}{6}$.

the isolated points belonged to one trajectories occupy most of the space in the nutshell-shaped region. The energy is so high enough that each turn of the orbits jump from one part to another of the domain without regularity. Therefore, the system is mostly chaotic at high energy level.

Fig21 shows the 3D state orbits of the two energy level



(a) $E = \frac{1}{8}$



(b) $E = \frac{1}{6}$

FIG. 21: 3D state orbits created for two different energies: a)$E = \frac{1}{8}$, b)$E = \frac{1}{6}$. $x_0 = y_0 + \dot{y}_0 = 0$ for both plot, and different $\dot{x}_0$ for corresponding energy level

above. When $E = \frac{1}{8}$, the state orbit is still confined to a torus, but it is obviously less regular than the system with $E = \frac{1}{12}$. In comparison, the trajectory is no longer confined to a torus when $E = \frac{1}{6}$. It fills out the entire nut-shaped region of the Poincaré section in 3D space[1].

4. CONCLUSION

For DDP system, as the driving strength γ increases over 1, we first saw the periodicity first doubles, doubles again with a higher value of γ, and finally the system is chaotic. Chaotic system is sensitive to initial conditions that even a small differences at $t = 0$ will lead to very large deviations.

For Hénon-Heiles system,when the total energy level is much smaller than 1, the system is nonlinear but the motion is bounded to some regular periodicity. As the total energy gets closer to 1, the system becomes mostly chaotic.

The results of the two systems confirms the canonical perturbation theory. With small nonlinear perturbation, the system can be approximated with the integrable linear system, but with large perturbation term, the motion of the system is unpredictable that we can not solve analytically.

5. ACKNOWLEDGEMENT

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Appendix A: Code and Animation

This work are in two progeams:*ddp.py*(DDP) and *HenonHelies.py*. Both programs take a really long time to run because everything I did for DDP or Hénon-Heiles system is in one file. Simply run these python file in Terminal to see the results I included in this work.

I do have an animation for the DDP system to show the time progression of the 2D phase plot and Poincaré section. However, I have not found it add new information besides the plots in this work, except it helped me practice using **vpython** package. The animation uses the numerical solutions in Section 2 which have been saved as ".txt" files whose filenames correspond to the driving strength. To see the animation, run **animate_plot.py** in Terminal and type in the filename when the program asks for an input.