Assignment3

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1. 5.3 COMPUTATION OF ERROR FUNCTION

I used Simpson's method to evaluate the integration. To find out what the suitable number of slices N is, I evaluated the estimated error using Eq 5.24:

$$\epsilon = \frac{1}{90} h^4 [f'''(a) - f'''(b)] \tag{1}$$

$$where\ a = 0, b = 3\ and\ h = \frac{b-a}{N} \tag{2}$$

See 1 for the result of fractional error as a function of number of bins N plotted in logarithmic scale. The precision limit of my computer is at 10^{-15}. From fig. 1, after about N = 5000, the fractional error falls down below the 10^{-14} and the trend becomes flatter as N increases. Therefore I chose 5000 bins for computing the integrals.

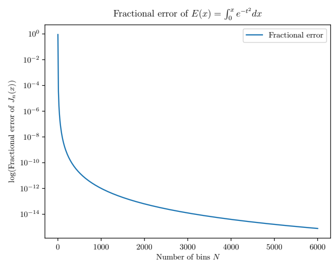

FIG. 1: fractional error as a function of number of bins N

Given the definition of **Erf** function, $E(x) = \int_0^x e^{-t^2} dx$ can be rewritten as $\frac{\sqrt{\pi}}{2} erf(x)$. Fig 2 shows the two methods using Simpson's integration method with 1000 bins and using **math.Erf(x)** from python library give very close results.

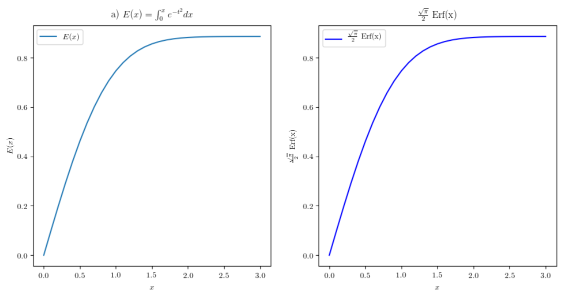

FIG. 2: Comparison of the result for $E(x) = \int_0^x e^{-t^2} dx$ using Simpson's integration method and **math.Erf(x)**

2. 5.4 DIFFRACTION LIMIT OF A TELESCOPE

a) I first defined the integrand as a function of m and x, $f(m, x)$. The output of the function is a function $f(\theta) = cos(m\theta - xsin\theta)$ for a specific pair of m and x input values. Then I used Simpson's integration method with 1000 slices to evaluate the integration form of Bessel function:

$$J_m(m, x) = \frac{1}{\pi} \int_0^\pi cos(m\theta - x \sin\theta) d\theta. \tag{3}$$

To plot $J_0(x), J_1(x)$ and $J_2(x)$ as a function of X from 0 to 20, I created a list to store the values of $J_m(m, x)$ for each m value at specific x values. I used 201 equally spaced x value from 0 to 20. In another word, I have three lists of 201 J_m values.

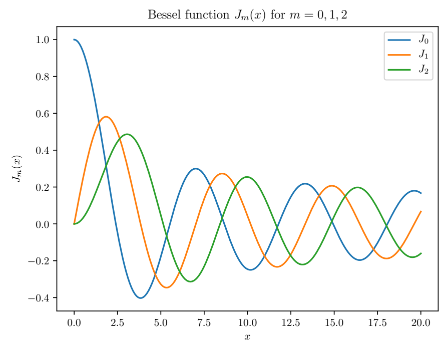

FIG. 3: Bessel function J_0, J_1, J_2 as a function of x from $x = 0$ to $x = 20$

b) To make a 2D density plot, I generated an 2D array of $I(r)$ at each (x, y) value. I created a 101 by 101 matrix to hold all coordinates from $-1\mu m$ to $1\mu m$ along both x and y axis.

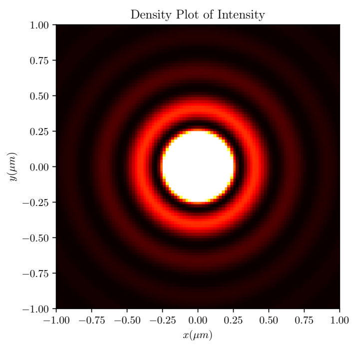

FIG. 4: Caption

c) We can rewrite the recursion definition as:

$$J_n(x) = \frac{2n}{x} J_{n-1}(x) - J_{n-2}(x). \tag{4}$$

I defined a recursion function $J(n)$. The base cases for this recursion are J_0 and J_1. It means if the input of the function n is 0 or 1, the function returns the results evaluated by integration. For other input value of n, the functions calls $J(n-1)$ and $J(n-1)$, and evaluated $J(n)$ in the recursion definition.

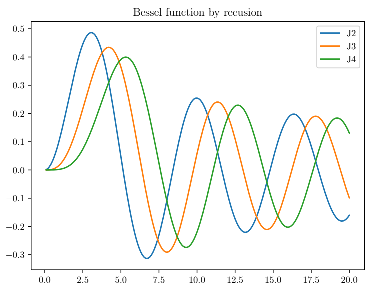

FIG. 5: Bessel function J_n evaluated using recursion definition for $n = 2, 3, 4$.

The error between using integration and recursion methods to find $J_m(x)$ from x=0 to x=20 for J_2, J_3 and J_4 is shown in Fig 6. It is very noisy. Although the error is relatively large for small x value, the error drops down below the precision limit of my computer very soon.

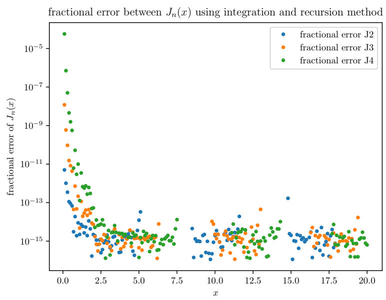

FIG. 6: Fractional error computed from $\frac{J_{integration} - J_{recursion}}{J_{integration}}$ plotted in logarithmic scale.

3. 5.10 ANHARMONIC OSCILLATOR

a) Find the period T: At $t = 0, x = a$ and $v = \frac{dx}{dt} = 0$.And since energy is conserved over time:

$$E = V(a) = \frac{1}{2} m (\frac{dx}{dt})^2 + V(x) \tag{5}$$

We can solve this differential equation. Rearrange the equation first :

$$dt = \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{V(a) - V(X)}} \tag{6}$$

When the particle reaches $x = 0$, it has gone through$\frac{1}{4}T$. Integrate both sides of the function from $x = 0$ to a and from $t = 0$ to $\frac{1}{4}T$:

$$\int_0^{\frac{1}{4}T} dt = \sqrt{\frac{m}{2}} \int_0^a \frac{dx}{\sqrt{V(a) - V(X)}} \tag{7}$$

$$T = \sqrt{8m} \int_0^a \frac{dx}{\sqrt{V(a) - V(X)}} \tag{8}$$

b) I used Simpson's integration with 1000 bins. See 4.

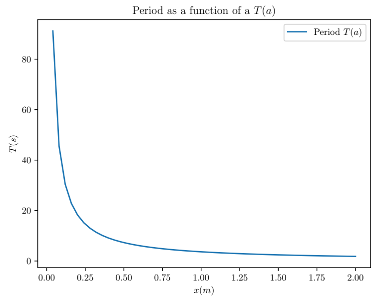

FIG. 7: The period for amplitudes ranging from a = 0 to a = 2 for a particle with mass = 1, in a $V(x) = x^4$ potential well

c) The restoring force in this anharmonic oscillator is nonlinear: $F_{restore} = -\frac{V(x)}{dx} = -4x^3$. As a result, increasing the amplitude increases the acceleration cubically. Plugging $V(x) = x^4$, and evaluate Eq.8 in **Mathematica**, I found:

$$T = \frac{Constant}{a} \tag{9}$$

The period is inverse proportional to the amplitude. P Therefore, as the amplitude increases, the oscillator gets faster. As a approaches 0, the potential well is approximately flat, and the oscillator hardly oscillates for small amplitude. Thus, the period approaches inf as the amplitude goes to zero.

4. SURVEY QUESTION

This homework took 12 hours. I learned to use my own python library, plotting 2D density plot from a 2D array and recursion. I also learned nested *for* loops can take a long time to run, but I was not able to improve the run time this time. I will try to keep this problem in consideration in the future. The most interested problem on is one on diffraction limit of a telescope because it is I found it is important to know the general "flow chart" of the program before coding, and I did some recursion as well. Moreover, it was a relatively long program, I found some tips on the process of debugging: checking variables in certain points in the program helps me narrow down the issues.